The geometry of equating things:

1. Determine all of the values of x which satisfy each of the following equations:

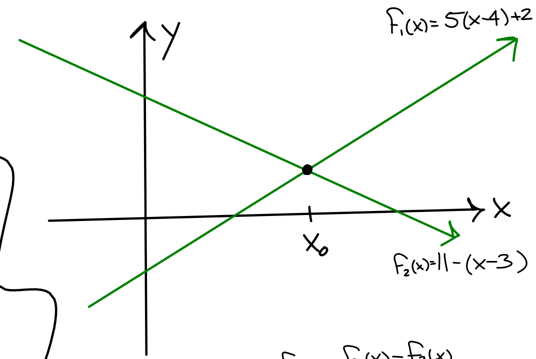
(a) $5(x-4)+2 = 11-(x-3)$

$\Rightarrow 5x-20+2 = 11-x-(-3)$

$\Rightarrow 5x-18+x = -x+14+x$

$\Rightarrow 6x-18+18 = 14+18$

$\Rightarrow \frac{6x}{6} = \frac{32}{6} \Rightarrow x = 32/6 \Rightarrow \boxed{x = 16/3.}$



(b) $4x^2 = 5x-1$

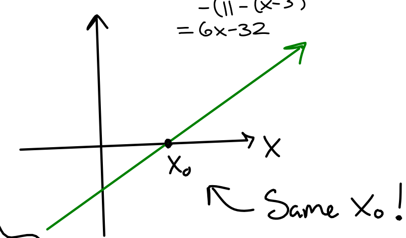
$\Rightarrow 4x^2-(5x-1) = 5x-1-(5x-1)$

$\Rightarrow 4x^2-5x+1 = 0$

$\Rightarrow (4x-1)(x-1) = 0$ (guess & check) $\Rightarrow \begin{cases} 4x-1=0, \text{ or} \\ x-1=0 \end{cases}$

$\boxed{\Rightarrow \begin{cases} x = 1/4, \text{ or} \\ x = 1. \end{cases}}$

$f(x) = f_1(x)-f_2(x)$
$= 5(x-4)+2$
$-(11-(x-3))$
$= 6x-32$



(c) $1 = -2+\sqrt{2x+7}$

$\Rightarrow 1+2 = -2+\sqrt{2x+7}+2$

$\Rightarrow (3)^2 = (\sqrt{2x+7})^2$

$\Rightarrow \frac{9-7}{2} = \frac{2x+7-7}{2} \Rightarrow \boxed{1 = x.}$

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2. Determine the domain of:

(a) $f(x) = \sqrt{x^2-3x-4}+5$

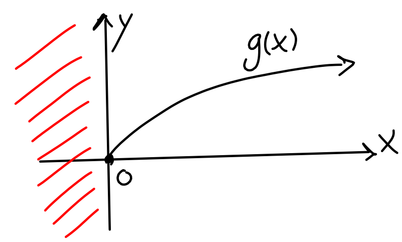
Note $g(x) = \sqrt{x}$ is defined only for $0 \le x < \infty$ (Since eg. $\sqrt{-1}$ is not a real number)

$\Rightarrow \text{dom}(f) = \{x \in \mathbb{R} \mid x^2-3x-4 > 0\}$

and $x^2-3x-4 = 0 \Rightarrow (x-4)(x+1) = 0 \Rightarrow x = 4, -1$

We check: sign of f(x):

$\boxed{\begin{array}{l} \Rightarrow x^2-3x-4 > 0 \text{ iff} \\ x \in (-\infty,-1) \cup (4,\infty). \end{array}}$ = dom(f).



(b) $g(x) = \dfrac{x^2-16}{x^2+9x+20}$

Note $h(x) = \frac{1}{x}$ is defined iff $x \neq 0$.

$\Rightarrow \text{dom}(g) = \{x \in \mathbb{R} \mid x^2+9x+20 \neq 0\}$, and

$x^2+9x+20 = 0 \Rightarrow (x+5)(x+4) = 0 \Rightarrow x = -4, -5$

$\Rightarrow \text{dom}(g) = \{x \in \mathbb{R} \mid x \neq -4, -5\} = \boxed{(-\infty,-5) \cup (-5,-4) \cup (-4,\infty).}$

3. Compute the following limits **with algebraic justification**.
"Bald" answers and "plug & chug" / "guess & check" will receive little or no credit.
If a limit Does Not Exist, mark it **DNE** and explain WHY the limit does not exist.
Calculating the limit using L'Hôpital's Rule will receive **NO CREDIT**.
(If you don't know what that means, don't worry.)

(a) $\lim_{x\to 4} \boxed{\dfrac{2x^2-2x-3}{3x^2-8x+5}}$

define as f(x)

If f(4) exists, we're done!

$f(4) = \dfrac{32-8-3}{48-32+5} = \dfrac{21}{21} \boxed{= 1.}$

Why? Polynomials are cts, so are products of them, and so are quotients at pts where the denominator is nonzero.

(b) $\lim_{x\to -3} \boxed{\dfrac{x^3+2x^2-3x}{3x^2+5x-12}}$

$:= f(x)$

$f(-3) = \dfrac{-27+18+9}{27-15-12}$ "=" $\dfrac{0}{0}$, not good!

But this means $x=-3$ is a factor of both the numerator and the denominator.

So

$$\begin{array}{r|l} & x^2 - x \\ x+3 & x^3+2x^2-3x \\ & x^3+3x^2 \\ \hline & -x^2 \\ & -x^2-3x \\ \hline & 0 \end{array} \Rightarrow x^3+2x^2-3x = (x^2-x)(x+3) \Rightarrow f(x) = \frac{x^2-x}{3x-4} \Rightarrow \boxed{f(-3) = \frac{12}{-13}.}$$

$$\begin{array}{r|l} & 3x - 4 \\ x+3 & 3x^2+5x-12 \\ & 3x^2+9x \\ \hline & -4x-12 \\ & -4x-12 \\ \hline & 0 \end{array} \Rightarrow 3x^2+5x-12 = (3x-4)(x+3)$$

(c) $\lim_{x\to 5} \dfrac{x^2-5x-6}{x^2-2x-15}$

Note that denom $= 0$ at $x=5$, but numerator $\neq 0$. So there's no common factor of $(x-5)$ to divide out this time.

So the limit could be "$\pm\infty$". But note that

$x \in (5,6) \Rightarrow \begin{cases} x^2-5x-6 < 6^2-5\cdot 6-6 = 0 \to \text{numerator is negative} \\ x^2-2x-15 > 5^2-2\cdot 5-15 = 0 \to \text{denom is positive} \end{cases} \Rightarrow \lim_{x\to 5^+} f(x) = -\infty$

$x \in (4,5) \Rightarrow \begin{cases} x^2-5x-6 < 5^2-5\cdot 5-6 < 0 \\ x^2-2x-15 < 5^2-2\cdot 5-15 < 0 \end{cases} \Rightarrow \lim_{x\to 5^-} f(x) = +\infty$

Not equal

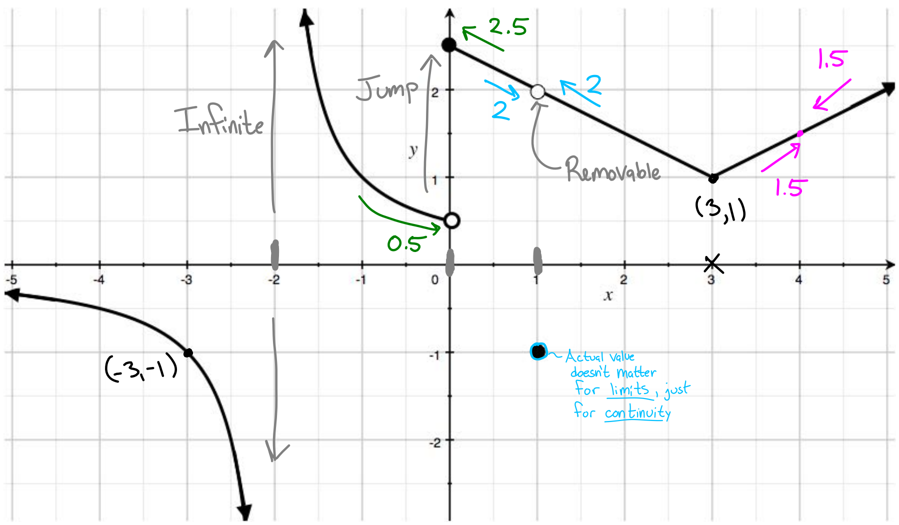
$\boxed{\text{So the limit DNE.}}$

(d) $\lim_{x\to 2} \underbrace{\dfrac{|x-2|}{x^2+3x-10}}_{:= f(x)}$

$f(2) = \dfrac{12-21}{4+12-10} = \dfrac{0}{6}$ $\boxed{= 0.}$

okay because it's not a $\frac{1}{0}$ or $\frac{0}{0}$ situation

4. Consider the graph $y = f(x)$ below.



Compute the following limits. If a limit **D**oes **N**ot **E**xist, explain why.

(a) $\lim_{x\to 0} f(x)$ $\quad \lim_{x\to 0^-} f(x) = 0.5 \neq 2.5 = \lim_{x\to 0^+} f(x)$, so $\boxed{\text{DNE.}}$

(b) $\lim_{x\to 1} f(x)$ $\quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) \boxed{= 2.}$

Note $\lim_{x\to 1} f(x) = 2 \neq f(1) = -1$, so f is not continuous at $x=1$.

(c) $\lim_{x\to 4} f(x)$ $\quad \lim_{x\to 4^+} f(x) = \lim_{x\to 4^-} f(x) \boxed{= 1.5\ .}$

(d) State the **limit definition** of continuity.

$\boxed{f \text{ is cts at } p \text{ iff } \lim_{x\to p} f(x) = f(p).}$

(e) Find and classify each discontinuity of $f(x)$ as **removable**, **jump**, or **infinite**. Your answer should include a limit justification.

See graph + above limits

- $\lim_{x\to c}$ exists but $\neq f(c)$: Removable
- $\lim_{x\to c^+} \neq \lim_{x\to c^-}$: Jump
- $\lim_{x\to c^+} = \pm\infty$ or $\lim_{x\to c^-} = \pm\infty$: Inf.

(f) Find the average rate of change of $f(x)$ on $[-3, 3]$

Avg rate of change $= \frac{\Delta y}{\Delta x} = \frac{1-(-1)}{3-(-3)} = \frac{2}{6} \boxed{= \frac{1}{3}.}$

(g) Where is $f(x)$ not differentiable?
Cite how you know it is not differentiable.

Theorem: $\cdot$ f differentiable at p $\Rightarrow$ f cts at p
$\cdot$ f not cts at p $\Rightarrow$ f not differentiable at p.

We know f is not cts at $x = -2, 0, 1$. Can f fail to be differentiable elsewhere? Answer: yes, $x=3$, because the slope of the tangent line is negative at 3^- and positive at 3^+, so...

$f'(3) = \lim_{x\to 3} \frac{f(x)-f(3)}{x-3}$ DNE.

5. Use the *Intermediate Value Theorem* (if applicable) to show that there is a solution on $[0, 1]$ to

$$x^5 + 2x = 1$$

Be sure to justify that you are allowed to use IVT by checking the necessary hypotheses.

Let $f(x) = x^5 + 2x - 1$. Then f is cts since it's a polynomial, and

$$\begin{cases} f(0) = -1 < 0 \\ f(1) = 2 > 0 \end{cases} \Rightarrow f(x) = 0 \text{ for some point } p \text{ such that } 0 < p < 1.$$

$\Rightarrow f(p) = 0 \Rightarrow p^5 - 2p - 1 = 0 \Rightarrow p^5 - 2p = 1 \Rightarrow p$ is a soln to the original equation.

6. Derivative by the Definition

(a) State the limit definition of the derivative.

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p} \quad \text{or} \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$(a-b)(a+b) = a^2 - b^2$

(b) <u>**Using the LIMIT DEFINITION**</u> of the derivative, show that

$$\text{if } f(x) = \sqrt{4 - 3x^2}, \text{ then } f'(x) = \frac{-3x}{\sqrt{4 - 3x^2}}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h}\left[\left(\sqrt{4-3(x+h)^2} - \sqrt{4-3x^2}\right)\overbrace{\frac{\left(\sqrt{4-3(x+h)^2} + \sqrt{4-3x^2}\right)}{\left(\sqrt{4-3(x+h)^2} + \sqrt{4-3x^2}\right)}}^{=1}\right] = \lim_{h \to 0} \frac{1}{h}\left[\frac{4-3(x+h)^2 - (4-3x^2)}{\sqrt{4-3(x+h)^2} + \sqrt{4-3x^2}}\right] = \lim_{h \to 0} \frac{1}{h}\left[\frac{4-3(x^2+2xh+h^2) - 4 + 3x^2}{\sqrt{4-3(x+h)^2} + \sqrt{4-3x^2}}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\cancel{4} \cancel{-3x^2} - 6xh - 3h^2 \cancel{-4} \cancel{+3x^2}}{\sqrt{4-3(x+h)^2} + \sqrt{4-3x^2}}\right] = \lim_{h \to 0} \cancel{\frac{1}{h}}\left[\frac{-6x\cancel{h} - 3h^{\cancel{2}\,1}}{\sqrt{4-3(x+h)^2} + \sqrt{4-3x^2}}\right] = \lim_{h \to 0}\left[\frac{-6x \cancel{-3h}}{\sqrt{4-3(x+\cancel{h})^2} + \sqrt{4-3x^2}}\right] = \frac{-6x}{2\sqrt{4-3x^2}} \boxed{= \frac{-3x}{\sqrt{4-3x^2}}.}$$

ZERO CREDIT will be given for using the power rule or any other differentiation "shortcut".

(c) Find an equation of the line tangent to $f(x)$ at $x = 1$

$m :=$ Slope of tangent line at $p = f'(p)$.

$$p = 1 \Rightarrow m = f'(1) = \left.\frac{-3}{\sqrt{4-x^2}}\right|_{x=1} = \frac{-3}{\sqrt{4-1}} = \frac{-3}{\sqrt{3}} \boxed{= -\sqrt{3}.}$$