JUST DO IT: A COLLECTION OF HARTSHORNE PROBLEMS

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Contents

1. I: Varieties

1.1. I.1: Affine Varieties.

1.1.1. *1.1.*

(a) Let Y be the plane curve $y = x^2$ (i.e., Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

(c) * Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

1.1.2. *1.2 The Twisted Cubic Curve.* Let $Y \subseteq \mathbf{A}^3$ be the set $Y = \left\{(t, t^2, t^3) \;\middle|\; t \in k\right\}$.

- Show that Y is an affine variety of dimension 1.
- Find generators for the ideal $I(Y)$.
- Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

We say that Y is given by the *parametric representation* $x = t.y = t^2, z = t^3$.

> Useful facts: $\sqrt{I} = \sqrt{\prod p_i^{a_i}} = \prod p_i$ in a UFD when I is a principal ideal factored into irreducibles. An ideal is also radical iff the quotient is reduced, and $\langle f \rangle$ is radical when f is irreducible.

1.1.3. *1.3.* Let Y be the algebraic set in $\mathbf{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

1.1.4. *1.4.* If we identify $\mathbf{A}^2$ with $\mathbf{A}^1 \times \mathbf{A}^1$ in the natural way, show that the Zariski topology on $\mathbf{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbf{A}^1$.

1.1.5. *1.5.* Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbf{A}^n$. for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.

1.1.6. *1.6.* Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

1.1.7. *1.7.*

(a) Show that the following conditions are equivalent for a topological space X :
   - X is noetherian:
   - Every nonempty family of closed subsets has a minimal element:
   - X satisfies the ascending chain condition for open subsets:
   - Every nonempty family of open subsets has a maximal element.

(b) A noetherian topological space is quasi-compact, i.e., every open cover has a finite subcover.

(c) Any subset of a noetherian topological space is noetherian in its induced topology.

(d) A noetherian space which is also Hausdorff must be a finite set with the discrete topology.

1.1.8. *1.8.* Let Y be an affine variety of dimension r in $\mathbf{A}^n$. Let H be a hypersurface in $\mathbf{A}^n$, and assume that $Y \nsubseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$.[1]

1.1.9. *1.9.* Let $a \subseteq A = k[x_1, \ldots, x_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $Z(a)$ has dimension $\geqslant n - r$.

[1] Use (b) above.

1.1.10. *1.10.*

(a) If Y is any subset of a topological space X, then $\dim Y \leqslant \dim X$.
(b) If X is a topological space which is covered by a family of open subsets $\{L_1;$, then $\dim X = \sup \dim U_i$.
(c) Give an example of a topological space X and a dense open subset U with $\dim L' < \dim X$.
(d) If Y is a closed subset of an irreducible finite-dimensional topological space X, and if $\dim Y = \dim X$, then $Y = X$.
(e) Give an example of a noetherian topological space of infinite dimension.

1.1.11. *1.11 *.* Let $Y \subseteq \mathbf{A}^3$ be the curve given parametrically by $x = t^3, y = t^4, z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x, y; -]$ which cannot be generated by 2 elements. We say Y is not a local complete intersection-cf. (Ex. 2.17).

1.1.12. *1.12.* Give an example of an irreducible polynomial $f \in \mathbf{R}[x, y]$. whose zero set $Z(f)$ in $\mathbf{A}^2_{\mathbf{R}}$ is not irreducible (cf. 1.4.2).

1.2. **I.2: Projective Varieties.**

1.2.1. *2.1.* Prove the "homogeneous Nullstellensatz," which says if $a \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(a)$ in $\mathbf{P}^n$, then $f^u \in a$ for some $q > 0$. [2]

1.2.2. *2.2.* For a homogeneous ideal $a \subseteq S$, show that the following conditions are equivalent:

(i) $Z(a) = \varnothing$ (the empty set);
(ii) $\sqrt{a} =$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;
(iii) $a \supseteq S_d$ for some $d > 0$.

1.2.3. *2.3.*

(a) If $T_1 \subseteq T_2$ are subsets of S^h. then $Z(T_1) \supseteq Z(T_2)$.
(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbf{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.
(c) For any two subsets Y_1, Y_2 of $\mathbf{P}^n, I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
(d) If $a \subseteq S$ is a homogeneous ideal with $Z(a) \neq \varnothing$. then $I(Z(a)) = \sqrt{a}$.
(e) For any subset $Y \subseteq \mathbf{P}^n, Z(I(Y)) = \overline{Y}$.

1.2.4. *2.4.*

(a) There is a 1-1 inclusion-reversing correspondence between algebraic sets in $\mathbf{P}^n$. and homogeneous radical ideals of S not equal to S_+ given by $Y \mapsto I(Y)$ and $a \mapsto Z(a)$. [3]
(b) An algebraic set $Y \subseteq \mathbf{P}^n$ is irreducible if and only if $I(Y')$ is a prime ideal.
(c) Show that $\mathbf{P}^n$ itself is irreducible.

1.2.5. *2.5.*

(a) $\mathbf{P}^n$ is a noetherian topological space.
(b) Every algebraic set in P^n can be written uniquely as a finite union of irreducible algebraic sets. no one containing another. These are called its irreducible components.

[2]Hint: Interpret the problem in terms of the affine $(n+1)$-space whose affine coordinate ring is S, and use the usual Nullstellensatz, (1.3A).

[3]Note: Since S_+does not occur in this correspondence, it is sometimes called the **irrelevant maximal ideal of** S.

1.2.6. *2.6.* If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$.[4]

1.2.7. *2.7.*

(a) $\dim \mathbf{P}^n = n$.
(b) If $Y \subseteq \mathbf{P}^n$ is a quasi-projective variety, then $\dim Y = \dim \overline{Y}$.[5]

1.2.8. *2.8.* A projective variety $Y \subseteq \mathbf{P}^n$ has dimension $n-1$ if and only if it is the zero set of a single irreducible homogeneous polynomial f of positive degree. Y is called a hypersurface in $\mathbf{P}^n$.

1.2.9. *2.9 Projective Closure of an Affine Variety.* If $Y \subseteq \mathbf{A}^n$ is an affine variety, we identify $\mathbf{A}^n$ with an open set $U_0 \subseteq \mathbf{P}^n$ by the homeomorphism φ_0. Then we can speak of $\overline{Y}$, the closure of Y in $\mathbf{P}^n$, which is called the projective closure of Y.

(a) Show that $I(\overline{Y})$ is the ideal generated by $\beta(I(Y))$, using the notation of the proof of (2.2).
(b) Let $Y \subseteq \mathbf{A}^3$ be the twisted cubic of (Ex. 1.2). Its projective closure $\overline{Y} \subseteq \mathbf{P}^3$ is called the twisted cubic curve in $\mathbf{P}^3$. Find generators for $I(Y)$ and $I(\overline{Y})$, and use this example to show that if $f_1, \ldots, f_r$ generate $I(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ do not necessarily generate $I(\overline{Y})$.

1.2.10. *2.10 The Cone Over a Projective Variety (Fig. 1).* Let $Y \subseteq \mathbf{P}^n$ be a nonempty algebraic set, and let $\theta : \mathbf{A}^{n+1} - \{(0, \ldots, 0)\} \to \mathbf{P}^n$ be the map which sends the point with affine coordinates $(a_0, \ldots, a_n)$ to the point with homogeneous coordinates $(a_0, \ldots, a_n)$. We define the affine cone over Y to be

(a) Show that $C(Y)$ is an algebraic set in $\mathbf{A}^{n+1}$, whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0, \ldots, x_n]$.
(b) $C(Y)$ is irreducible if and only if Y is.
(c) $\dim C(Y) = \dim Y + 1$.

Sometimes we consider the projective closure $\overline{C(Y)}$ of $C(Y)$ in $\mathbf{P}^{n+1}$. This is called the **projective cone** over Y.

[4]Hint: Let $\varphi_i : U_i \to \mathbf{A}^n$ be the homeomorphism of (2.2), let Y_t be the affine variety $\varphi_t(Y \cap U_i)$, and let $A(Y_i)$ be its affine coordinate ring. Show that $A(Y_t)$ can be identified with the subring of elements of degree 0 of the localized ring $S(Y)_{x_i}$. Then show that $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. Now use (1.7), (1.8A), and (Ex 1.10), and look at transcendence degrees. Conclude also that $\dim Y = \dim Y_i$ whenever Y_i is nonempty.

[5]Hint: Use (Ex. 2.6) to reduce to (1.10).

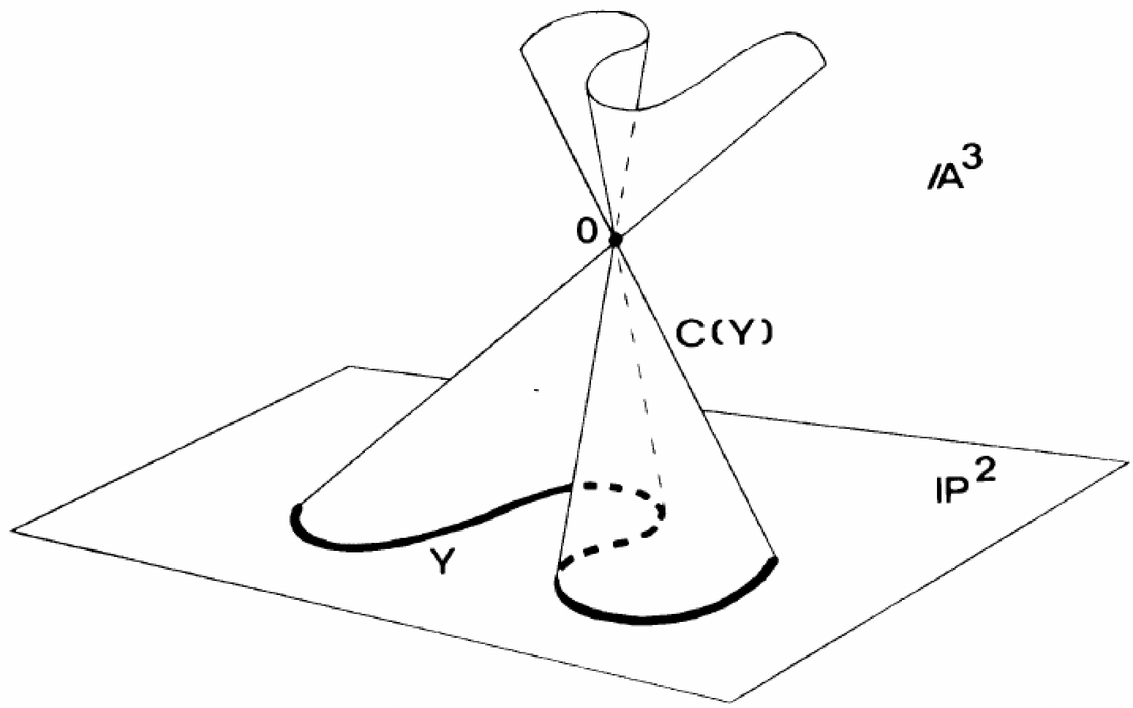

Figure 1. The cone over a curve in $\mathbf{P}^2$.

1.2.11. *2.11 Linear Varieties in* $\mathbf{P}^n$. A hypersurface defined by a linear polynomial is called a hyperplane.

(a) Show that the following two conditions are equivalent for a variety Y in $\mathbf{P}^n$:
   (i) $I(Y)$ can be generated by linear polynomials.
   (ii) Y can be written as an intersection of hyperplanes.
   In this case we say that Y is a **linear variety** in $\mathbf{P}^n$.

(b) If Y is a linear variety of dimension r in $\mathbf{P}^n$, show that $I(Y)$ is minimally generated by $n-r$ linear polynomials.

(c) Let Y, Z be linear varieties in $\mathbf{P}^n$, with $\dim Y = i, \dim Z =$ s. If $r+s-n \geqslant 0$, then $Y \cap Z \neq \varnothing$. Furthermore, if $Y \cap Z \neq \varnothing$, then $Y \cap Z$ is a linear variety of dimension $\geqslant r+s-n$. [6]

1.2.12. *2.12 The d-uple Embedding.* For given $n, d > 0$, let $M_0, M_1, \ldots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \ldots, x_n$, where $N = \binom{n+d}{n} - 1$. We define a mapping $\rho_d : \mathbf{P}^n \to \mathbf{P}^N$ by sending the point $P = (a_0, \ldots, a_n)$ to the point $\rho_d(P) = (M_0(a), \ldots, M_N(a))$ obtained by substituting the a_t in the monomials M_J. This is called the d-uple embedding of $\mathbf{P}^n$ in $\mathbf{P}^N$. For example, if $n=1, d=2$, then $N=2$, and the image Y of the 2-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^2$ is a conic.

(a) Let $\theta : k[y_0, \ldots, y_v] \to k[x_0, \ldots, x_n]$ be the homomorphism defined by sending y_i to M_i, and let a be the kernel of θ. Then a is a homogeneous prime ideal, and so Z (a) is a projective variety in $\mathbf{P}^N$.

[6] Think of $\mathbf{A}^{n+1}$ as a vector space over k, and work with its subspaces.

(b) Show that the image of ρ_d is exactly $Z(a)$. [7]
(c) Now show that ρ_d is a homeomorphism of $\mathbf{P}^n$ onto the projective variety Z (a).
(d) Show that the twisted cubic curve in $\mathbf{P}^3$ (Ex. 2.9) is equal to the 3-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^3$, for suitable choice of coordinates.

1.2.13. *2.13.* Let Y be the image of the 2-uple embedding of $\mathbf{P}^2$ in $\mathbf{P}^5$. This is the Veronese surface. If $Z \subseteq Y$ is a closed curve (a **curve** is a variety of dimension 1), show that there exists a hypersurface $V \subseteq \mathbf{P}^5$ such that $V \cap Y = Z$.

1.2.14. *2.14 The Segre Embedding.* Let $\psi : \mathbf{P}^r \times \mathbf{P}^s \to \mathbf{P}^N$ be the map defined by sending the ordered pair $(a_0, \ldots, a_r) \times (b_0, \ldots, b_s)$ to $(\ldots, a_i b_j, \ldots)$ in lexicographic order. where $N = rs + r + s$. Note that ψ is well-defined and injective. It is called the Segre embedding. Show that the image of ψ is a subvariety of $\mathbf{P}^N$. [8]

1.2.15. *2.15 The Quadric Surface in $\mathbf{P}^3$ (Fig. 2).* Consider the surface Q (a surface is a variety of dimension 2) in $\mathbf{P}^3$ defined by the equation $xy - zw = 0$.

(a) Show that Q is equal to the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^3$. for suitable choice of coordinates.
(b) Show that Q contains two families of lines (a line is a linear variety of dimension 1) $\{L_t\}, \{M_t\}$, each parametrized by $t \in \mathbf{P}^1$. with the properties that if $L_t \neq L_u$. then $L_t \cap L_u = \varnothing$: if $M_t \neq M_u, M_t \cap M_u = \varnothing$, and for all t, u, $L_t \cap M_u =$ one point.
(c) Show that Q contains other curves besides these lines, and deduce that the Zariski topology on Q is not homeomorphic via ψ to the product topology on $\mathbf{P}^1 \times \mathbf{P}^1$ (where each $\mathbf{P}^1$ has its Zariski topology).

[7] One inclusion is easy. The other will require some calculation.

[8] Hint: Let the homogeneous coordinates of $\mathbf{P}^N$ be $\left\{ z_{ij} \;\middle|\; 0 \leq i, j \leq r \right\}$ and let a be the kernel of the homomorphism $k[\{z_{ij}\}] \to k[x_0, \cdots, x_r, y_0, \cdots, y_s]$, which sends z_{ij} to $x_i y_j$. Then show that $\operatorname{im} \psi = Z(a)$.

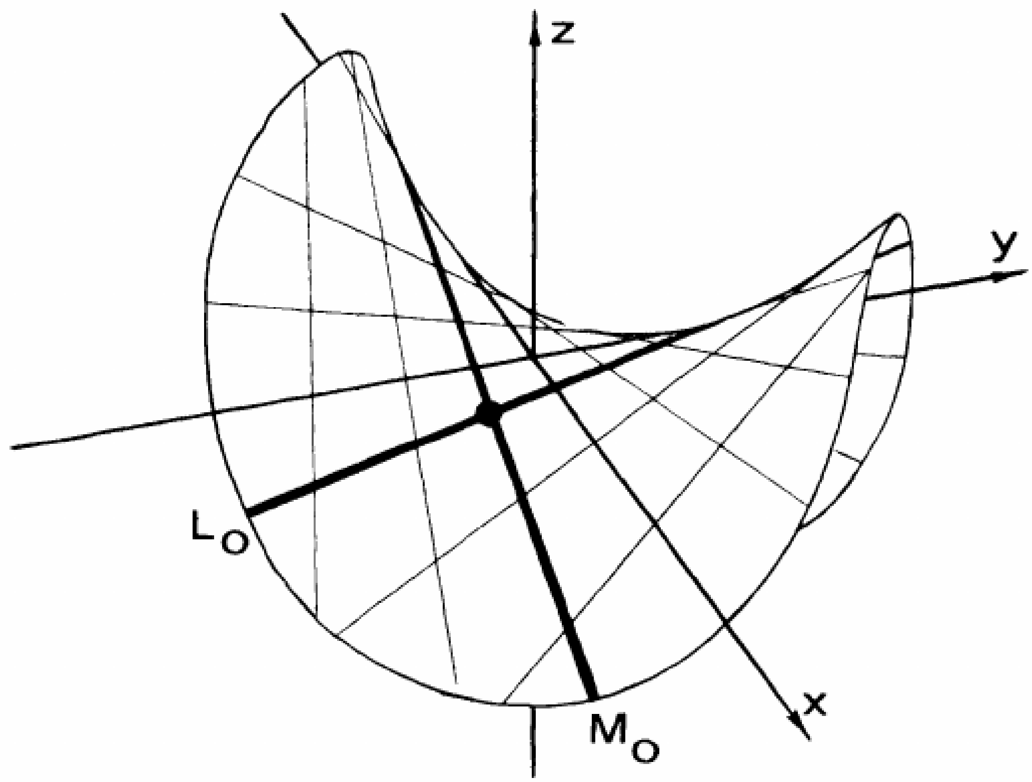

Figure 2. The quadric surface in $\mathbf{P}^3$.

1.2.16. *2.16.*

(a) The intersection of two varieties need not be a variety. For example, let Q_1 and Q_2 be the quadric surfaces in $\mathbf{P}^3$ given by the equations $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Show that $Q_1 \cap Q_2$ is the union of a twisted cubic curve and a line.
(b) Even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals. For example, let C be the conic in $\mathbf{P}^2$ given by the equation $xy - zw = 0$. Let L be the line given by $y = 0$. Show that $C \cap L$ consists of one point P, but that $I(C) + I(L) \neq I(P)$.

1.2.17. *2.17 Complete intersections.* A variety Y of dimension r in $\mathbf{P}^n$ is a (strict) complete intersection if $I(Y)$ can be generated by $n - r$ elements. Y is a set-theoretic complete intersection if Y can be written as the intersection of $n - r$ hypersurfaces.

(a) Let Y be a variety in $\mathbf{P}^n$, let $Y = Z(a)$; and suppose that a can be generated by q elements. Then show that $\dim Y \geqslant n - q$.
(b) Show that a strict complete intersection is a set-theoretic complete intersection.
(c) * The converse of (b) is false. For example let Y be the twisted cubic curve in $\mathbf{P}^3$ (Ex. 2.9). Show that $I(Y)$ cannot be generated by two elements. On the other hand, find hypersurfaces $\mathrm{H}_1, \mathrm{H}_2$ of degrees 2,3 respectively, such that $Y = H_1 \cap H_2$.
(d) ** It is an unsolved problem whether every closed irreducible curve in $\mathbf{P}^3$ is a set-theoretic intersection of two surfaces. See Hartshorne [1] and Hartshorne [5.III, section 5] for commentary.

1.3. I.3: Morphisms.

1.3.1. *3.1.*

1. Show that any conic in $\mathbf{A}^2$ is isomorphic either to $\mathbf{A}^1$ or $\mathbf{A}^1 \setminus \{0\}$ (cf. Ex.1.1).
2. Show that $\mathbf{A}^1$ is not isomorphic to any proper open subset of itself.[9]
3. Any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.
4. We will see later (Ex. 4.8) that any two curves are homeomorphic. But show now that $\mathbf{A}^2$ is not even homeomorphic to $\mathbb{P}^2$.
5. If an affine variety is isomorphic to a projective variety, then it consists of only one point.

1.3.2. *3.2.* A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism.

1. For example, let $\varphi : \mathbf{A}^1 \to \mathbf{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a bijective bicontinuous morphism of $\mathbf{A}^1$ onto the curve $y^2 = x^3$, but that φ is not an isomorphism.
2. For another example. let the characteristic of the base field k be $p > 0$, and define a map $\rho : \mathbf{A}^1 \to \mathbf{A}^1$ by $t \mapsto t^p$. Show that φ is bijective and bicontinuous but not an isomorphism. This is called the Frobenius morphism.

1.3.3. *3.3.*

1. Let $\varphi : X \to Y$ be a morphism. Then for each $P \in X, \varphi$ induces a homomorphism of local rings $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,Y}$.
2. Show that a morphism φ is an isomorphism if and only if φ is a homeomorphism, and the induced map φ_P^* on local rings is an isomorphism, for all $P \in X$.
3. Show that if $\varphi(X)$ is dense in Y, then the map ρ_P^* is injective for all $P \in X$.

1.3.4. *3.4.* Show that the d-uple embedding of $\mathbb{P}^n$(Ex.2.12) is an isomorphism onto its image.

1.3.5. *3.5.* By abuse of language, we will say that a variety "is affine" if it is isomorphic to an affine variety. If $H \subseteq \mathbb{P}^n$ is any hypersurface. show that $\mathbb{P}^n - H$ is affine.[10]

1.3.6. *3.6 There are quasi-affine varieties which are not affine.* For example, show that $\mathrm{I} = \mathbf{A}^2 \setminus \{(0,0)\}$ is not affine.[11]

1.3.7. *3.7.*

1. Show that any two curves in $\mathbb{P}^2$ have a nonempty intersection.
2. More generally, show that if $Y \subseteq \mathbb{P}^n$ is a projective variety of dimension $\geqslant 1$. and if H is a hypersurface. then $Y \cap H \neq \varnothing$.[12]

1.3.8. *3.8.* Let H_1 and H, be the hyperplanes in $\mathbb{P}^n$ defined by $x_1 = 0$ and $x_, = 0$, with $i \neq j$. Show that any regular function on $\mathbb{P}^n - (H_1 \cap H_1)$ is constant.[13]

1.3.9. *3.9.* The homogeneous coordinate ring of a projective variety is not invariant under isomorphism. For example, let $X = \mathbb{P}^1$. and let Y be the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$. Then $X \cong Y($ Ex. 3.4). But show that $S(X) \equiv S(Y)$.

[9]Use (b) above.
[10]Use (c) above.
[11]Use induction on $\dim X$.
[12]Note: We will give another proof of this result using sheaves of ideals later (V.10).
[13]This gives an alternate proof of (3.4a) in the case $Y = \mathbb{P}^n$.

1.3.10. *3.10 Subvarieties.* A subset of a topological space is locally closed if it is an open subset of its closure. or. equivalently. if it is the intersection of an open set with a closed set.
If X is a quasi-affine or quasi-projective variety and Y is an irreducible locally closed subset. then I is also a quasi-affine (respectively, quasi-projective) variety by virtue of being a locally closed subset of the same affine or projective space. We call this the induced structure on Y. and we call Y a subvariety of X.
Now let $\varphi : X \to Y$ he a morphism. let $X' \subseteq X$ and $Y' \subseteq Y$ be irreducible locally closed subsets such that $\varphi(X') \subseteq Y'$. Show that $\left.\varphi\right|_X : X' \to Y'$ is a morphism.

1.3.11. *3.11.* Let X be any variety and let $P \in X$. Show there is a 1-1 correspondence between the prime ideals of the local ring $\mathcal{O}_P$ and the closed subvarieties of X containing P.

1.3.12. *3.12.* If P is a point on a variety X, then $\dim \mathcal{O}_P = \dim X$.[14]

1.3.13. *3.13 The Local Ring of a Subvariety.* Let $Y \subseteq X$ be a subvariety. Let $\mathcal{O}_{Y,X}$ be the set of equivalence classes $\langle L, f\rangle$ where $L \subseteq X$ is open. $L \cap Y \neq \varnothing$, and f is a regular function on L. We say $\langle L, f\rangle$ is equivalent to $\langle V, g\rangle$ if $f = g$ on $U \cap V$.
Show that $\mathcal{O}_{Y,X}$ is a local ring, with residue field $K(Y)$ and dimension $= \dim \mathrm{X} - \dim Y$. It is the local ring of Y on X. Note if $Y = P$ is a point we get $\mathcal{O}_P$. and if $Y = X$ we get $K(X)$. Note also that if Y is not a point, then $K(Y)$ is not algebraically closed, so in this way we get local rings whose residue fields are not algebraically closed.

1.3.14. *3.14 Projection from a Point.* Let $\mathbb{P}^n$ be a hyperplane in $\mathbb{P}^{n+1}$ and let $P \in \mathbb{P}^{n+1} - \mathbb{P}^n$. Define a mapping $\varphi : \mathbb{P}^{n+1} \setminus \{P\} \to \mathbb{P}^n$ by $\varphi(Q) =$ the intersection of the unique line containing P and Q with $\mathbb{P}^n$.

1. Show that φ is a morphism.
2. Let $Y \subseteq \mathbb{P}^3$ be the twisted cubic curve which is the image of the 3-uple embedding of $\mathbb{P}^1$ (Ex. 2.12). If t, u are the homogeneous coordinates on $\mathbb{P}^1$. we say that Y is the curve given parametrically by $(x, y, z, w) = (t^3, t^2u, tu^2, u^3)$. Let $P = (0, 0, 1, 0)$, and let $\mathbb{P}^2$ be the hyperplane $z = 0$. Show that the projection of Y from P is a cuspidal cubic curve in the plane, and find its equation.

1.3.15. *3.15 Products of Affine Varieties.* Let $X \subseteq \mathbf{A}^n$ and $Y \subseteq \mathbf{A}^m$ be affine varieties.

1. Show that $X \times Y \subseteq \mathbf{A}^{n+m}$ with its induced topology is irreducible.[15] The affine variety $X \times Y$ is called the product of X and Y. Note that its topology is in general not equal to the product topology (Ex. 1.4).
2. Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$.
3. Show that $X \times Y$ is a product in the category of varieties, i.e., show
   - the projections $X \times Y \to X$ and $X \times Y \to Y$ are morphisms, and
   - given a variety Z, and the morphisms $Z \to X, Z \to Y$. there is a unique morphism $Z \to X \times Y$ making a commutative diagram

[14] Hint: Reduce 10 the affine case and use (3.2c)

[15] Hint: Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let $X_1 = \left\{ x \in X \;\middle|\; x \times Y \subseteq Z_1 \right\}, i = 1, 2.$ Show that $X = X_1 \cup X_2$ and X_1, X_2 are closed. Then $X = X_1$ or X_2 so $X \times Y = Z_1$ or Z_2.

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr] \arrow[drrr] & & X \times Y \arrow[dl] \arrow[dr] & \\ & X & & Y \end{tikzcd}$$

FIGURE 1. Link to Diagram

4. Show that $\dim X \times Y = \dim X + \dim Y$.

1.3.16. *3.16 Products of Quasi-Projective Varieties.* Use the Segre embedding (Ex. 2.14) to identify $\mathbb{P}^n \times \mathbb{P}^m$ with its image and hence give it a structure of projective varieties. Now for any two quasi-projective varieties $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$, consider $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$.

1. Show that $X \times Y$ is a quasi-projective variety.
2. If X, Y are both projective, show that $X \times Y$ is projective.
3. Show that $X \times Y$ is a product in the category of varieties.

1.3.17. *3.17 Normal Varieties.* A variety Y is normal at a point $P \in Y$ if $\mathcal{O}_P$ is an integrally closed ring. Y is normal if it is normal at every point.

1. Show that every conic in $\mathbb{P}^2$ is normal.
2. Show that the quadric surfaces Q_1, Q_2 in P^3 given by equations $Q_1 : xy = zw$; $Q_2 : xy = z^2$ are normal. (cf. (II. Ex. 6.4) for the latter.)
3. Show that the cuspidal cubic $y^2 = x^3$ in $\mathbf{A}^2$ is not normal.
4. If Y is affine, then Y is normal $\Leftrightarrow A(Y)$ is integrally closed.
5. Let Y be an affine variety. Show that there is a normal affine variety $\tilde{Y}$, and a morphism $\pi : \tilde{Y} \to Y$, with the property that whenever Z is a normal variety, and $\varphi : Z \to Y$ is a **dominant** morphism (i.e., $\varphi(Z)$ is dense in Y), then there is a unique morphism $\theta : Z \to \tilde{Y}$ such that $\varphi = \pi \quad \theta$. $\tilde{Y}$ is called the **normalization** of Y. You will need (3.9 A) above.

1.3.18. *3.18 Projectively Normal Varieties.* A projective variety $Y \subseteq \mathrm{P}^n$ is projectively normal (with respect to the given embedding) if its homogeneous coordinate ring $S(Y)$ is integrally closed.

1. If Y is projectively normal, then Y is normal.
2. There are normal varieties in projective space which are not projectively normal. For example, let Y be the twisted quartic curve in $\mathbb{P}^3$ given parametrically by $(x, y : z, w) = (t^4, t^3u, tu^3, u^4)$. Then Y is normal but not projectively normal. See (III, Ex. 5.6) for more examples.
3. Show that the twisted quartic curve Y above is isomorphic to $\mathbb{P}^1$. which is projectively normal. Thus projective normality depends on the embedding.

1.3.19. *3.19 Automorphisms of* $\mathbf{A}^n$. Let $\varphi : \mathbf{A}^n \to \mathbf{A}^n$ be a morphism of $\mathbf{A}^n$ to $\mathbf{A}^n$ given by n polynomials $f_1 \ldots . f_n$ of n variables $x_1, \ldots x_n$. Let $J = \det \left[\frac{\partial f_i}{\partial x_j}\right]$ be the Jacobian polynomial of φ.

1. If φ is an isomorphism (in which case we call φ an automorphism of $\mathbf{A}^n$) show that J is a nonzero constant polynomial.
2. ** The converse of 1.is an unsolved problem, even for $n = 2$. See, for example, Vitushkin.

1.3.20. *3.20.* Let Y be a variety of dimension $\geqslant 2$, and let $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.

1. Show that f extends to a regular function on Y.
2. Show this would be false for $\dim Y = 1$. See (III. Ex. 3.5) for generalization.

1.3.21. *3.21. Group Varieties.* A group variety consists of a variety Y together with a morphism $\mu : Y \times Y \to Y$. such that the set of points of Y with the operation given by μ is a group. and such that the inverse map $y^{-y^{-1}}$ is also a morphism of $Y \to Y$.

1. The additive group $\mathbf{G}_a$ is given by the variety $\mathbf{A}^1$ and the morphism $\mu : \mathbf{A}^2 \to \mathbf{A}^1$ defined by $\mu(a, b) = a + b$. Show it is a group variety.
2. The multiplicative croup $\mathbf{G}_m$ is given by the variety $\mathbf{A}^1 \backslash \{0\}$, and the morphism $\mu(a, b) = ab$. Show | in a group variety.
3. If G is a group variety, and X is any variety. show that the set $\operatorname{Hom}(X, G)$ has a natural group structure.
4. For any variety X, show that $\operatorname{Hom}(X, \mathbf{G}_a)$ is isomorphic to (' (X) as a group under addition.
5. For any variety X, show that $\operatorname{Hom}(X, \mathbf{G}_m)$ is isomorphic to the group of units in $\mathcal{O}(X)$, under multiplication.

1.4. I.4: Rational Maps.

1.4.1. *4.1.* If f and g are regular functions on open subsets U and V of a variety X, and if $f = g$ on $U \cap V$. show that the function which is f on U and g on V is a regular function on $U \cup V$. Conclude that if f is a rational function on X, then there is a largest open subset U of X on which f is represented by a regular function. We say that f is defined at the points of U.

1.4.2. *4.2. Same problem for rational maps.* If φ is a rational map of X to Y, show there is a largest open set on which φ is represented by a morphism. We say the rational map is defined at the points of that open set.

1.4.3. *4.3.*

1. Let f be the rational function on $\mathbb{P}^2$ given by $f = x_1/x_0$. Find the set of points where f is defined and describe the corresponding regular function.
2. Now think of this function as a rational map from $\mathbb{P}^2$ to $\mathbf{A}^1$. Embed $\mathbf{A}^1$ in $\mathbb{P}^1$, and let $\varphi : \mathbb{P}^2 \to \mathbb{P}^1$ be the resulting rational map. Find the set of points where φ is defined, and describe the corresponding morphism.

1.4.4. *4.4.* A variety Y is rational if it is birationally equivalent to $\mathbb{P}^n$ for some n (or, equivalently by (4.5), if $K(Y)$ is a pure transcendental extension of k).

1. Any conic in $\mathbb{P}^2$ is a rational curve.
2. The cuspidal cubic $y^2 = x^3$ is a rational curve.
3. Let Y be the nodal cubic curve $y^2 z = x^2(x + z)$ in $\mathbb{P}^2$. Show that the projection φ from the point $P = (0, 0, 1)$ to the line $z = 0$ (Ex. 3.14) induces a birational map from Y to $\mathbb{P}^1$. Thus Y is a rational curve.

1.4.5. *4.5.* Show that the quadric surface $Q : xy = zw$ in $\mathbb{P}^3$ is birational to $\mathbb{P}^2$, but not isomorphic to $\mathbb{P}^2$ (cf. Ex. 2.15).

1.4.6. *4.6. Plane Cremona Transformations.* A birational map of $\mathbb{P}^2$ into itself is called a plane Cremona transformation. We give an example, called a quadratic transformation. It is the rational map $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ given by $(a_0, a_1, a_2) \to (a_1 a_2, a_0 a_2, a_0 a_1)$ when no two of a_0, a_1, a_2 are 0.

1. Show that φ is birational, and is its own inverse.
2. Find open sets $U, V \subseteq \mathbb{P}^2$ such that $\varphi : U \to V$ is an isomorphism.
3. Find the open sets where φ and φ^{-1} are defined. and describe the corresponding morphisms. See also (Chapter V, 4.2.3).

1.4.7. *4.7.* Let X and Y be two varieties. Suppose there are points $P \in X$ and $Q \in Y$ such that the local rings $\mathcal{O}_{P,X}$ and $\mathcal{O}_{Q,Y}$ are isomorphic as k-algebras. Then show that there are open sets $P \in U \subseteq X$ and $Q \in V \subseteq Y$ and an isomorphism of U to V which sends P to Q.

1.4.8. *4.8.*

1. Show that any variety of positive dimension over k has the same cardinality as k.[16]
2. Deduce that any two curves over k are homeomorphic (cf. Ex. 3.1).

1.4.9. *4.9.* Let X be a projective variety of dimension r in P^n. with $n \geqslant r+2$. Show that for suitable choice of $P \notin X$. and a linear $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$. the projection from P to $\mathbb{P}^{n-1}$ (Ex. 3.14) induces a birational morphism of X onto its image $X' \subseteq \mathbb{P}^{n-1}$. You will need to use (4.6A). (4.7A). and (4.8A). This shows in particular that the birational map of (4.9) can be obtained by a finite number of such projections.

1.4.10. *4.10.* Let Y be the cuspidal cubic curve $y^2 = x^3$ in $\mathbf{A}^2$. Blow up the point $O = (0.0)$. Let E be the exceptional curve. and let $\tilde{Y}$ be the strict transform of Y. Show that E meets $\tilde{Y}$ in one point. and that $\tilde{Y} \cong \mathbf{A}^1$. In this case the morphism $\rho : \tilde{Y} \to Y$ is bijective and bicontinuous. but it is not an isomorphism.

1.5. **I.5: Nonsingular Varieties.**

1.5.1. *5.1.* Locate the singular points and sketch the following curves in $\mathbb{A}^2$ (assume char $k \neq 2$). Which is which in Figure 4 ?

1. $x^2 = x^4 + y^4$:
2. $xy = x^6 + y^6$:
3. $x^3 = y^2 + x^4 + y^4$:
4. $x^2y + xy^2 = x^4 + y^4$.

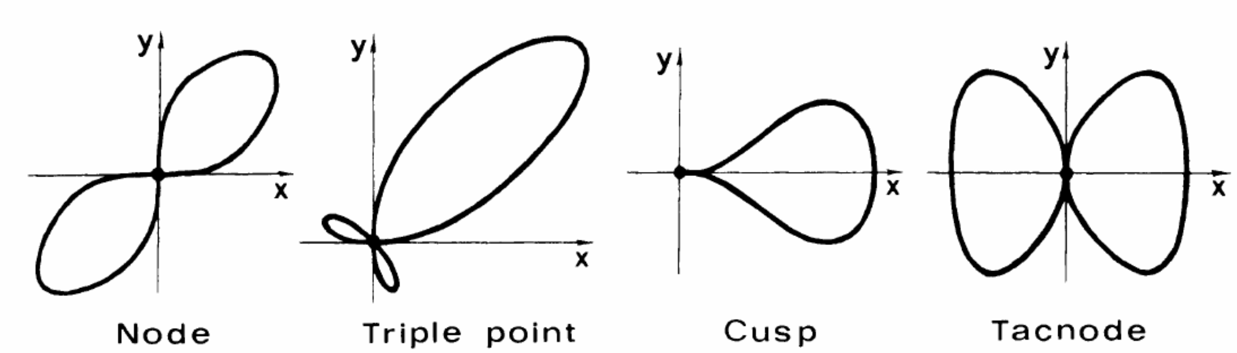

Figure 4. Singularities of plane curves.

1.5.2. *5.2.* Locate the singular points and describe the singularities of the following surfaces in $\mathbb{A}^3$ (assume char $k \neq 2$). Which is which in Figure 5?

1. $xy^2 = z^2$
2. $x^2 + y^2 = z^2$
3. $xy + x^3 + y^3 = 0$.

[16]Use (b) above.

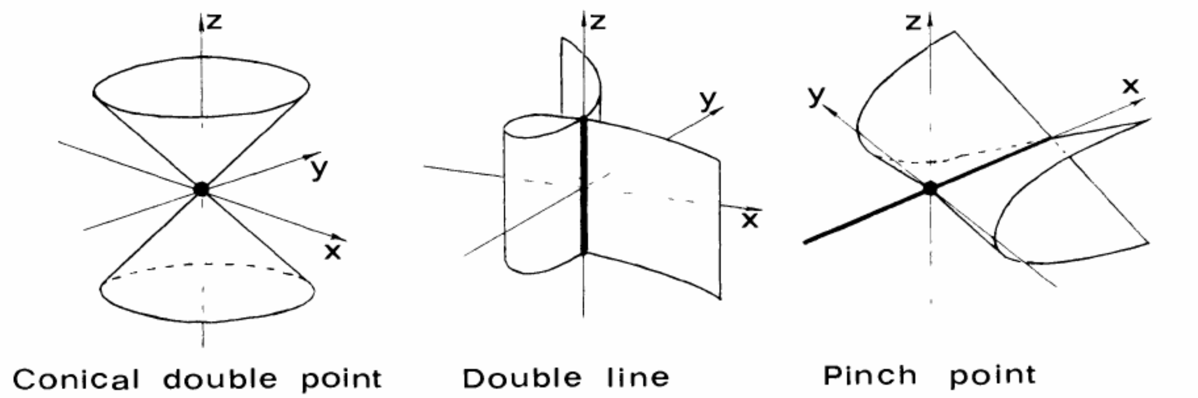

Figure 5. Surface singularities.

1.5.3. *5.3. Multiplicities.* Let $Y \subseteq \mathbb{A}^2$ be a curve defined by the equation $f(x, y) = 0$. Let $P = (a, b)$ be a point of $\mathbb{A}^2$. Make a linear change of coordinates so that P becomes the point $(0, 0)$. Then write f as a sum $f = f_0 + f_1 + \ldots + f_d$, where f_i is a homogeneous polynomial of degree i in x and y. Then we define the multiplicity of P on Y, denoted $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. (Note that $P \in Y \Leftrightarrow \mu_P(Y) > 0$.) The linear factors of f_r are called the tangent directions at P.

1. Show that $\mu_P(Y) = 1 \iff P$ is a nonsingular point of Y.
2. Find the multiplicity of each of the singular points in (Ex. 5.1) above.

1.5.4. *5.4. Intersection Multiplicity.* If $Y, Z \subseteq \mathbb{A}^2$ are two distinct curves, given by equations $f = 0, g = 0$, and if $P \in Y \cap Z$, we define the intersection multiplicity $(Y \cdot Z)_P$ of Y and Z at P to be the length of the $\mathcal{O}_P$-module $\mathcal{O}_P / \langle f, g\rangle$.

1. Show that $(Y \cdot Z)_P$ is finite, and $(Y \cdot Z)_P \geqslant \mu_P(Y) \cdot \mu_P(Z)$.
2. If $P \in Y$, show that for almost all lines L through P (i.e., all but a finite number), $(L \cdot Y)_P = \mu_P(Y)$.
3. If Y is a curve of degree d in $\mathbb{P}^2$, and if L is a line in $\mathbb{P}^2, L \neq Y$, show that $(L \cdot Y) = d$. Here we define $(L \cdot Y) = \sum (L \cdot Y)_P$ taken over all points $P \in L \cap Y$, where $(L \cdot Y)_p$ is defined using a suitable affine cover of $\mathbb{P}^2$.

1.5.5. *5.5.* For every degree $d > 0$, and every $p = 0$ or a prime number, give the equation of a nonsingular curve of degree d in $\mathbb{P}^2$ over a field k of characteristic p.

1.5.6. *5.6. Blowing Up Curve Singularities.*

1. Let Y be the cusp or node of (Ex. 5.1). Show that the curve $\tilde{Y}$ obtained by blowing up Y at $O = (0, 0)$ is nonsingular (cf. (4.9.1) and (Ex. 4.10)).
2. We define a node (also called ordinary double point) to be a double point (i.e., a point of multiplicity 2) of a plane curve with distinct tangent directions (Ex. 5.3). If P is a node on a plane curve Y, show that $\varphi^{-1}(P)$ consists of two distinct nonsingular points on the blown-up curve $\tilde{Y}$. We say that "blowing up P resolves the singularity at P".
3. Let $P \in Y$ be the tacnode of (Ex. 5.1). If $\varphi : \tilde{Y} \to Y$ is the blowing-up at P. show that $\rho^{-1}(P)$ is a node. Using 2. we see that the tacnode can be resolved by two successive blowings-up.
4. Let Y be the plane curve $y^3 = x^5$, which has a "higher order cusp" at O. Show that O is a triple point: that blowing up O gives rise to a double point (what kind?) and that one further blowing up resolves the singularity.

Note: We will see later (V, 3.8) that any singular point of a plane curve can be resolved by a finite sequence of successive blowings-up.

1.5.7. *5.7.* Let $Y \subseteq \mathbb{P}^2$ be a nonsingular plane curve of degree > 1, defined by the equation $f(x, y, z) = 0$. Let $X \subseteq \mathbb{A}^3$ be the affine variety defined by f (this is the cone over Y; see (Ex. 2.10)). Let P be the point $(0, 0, 0)$, which is the vertex of the cone. Let $\varphi : \tilde{X} \to X$ be the blowing-up of X at P.

1. Show that X has just one singular point, namely P.
2. Show that $\tilde{X}$ is nonsingular (cover it with open affines).
3. Show that $\varphi^{-1}(P)$ is isomorphic to Y.

1.5.8. *5.8.* Let $Y \subseteq \mathbb{P}^n$ be a projective variety of dimension r. Let $f_1, \ldots, f_t \in S = k[x_0, \ldots, x_n]$ be homogeneous polynomials which generate the ideal of Y. Let $P \in Y$ be a point, with homogeneous coordinates $P = (a_0, \ldots, a_n)$. Show that P is nonsingular on Y if and only if the rank of the matrix $\left[\frac{\partial f_i}{\partial x_j}(a_0, \cdots, a_n)\right]$ is $n - r$.[17]

1.5.9. *5.9.* Let $f \in k[x, y; z]$ be a homogeneous polynomial, let $Y = Z(f) \subseteq \mathbb{P}^2$ be the algebraic set defined by f, and suppose that for every $P \in Y$, at least one of $\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P), \frac{\partial f}{\partial z}(P)$ is nonzero. Show that f is irreducible (and hence that Y is a nonsingular variety). [18]

1.5.10. *5.10.* For a point P on a variety X. let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_P$. We define the Zariski tangent space $T_P(X)$ of X at P to be the dual k-vector space of $\mathfrak{m}/\mathfrak{m}^2$.

1. For any point $P \in X$. $\dim T_P(X) \geqslant \dim X$. with equality if and only if P is nonsingular.
2. For any morphism $\varphi : X \to Y$, there is a natural induced k-linear map $T_P(\varphi) : T_P(X) \to T_{\varphi(P)}(Y)$
3. If φ is the vertical projection of the parabola $x = y^2$ onto the x-axis, show that the induced map $T_0(\varphi)$ of tangent spaces at the origin is the zero map.

1.5.11. *5.11. The Elliptic Quartic Curve in* $\mathbb{P}^3$*.* Let Y be the algebraic set in $\mathbb{P}^3$ defined by the equations $x^2 - xz - yw = 0$ and $yz - xw - zw = 0$. Let P be the point $(x, y, z, w) = (0, 0, 0, 1)$. and let φ denote the projection from P to the plane $w = 0$. Show that φ induces an isomorphism of $Y - P$ with the plane cubic curve $y^2z - x^3 + xz^2 = 0$ minus the point $(1, 0, -1)$. Then show that Y is an irreducible nonsingular curve. It is called the elliptic quartic curve in $\mathbb{P}^3$. Since it is defined by two equations it is another example of a complete intersection (Ex. 2.17).

1.5.12. *5.12. Quadric Hypersurfaces.* Assume char $k \neq 2$. and let f be a homogeneous polynomial of degree 2 in $x_0 \ldots\ldots x_n$.

1. Show that after a suitable linear change of variables, f can be brought into the form $f = x_0^2 + \ldots + x_r^2$ for some $0 \leqslant r \leqslant n$.
2. Show that f is irreducible if and only if $r \geqslant 2$.
3. Assume $r \geqslant 2$, and let Q be the quadric hypersurface in $\mathbb{P}^n$ defined by f. Show that the singular locus $Z = \operatorname{Sing} Q$ of Q is a linear variety (Ex. 2.11) of dimension $n - r - 1$. In particular, Q is nonsingular if and only if $r = n$.
4. In case $r < n$, show that Q is a cone with axis Z over a nonsingular quadric hypersurface $Q' \subseteq \mathbb{P}^r$.[19]

[17]Use (b) above.
[18]Use (c) above.
[19]Use induction on $\dim X$.

1.5.13. *5.13.* It is a fact that any regular local ring is an integrally closed domain (Matsumura [2.Th.36, p.121]). Thus we see from (5.3) that any variety has a nonempty open subset of normal points (Ex. 3.17). In this exercise, show directly (without using (5.3)) that the set of nonnormal points of a variety is a proper closed subset (you will need the finiteness of integral closure: see (3.9A)).

1.5.14. *5.14. Analytically Isomorphic Singularities.*

1. If $P \in Y$ and $Q \in Z$ are analytically isomorphic plane curve singularities, show that the multiplicities $\mu_P(Y)$ and $\mu_Q(Z)$ are the same (Ex. 5.3).
2. Generalize the example in the text (5.6.3) to show that if $f = f_r + f_{r+1} + \ldots \in k[[x, y]]$, and if the leading form f_r of f factors as $f_r = g_s h_t$, where g_s, h_t are homogeneous of degrees s and t respectively, and have no common linear factor, then there are formal power series in $k[[x, y]]$ such that $f = gh$.
3. Let Y be defined by the equation $f(x, y) = 0$ in $\mathbb{A}^2$, and let $P = (0, 0)$ be a point of multiplicity r on Y, so that when f is expanded as a polynomial in x and y, we have $f = f_r +$ higher terms. We say that P is an ordinary r-fold point if f_r is a product of r distinct linear factors. Show that any two ordinary double points are analytically isomorphic. Ditto for ordinary triple points. But show that there is a one-parameter family of mutually nonisomorphic ordinary 4-fold points.
4. * Assume char $k \neq 2$. Show that any double point of a plane curve is analytically isomorphic to the singularity at $(0, 0)$ of the curve $y^2 = x^r$, for a uniquely determined $r \geqslant 2$. If $r = 2$ it is a node (Ex. 5.6). If $r = 3$ we call it a cusp: if $r = 4$ a tacnode. See (V, 3.9.5) for further discussion.

1.5.15. *5.15. Families of Plane Curves.* A homogeneous polynomial f of degree d in three variables x, y, z has $\binom{d+2}{2}$ coefficients. Let these coefficients represent a point in $\mathbb{P}^N$. where $N = \binom{d+2}{2} - 1 = \frac{1}{2}d(d + 3)$.

1. Show that this gives a correspondence between points of $\mathbb{P}^N$ and algebraic sets in $\mathbb{P}^2$ which can be defined by an equation of degree d. The correspondence is 1-1 except in some cases where f has a multiple factor.
2. Show under this correspondence that the (irreducible) nonsingular curves of degree d correspond 1-1 to the points of a nonempty Zariski-open subset of $\mathbb{P}^N$. [20]

1.6. **I.6: Nonsingular Curves.**

1.6.1. *6.1.* Recall that a curve is rational if it is birationally equivalent to $\mathbf{P}^1$(Ex.4.4). Let Y be a nonsingular rational curve which is not isomorphic to $\mathbf{P}^1$.

(a) Show that Y is isomorphic to an open subset of $\mathbf{A}^1$.
(b) Show that Y is affine.
(c) Show that $A(Y)$ is a unique factorization domain.

1.6.2. *6.2. An Elliptic Curve.* Let Y be the curve $y^2 = x^3 - x$ in $\mathbf{A}^2$, and assume that the characteristic of the base field k is $\neq 2$. In this exercise we will show that Y is not a rational curve, and hence $K(Y)$ is not a pure transcendental extension of k.

(a) Show that Y is nonsingular, and deduce that $A = A(Y) \simeq k[x, y]/\left(y^2 - x^3 + x\right)$ is an integrally closed domain.
(b) Let $k[x]$ be the subring of $K = K(Y)$ generated by the image of x in A. Show that $k[x]$ is a polynomial ring, and that A is the integral closure of $k[x]$ in K.

[20] Note: We will give another proof of this result using sheaves of ideals later (V.10).

(c) Show that there is an automorphism $\sigma : A \to A$ which sends y to $-y$ and leaves x fixed. For any $a \in A$, define the norm of a to be $N(a) = a \cdot \sigma(a)$. Show that $N(a) \in k[x], N(1) = 1$, and $N(ab) = N(a) \cdot N(b)$ for any $a, b \in A$.

(d) Using the norm, show that the units in A are precisely the nonzero elements of k. Show that x and y are irreducible elements of A. Show that A is not a unique factorization domain.

(e) Prove that Y is not a rational curve (Ex. 6.1). See (II, 8.20.3) and (III, Ex. 5.3) for other proofs of this important result.

1.6.3. *6.3.* Show by example that the result of (6.8) is false if either (a) $\dim X \geqslant 2$, or (b) Y is not projective.

1.6.4. *6.4.* Let Y be a nonsingular projective curve. Show that every nonconstant rational function f on Y defines a surjective morphism $\varphi : Y \to \mathbf{P}^1$, and that for every $P \in \mathbf{P}^1$, $\varphi^{-1}(P)$ is a finite set of points.

1.6.5. *6.5.* Let X be a nonsingular projective curve. Suppose that X is a (locally closed) subvariety of a variety Y (Ex. 3.10). Show that X is in fact a closed subset of Y. See (II, Ex. 4.4) for generalization.

1.6.6. *6.6. Automorphisms of* $\mathbf{P}^1$. Think of $\mathbf{P}^1$ as $\mathbf{A}^1 \cup \{\infty\}$. Then we define a fractional linear transformation of $\mathbf{P}^1$ by sending $x \mapsto (ax+b)/(cx+d)$, for $a, b, c, d \in k$, and $ad - bc \neq 0$.

(a) Show that a fractional linear transformation induces an automorphism of $\mathbf{P}^1$ (i.e., an isomorphism of $\mathbf{P}^1$ with itself). We denote the group of all these fractional linear transformations by PGL(1).

(b) Let Aut $\mathbf{P}^1$ denote the group of all automorphisms of $\mathbf{P}^1$. Show that Aut $\mathbf{P}^1 \simeq$ Aut $k(x)$, the group of k-automorphisms of the field $k(x)$.

(c) Now show that every automorphism of $k(x)$ is a fractional linear transformation, and deduce that PGL(1) $\to$ Aut $\mathbf{P}^1$ is an isomorphism.

Note: We will see later (II. 7.1.1) that a similar result holds for $\mathbf{P}^n$: every automorphism is given by a linear transformation of the homogeneous coordinates.

1.6.7. *6.7.* Let $P_1, \ldots, P_r, Q_1, \ldots, Q_s$ be distinct points of $\mathbf{A}^1$. If $\mathbf{A}^1 - \{P_1, \ldots, P_r\}$ is isomorphic to $\mathbf{A}^1 - \{Q_1, \ldots, Q_\diamond\}$, show that $r = s$. Is the converse true? Cf. (Ex. 3.1).

1.7. **I.7: Intersections in Projective Space.**

1.7.1. *7.1.*

1. Find the degree of the d-uple embedding of $\mathbf{P}^n$ in $\mathbf{P}^N$ (Ex. 2.12).[21]
2. Find the degree of the Segre embedding of $\mathbf{P}^r \times \mathbf{P}^s$ in $\mathbf{P}'$(Ex. 2.14).[22]

1.7.2. *7.2.* Let Y be a variety of dimension r in $\mathbf{P}^n$, with Hilbert polynomial P_Y. We define the arithmetic genus of Y to be $p_a(Y) = (-1)^r \left(P_Y(0) - 1\right)$. This is an important invariant which (as we will see later in (III, Ex. 5.3)) is independent of the projective embedding of Y.

1. Show that $p_a\left(\mathbf{P}^n\right) = 0$.
2. If Y is a plane curve of degree d, show that $p_a(Y) = \frac{1}{2}(d-1)(d-2)$.
3. More generally, if H is a hypersurface of degree d in $\mathbf{P}^n$, then $p_a(H) = \binom{d-1}{n}$.
4. If Y is a complete intersection (Ex. 2.17) of surfaces of degrees a, h in $\mathbf{P}^3$, then $p_a(Y) = \frac{1}{2}ab(a+b-4)+1$.

[21] Answer: d^n.
[22] Answer: $\binom{r+s}{s}$.

(e) Let $Y^r \subseteq \mathbf{P}^n, Z^s \subseteq \mathbf{P}^m$ be projective varieties, and embed $Y \times Z \subseteq \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^N$ by the Segre embedding. Show that

1.7.3. *7.3. The Dual Curve.* Let $Y \subseteq \mathbf{P}^2$ be a curve. We regard the set of lines in $\mathbf{P}^2$ as another projective space, $\left(\mathbf{P}^2\right)^*$. by taking (a_0, a_1, a_2) as homogeneous coordinates of the line $L : a_0x_0 + a_1x_1 + a_2x_2 = 0$. For each nonsingular point $P \in Y$, show that there is a unique line $T_P(Y)$ whose intersection multiplicity with Y at P is > 1. This is the tangent line to Y at P.
Show that the mapping $P \mapsto T_P(Y)$ defines a morphism of $\operatorname{Reg} Y$ (the set of nonsingular points of Y) into $\left(\mathbf{P}^2\right)^*$. The closure of the image of this morphism is called the dual curve $Y^* \subseteq \left(\mathbf{P}^2\right)^*$ of Y.

1.7.4. *7.4.* Given a curve Y of degree d in $\mathbf{P}^2$, show that there is a nonempty open subset U of $\left(\mathbf{P}^2\right)^*$ in its Zariski topology such that for each $L \in U, L$ meets Y in exactly d points. [23]
This result shows that we could have defined the degree of Y to be the number d such that almost all lines in $\mathbf{P}^2$ meet Y in d points, where "almost all" refers to a nonempty open set of the set of lines, when this set is identified with the dual projective space $\left(\mathbf{P}^2\right)^*$

1.7.5. *7.5.*

1. Show that an irreducible curve Y of degree $d > 1$ in $\mathbf{P}^2$ cannot have a point of multiplicity $\geqslant d$(Ex. 5.3).
2. If Y is an irreducible curve of degree $d > 1$ having a point of multiplicity $d - 1$. then Y is a rational curve (Ex. 6.1).

1.7.6. *7.6. Linear Varieties.* Show that an algebraic set Y of pure dimension r (i.e., every irreducible component of Y has dimension r) has degree 1 if and only if Y is a linear variety (Ex. 2.11).[24]

1.7.7. *7.7.* Let Y be a variety of dimension r and degree $d > 1$ in $\mathbf{P}^n$. Let $P \in Y$ be a nonsingular point. Define X to be the closure of the union of all lines PQ, where $Q \in Y, Q \neq P$.

1. Show that X is a variety of dimension $r + 1$.
2. Show that $\deg X < d$. [25]

1.7.8. *7.8.* Let $Y^r \subseteq \mathbf{P}^n$ be a variety of degree 2. Show that Y is contained in a linear subspace L of dimension $r + 1$ in $\mathbf{P}^n$. Thus Y is isomorphic to a quadric hypersurface in $\mathbf{P}^{r+1}$(Ex.5.12)

2. II: Schemes

2.1. **II.1: Sheaves.**

2.1.1. *II.1.1.* Let A be an abelian group, and define the **constant presheaf** associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$. with restriction maps the identity. Show that the constant sheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.

[23]Hint: Show that the set of lines in $\left(\mathbf{P}^2\right)^*$ which are either tangent to Y or pass through a singular point of Y is contained in a proper closed subset.

[24]Hint: First, use (7.7) and treat the case $\dim Y = 1$. Then do the general case by cutting with a hyperplane and using induction.

[25]Hint: Use induction on $\dim Y$.

2.1.2. *II.1.2.*

a. For any morphism of sheaves $\varphi : \mathcal{F} \to \mathscr{G}$, show that for each point P, $(\ker \varphi)_P = \ker(\varphi_P)$ and $(\operatorname{im} \varphi)_P = \operatorname{im}(\varphi_P)$.
b. Show that φ is injective (respectively, surjective) if and only if the induced map on the stalks φ_P is injective (respectively, surjective) for all P.
c. Show that a sequence of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

2.1.3. *II.1.3.*

a. Let $\varphi : \mathcal{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathcal{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$ for all i.
b. Give an example of a surjective morphism of sheaves $\varphi : \mathcal{F} \to \mathscr{G}$, and an open set U such that $\varphi(U) : \mathcal{F}(U) \to \mathscr{G}(U)$ is not surjective.

2.1.4. *II.1.4.*

a. Let $\varphi : \mathcal{F} \to \mathscr{G}$ be a morphism of presheaves such that $\varphi(U) : \mathcal{F}(U) \to \mathscr{G}(U)$ is injective for each U. Show that the induced map $\varphi^+ : \mathcal{F}^+ \to \mathscr{G}^+$of associated sheaves is injective.
b. Use part (a) to show that if $\varphi : \mathcal{F} \to \mathscr{G}$ is a morphism of sheaves, then $\operatorname{im} \varphi$ can be naturally identified with a subsheaf of $\mathscr{G}$. as mentioned in the text.

2.1.5. *II.1.5.* Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.

2.1.6. *II.1.6.*

a. Let $\mathcal{F}'$ be a subsheaf of a sheaf $\tilde{\mathcal{F}}$. Show that the natural map of $\tilde{\mathcal{F}}$ to the quotient sheaf $\mathcal{F}/\mathcal{F}'$ is surjective, and has kernel $\mathcal{F}'$. Thus there is an exact sequence
b. Conversely, if is an exact sequence, show that $\mathcal{F}'$ is isomorphic to a subsheaf of $\mathcal{F}$, and that $\mathcal{F}''$ is isomorphic to the quotient of $\mathcal{F}$ by this subsheaf.

2.1.7. *II.1.7.* Let $\varphi : \mathcal{F} \to \mathscr{G}$ be a morphism of sheaves.

a. Show that $\operatorname{im} \varphi \cong \mathcal{F}/\ker \varphi$.
b. Show that $\operatorname{coker} \varphi \cong \mathscr{G}/\operatorname{im} \varphi$.

2.1.8. *II.1.8.* For any open subset $U \subseteq X$, show that the functor $\Gamma(U.\cdot)$ from sheaves on X to abelian groups is a left exact functor, i.e.. if is an exact sequence of sheaves, then is an exact sequence of groups.
The functor $\Gamma(U, -)$ need not be exact: see (Ex. 1.21) below.

2.1.9. *II.1.9. Direct Sum.* Let $\mathcal{F}$ and $\mathscr{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathcal{F}(U) \oplus \mathscr{G}(U)$ is a sheaf. It is called the **direct sum** of $\mathcal{F}$ and $\mathscr{G}$, and is denoted by $\mathcal{F} \oplus \mathscr{G}$.
Show that it plays the role of direct sum and of direct product in the category of sheaves of abelian groups on X.

2.1.10. *II.1.10. Direct Limit.* Let $\{\mathcal{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the direct limit of the system $\{\mathcal{F}_i\}$, denoted $\lim \mathcal{F}_i$, to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$.
Show that this is a direct limit in the category of sheaves on X, i.e., that it has the following universal property: given a sheaf $\mathscr{G}$, and a collection of morphisms $\mathcal{F}_i \to \mathscr{G}$. compatible with the maps of the direct system, then there exists a unique map $\varinjlim \mathcal{F}_i \to \mathscr{G}$ such that for each i, the original map $\mathscr{F} \to \mathscr{G}$ is obtained by composing the maps $\mathcal{F}_i \to \varinjlim \mathcal{F}_i \to \mathscr{G}$.

2.1.11. *II.1.11.* Let $\{\mathcal{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In this case show that the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$ is already a sheaf. In particular, $\Gamma\left(X, \varinjlim \mathcal{F}_i\right) = \varinjlim \Gamma\left(X, \mathcal{F}_i\right)$

2.1.12. *II.1.12. Inverse Limit.* Let, $\{\mathcal{F}_i\}$ be an inverse system of sheaves on X. Show that the presheaf $U \mapsto \varprojlim \mathcal{F}_i\left(U\right)$ is a sheaf. It is called the **inverse limit** of the system $\{\mathcal{F}_i\}$, and is denoted by $\varprojlim \mathcal{F}_i$. Show that it has the universal property of an inverse limit in the category of sheaves.

2.1.13. *II.1.13. Espace Etale of a Presheaf.* Given a presheaf $\mathcal{F}$ on X, we define a topological space $\mathrm{Spe}(\mathcal{F})$, called the **espace etale** of $\mathcal{F}$, as follows[26]. As a set, $\mathrm{Spe} = \bigcup_{p \in X} \mathcal{F}_P$. We define a projection map $\pi : \mathrm{Spe}(\mathcal{F}) \to X$ by sending $s \in \mathcal{F}_p$ to P. For each open set $U \subseteq X$ and each section $s \in \mathcal{F}\left(U\right)$, we obtain a map $\bar{s} : U \to \mathrm{Spe}(\mathcal{F})$ by sending $P \mapsto s_P$, its germ at P.
This map has the property that $\pi \circ \bar{s} = \mathrm{id}$, in other words, it is a "section" of π over U. We now make $\mathrm{Spe}(\mathcal{F})$ into a topological space by giving it the strongest topology such that all the maps $\bar{s} : U \to \mathrm{Spe}(\mathcal{F})$ for all U. and all $s \in \mathcal{F}\left(U\right)$, are continuous.
Now show that the sheaf $\mathcal{F}^+$associated to $\mathcal{F}$ can be described as follows: for any open set $U \subseteq X, \mathcal{F}^+\left(U\right)$ is the set of continuous sections of $\mathrm{Spe}(\mathcal{F})$ over U.
In particular, the original presheaf $\mathcal{F}$ was a sheaf if and only if for each $U, \mathcal{F}(U)$ is equal to the set of all continuous sections of $\mathrm{Spe}(\mathcal{F})$ over U.

2.1.14. *II.1.14. Support.* Let $\mathcal{F}$ be a sheaf on X, and let $s \in \mathcal{F}(U)$ be a section over an open set U. The **support of** s, denoted $\operatorname{supp} s$, is defined to be $\left\{P \in U \;\middle|\; s_P \neq 0\right\}$, where s_P denotes the germ of s in the stalk $\mathcal{F}_P$. Show that Supp s is a closed subset of U.
We define the support of $\mathcal{F}$, $\operatorname{supp} \mathcal{F}$, to be $\left\{P \in X \;\middle|\; \cdot \mathcal{F}_P \neq 0\right\}$. It need not be a closed subset.

2.1.15. *II.1.15. Sheaf $\mathcal{H}om$.* Let $\mathcal{F}, \mathscr{G}$ be sheaves of abelian groups on X. For any open set $U \subseteq X$, show that the set $\mathrm{Hom}(\mathcal{F}|_U, \mathscr{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of abelian group. Show that the presheaf $U \mapsto \mathrm{Hom}\left(\mathcal{F}|_U, \mathscr{G}|_U\right)$ is a sheaf. It is called the **sheaf of local morphisms of $\mathcal{F}$ into $\mathscr{G}$, "sheaf hom" for short**, and is denoted $\mathcal{H}om(\mathcal{F}, \mathscr{G})$.

2.1.16. *II.1.16. Flasque Sheaves.* A sheaf $\mathcal{F}$ on a topological space X is **flasque** if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

a. Show that a constant sheaf on an irreducible topological space is flasque. See (I, §1) for irreducible topological spaces.
b. If is an exact sequence of sheaves, and if $\mathcal{F}'$ is flasque, then for any open set L, the sequence of abelian groups is also exact.
c. If is an exact sequence of sheaves, and if $\mathcal{F}'$ and $\mathcal{F}$ are flasque, then $\mathcal{F}''$ is flasque.
d. If $f : X \to Y$ is a continuous map, and if $\mathcal{F}$ is a flasque sheaf on X, then $f_* \mathcal{F}$ is a flasque sheaf on Y.

[26]This exercise is included only to establish the connection between our definition of a sheaf and another definition often found in the literature. See for example Godement (1. Ch. II, 1.2).

e. Let $\mathcal{F}$ be any sheaf on X. We define a new sheaf $\mathscr{G}$, called the sheaf of **discontinuous sections** of $\mathcal{F}$ as follows. For each open set $U \subseteq X, \mathscr{G}(U)$ is the set of maps $s : U \to \bigcup_{P \in U} \mathcal{F}_P$ such that for each $P \in U, s(P) \in \mathcal{F}_P$. Show that $\mathscr{G}$ is a flasque sheaf, and that there is a natural injective morphism of $\mathcal{F}$ to $\mathscr{G}$.

2.1.17. *II.1.17. Skyscraper Sheaves.* Let X be a topological space, let P be a point, and let A be an abelian group. Define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U) = A$ if $P \in U', 0$ otherwise. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \{P\}^-$, and 0 elsewhere, where $\{P\}^-$denotes the closure of the set consisting of the point P. Hence the name "skyscraper sheaf."
Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\{P\}^-$, and $i : \{P\}^- \to X$ is the inclusion.

2.1.18. *II.1.18. Adjoint Property of f^{-1}.* Let $f : X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathcal{F}$ on X there is a natural map $f^{-1} f_* \mathcal{F} \to \mathcal{F}$, and for any sheaf $\mathscr{G}$ on Y there is a natural map $\mathscr{G} \to f_* f^{-1} \mathscr{G}$.
Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathcal{F}$ on X and $\mathscr{G}$ on Y, Hence we say that f^{-1} is a **left adjoint** of f_*, and that f_* is a **right adjoint** of f^{-1}.

2.1.19. *II.1.19. Extending a Sheaf by Zero.* Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion, let $U = X - Z$ be the complementary open subset, and let $j : U \to X$ be its inclusion.

a. Let $\mathcal{F}$ be a sheaf on Z. Show that the stalk $(i_* \mathcal{F})_P$ of the direct image sheaf on X is $\mathcal{F}_P$ if $P \in Z, 0$ if $P \notin Z$. Hence we call $i_* \cdot \mathcal{F}$ the sheaf obtained by **extending $\mathcal{F}$ by zero outside Z.** By abuse of notation we will sometimes write $\mathcal{F}$ instead of $i_* \mathcal{F}$, and say "consider $\mathcal{F}$ as a sheaf on X," when we mean "consider $i_* \mathcal{F}$.
b. Now let $\mathcal{F}$ be a sheaf on U. Let $j_!(\mathcal{F})$ be the sheaf on X associated to the presheaf $V \mapsto \mathcal{F}(V)$ if $V \subseteq U, V \mapsto 0$ otherwise. Show that the stalk $(j_!(\mathcal{F}))_P$ is equal to $\mathcal{F}_P$ if $P \in U, 0$ if $P \notin U$, and show that $j_! \mathcal{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathcal{F}$. We call $j_! \mathcal{F}$ the sheaf obtained by **extending $\mathcal{F}$ by zero outside U.**
c. Now let $\mathcal{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X,

2.1.20. *II.1.20. Subsheaf with Supports.* Let Z be a closed subset of X, and let $\mathcal{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathcal{F})$ to be the subgroup of $\Gamma(X, \mathcal{F})$ consisting of all sections whose support (Ex. 1.14) is contained in Z.

a. Show that the presheaf $V \mapsto \Gamma_{\mathrm{Z} \cap V}\left(V, \mathcal{F}|_V\right)$ is a sheaf. It is called the **subsheaf of $\mathcal{F}$ with supports in** Z, and is denoted by $\mathscr{H}_Z^0(\mathcal{F})$.
b. Let $U = X - Z$, and let $j : U \to X$ be the inclusion. Show there is an exact sequence of sheaves on X Furthermore, if $\mathcal{F}$ is flasque, the map $\mathcal{F} \to i_*(\mathcal{F}|_U)$ is surjective.

2.1.21. *II.1.21. Some Examples of Sheaves on Varieties.* Let X be a variety over an algebraically closed field k, as in Ch. I. Let $\mathcal{O}_X$ be the sheaf of regular functions on X (See (1.0 .1)).

a. Let Y be a closed subset of X. For each open set $U \subseteq X$, let $\mathscr{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathscr{I}_Y(U)$ is a sheaf. It is called the sheaf of ideals $\mathcal{I}_Y$ of Y, and it is a subsheaf of the sheaf of rings $\mathcal{O}_X$.
b. If Y is a subvariety, then the quotient sheaf $C_1 \mathcal{T}_,$is isomorphic to $i_*\left(C_1\right)$, where $i : Y \to X$ is the inclusion, and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.

c. Now let $X = \mathbf{P}^1$, and let Y be the union of two distinct points $P, Q \in X$. Then there is an exact sequence of sheaves on X where $\mathcal{F} = i_* \mathcal{O}_P \oplus i_* \mathcal{O}_Q$: Show however that the induced map on global sections $\Gamma(X; \mathcal{O}_X) \to \Gamma(X; \mathcal{F})$ is not surjective. This shows that the global section functor $\Gamma(X, \cdot)$ is not exact (cf. (Ex. 1.8) which shows that it is left exact).

d. Again let $X = \mathbf{P}^1$, and let $\mathcal{O}$ be the sheaf of regular functions. Let $\mathscr{K}$ be the constant sheaf on X associated to the function field K of X. Show that there is a natural injection $\mathcal{O} \to \mathscr{K}$. Show that the quotient sheaf $\mathscr{K}/\mathcal{O}$ is isomorphic to the direct sum of sheaves $\sum_{P \in X} i_P(I_P)$, where I_P is the group $K/\mathcal{O}_P$, and $i_P(I_P)$ denotes the skyscraper sheaf (Ex. 1.17) given by I_P at the point P.

e. Finally show that in the case of (d) the sequence is exact.[27]

2.1.22. *II.1.22. Glueing Sheaves.* Let X be a topological space, let $\mathfrak{U} = \{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathcal{F}_i$ on U_i, and for each i, j an isomorphism such that

1. for each $i, \varphi_{ii} = \mathrm{id}$, and
2. for each i, j, k, $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$.

Then there exists a unique sheaf $\mathcal{F}$ on X, together with isomorphisms $\psi_i : \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$, such that for each $i, j, \psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathcal{F}$ is obtained by **glueing** the sheaves $\mathcal{F}_i$ via the isomorphisms φ_i.

2.2. II.2: Schemes.

2.2.1. *II.2.1.* Let A be a ring, let $X = \operatorname{Spec} A$, let $f \in A$ and let $D(f) \subseteq X$ be the open complement of $V(f)$. Show that the locally ringed space $\left(D(f), \mathcal{O}_X|_{D(f)}\right)$ is isomorphic to spec A_f.

2.2.2. *II.2.2.* Let $(X, \mathcal{O}_X)$ be a scheme, and let $U \subseteq X$ be any open subset. Show that $(U, \mathcal{O}_X|_U)$ is a scheme. We call this the **induced scheme structure** on the open set U, and we refer to $(U, \mathcal{O}_X|_U)$ as an **open subscheme** of X.

2.2.3. *II.2.3 Reduced Schemes.* A scheme $(X, \mathcal{O}_X)$ is **reduced** if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

a. Show that $(X, \mathcal{O}_X)$ is reduced if and only if for every $P \in X$, the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements.

b. Let $(X, \mathcal{O}_X)$ be a scheme. Let $(\mathcal{O}_X)_{\text{red}}$ be the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)_{\text{red}}$, where for any ring A, we denote by A_{red} the quotient of A by its ideal of nilpotent elements. Show that $(X, (\mathcal{O}_X)_{\text{red}})$ is a scheme. We call it the **reduced scheme** associated to X, and denote it by X_{red}. Show that there is a morphism of schemes $X_{\text{red}} \to X$, which is a homeomorphism on the underlying topological spaces.

c. Let $f : X \to Y$ be a morphism of schemes, and assume that X is reduced. Show that there is a unique morphism $g : X \to Y_{\text{red}}$ such that f is obtained by composing g with the natural map $Y_{\text{red}} \to Y$.

2.2.4. *II.2.4.* Let A be a ring and let $(X, \mathcal{O}_X)$ be a scheme. Given a morphism $f : X \to \operatorname{Spec} A$, we have an associated map on sheaves $f^\sharp : \mathcal{U}_{\operatorname{Spec}(A)} \to f_* \mathcal{O}_X$. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. Thus there is a natural map Show that α is bijective (cf. (I, 3.5) for an analogous statement about varieties).

2.2.5. *II.2.5.* Describe Spec $\mathbf{Z}$, and show that it is a final object for the category of schemes, i.e., each scheme X admits a unique morphism to Spec $\mathbf{Z}$.

[27] This is an analogue of what is called the *first Cousin problem* in several complex variables. See Gunning and Rossi (1, p. 248)

2.2.6. *II.2.6.* Describe the spectrum of the zero ring, and show that it is an initial object for the category of schemes. (According to our conventions, all ring homomorphisms must take 1 to 1 . Since $0 = 1$ in the zero ring, we see that each ring R admits a unique homomorphism to the zero ring, but that there is no homomorphism from the zero ring to R unless $0 = 1$ in R.)

2.2.7. *II.2.7.* Let X be a scheme. For any $x \in X$, let $\mathcal{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the residue field of x on X to be the field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of $\operatorname{Spec} K$ to X it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$.

2.2.8. *II.2.8.* Let X be a scheme. For any point $x \in X$, we define the **Zariski tangent space** T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\varepsilon]/\varepsilon^2$ be the ring of dual numbers over k. Show that to give a k-morphism of $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ to X is equivalent to giving a point $x \in X$, rational over k (i.e., such that $k(x) = k$), and an element of T_x.

2.2.9. *II.2.9.* If X is a topological space, and Z an irreducible closed subset of X, a **generic point** for Z is a point ζ such that $Z = \{\zeta\}^-$. If X is a scheme, show that every (nonempty) irreducible closed subset has a unique generic point.

2.2.10. *II.2.10.* Describe $\operatorname{Spec} \mathbf{R}[x]$. How does its topological space compare to the set $\mathbf{R}$? To $\mathbf{C}$?

2.2.11. *II.2.11.* Let $k = \mathbf{F}_p$ be the finite field with p elements. Describe $\operatorname{Spec} k[x]$. What are the residue fields of its points? How many points are there with a given residue field?

2.2.12. *II.2.12 Glueing Lemma.* Generalize the glueing procedure described in the text (2.3.5) as follows. Let $\{X_i\}$ be a family of schemes (possible infinite). For each $i \neq j$, suppose given an open subset $U_{ij} \subseteq X_i$, and let it have the induced scheme structure (Ex. 2.2). Suppose also given for each $i \neq j$ an isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$ such that

(1) for each $i, j, \varphi_{ji} = \varphi_{ij}^{-1}$, and
(2) for each i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$.

Then show that there is a scheme X, together with morphisms $\psi_i : X_i \to X$ for each i, such that

(1) ψ_i is an isomorphism of X_i onto an open subscheme of X
(2) the $\psi_i(X_i)$ cover X,
(3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and (4) $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}.

We say that X is obtained by **glueing** the schemes X_i along the isomorphisms φ_{ij}. An interesting special case is when the family X_i is arbitrary, but the U_{ij} and φ_{ij} are all empty. Then the scheme X is called the **disjoint union** of the X_i, and is denoted $\coprod X_i$.

2.2.13. *II.2.13.* A topological space is **quasi-compact** if every open cover has a finite subcover.

a. Show that a topological space is noetherian (I.1) if and only if every open subset is quasi-compact.
b. If X is an affine scheme, show that $\operatorname{sp}(X)$ is quasi-compact, but not in general noetherian. We say a scheme X is quasi-compact if $\operatorname{sp}(X)$ is.
c. If A is a noetherian ring, show that $\operatorname{sp}(\operatorname{Spec} A)$ is a noetherian topological space.
d. Give an example to show that $\operatorname{sp}(\operatorname{Spec} A)$ can be noetherian even when A is not.

2.2.14. *II.2.14.*

a. Let S be a graded ring. Show that $\operatorname{Proj} S=\varnothing$ if and only if every element of $S_{+}$is nilpotent.

b. Let $\varphi: S \rightarrow T$ be a graded homomorphism of graded rings (preserving degrees). Let $U=\left\{\mathfrak{p} \in T \mid \mathfrak{p} \nsupseteq \varphi\left(S_{+}\right)\right\}$. Show that U is an open subset of $\operatorname{Proj} T$, and show that φ determines a natural morphism $f: U \rightarrow \operatorname{Proj} S$.

c. The morphism f can be an isomorphism even when φ is not. For example, suppose that $\varphi_{d}: S_{d} \rightarrow T_{d}$ is an isomorphism for all $d \geqslant d_{0}$, where d_{0} is an integer. Then show that $U=\operatorname{Proj} T$ and the morphism $f: \operatorname{Proj} T \rightarrow \operatorname{Proj} S$ is an isomorphism.

d. Let V be a projective variety with homogeneous coordinate ring S (See I.2). Show that $t(V) \cong \operatorname{Proj} S$.

2.2.15. *II.2.15.*

a. Let V be a variety over the algebraically closed field k. Show that a point $P \in t(V)$ is a closed point if and only if its residue field is k.

b. If $f: X \rightarrow Y$ is a morphism of schemes over k, and if $P \in X$ is a point with residue field k, then $f(P) \in Y$ also has residue field k.

c. Now show that if V, W are any two varieties over k, then the natural map is bijective. (Injectivity is easy. The hard part is to show it is surjective.)

2.2.16. *II.2.16.* Let X be a scheme, let $f \in \Gamma\left(X, \mathcal{O}_{X}\right)$, and define X_{f} to be the subset of points $x \in X$ such that the stalk f_{x} of f at x is not contained in the maximal ideal $\mathfrak{m}_{x}$ of the local ring $\mathcal{O}_{x}$.

a. If $U=\operatorname{Spec} B$ is an open affine subscheme of X, and if $\bar{f} \in B=\Gamma\left(U,\left.\mathcal{O}_{X}\right|_{U}\right)$ is the restriction of f, show that $U \cap X_{f}=D(\bar{f})$. Conclude that X_{f} is an open subset of X.

b. Assume that X is quasi-compact. Let $A=\Gamma\left(X, \mathcal{O}_{X}\right)$, and let $a \in A$ be an element whose restriction to X_{f} is 0 . Show that for some $n>0, f^{n} a=0$.

Hint: Use an open affine cover of X.

c. Now assume that X has a finite cover by open affines U_{i} such that each intersection $U_{i} \cap U_{j}$ is quasi-compact. (This hypothesis is satisfied, for example, if $\operatorname{sp}(X)$ is noetherian.) Let $b \in \Gamma\left(X_{f}, \mathcal{O}_{X_{f}}\right)$. Show that for some $n>0, f^{n} b$ is the restriction of an element of A.

d. With the hypothesis of (c), conclude that $\Gamma\left(X_{f}, \mathcal{O}_{X_{f}}\right) \cong A_{f}$.

2.2.17. *II.2.17 A Criterion for Affineness.*

a. Let $f: X \rightarrow Y$ be a morphism of schemes, and suppose that Y can be covered by open subsets U_{i}, such that for each i, the induced map $f^{-1}\left(U_{i}\right) \rightarrow U_{i}$ is an isomorphism. Then f is an isomorphism.

b. A scheme X is affine if and only if there is a finite set of elements $f_{1}, \ldots, f_{r} \in A=\Gamma\left(X, \mathcal{O}_{X}\right)$, such that the open subsets $X_{f_{i}}$ are affine, and $f_{1}, \ldots, f_{r}$ generate the unit ideal in A.[28]

2.2.18. *II.2.18.* In this exercise, we compare some properties of a ring homomorphism to the induced morphism of the spectra of the rings.

a. Let A be a ring, $X=\operatorname{Spec} A$, and $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.

[28] Hint: Use (Ex. 2.4) and (Ex. 2.16d) above.

b. Let $\varphi : A \to B$ be a homomorphism of rings, and let $f : Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ be the induced morphism of affine schemes. Show that φ is injective if and only if the map of sheaves $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. Show furthermore in that case f is dominant, i.e., $f(Y)$ is dense in X.

c. With the same notation, show that if φ is surjective, then f is a homeomorphism of Y onto a closed subset of X, and $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.

d. Prove the converse to (c), namely, if $f : Y \to X$ is a homeomorphism onto a closed subset, and $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, then φ is surjective.[29]

2.2.19. *II.2.19.* Let A be a ring. Show that the following conditions are equivalent:

(i) $\operatorname{Spec} A$ is disconnected;

(ii) there exist nonzero elements $e_1, e_2 \in A$ such that $e_1 e_2 = 0, e_1^2 = e_1, e_2^2 = e_2$, $e_1 + e_2 = 1$ (these elements are called **orthogonal idempotents**);

(iii) A is isomorphic to a direct product $A_1 \times A_2$ of two nonzero rings.

2.3. II.3: First Properties of Schemes.

2.3.1. *II.3.1.* Show that a morphism $f : X \to Y$ is locally of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of $Y, f^{-1}(V)$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.

2.3.2. *II.3.2.* A morphism $f : X \to Y$ of schemes is **quasi-compact** if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i.
Show that f is quasicompact if and only if for every open affine subset $V \subseteq Y, f^{-1}(V)$ is quasi-compact.

2.3.3. *II.3.3.*

a. Show that a morphism $f : X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact.

b. Conclude from this that f is of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of $Y, f^{-1}(V)$ can be covered by a finite number of open affines $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.

c. Show also if f is of finite type, then for every open affine subset $V = \operatorname{Spec} B \subseteq Y$, and for every open affine subset $U = \operatorname{Spec} A \subseteq f^{-1}(V), A$ is a finitely generated B-algebra.

2.3.4. *II.3.4.* Show that a morphism $f : X \to Y$ is finite if and only if for every open affine subset $V = \operatorname{Spec} B$ of $Y, f^{-1}(V)$ is affine, equal to $\operatorname{Spec} A$, where A is a finite B-module.

2.3.5. *II.3.5.* A morphism $f : X \to Y$ is **quasi-finite** if for every point $y \in Y, f^{-1}(y)$ is a finite set.

a. Show that a finite morphism is quasi-finite.

b. Show that a finite morphism is closed, i.e., the image of any closed subset is closed.

c. Show by example that a surjective, finite-type, quasi-finite morphism need not be finite.

2.3.6. *II.3.6.* Let X be an integral scheme. Show that the local ring $\mathcal{O}_\xi$ of the generic point ξ of X is a field. It is called the **function field** of X, and is denoted by $K(X)$.
Show also that if $U = \operatorname{Spec} A$ is any open affine subset of X, then $K(X)$ is isomorphic to the quotient field of A.

[29] Hint: Consider $X' = \operatorname{Spec}(A/\ker\varphi)$ and use (b) and (c).

2.3.7. *II.3.7.* A morphism $f : X \to Y$, with Y irreducible, is **generically finite** if $f^{-1}(\eta)$ is a finite set, where η is the generic point of Y. A morphism $f : X \to Y$ is **dominant** if $f(X)$ is dense in Y. Now let $f : X \to Y$ be a dominant, generically finite morphism of finite type of integral schemes. Show that there is an open dense subset $U \subseteq Y$ such that the induced morphism $f^{-1}(U) \to U$ is finite.[30]

2.3.8. *II.3.8. Normalization.* A scheme is **normal** if all of its local rings are integrally closed domains. Let X be an integral scheme. For each open affine subset $U = \operatorname{Spec} A$ of X, let $\tilde{A}$ be the integral closure of A in its quotient field, and let $\tilde{U} = \operatorname{Spec} \tilde{A}$.
Show that one can glue the schemes $\tilde{U}$ to obtain a normal integral scheme $\tilde{X}$, called the **normalization** of X.
Show also that there is a morphism $\tilde{X} \to X$, having the following universal property: for every normal integral scheme Z, and for every dominant morphism $f : Z \to X$, f factors uniquely through $\tilde{X}$. If X is of finite type over a field k, then the morphism $\tilde{X} \to X$ is a finite morphism. This generalizes (I, Ex. 3.17).

2.3.9. *II.3.9. The Topological Space of a Product.* Recall that in the category of varieties, the Zariski topology on the product of two varieties is not equal to the product topology (I, Ex. 1.4). Now we see that in the category of schemes, the underlying point set of a product of schemes is not even the product set.

a. Let k be a field, and let $\mathbf{A}_k^1 = \operatorname{Spec} k[x]$ be the affine line over k. Show that $\mathbf{A}_k^1 \underset{\operatorname{Spec} k}{\times} \mathbf{A}_k^1 \cong \mathbf{A}_k^2$, and show that the underlying point set of the product is not the product of the underlying point sets of the factors (even if k is algebraically closed).
b. Let k be a field, let s and t be indeterminates over k. Then $\operatorname{Spec} k(s)$, $\operatorname{Spec} k(t)$, and $\operatorname{Spec} k$ are all one-point spaces. Describe the product scheme $\operatorname{Spec}\, k(s) \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} k(t)$.

2.3.10. *II.3.10. Fibres of a Morphism.*

a. If $f : X \to Y$ is a morphism, and $y \in Y$ a point, show that $\operatorname{sp}(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.
b. Let $X = \operatorname{Spec} k[s,t]/\left(s - t^2\right)$, let $Y = \operatorname{Spec} k[s]$, and let $f : X \to Y$ be the morphism defined by sending $s \to s$.
   - If $y \in Y$ is the point $a \in k$ with $a \neq 0$, show that the fibre X_y consists of two points, with residue field k.
   - If $y \in Y$ corresponds to $0 \in k$, show that the fibre X_y is a nonreduced one-point scheme.
   - If η is the generic point of Y, show that X_η is a one-point scheme, whose residue field is an extension of degree two of the residue field of η. (Assume k algebraically closed.)

2.3.11. *II.3.11. Closed Subschemes.*

a. Closed immersions are stable under base extension: if $f : Y \to X$ is a closed immersion, and if $X' \to X$ is any morphism, then $f' : Y \times_X X' \to X'$ is also a closed immersion.

[30]Hint: First show that the function field of X is a finite field extension of the function field of Y.

(b) * If Y is a closed subscheme of an affine scheme $X = \operatorname{Spec} A$, then Y is also affine, and in fact Y is the closed subscheme determined by a suitable ideal $\mathfrak{a} \subseteq A$ as the image of the closed immersion $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$.[31][32]

c. Let Y be a closed subset of a scheme X, and give Y the reduced induced subscheme structure. If Y' is any other closed subscheme of X with the same underlying topological space, show that the closed immersion $Y \to X$ factors through Y'. We express this property by saying that **the reduced induced structure is the smallest subscheme structure on a closed subset**.

d. Let $f : Z \to X$ be a morphism. Then there is a unique closed subscheme Y of X with the following property: the morphism f factors through Y, and if Y' is any other closed subscheme of X through which f factors, then $Y \to X$ factors through Y' also. We call Y the **scheme-theoretic image** of f. If Z is a reduced scheme, then Y is just the reduced induced structure on the closure of the image $f(Z)$.

2.3.12. *II.3.12. Closed Subschemes of Proj S.*

a. Let $\varphi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set U of (Ex. 2.14) is equal to $\operatorname{Proj} T$, and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is a closed immersion.

b. If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X = \operatorname{Proj} S$ defined as image of the closed immersion $\operatorname{Proj} S/I \to X$. Show that different homogeneous ideals can give rise to the same closed subscheme. For example, let d_0 be an integer, and let $I' = \bigoplus_{d \geqslant d_0} I_d$. Show that I and I' determine the same closed subscheme.[33]

2.3.13. *II.3.13. Properties of Morphisms of Finite Type.*

a. A closed immersion is a morphism of finite type.

b. A quasi-compact open immersion (Ex. 3.2) is of finite type.

c. A composition of two morphisms of finite type is of finite type.

d. Morphisms of finite type are stable under base extension.

e. If X and Y are schemes of finite type over S, then $X \times_S Y$ is of finite type over S.

f. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two morphisms, and if f is quasi-compact, and $g \circ f$ is of finite type, then f is of finite type.

g. If $f : X \to Y$ is a morphism of finite type, and if Y is noetherian, then X is noetherian.

2.3.14. *II.3.14.* If X is a scheme of finite type over a field, show that the closed points of X are dense. Give an example to show that this is not true for arbitrary schemes.

2.3.15. *II.3.15.* Let X be a scheme of finite type over a field k (not necessarily algebraically closed).

a. Show that the following three conditions are equivalent (in which case we say that X is **geometrically irreducible**):

---

[31] Hints: First show that Y can be covered by a finite number of open affine subsets of the form $D(f_i) \cap Y$, with $f_i \in A$. By adding some more f_i with $D(f_i) \cap Y = \varnothing$, if necessary, show that we may assume that the $D(f_i)$ cover X. Next show that $f_1, \ldots, f_r$ generate the unit ideal of A. Then use (Ex. 2.17b) to show that Y is affine, and (Ex. 2.18d) to show that Y comes from an ideal $\mathfrak{a} \subseteq A$.]

[32] Note: We will give another proof of this result using sheaves of ideals later (V.10).

[33] We will see later (5.16) that every closed subscheme of X comes from a homogeneous ideal I of S (at least in the case where S is a polynomial ring over S_0).

- i: $X \underset{k}{\times} \bar{k}$ is irreducible, where $\bar{k}$ denotes the algebraic closure of k.[34]
- ii: $X \underset{k}{\times} k_s$ is irreducible, where k_s denotes the separable closure of k.
- iii: $X \underset{k}{\times} K$ is irreducible for every extension field K of k.

b. Show that the following three conditions are equivalent (in which case we say X is **geometrically reduced**):
- i: $X \underset{k}{\times} \bar{k}$ is reduced.
- ii: $X \underset{k}{\times} k_p$ is reduced, where k_p denotes the perfect closure of k.
- iii: $X \underset{k}{\times} K$ is reduced for all extension fields K of k.

c. We say that X is **geometrically integral** if $X \underset{k}{\times} \bar{k}$ is integral. Give examples of integral schemes which are neither geometrically irreducible nor geometrically reduced.

2.3.16. *II.3.16. Noetherian Induction.* Let X be a noetherian topological space, and let $\mathscr{P}$ be a property of closed subsets of X. Assume that for any closed subset Y of X, if $\mathscr{P}$ holds for every proper closed subset of Y, then $\mathscr{P}$ holds for Y. (In particular, $\mathscr{P}$ must hold for the empty set.) Then $\mathscr{P}$ holds for X.

2.3.17. *II.3.17. Zariski Spaces.* A topological space X is a **Zariski space** if it is noetherian and every (nonempty) closed irreducible subset has a unique generic point (Ex. 2.9).
For example, let R be a discrete valuation ring, and let $T = \operatorname{sp}(\operatorname{Spec} R)$. Then T consists of two points $t_0 =$ the maximal ideal, $t_1 =$ the zero ideal. The open subsets are $\varnothing, \{t_1\}$, and T. This is an irreducible Zariski space with generic point t_1.

a. Show that if X is a noetherian scheme, then $\operatorname{sp}(X)$ is a Zariski space.
b. Show that any minimal nonempty closed subset of a Zariski space consists of one point. We call these closed points.
c. Show that a Zariski space X satisfies the axiom T_0 :given any two distinct points of X, there is an open set containing one but not the other.
d. If X is an irreducible Zariski space, then its generic point is contained in every nonempty open subset of X of x_1, or that x_1 is a generization of x_0. Now let X be a Zariski space.
   - Show that the minimal points, for the partial ordering determined by $x_1 > x_0$ if $x_1 \leftrightarrow x_0$, are the closed points, and the maximal points are the generic points of the irreducible components of X.
   - Show also that a closed subset contains every specialization of any of its points. (We say closed subsets are **stable under specialization**.) Similarly, open subsets are stable under generization.
e. Let t be the functor on topological spaces introduced in the proof of (2.6). If X is a noetherian topological space, show that $t(X)$ is a Zariski space. Furthermore X itself is a Zariski space if and only if the map $\alpha : X \to t(X)$ is a homeomorphism.

2.3.18. *II.3.18 Constructible Sets.* Let X be a Zariski topological space. A constructible subset of X is a subset which belongs to the smallest family $\mathscr{F}$ of subsets such that (1) every open subset is in $\mathscr{F}$, (2) a finite intersection of elements of $\mathscr{F}$ is in $\mathscr{F}$, and (3) the complement of an element of $\mathfrak{F}$ is in $\mathfrak{F}$.

[34]By abuse of notation, we write $X \underset{k}{\times} \bar{k}$ to denote $X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} \bar{k}$.

a. A subset of X is locally closed if it is the intersection of an open subset with a closed subset. Show that a subset of X is constructible if and only if it can be written as a finite disjoint union of locally closed subsets.

b. Show that a constructible subset of an irreducible Zariski space X is dense if and only if it contains the generic point. Furthermore, in that case it contains a nonempty open subset.

c. A subset S of X is closed if and only if it is constructible and stable under specialization. Similarly, a subset T of X is open if and only if it is constructible and stable under generization.

d. If $f : X \to Y$ is a continuous map of Zariski spaces, then the inverse image of any constructible subset of Y is a constructible subset of X.

2.3.19. *II.3.19.* Let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Then the image of any constructible subset of X is a constructible subset of Y. In particular, $f(X)$, which need not be either open or closed, is a constructible subset of Y.[35]
Prove this theorem in the following steps.

a. Reduce to showing that $f(X)$ itself is constructible, in the case where X and Y are affine, integral noetherian schemes, and f is a dominant morphism.

b. * In that case, show that $f(X)$ contains a nonempty open subset of Y by using the following result from commutative algebra: let $A \subseteq B$ be an inclusion of noetherian integral domains, such that B is a finitely generated A-algebra. Then given a nonzero element $b \in B$, there is a nonzero element $a \in A$ with the following property: if $\varphi : A \to K$ is any homomorphism of A to an algebraically closed field K, such that $\varphi(a) \neq 0$, then φ extends to a homomorphism φ' of B into K, such that $\varphi'(b) \neq 0$.[36]

c. Now use noetherian induction on Y to complete the proof.

d. Give some examples of morphisms $f : X \to Y$ of varieties over an algebraically closed field k, to show that $f(X)$ need not be either open or closed.

2.3.20. *II.3.20. Dimension.* Let X be an integral scheme of finite type over a field k (not necessarily algebraically closed). Use appropriate results from I.1 to prove the following.

a. For any closed point $P \in X, \dim X = \dim \mathcal{O}_P$, where for rings, we always mean the Krull dimension.

b. Let $K(X)$ be the function field of X (Ex. 3.6). Then

c. If Y is a closed subset of X, then

d. If Y is a closed subset of X, then

e. If U is a nonempty open subset of X, then $\dim U = \dim X$.

f. If $k \subseteq k'$ is a field extension, then every irreducible component of $X' = X \underset{k}{\times} k'$ has dimension $= \dim X$.

2.3.21. *II.3.21.* Let R be a discrete valuation ring containing its residue field k. Let $X = \text{Spec } R[t]$ be the affine line over Spec R. Show that statements (a), (d), (e) of (Ex. 3.20) are false for X.

[35]The real importance of the notion of constructible subsets derives from the following theorem of Chevalley-see Cartan and Chevalley (1, exposé 7) and see also Matsumura (2, Ch. 2, 6).

[36]Hint: Prove this algebraic result by induction on the number of generators of B over A. For the case of one generator, prove the result directly. In the application, take $b = 1$.

2.3.22. *3.22. * Dimension of the Fibres of a Morphism.* Let $f : X \to Y$ be a dominant morphism of integral schemes of finite type over a field k.

a. Let Y' be a closed irreducible subset of Y, whose generic point η' is contained in $f(X)$. Let Z be any irreducible component of $f^{-1}(Y')$, such that $\eta' \in f(Z)$, and show that $\operatorname{codim}(Z, X) \leqslant \operatorname{codim}(Y', Y)$.
b. Let $e = \dim X - \dim Y$ be the relative dimension of X over Y. For any point $y \in f(X)$, show that every irreducible component of the fibre X_y has dimension $\geqslant e$.[37]
c. Show that there is a dense open subset $U \subseteq X$, such that for any $y \in f(U)$, $\dim U_y = e$.[38]
d. Going back to our original morphism $f : X \to Y$, for any integer h, let E_h be the set of points $x \in X$ such that, letting $y = f(x)$, there is an irreducible component Z of the fibre X_y, containing x, and having $\dim Z \geqslant h$. Show that
   - 1) $E_e = X$[39];
   - 2) if $h > e$, then E_h is not dense in X[40]; and
   - 3) E_h is closed, for all h[41].
e. Prove the following theorem of Chevalley-see Cartan and Chevalley (1, exposé 8): For each integer h, let C_h be the set of points $y \in Y$ such that $\dim X_y = h$. Then the subsets C_h are constructible, and C_e contains an open dense subset of Y.

2.3.23. *II.3.23.* If V, W are two varieties over an algebraically closed field k, and if $V \times W$ is their product, as defined in (I, Ex. 3.15, 3.16), and if t is the functor of (2.6), then $t(V \times W) = t(V) \underset{\operatorname{Spec} k}{\times} t(W)$.

2.4. **II.4: Separated and Proper Morphisms.**

2.4.1. *II.4.1.* Show that a finite morphism is proper.

2.4.2. *II.4.2.* Let S be a scheme, let X be a reduced scheme over S, and let Y be a separated scheme over S. Let f and g be two S-morphisms of X to Y which agree on an open dense subset of X. Show that $f = g$. Give examples to show that this result fails if either

a. X is nonreduced, or
b. Y is nonseparated.[42]

2.4.3. *II.4.3.* Let X be a separated scheme over an affine scheme S. Let U and V be open affine subsets of X. Then $U \cap V$ is also affine. Give an example to show that this fails if X is not separated.

---

[37] Hint: Let $Y' = \{y\}^-$, and use (a) and (Ex. 3.20b).

[38] Hint: First reduce to the case where X and Y are affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then A is a finitely generated B-algebra. Take $t_1, \ldots, t_e \in A$ which form a transcendence base of $K(X)$ over $K(Y)$, and let $X_1 = \operatorname{Spec} B[t_1, \ldots, t_e]$. Then X_1 is isomorphic to affine e-space over Y, and the morphism $X \to X_1$ is generically finite. Now use (Ex. 3.7) above.

[39] Use (b) above.

[40] Use (c) above.

[41] Use induction on $\dim X$.

[42] Hint: Consider the map $h : X \to Y \times {}_S Y$ obtained from f and g.

2.4.4. *II.4.4. The image of a proper scheme is proper.* Let $f : X \to Y$ be a morphism of separated schemes of finite type over a noetherian scheme S. Let Z be a closed subscheme of X which is proper over S. Show that $f(Z)$ is closed in Y, and that $f(Z)$ with its image subscheme structure (Ex. 3.11d) is proper over S.[43]

2.4.5. *II.4.5.* Let X be an integral scheme of finite type over a field k, having function field K. We say that a valuation of K/k (see I, §6) has **center** x on X if its valuation ring R dominates the local ring $\mathcal{O}_{x,X}$.

a. If X is separated over k, then the center of any valuation of K/k on X (if it exists) is unique.
b. If X is proper over k, then every valuation of K/k has a unique center on X.[44]
c. * Prove the converses of (a) and (b).[45]
d. If X is proper over k, and if k is algebraically closed, show that $\Gamma(X, \mathcal{O}_X) = k$. This result generalizes (I, 3.4a).[46]

2.4.6. *II.4.6.* Let $f : X \to Y$ be a proper morphism of affine varieties over k. Then f is a finite morphism.[47]

2.4.7. *II.4.7. Schemes Over* **R**. For any scheme X_0 over $\mathbf{R}$, let $X = X_0 \times_{\mathbf{R}} \mathbf{C}$. Let $\alpha : \mathbf{C} \to \mathbf{C}$ be complex conjugation, and let $\sigma : X \to X$ be the automorphism obtained by keeping X_0 fixed and applying α to $\mathbf{C}$. Then X is a scheme over $\mathbf{C}$, and σ is a semi-linear automorphism, in the sense that we have a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} \mathbf{C} \arrow[r, "\alpha"] & \operatorname{Spec} \mathbf{C} \end{tikzcd}$$

FIGURE 2. Link to Diagram

Since $\sigma^2 = \mathrm{id}$, we call σ an involution.

a. Now let X be a separated scheme of finite type over $\mathbf{C}$, let σ be a semilinear involution on X, and assume that for any two points $x_1, x_2 \in X$, there is an open affine subset containing both of them. (This last condition is satisfied for example if X is quasi-projective.) Show that there is a unique separated scheme X_0 of finite type over $\mathbf{R}$, such that $X_0 \times_{\mathbf{R}} \mathbf{C} \cong X$, and such that this isomorphism identifies the given involution of X with the one on $X_0 \times_{\mathbf{R}} \mathbf{C}$ described above.

   For the following statements, X_0 will denote a separated scheme of finite type over $\mathbf{R}$, and X, σ will denote the corresponding scheme with involution over $\mathbf{C}$.
b. Show that X_0 is affine if and only if X is.

---

[43]Hint: Factor f into the graph morphism $\Gamma_f : X \to X \times_s Y$ followed by the second projection p_2, and show that Γ_f is a closed immersion.

[44]Note: if X is a variety over k, the criterion of (b) is sometimes taken as the definition of a complete variety.

[45]Hint: While parts (a) and (b) follow quite easily from (4.3) and (4.7), their converses will require some comparison of valuations in different fields.

[46]Hint: Let $a \in \Gamma(X, \mathcal{O}_X)$, with $a \notin k$. Show that there is a valuation ring R of K/k with $a^{-1} \in \mathfrak{m}_R$. Then use (b) to get a contradiction.

[47]Hint: Use (4.11A).

c. If X_0, Y_0 are two such schemes over $\mathbf{R}$, then to give a morphism $f_0 : X_0 \to Y_0$ is equivalent to giving a morphism $f : X \to Y$ which commutes with the involutions, i.e., $f \circ \sigma_X = \sigma_Y \circ f$.

d. If $X \cong \mathbf{A}^1_{\mathbf{C}}$, then $X_0 \cong \mathbf{A}^1_{\mathbf{R}}$.

e. If $X \cong \mathbf{P}^1_{\mathbf{C}}$, then either $X_0 \cong \mathbf{P}^1_{\mathbf{R}}$, or X_0 is isomorphic to the conic in $\mathbf{P}^2_{\mathbf{R}}$ given by the homogeneous equation $x_0^2 + x_1^2 + x_2^2 = 0$.

2.4.8. *II.4.8.* Let $\mathscr{P}$ be a property of morphisms of schemes such that:

a. a closed immersion has $\mathscr{P}$;

b. a composition of two morphisms having $\mathscr{P}$ has $\mathscr{P}$;

c. $\mathscr{P}$ is stable under base extension.

Then show that:

d. a product of morphisms having $\mathscr{P}$ has $\mathscr{P}$;

e. if $f : X \to Y$ and $g : Y \to Z$ are two morphisms, and if $g \circ f$ has $\mathscr{P}$ and g is separated, then f has $\mathscr{P}$;[48]

f. If $f : X \to Y$ has $\mathscr{P}$, then $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ has $\mathscr{P}$.

2.4.9. *II.4.9.* Show that a composition of projective morphisms is projective.[49] Conclude that projective morphisms have properties (a)-(f) of (Ex. 4.8) above.

2.4.10. *II.4.10 * Chow's Lemma.* This result says that proper morphisms are fairly close to projective morphisms.
Let X be proper over a noetherian scheme S. Then there is a scheme X' and a morphism $g : X' \to X$ such that X' is projective over S, and there is an open dense subset $U \subseteq X$ such that g induces an isomorphism of $g^{-1}(U)$ to U. Prove this result in the following steps.

a. Reduce to the case X irreducible.

b. Show that X can be covered by a finite number of open subsets $U_i, i = 1, \ldots, n$, each of which is quasi-projective over S. Let $U_i \to P_i$ be an open immersion of U_i into a scheme P_i which is projective over S.

c. Let $U = \bigcap U_i$, and consider the map deduced from the given maps $U \to X$ and $U \to P_i$. Let X' be the closed image subscheme structure[50] $f(U)^-$. Let $g : X' \to X$ be the projection onto the first factor, and let be the projection onto the product of the remaining factors. Show that h is a closed immersion, hence X' is projective over S.

d. Show that $g^{-1}(U) \to U$ is an isomorphism, thus completing the proof.

2.4.11. *II.4.11.* If you are willing to do some harder commutative algebra, and stick to noetherian schemes, then we can express the valuative criteria of separatedness and properness using only **discrete** valuation rings.

a. If $\mathcal{O}, \mathfrak{m}$ is a noetherian local domain with quotient field K, and if L is a finitely generated field extension of K, then there exists a discrete valuation ring R of L dominating $\mathcal{O}$.

Prove this in the following steps.

- By taking a polynomial ring over $\mathcal{O}$, reduce to the case where L is a finite extension field of K.

[48] Hint: For (e), consider the graph morphism $\Gamma_f : X \to X \times_Z Y$ and note that it is obtained by base extension from the diagonal morphism $\Delta : Y \to Y \times_Z Y$.

[49] Hint: Use the Segre embedding defined in (I, Ex. 2.14) and show that it gives a closed immersion $\mathbf{P}^r \times \mathbf{P}^s \to \mathbf{P}^{r+r+s}$..

[50] See Ex. 3.11d.

- Then show that for a suitable choice of generators $x_1, \ldots, x_n$ of m, the ideal $\mathfrak{a} = (x_1)$ in $\mathcal{O}' = \mathbb{C}[x_2/x_1, \ldots, x_n/x_1]$ is not equal to the unit ideal.
- Then let $\mathfrak{p}$ be a minimal prime ideal of a, and let $\mathcal{O}'_p$ be the localization of $\mathcal{O}'$ at $\mathfrak{p}$. This is a noetherian local domain of dimension 1 dominating $\mathcal{O}$.
- Let $\tilde{\mathcal{O}}'_p$ be the integral closure of $\mathcal{O}'_p$ in L. Use the theorem of Krull-Akizuki[51] to show that $\tilde{U}'_{\mathrm{p}}$ is noetherian of dimension 1.
- Finally, take R to be a localization of $\tilde{\mathcal{O}}'_p$ at one of its maximal ideals.

b. Let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Show that f is separated (respectively, proper) if and only if the criterion of (4.3) (respectively, (4.7)) holds for all discrete valuation rings.

2.4.12. *II.4.12 Examples of Valuation Rings.* Let k be an algebraically closed field.

a. If K is a function field of dimension 1 over k then every valuation ring of K/k (except for K itself) is discrete. Thus the set of all of them is just the abstract nonsingular curve C_K of $(I, §6)$.

b. If K/k is a function field of dimension two, there are several different kinds of valuations. Suppose that X is a complete nonsingular surface with function field K.

- If Y is an irreducible curve on X, with generic point x_1, then the local ring $R = \mathcal{O}_{x_1,X}$ is a discrete valuation ring of K/k with center at the (nonclosed) point x_1 on X.
- If $f : X' \to X$ is a birational morphism, and if Y' is an irreducible curve in X' whose image in X is a single closed point x_0, then the local ring R of the generic point of Y' on X' is a discrete valuation ring of K/k with center at the closed point x_0 on X.
- Let $x_0 \in X$ be a closed point. Let $f : X_1 \to X$ be the blowing-up of x_0 (I, §4) and let $E_1 = f^{-1}(x_0)$ be the exceptional curve. Choose a closed point $x_1 \in E_1$, let $f_2 : X_2 \to X_1$ be the blowing-up of x_1, and let $E_2 = f_2^{-1}(x_1)$ be the exceptional curve. Repeat.

  In this manner we obtain a sequence of varieties X_i with closed points x_i chosen on them, and for each i, the local ring $\mathcal{O}_{x_{i+1},X_{i+1}}$ dominates $\mathcal{O}_{x_i,X_i}$. Let $R_0 = \bigcup_{i=0}^{\infty} \mathcal{O}_{x_i,X_i}$. Then R_0 is a local ring, so it is dominated by some valuation ring R of K/k by (I, 6.1A).

  Show that R is a valuation ring of K/k, and that it has center x_0 on X. When is R a discrete valuation ring?[52]

2.5. **II.5: Sheaves of Modules.**

2.5.1. *II.5.1.* Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. We define the dual of $\mathcal{E}$, denoted $\mathcal{E}^\vee$, to be the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.

a. Show that $(\mathcal{E}^\vee)^\vee \cong \mathcal{E}$.

b. For any $\mathcal{O}_X$-module $\mathcal{F}$,

c. (Projection Formula). If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathcal{F}$ is an $\mathcal{O}_X$-module, and if $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then there is a natural isomorphism

2.5.2. *II.5.2.* Let R be a discrete valuation ring with quotient field K, and let $X = \operatorname{Spec} R$.

a. To give an $\mathcal{O}_X$-module is equivalent to giving an R-module M, a K-vector space L, and a homomorphism $\rho : M \otimes_R K \to L$.

b. That $\mathcal{O}_X$-module is quasi-coherent if and only if ρ is an isomorphism.

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[51]See Nagata 7, p. 115.

[52]Note. We will see later (V, Ex. 5.6) that in fact the R_0 of (3) is already a valuation ring itself, so $R_0 = R$. Furthermore, every valuation ring of K/k (except for K itself) is one of the three kinds just described.

2.5.3. *II.5.3.* Let $X = \operatorname{Spec} A$ be an affine scheme. Show that the functors $\sim$ and Γ are adjoint, in the following sense: for any A-module M, and for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, there is a natural isomorphism

2.5.4. *II.5.4.* Show that a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on a scheme X is quasi-coherent if and only if every point of X has a neighborhood U, such that $\mathcal{F}|_U$ is isomorphic to a cokernel of a morphism of free sheaves on U. If X is noetherian, then $\mathcal{F}$ is coherent if and only if it is locally a cokernel of a morphism of free sheaves of finite rank. (These properties were originally the definition of quasi-coherent and coherent sheaves.)

2.5.5. *II.5.5.* Let $f : X \to Y$ be a morphism of schemes.

a. Show by example that if $\mathcal{F}$ is coherent on X, then $f_*\mathcal{F}$ need not be coherent on Y, even if X and Y are varieties over a field k.
b. Show that a closed immersion is a finite morphism (§3).
c. If f is a finite morphism of noetherian schemes, and if $\mathcal{F}$ is coherent on X, then $f_*\mathcal{F}$ is coherent on Y.

2.5.6. *II.5.6. Support.* Recall the notions of support of a section of a sheaf, support of a sheaf, and subsheaf with supports from (Ex. 1.14) and (Ex. 1.20).

a. Let A be a ring, let M be an A-module, let $X = \operatorname{Spec} A$, and let $\mathcal{F} = \tilde{M}$. For any $m \in M = \Gamma(X, \mathcal{F})$, show that $\operatorname{supp} m = V(\operatorname{Ann} m)$, where $\operatorname{Ann} m$ is the annihilator of $m = \{a \in A \mathrel{\Big|} am = 0\}$.
b. Now suppose that A is noetherian, and M finitely generated. Show that $\operatorname{supp} \mathcal{F} = V(\operatorname{Ann} M)$.
c. The support of a coherent sheaf on a noetherian scheme is closed.
d. For any ideal $\mathfrak{a} \subseteq A$, we define a submodule $\Gamma_{\mathfrak{a}}(M)$ of M by Assume that A is noetherian, and M any A-module. Show that $\Gamma_{\mathfrak{a}}(M)^{\sim} \cong \mathcal{H}^0_Z(\mathcal{F})$, where $Z = V(\mathfrak{a})$ and $\mathcal{F} = \tilde{M}$.[53]
e. Let X be a noetherian scheme, and let Z be a closed subset. If $\mathcal{F}$ is a quasicoherent (respectively, coherent) $\mathcal{O}_X$-module, then $\mathcal{H}^0_Z(\mathcal{F})$ is also quasicoherent (respectively, coherent).

2.5.7. *II.5.7.* Let X be a noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf.

a. If the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_x$-module for some point $x \in X$, then there is a neighborhood U of x such that $\mathcal{F}|_U$ is free.
b. $\mathcal{F}$ is locally free if and only if its stalks $\mathcal{F}_x$ are free $\mathcal{O}_x$-modules for all $x \in X$.
c. $\mathcal{F}$ is invertible (i.e., locally free of rank 1) if and only if there is a coherent sheaf $\mathcal{G}$ such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$.[54]

2.5.8. *II.5.8.* Again let X be a noetherian scheme, and $\mathcal{F}$ a coherent sheaf on X. We will consider the function where $k(x) = \mathcal{O}_x/m_x$ is the residue field at the point x. Use Nakayama's lemma to prove the following results.

a. The function φ is upper semi-continuous, i.e., for any $n \in \mathbf{Z}$, the set $\{x \in X \mathrel{\Big|} \varphi(x) \geqslant n\}$ is closed.
b. If $\mathcal{F}$ is locally free, and X is connected, then φ is a constant function.
c. Conversely, if X is reduced, and φ is constant, then $\mathcal{F}$ is locally free.

[53] Hint: Use (Ex. 1.20) and (5.8) to show a priori that $\mathcal{H}^0_Z(\mathcal{F})$ is quasi-coherent. Then show that $\Gamma_{\mathrm{a}}(M) \cong \Gamma_Z(\mathcal{F})$.

[54] This justifies the terminology invertible: it means that $\mathcal{F}$ is an invertible element of the monoid of coherent sheaves under the operation $\otimes$.

2.5.9. *II.5.9.* Let S be a graded ring, generated by S_1 as an S_0-algebra, let M be a graded S module, and let $X = \text{Proj S}$.

a. Show that there is a natural homomorphism $\alpha : M \to \Gamma_*(\tilde{M})$.
b. Assume now that $S_0 = A$ is a finitely generated k-algebra for some field k, that S_1 is a finitely generated A-module, and that M is a finitely generated S-module. Show that the map α is an isomorphism in all large enough degrees, i.e., there is a $d_0 \in \mathbf{Z}$ such that for all $d \geqslant d_0, \alpha_d : M_d \to \Gamma(X, \tilde{M}(d))$ is an isomorphism.[55]
c. With the same hypotheses, we define an equivalence relation $\approx$ on graded S-modules by saying $M \approx M'$ if there is an integer d such that $M_{\geqslant d} \cong M'_{\geqslant d}$. Here $M_{\geqslant d} = \bigoplus_{n \geqslant d} M_n$. We will say that a graded S-module M is **quasifinitely generated** if it is equivalent to a finitely generated module.

   Now show that the functors $\sim$ and Γ_* induce an equivalence of categories between the category of quasi-finitely generated graded S-modules modulo the equivalence relation $\approx$, and the category of coherent $\mathcal{O}_X$-modules.

2.5.10. *II.5.10.* Let A be a ring, let $S = A\left[x_0, \ldots, x_r\right]$ and let $X = \operatorname{Proj} S$. We have seen that a homogeneous ideal I in S defines a closed subscheme of X (Ex. 3.12), and that conversely every closed subscheme of X arises in this way (5.16).

a. For any homogeneous ideal $I \subseteq S$, we define the saturation $\bar{I}$ of I to be $\left\{ s \in S \;\middle|\; \text{for each } i = 0, \ldots, r, \text{ there is an } n \text{ such that } x_i^n s \in I \right\}$. We say that I is saturated if $I = \bar{I}$. Show that $\bar{I}$ is a homogeneous ideal of S.
b. Two homogeneous ideals I_1 and I_2 of S define the same closed subscheme of X if and only if they have the same saturation.
c. If Y is any closed subscheme of X, then the ideal $\Gamma_*\left(\mathcal{I}_Y\right)$ is saturated. Hence it is the largest homogeneous ideal defining the subscheme Y.
d. There is a 1-1 correspondence between saturated ideals of S and closed subschemes of X.

2.5.11. *II.5.11.* Let S and T be two graded rings with $S_0 = T_0 = A$. We define the **Cartesian product** $S \underset{A}{\times} T$ to be the graded ring $\bigoplus_{d \geqslant 0} S_d \otimes_A T_d$. If $X = \operatorname{Proj} S$ and $Y = \operatorname{Proj} T$, show that $\operatorname{Proj}\left(S \times{}_A T\right) \cong X \times{}_A Y$, and show that the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj}\left(S \times{}_A T\right)$ is isomorphic to the sheaf $p_1^*\left(\mathcal{O}_X(1)\right) \otimes p_2^*\left(\mathcal{O}_Y(1)\right)$ on $X \times Y$.
The Cartesian product of rings is related to the **Segre embedding** of projective spaces (I, Ex. 2.14) in the following way. If $x_0, \ldots, x_r$ is a set of generators for S_1 over A, corresponding to a projective embedding $X \hookrightarrow \mathbf{P}_A^r$, and if $y_0, \ldots, y_s$ is a set of generators for T_1, corresponding to a projective embedding $Y \hookrightarrow P_A^s$, then $\{x_i \otimes y_j\}$ is a set of generators for $\left(S \times{}_A T\right)_1$, and hence defines a projective embedding $\operatorname{Proj}(S \underset{A}{\times} T) \hookrightarrow \mathbf{P}_A^N$, with $N = rs + r + s$. This is just the image of $X \times Y \subseteq \mathbf{P}^r \times \mathbf{P}^s$ in its Segre embedding.

2.5.12. *II.5.12.*

a. Let X be a scheme over a scheme Y, and let $\mathcal{L}, \mathcal{M}$ be two very ample invertible sheaves on X. Show that $\mathcal{L} \otimes \mathcal{M}$ is also very ample.[56]
b. Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of schemes. Let $\mathcal{L}$ be a very ample invertible sheaf on X relative to Y, and let $\mathcal{M}$ be a very ample invertible sheaf on Y relative to Z. Show that $\mathcal{L} \otimes f^* \mathcal{M}$ is a very ample invertible sheaf on X relative to Z.

[55] Hint: Use the methods of the proof of (5.19).
[56] Hint: Use a Segre embedding.

2.5.13. *II.5.13.* Let S be a graded ring, generated by S_1 as an S_0-algebra. For any integer $d>0$, let $S^{(d)}$ be the graded ring $\bigoplus_{n \geqslant 0} S_n^{(d)}$ where $S_n^{(d)}=S_{nd}$. Let $X=\operatorname{Proj} S$. Show that $\operatorname{Proj} S^{(d)} \cong X$, and that the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj} S^{(d)}$ corresponds via this isomorphism to $\mathcal{O}_X(d)$.
This construction is related to the d-uple embedding (I, Ex. 2.12) in the following way. If $x_0, \ldots, x_r$ is a set of generators for S_1, corresponding to an embedding $X \hookrightarrow \mathbf{P}_A^r$, then the set of monomials of degree d in the x_i is a set of generators for $S_1^{(d)}=S_d$. These define a projective embedding of $\operatorname{Proj} S^{(d)}$ which is none other than the image of X under the d-uple embedding of $\mathbf{P}_A^r$.

2.5.14. *II.5.14.* Let A be a ring, and let X be a closed subscheme of $\mathbf{P}_A^r$. We define the **homogeneous coordinate ring** $S(X)$ of X for the given embedding to be $A\left[x_0, \ldots, x_r\right]/I$, where I is the ideal $\Gamma_*\left(\mathcal{I}_X\right)$ constructed in the proof of (5.16). Of course if A is a field and X a variety, this coincides with the definition given in (I, §2)! Recall that a scheme X is **normal** if its local rings are integrally closed domains.
A closed subscheme $X \subseteq \mathbf{P}_A^r$ is **projectively normal** for the given embedding, if its homogeneous coordinate ring $S(X)$ is an integrally closed domain (cf. (I, Ex. 3.18)).
Now assume that k is an algebraically closed field, and that X is a connected, normal closed subscheme of $\mathbf{P}_k^r$. Show that for some $d>0$, the d-uple embedding of X is projectively normal, as follows.

a. Let S be the homogeneous coordinate ring of X, and let $S^{\prime}=\bigoplus_{n \geqslant 0} \Gamma\left(X, \mathcal{O}_X(n)\right)$. Show that S is a domain, and that $S^{\prime}$ is its integral closure.[57]
b. Use (Ex. 5.9) to show that $S_d=S_d^{\prime}$ for all sufficiently large d.
c. Show that $S^{(d)}$ is integrally closed for sufficiently large d, and hence conclude that the d-uple embedding of X is projectively normal.
d. As a corollary of (a), show that a closed subscheme $X \subseteq \mathbf{P}_A^r$ is projectively normal if and only if it is normal, and for every $n \geqslant 0$ the natural map $\Gamma\left(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(n)\right) \rightarrow \Gamma\left(X, \mathcal{O}_X(n)\right)$ is surjective.

2.5.15. *II.5.15. Extension of Coherent Sheaves.* We will prove the following theorem in several steps: Let X be a noetherian scheme, let U be an open subset, and let $\mathcal{F}$ be a coherent sheaf on U. Then there is a coherent sheaf $\mathcal{F}^{\prime}$ on X such that $\left.\mathcal{F}^{\prime}\right|_U \cong \mathcal{F}$.

a. On a noetherian affine scheme, every quasi-coherent sheaf is the union of its coherent subsheaves. We say a sheaf $\mathcal{F}$ is the union of its subsheaves $\mathcal{F}$ if for every open set U, the group $\mathcal{F}(U)$ is the union of the subgroups $\mathcal{F}_\alpha(U)$.
b. Let X be an affine noetherian scheme, U an open subset, and $\mathcal{F}$ coherent on U. Then there exists a coherent sheaf $\mathcal{F}^{\prime}$ on X with $\left.\mathcal{F}^{\prime}\right|_U \cong \mathcal{F}$.[58]
c. With $X, U, \mathcal{F}$ as in (b), suppose furthermore we are given a quasi-coherent sheaf $\mathcal{G}$ on X such that $\mathcal{F} \subseteq \left.\mathcal{G}\right|_U$. Show that we can find $\mathcal{F}^{\prime}$ a coherent subsheaf of $\mathcal{G}$, with $\left.\mathcal{F}^{\prime}\right|_U \cong \mathcal{F}$.[59]
d. Now let X be any noetherian scheme, U an open subset, $\mathcal{F}$ a coherent sheaf on U, and $\mathcal{G}$ a quasi-coherent sheaf on X such that $\mathcal{F} \subseteq \left.\mathcal{G}\right|_U$. Show that there is a coherent subsheaf $\mathcal{F}^{\prime} \subseteq \mathcal{G}$ on X with $\left.\mathcal{F}^{\prime}\right|_U \cong \mathcal{F}$. Taking $\mathcal{G}=i_* \mathcal{F}$ proves the result announced at the beginning.[60]

[57]Hint: First show that X is integral. Then regard $S^{\prime}$ as the global sections of the sheaf of rings $\mathcal{S}=\bigoplus_{n \geq 0} \mathcal{O}_X(n)$ on X, and show that $\mathcal{S}$ is a sheaf of integrally closed domains.

[58]Hint: Let $i: U \rightarrow X$ be the inclusion map. Show that $i_* \mathcal{F}$ is quasi-coherent, then use (a).

[59]Hint: Use the same method, but replace $i_* \mathcal{F}$ by $\rho^{-1}\left(i_* \mathcal{F}\right)$, where ρ is the natural map $\mathcal{G} \rightarrow i_*\left(\left.\mathcal{G}\right|_U\right)$.

[60]Hint: Cover X with open affines, and extend over one of them at a time.

e. As an extra corollary, show that on a noetherian scheme, any quasi-coherent sheaf $\mathcal{F}$ is the union of its coherent subsheaves.[61]

2.5.16. *II.5.16. Tensor Operations on Sheaves.* First we recall the definitions of various tensor operations on a module. Let A be a ring, and let M be an A-module.

- Let $T^n(M)$ be the tensor product $M \otimes \ldots \otimes M$ of M with itself n times, for $n \geqslant 1$. For $n = 0$ we put $T^0(M) = A$. Then $T(M) = \bigoplus_{n \geqslant 0} T^n(M)$ is a (noncommutative) A-algebra, which we call the **tensor algebra** of M.
- We define the **symmetric algebra** $S(M) = \bigoplus_{n \geqslant 0} S^n(M)$ of M to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes y - y \otimes x$, for all $x, y \in M$. Then $S(M)$ is a commutative A-algebra. Its component $S^n(M)$ in degree n is called the nth **symmetric product** of M. We denote the image of $x \otimes y$ in $S(M)$ by xy, for any $x, y \in M$. As an example, note that if M is a free A-module of rank r, then $S(M) \cong A\left[x_1, \ldots, x_r\right]$
- We define the **exterior algebra** $\bigwedge(M) = \bigoplus_{n \geqslant 0} \bigwedge^n(M)$ of M to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes x$ for $x \in M$. Note that this ideal contains all expressions of the form $x \otimes y + y \otimes x$, so that $\bigwedge(M)$ is a skew commutative graded A-algebra. This means that if $u \in \bigwedge^r(M)$ and $v \in \bigwedge^s(M)$, then $u \wedge v = (-1)^{rs} v \wedge u$ (here we denote by $\wedge$ the multiplication in this algebra; so the image of $x \otimes y$ in $\bigwedge^2(M)$ is denoted by $x \wedge y$). The nth component $\bigwedge^n(M)$ is called the nth **exterior power** of M.

Now let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the tensor algebra, symmetric algebra, and exterior algebra of $\mathcal{F}$ by taking the sheaves associated to the presheaf, which to each open 'set U assigns the corresponding tensor operation applied to $\mathcal{F}(U)$ as an $\mathcal{O}_X(U)$-module. The results are $\mathcal{O}_x$-algebras, and their components in each degree are $\mathcal{O}_x$-modules.

a. Suppose that $\mathcal{F}$ is locally free of rank n. Then $T^r(\mathcal{F}), S^r(\mathcal{F})$, and $\bigwedge^r(\mathcal{F})$ are also locally free, of ranks n^r, $\left(\begin{array}{c} n+r-1 \\ n-1 \end{array}\right)$, and $\left(\begin{array}{c} r \\ r \end{array}\right)$ respectively.

b. Again let $\mathcal{F}$ be locally free of rank n. Then the multiplication map $\bigwedge^r \mathcal{F} \otimes \bigwedge^{n-r} \mathcal{F} \rightarrow \bigwedge^n \cdot \mathcal{F}$ is a perfect pairing for any r, i.c., it induces an isomorphism of $\bigwedge^r \mathcal{F}$ with $\left(\bigwedge^{n-r} \mathcal{F}\right)^{\ulcorner} \otimes \bigwedge^n \mathcal{F}$. As a special case, note if $\mathcal{F}$ has rank 2 , then $\mathcal{F} \cong \mathcal{F}^{\top} \otimes \bigwedge^2 \mathcal{F}$.

c. Let $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ be an exact sequence of locally free sheaves. Then for any r there is a finite filtration of $S^r(\mathcal{F})$, with quotients for each p.

d. Same statement as (c), with exterior powers instead of symmetric powers. In particular, if $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ have ranks n', n, n'' respectively, there is an isomorphism

e. Let $f : X \rightarrow Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be an $\mathcal{O}_Y$-module. Then f^* commutes with all the tensor operations on $\mathcal{F}$, i.e., $f^*\left(S^n(\mathcal{F})\right) = S^n\left(f^* \mathcal{F}\right)$ etc.

2.5.17. *II.5.17. Affine Morphisms.* A morphism $f : X \rightarrow Y$ of schemes is **affine** if there is an open affine cover $\{V_i\}$ of Y such that $f^{-1}\left(V_i\right)$ is affine for each i.

a. Show that $f : X \rightarrow Y$ is an affine morphism if and only if for every open affine $V \subseteq Y, f^{-1}(V)$ is affine[62]

b. An affine morphism is quasi-compact and separated. Any finite morphism is affine.

c. Let Y be a scheme, and let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras (i.e., a sheaf of rings which is at the same time a quasi-coherent sheaf of $\mathcal{O}_Y$-modules). Show that there

[61] Hint: If s is a section of $\mathcal{F}$ over an open set U, apply (d) to the subsheaf of $\left.\mathcal{F}\right|_U$ generated by s.

[62] Hint: Reduce to the case Y affine, and use (Ex. 2.17).

is a unique scheme X, and a morphism $f : X \to Y$, such that for every open affine $V \subseteq Y, f^{-1}(V) \cong \operatorname{Spec} \mathcal{A}(V)$, and for every inclusion $U \hookrightarrow V$ of open affines of Y, the morphism $f^{-1}(U) \hookrightarrow f^{-1}(V)$ corresponds to the restriction homomorphism $\mathcal{A}(V) \to \mathcal{A}(U)$. The scheme X is called $\operatorname{Spec} \mathcal{A}$.[63]

d. If $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_Y$-algebra, then $f : X = \operatorname{Spec} \mathcal{A} \to Y$ is an affine morphism, and $\mathcal{A} \cong f_* \mathcal{O}_X$. Conversely, if $f : X \to Y$ is an affine morphism, then $\mathcal{A} = f_* \mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras, and $X \cong \operatorname{Spec} \mathcal{A}$.

e. Let $f : X \to Y$ be an affine morphism, and let $\mathcal{A} = f_* \mathcal{O}_X$. Show that f_* induces an equivalence of categories from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of quasi-coherent $\mathcal{A}$-modules (i.e., quasi-coherent $\mathcal{O}_Y$-modules having a structure of $\mathcal{A}$-module).[64]

2.5.18. *II.5.18. Vector Bundles.* Let Y be a scheme. A **(geometric) vector bundle** of rank n over Y is a scheme X and a morphism $f : X \to Y$, together with additional data consisting of an open covering $\{U_i\}$ of Y, and isomorphisms $\psi_i : f^{-1}(U_i) \to \mathbf{A}^n_{U_i}$, such that for any i, j, and for any open affine subset $V = \operatorname{Spec} A \subseteq U_i \cap U_j$, the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of $\mathbf{A}^n_V = \operatorname{Spec} A[x_1, \ldots, x_n]$ is given by a *linear* automorphism θ of $A[x_1, \ldots, x_n]$, i.e., $\theta(a) = a$ for any $a \in A$, and $\theta(x_i) = \sum a_{ij} x_j$ for suitable $a_{ij} \in A$.

An **isomorphism** of one vector bundle of rank n to another one is an isomorphism $g : X \to X'$ of the underlying schemes, such that $f = f' \circ g$, and such that X, f, together with the covering of Y consisting of all the U_i and U_i', and the isomorphisms ψ_i and $\psi_i' \circ g$, is also a vector bundle structure on X.

a. Let $\mathcal{E}$ be a locally free sheaf of rank n on a scheme Y. Let $S(\mathcal{E})$ be the symmetric algebra on $\mathcal{E}$, and let $X = \operatorname{Spec} S(\mathcal{E})$, with projection morphism $f : X \to Y$. For each open affine subset $U \subseteq Y$ for which $\left.\mathcal{E}\right|_U$ is free, choose a basis of $\mathcal{E}$, and let $\psi : f^{-1}(U) \to \mathbf{A}^n_U$ be the isomorphism resulting from the identification of $S(\mathcal{E}(U))$ with $\mathcal{O}(U)[x_1, \ldots, x_n]$.

Then $(X, f, \{U\}, \{\psi\})$ is a vector bundle of rank n over Y, which (up to isomorphism) does not depend on the bases of $\mathcal{E}_U$ chosen. We call it the geometric vector bundle associated to $\mathcal{E}$, and denote it by $\mathbf{V}(\mathcal{E})$.

b. For any morphism $f : X \to Y$, a section of f over an open set $U \subseteq Y$ is a morphism $s : U \to X$ such that $f \circ s = \mathrm{id}_U$. It is clear how to restrict sections to smaller open sets, or how to glue them together, so we see that the presheaf $U \mapsto \{$ set of sections of f over $U\}$ is a sheaf of sets on Y, which we denote by $\mathcal{S}(X/Y)$.

Show that if $f : X \to Y$ is a vector bundle of rank n, then the sheaf of sections $\mathcal{S}(X/Y)$ has a natural structure of $\mathcal{O}_Y$-module, which makes it a locally free $\mathcal{O}_Y$-module of rank n.[65]

c. Again let $\mathcal{E}$ be a locally free sheaf of rank n on Y, let $X = \mathbf{V}(\mathcal{E})$, and let $\mathcal{S} = \mathcal{S}(X/Y)$ be the sheaf of sections of X over Y. Show that $\mathcal{S} \cong \mathcal{E}^2$, as follows. Given a section $s \in \Gamma(V, \mathcal{E}^\curlyvee)$ over any open set V, we think of s as an element of $\operatorname{Hom}(\left.\mathcal{E}\right|_V, \mathcal{O}_V)$. So s determines an $\mathcal{O}_V$-algebra homomorphism $S(\left.\mathcal{E}\right|_V) \to \mathcal{O}_V$.

This determines a morphism of spectra $V = \operatorname{Spec} \mathcal{O}_V \to \operatorname{Spec} S(\left.\mathcal{E}\right|_V) = f^{-1}(V)$, which is a section of X/Y. Show that this construction gives an isomorphism of $\mathcal{E}^2$ to $\mathcal{S}$.

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[63]Hint: Construct X by glueing together the schemes $\operatorname{Spec} \mathcal{A}(V)$, for V open affine in Y.

[64]Hint: For any quasi-coherent $\mathcal{A}$-module $\mathcal{M}$, construct a quasi-coherent $\mathcal{O}_X$-module $\tilde{\mathcal{M}}$, and show that the functors f_* and $\sim$ are inverse to each other.

[65]Hint: It is enough to define the module structure locally, so we can assume $Y = \operatorname{Spec} A$ is affine, and $X = \mathbf{A}^n_Y$. Then a section $s : Y \to X$ comes from an A-algebra homomorphism $\theta : A[x_1, \ldots, x_n] \to A$, which in turn determines an ordered n-tuple $\langle \theta(x_1), \ldots, \theta(x_n) \rangle$ of elements of A. Use this correspondence between sections s and ordered n-tuples of elements of A to define the module structure.]

d. Summing up, show that we have established a one-to-one correspondence between isomorphism classes of locally free sheaves of rank n on Y, and isomorphism classes of vector bundles of rank n over Y. Because of this, we sometimes use the words "locally free sheaf" and "vector bundle" interchangeably, if no confusion seems likely to result.

2.6. **II.6: Divisors.** In this section we will consider schemes satisfying the following condition: $(*)X$ is a noetherian integral separated scheme which is regular in codimension one.

2.6.1. *II.6.1.* Let X be a scheme satisfying $(*)$. Then $X \times \mathbf{P}^n$ also satisfies $(*)$, and $\mathrm{Cl}\,(X \times \mathbf{P}^n) \cong (\mathrm{Cl}\,X) \times \mathbf{Z}$

2.6.2. *II.6.2. * Varieties in Projective Space.* Let k be an algebraically closed field, and let X be a closed subvariety of $\mathbf{P}^n_k$ which is nonsingular in codimension one (hence satisfies $(*)$). For any divisor $D = \sum n_i Y_i$ on X, we define the **degree** of D to be $\sum n_i \deg Y_i$, where $\deg Y_i$ is the degree of Y_i, considered as a projective variety itself (I, §7).

a. Let V be an irreducible hypersurface in $\mathbf{P}^n$ which does not contain X, and let Y_i be the irreducible components of $V \cap X$. They all have codimension 1 by (I, Ex. 1.8). For each i, let f_i be a local equation for V on some open set U_i of $\mathbf{P}^n$ for which $Y_i \cap U_i \neq \varnothing$, and let $n_i = v_{Y_i}(\overline{f}_i)$, where $\overline{f}_i$ is the restriction of f_i to $U_i \cap X$.

   Then we define the **divisor** $V.X$ to be $\sum n_i Y_i$. Extend by linearity, and show that this gives a well-defined homomorphism from the subgroup of Div $\mathbf{P}^n$ consisting of divisors, none of whose components contain X, to $\mathrm{Div}\,X$.

b. If D is a principal divisor on $\mathbf{P}^n$, for which $D.X$ is defined as in (a), show that $D.X$ is principal on X. Thus we get a homomorphism $\mathrm{Cl}\,\mathbf{P}^n \to \mathrm{Cl}\,X$.

c. Show that the integer n_i defined in (a) is the same as the intersection multiplicity $i\,(X, V; Y_i)$ defined in (I, §7). Then use the generalized Bézout theorem (I, 7.7) to show that for any divisor D on $\mathbf{P}^n$, none of whose components contain X,

d. If D is a principal divisor on X, show that there is a rational function f on $\mathbf{P}^n$ such that $D = (f).X$. Conclude that $\deg D = 0$. Thus the degree function defines a homomorphism deg: $\mathrm{Cl}\,X \to \mathbf{Z}$.[66] Finally, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Cl}\,\mathbf{P}^n \arrow[r] \arrow[d, "\deg,\cong"'] & \mathrm{Cl}\,X \arrow[d, "\deg"] \\
\mathbf{Z} \arrow[r, "\cdot\deg(X)"] & \mathbf{Z}
\end{tikzcd}$$

FIGURE 3. Link to Diagram

and in particular, we see that the map $\mathrm{Cl}\,\mathbf{P}^n \to \mathrm{Cl}\,X$ is injective.

2.6.3. *II.6.3. * Cones.* In this exercise we compare the class group of a projective variety V to the class group of its cone (I, Ex. 2.10). So let V be a projective variety in $\mathbf{P}^n$, which is of dimension $\geqslant 1$ and nonsingular in codimension 1 . Let $X = C(V)$ be the affine cone over V in $\mathbf{A}^{n+1}$, and let $\overline{X}$ be its projective closure in $\mathbf{P}^{n+1}$. Let $P \in X$ be the vertex of the cone.

a. Let $\pi : \overline{X} - P \to V$ be the projection map. Show that V can be covered by open subsets U_i such that $\pi^{-1}(U_i) \cong U_i \times \mathbf{A}^1$ for each i, and then show as in (6.6) that $\pi^* : \mathrm{Cl}\,V \to \mathrm{Cl}(\overline{X} - P)$ is an isomorphism. Since $\mathrm{Cl}\,\overline{X} \cong \mathrm{Cl}(\overline{X} - P)$, we have also $\mathrm{Cl}\,V \cong \mathrm{Cl}\,\overline{X}$.

[66] This gives another proof of (6.10), since any complete nonsingular curve is projective.

b. We have $V \subseteq \overline{X}$ as the hyperplane section at infinity. Show that the class of the divisor V in $\operatorname{Cl} \overline{X}$ is equal to π^* (class of $V.H$) where H is any hyperplane of $\mathbf{P}^n$ not containing V. Thus conclude using (6.5) that there is an exact sequence where the first arrow sends $1 \mapsto V.H$, and the second is π^* followed by the restriction to $X - P$ and inclusion in X. (The injectivity of the first arrow follows from the previous exercise.)

c. Let $S(V)$ be the homogeneous coordinate ring of V (which is also the affine coordinate ring of X). Show that $S(V)$ is a unique factorization domain if and only if
   - V is projectively normal (Ex. 5.14), and
   - $\operatorname{Cl} V \cong \mathrm{Z}$ and is generated by the class of $V.H$.

d. Let $\mathcal{O}_P$ be the local ring of P on X. Show that the natural restriction map induces an isomorphism $\operatorname{Cl} X \to \operatorname{Cl}(\operatorname{Spec} \mathcal{O}_P)$.

2.6.4. *II.6.4.* Let k be a field of characteristic $\neq 2$. Let $f \in k[x_1, \ldots, x_n]$ be a square-free nonconstant polynomial, i.e., in the unique factorization of f into irreducible polynomials, there are no repeated factors. Let $A = k[x_1, \ldots, x_n, z] / (z^2 - f)$. Show that A is an integrally closed ring.[67] Conclude that A is the integral closure of $k[x_1, \ldots, x_n]$ in K.

2.6.5. *II.6.5. * Quadric Hypersurfaces.* Let char $k \neq 2$, and let X be the affine quadric hypersurface[68]

a. Show that X is normal if $r \geqslant 2$ (use (Ex. 6.4)).

b. Show by a suitable linear change of coordinates that the equation of X could be written as $x_0 x_1 = x_2^2 + \ldots + x_r^2$. Now imitate the method of (6.5.2) to show that:
   (1) If $r = 2$, then $\operatorname{Cl} X \cong \mathbf{Z}/2\mathbf{Z}$;
   (2) If $r = 3$, then $\operatorname{Cl} X \cong \mathbf{Z}$ (use (6.6.1) and (Ex. 6.3) above);
   (3) If $r \geqslant 4$ then $\operatorname{Cl} X = 0$.

c. Now let Q be the projective quadric hypersurface in $\mathbf{P}^n$ defined by the same equation. Show that:
   (1) If $r = 2, \operatorname{Cl} Q \cong \mathbf{Z}$, and the class of a hyperplane section $Q.H$ is twice the generator;
   (2) If $r = 3, \operatorname{Cl} Q \cong \mathbf{Z} \oplus \mathbf{Z}$;
   (3) If $r \geqslant 4, \operatorname{Cl} Q \cong \mathbf{Z}$, generated by $Q.H$.

d. Prove Klein's theorem, which says that if $r \geqslant 4$, and if Y is an irreducible subvariety of codimension 1 on Q, then there is an irreducible hypersurface $V \subseteq \mathbf{P}^n$ such that $V \cap Q = Y$, with multiplicity one. In other words, Y is a complete intersection.[69]

2.6.6. *II.6.6.* Let X be the nonsingular plane cubic curve $y^2 z = x^3 - xz^2$ of (6.10.2).

a. Show that three points P, Q, R of X are collinear if and only if $P + Q + R = 0$ in the group law on X.

   Note that the point $P_0 = (0, 1, 0)$ is the zero element in the group structure on X.

b. A point $P \in X$ has order 2 in the group law on X if and only if the tangent line at P passes through P_0.

[67] Hint: The quotient field K of A is just $k(x_1, \ldots, x_n)[z] / (z^2 - f)$. It is a Galois extension of $k(x_1, \ldots, x_n)$ with Galois group $\mathbf{Z}/2\mathbf{Z}$ generated by $z \mapsto -z$. If $\alpha = g + hz \in K$, where $g, h \in k(x_1, \ldots, x_n)$, then the minimal polynomial of α is Now show that α is integral over $k[x_1, \ldots, x_n]$ if and only if $g, h \in k[x_1, \ldots, x_n]$.

[68] cf. (I, Ex. 5.12).

[69] Hint: First show that for $r \geqslant 4$, the homogeneous coordinate ring

$$S(Q) = k[x_0, \ldots, x_n] / \left(x_0^2 + \ldots + x_r^2\right)$$

is a UFD.

c. A point $P \in X$ has order 3 in the group law on X if and only if P is an inflection point.[70]

d. Let $k = \mathbf{C}$. Show that the points of X with coordinates in $\mathbf{Q}$ form a subgroup of the group X. Can you determine the structure of this subgroup explicitly?

2.6.7. *II.6.7. *.* Let X be the nodal cubic curve $y^2z = x^3 + x^2z$ in $\mathbf{P}^2$. Imitate (6.11.4) and show that the group of Cartier divisors of degree $0, \mathrm{CaCl}^{\circ} X$, is naturally isomorphic to the multiplicative group $\mathbf{G}_m$.

2.6.8. *II.6.8.*

a. Let $f : X \to Y$ be a morphism of schemes. Show that $\mathcal{L} \mapsto f^*\mathcal{L}$ induces a homomorphism of Picard groups, $f^* : \operatorname{Pic} Y \to \operatorname{Pic} X$.

b. If f is a finite morphism of nonsingular curves, show that this homomorphism corresponds to the homomorphism $f^* : \operatorname{Cl} Y \to \operatorname{Cl} X$ defined in the text, via the isomorphisms of (6.16).

c. If X is a locally factorial integral closed subscheme of $\mathbf{P}^n_k$, and if $f : X \to \mathbf{P}^n$ is the inclusion map, then f^* on Pic agrees with the homomorphism on divisor class groups defined in (Ex. 6.2) via the isomorphisms of (6.16).

2.6.9. *II.6.9. * Singular Curves.* Here we give another method of calculating the Picard group of a singular curve. Let X be a projective curve over k, let $\tilde{X}$ be its normalization, and let $\pi : \tilde{X} \to X$ be the projection map (Ex. 3.8). For each point $P \in X$, let $\mathcal{O}_P$ be its local ring, and let $\tilde{\mathcal{O}}_P$ be the integral closure of $\mathcal{O}_P$. We use a $*$ to denote the group of units in a ring.

a. Show there is an exact sequence[71]

b. Use (a) to give another proof of the fact that if X is a plane cuspidal cubic curve, then there is an exact sequence and if X is a plane nodal cubic curve, there is an exact sequence

2.6.10. *II.6.10. The Grothendieck Group $K(X)$.* Let X be a noetherian scheme. We define $K(X)$ to be the quotient of the free abelian group generated by all the coherent sheaves on X, by the subgroup generated by all expressions $\mathcal{F} - \mathcal{F}' - \mathcal{F}''$, whenever there is an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of coherent sheaves on X. If $\mathcal{F}$ is a coherent sheaf, we denote by $\gamma(\mathcal{F})$ its image in $K(X)$.

a. If $X = \mathbf{A}^1_k$, then $K(X) \cong \mathbf{Z}$.

b. If X is any integral scheme, and $\mathcal{F}$ a coherent sheaf, we define the **rank** of $\mathcal{F}$ to be $\dim_K \mathcal{F}_\xi$, where ξ is the generic point of X, and $K = \mathcal{O}_\xi$ is the function field of X.

   Show that the rank function defines a surjective homomorphism rank: $K(X) \to \mathbf{Z}$.

c. If Y is a closed subscheme of X, there is an exact sequence where the first map is extension by zero, and the second map is restriction.[72]

[70] An inflection point of a plane curve is a nonsingular point P of the curve, whose tangent line (I, Ex. 7.3) has intersection multiplicity $\geqslant 3$ with the curve at P.

[71] Hint: Represent Pic X and Pic $\tilde{X}$ as the groups of Cartier divisors modulo principal divisors, and use the exact sequence of sheaves on X

[72] Hint: For exactness in the middle, show that if $\mathcal{F}$ is a coherent sheaf on X, whose support is contained in Y, then there is a finite filtration $\mathcal{F} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \ldots \supseteq \mathcal{F}_n = 0$, such that each $\mathcal{F}_i/\mathcal{F}_{i+1}$ is an $\mathcal{O}_Y$-module. To show surjectivity on the right, use (Ex. 5.15).
For further information about $K(X)$, and its applications to the generalized Riemann-Roch theorem, see Borel-Serre [1], Manin [1], and Appendix A.

2.6.11. *II.6.11. *The Grothendieck Group of a Nonsingular Curve.* Let X be a nonsingular curve over an algebraically closed field k. We will show that $K(X) \cong \operatorname{Pic} X \oplus \mathbf{Z}$, in several steps.

a. For any divisor $D=\sum n_i P_i$ on X, let $\psi(D)=\sum n_i \gamma\left(k\left(P_i\right)\right) \in K(X)$, where $k\left(P_i\right)$ is the skyscraper sheaf k at P_i and 0 elsewhere. If D is an effective divisor, let $\mathcal{O}_D$ be the structure sheaf of the associated subscheme of codimension 1, and show that $\psi(D)=\gamma\left(\mathcal{O}_D\right)$. Then use (6.18) to show that for any $D, \psi(D)$ depends only on the linear equivalence class of D, so ψ defines a homomorphism $\psi: \operatorname{Cl} X \rightarrow K(X)$
b. For any coherent sheaf $\mathcal{F}$ on X, show that there exist locally free sheaves $\mathcal{E}_0$ and $\mathcal{E}_1$ and an exact sequence $0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0$. Let $r_0=\operatorname{rank} \mathcal{E}_0$, $r_1=\operatorname{rank} \mathcal{E}_1$, and define Here $\bigwedge$ denotes the exterior power (Ex. 5.16). Show that $\det \mathcal{F}$ is independent of the resolution chosen, and that it gives a homomorphism det: $K(X) \rightarrow \operatorname{Pic} X$. Finally show that if D is a divisor, then $\det(\psi(D))=\mathcal{L}(D)$.
c. If $\mathcal{F}$ is any coherent sheaf of rank r, show that there is a divisor D on X and an exact sequence where $\mathcal{T}$ is a torsion sheaf. Conclude that if $\mathcal{F}$ is a sheaf of rank r, then
d. Using the maps $\psi, \det, \operatorname{rank}$, and $1 \mapsto \gamma\left(\mathcal{O}_X\right)$ from $\mathbf{Z} \rightarrow K(X)$, show that $K(X) \cong \operatorname{Pic} X \oplus \mathbf{Z}$.

2.6.12. *II.6.12.* Let X be a complete nonsingular curve. Show that there is a unique way to define the degree of any coherent sheaf on X, $\deg \mathcal{F} \in \mathbf{Z}$, such that:

a. If D is a divisor, $\deg \mathcal{L}(D)=\deg D$;
b. If $\mathcal{F}$ is a torsion sheaf(meaning a sheaf whose stalk at the generic point is zero), then $\deg \mathcal{F}=\sum_{P \in X}$ length $\left(\mathcal{F}_P\right)$; and
c. If $0 \rightarrow \mathcal{F}^{\prime} \rightarrow \mathcal{F} \rightarrow \mathcal{F}^{\prime \prime} \rightarrow 0$ is an exact sequence, then $\deg \mathcal{F}=\deg \mathcal{F}^{\prime}+\deg \mathcal{F}^{\prime \prime}$.

2.7. **II.7: Projective Morphisms.**

2.7.1. *II.7.1.* Let $\left(X, \mathcal{O}_X\right)$ be a locally ringed space, and let $f: \mathcal{L} \rightarrow \mathcal{M}$ be a surjective map of invertible sheaves on X. Show that f is an isomorphism.[73]

2.7.2. *II.7.2.* Let X be a scheme over a field k. Let $\mathcal{L}$ be an invertible sheaf on X, and let $\left\{s_0, \ldots, s_n\right\}$ and $\left\{t_0, \ldots, t_m\right\}$ be two sets of sections of $\mathcal{L}$, which generate the same subspace $V \subseteq \Gamma(X, \mathcal{L})$, and which generate the sheaf $\mathcal{L}$ at every point. Suppose $n \leqslant m$.
Show that the corresponding morphisms $\varphi: X \rightarrow \mathbf{P}_k^n$ and $\psi: X \rightarrow \mathbf{P}_k^m$ differ by a suitable linear projection $\mathbf{P}^m-L \rightarrow \mathbf{P}^n$ and an automorphism of $\mathbf{P}^n$, where L is a linear subspace of $\mathbf{P}^m$ of dimension $m-n-1$.

2.7.3. *II.7.3.* Let $\varphi: \mathbf{P}_k^n \rightarrow \mathbf{P}_k^m$ be a morphism. Then:

a. Either $\varphi\left(\mathbf{P}^n\right)=\mathrm{pt}$ or $m \geqslant n$ and $\dim \varphi\left(\mathbf{P}^n\right)=n$;
b. In the second case, φ can be obtained as the composition of
   (1) a d-uple embedding $\mathbf{P}^n \rightarrow \mathbf{P}^N$ for a uniquely determined $d \geqslant 1$,
   (2) a linear projection $\mathbf{P}^N-\mathbf{L} \rightarrow \mathbf{P}^m$, and
   (3) an automorphism of $\mathbf{P}^m$.

   Also, φ has finite fibres.

[73] Hint: Reduce to a question of modules over a local ring by looking at the stalks.

2.7.4. *II.7.4.*

a. Use (7.6) to show that if X is a scheme of finite type over a noetherian ring A, and if X admits an ample invertible sheaf, then X is separated.

b. Let X be the affine line over a field k with the origin doubled (4.0.1). Calculate $\operatorname{Pic} X$, determine which invertible sheaves are generated by global sections, and then show directly (without using (a)) that there is no ample invertible sheaf on X.

2.7.5. *II.7.5.* Establish the following properties of ample and very ample invertible sheaves on a noetherian scheme X. $\mathcal{L}, \mathcal{M}$ will denote invertible sheaves, and for (d), (e) we assume furthermore that X is of finite type over a noetherian ring A.

a. If $\mathcal{L}$ is ample and $\mathcal{M}$ is generated by global sections, then $\mathcal{L} \otimes \mathcal{M}$ is ample.

b. If $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then $\mathcal{M} \otimes \mathcal{L}^n$ is ample for sufficiently large n.

c. If $\mathcal{L}, \mathcal{M}$ are both ample, so is $\mathcal{L} \otimes \mathcal{M}$.

d. If $\mathcal{L}$ is very ample and $\mathcal{M}$ is generated by global sections, then $\mathcal{L} \otimes \mathcal{M}$ is very ample.

e. If $\mathcal{L}$ is ample, then there is an $n_0 > 0$ such that $\mathcal{L}^n$ is very ample for all $n \geqslant n_0$.

2.7.6. *II.7.6. The Riemann-Roch Problem.* Let X be a nonsingular projective variety over an algebraically closed field, and let D be a divisor on X. For any $n > 0$ we consider the complete linear system $|nD|$. Then the Riemann-Roch problem is to determine $\dim |nD|$ as a function of n, and, in particular, its behavior for large n.
If $\mathcal{L}$ is the corresponding invertible sheaf, then $\dim |nD| = \dim \Gamma(X, \mathcal{L}^n) - 1$, so an equivalent problem is to determine $\dim \Gamma(X, \mathcal{L}^n)$ as a function of n.

a. Show that if D is very ample, and if $X \hookrightarrow \mathbf{P}^n_k$ is the corresponding embedding in projective space, then for all n sufficiently large, $\dim |nD| = P_X(n) - 1$, where P_X is the Hilbert polynomial of $X(\mathbf{I}, \S 7)$. Thus in this case $\dim |nD|$ is a polynomial function of n, for n large.

b. If D corresponds to a torsion element of $\operatorname{Pic} X$, of order r, then $\dim |nD| = 0$ if $r \,\big|\, n$ and -1 otherwise. In this case the function is periodic of period r. It follows from the general Riemann-Roch theorem that $\dim |nD|$ is a polynomial function for n large, whenever D is an ample divisor.[74]

In the case of algebraic surfaces, Zariski [7] has shown for any effective divisor D, that there is a finite set of polynomials $P_1, \ldots, P_r$, such that for all n sufficiently large, $\dim |nD| = P_{i(n)}(n)$, where $i(n) \in \{1, 2, \ldots, r\}$ is a function of n.

2.7.7. *II.7.7. Some Rational Surfaces.* Let $X = \mathbf{P}^2_k$, and let $|D|$ be the complete linear system of all divisors of degree 2 on X (conics). D corresponds to the invertible sheaf $\mathcal{O}(2)$, whose space of global sections has a basis $x^2, y^2, z^2, xy, xz, yz$, where x, y, z are the homogeneous coordinates of X.

a. The complete linear system $|D|$ gives an embedding of $\mathbf{P}^2$ in $\mathbf{P}^5$, whose image is the Veronese surface.[75]

b. Show that the subsystem defined by $x^2, y^2, z^2, y(x-z), (x-y)z$ gives a closed immersion of X into $\mathbf{P}^4$. The image is called the Veronese surface in $\mathbf{P}^4$. Cf. (IV, Ex. 3.11).

[74] See (IV, 1.3.2), (V, 1.6), and Appendix A.
[75] I, Ex. 2.13.

c. Let $\nu \subseteq |D|$ be the linear system of all conics passing through a fixed point P. Then ν gives an immersion of $U = X - P$ into $\mathbf{P}^4$. Furthermore, if we blow up P, to get a surface $\tilde{X}$, then this map extends to give a closed immersion of $\tilde{X}$ in $\mathbf{P}^4$.

Show that $\tilde{X}$ is a surface of degree 3 in $\mathbf{P}^4$, and that the lines in X through P are transformed into straight lines in $\tilde{X}$ which do not meet. $\tilde{X}$ is the union of all these lines, so we say $\tilde{X}$ is a **ruled surface** $(\mathrm{V}, 2.19.1)$.

2.7.8. *II.7.8.* Let X be a noetherian scheme, let $\mathcal{E}$ be a coherent locally free sheaf on X, and let $\pi : \mathbf{P}(\mathcal{E}) \to X$ be the corresponding projective space bundle. Show that there is a natural $1-1$ correspondence between sections of π (i.e., morphisms $\sigma : X \to \mathbf{P}(\mathcal{E})$ such that $\pi \circ \sigma = \mathrm{id}_X$) and quotient invertible sheaves $\mathcal{E} \to \mathcal{L} \to 0$ of $\mathcal{E}$.

2.7.9. *II.7.9.* Let X be a regular noetherian scheme, and $\mathcal{E}$ a locally free coherent sheaf of rank $\geqslant 2$ on X.

a. Show that $\operatorname{Pic} \mathbf{P}(\mathcal{E}) \cong \operatorname{Pic} X \times \mathbf{Z}$.
b. If $\mathcal{E}'$ is another locally free coherent sheaf on X, show that $\mathbf{P}(\mathcal{E}) \cong \mathbf{P}\left(\mathcal{E}'\right)$ (over X) if and only if there is an invertible sheaf $\mathcal{L}$ on X such that $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$.

2.7.10. *II.7.10.* $\mathbf{P}^n$*-Bundles Over a Scheme.* Let X be a noetherian scheme.

a. By analogy with the definition of a vector bundle (Ex. 5.18), define the notion of a projective n-space bundle over X, as a scheme P with a morphism $\pi : P \to X$ such that P is locally isomorphic to $U \times \mathbf{P}^n, U \subseteq X$ open, and the transition automorphisms on $\operatorname{Spec} A \times \mathbf{P}^n$ are given by A-linear automorphisms of the homogeneous coordinate ring $A\left[x_0, \ldots, x_n\right]$

   E.g., $x_i' = \sum a_{ij} x_j, a_{ij} \in A$.
b. If $\mathcal{E}$ is a locally free sheaf of rankof rank $n+1$ on X then $\mathbf{P}(\mathcal{E})$ is a $\mathbf{P}^n$-bundle over X.
c. * Assume that X is regular, and show that every $\mathbf{P}^n$-bundle P over X is isomorphic to $\mathbf{P}(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$ on X.[76]
d. Conclude (in the case X regular) that we have a 1-1 correspondence between $\mathbf{P}^n$-bundles over X, and equivalence classes of locally free sheaves $\mathcal{E}$ of rank $n+1$ under the equivalence relation $\mathcal{E} \sim \mathcal{E}'$ if and only if $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{M}$ for some invertible sheaf $\mathcal{M}$ on X.

2.7.11. *II.7.11.* On a noetherian scheme X, different sheaves of ideals can give rise to isomorphic blown up schemes.

a. If $\mathcal{I}$ is any coherent sheaf of ideals on X, show that blowing up $\mathcal{I}^d$ for any $d \geqslant 1$ gives a scheme isomorphic to the blowing up of $\mathcal{I}$ (cf. Ex. 5.13).
b. If $\mathcal{I}$ is any coherent sheaf of ideals, and if $\mathcal{J}$ is an invertible sheaf of ideals, then $\mathcal{I}$ and $\mathcal{I} \cdot \mathcal{J}$ give isomorphic blowings-up.
c. If X is regular, show that (7.17) can be strengthened as follows. Let $U \subseteq X$ be the largest open set such that $f : f^{-1} U \to U$ is an isomorphism. Then $\mathcal{I}$ can be chosen such that the corresponding closed subscheme Y has support equal to $X - U$

2.7.12. *II.7.12.* Let X be a noetherian scheme, and let Y, Z be two closed subschemes, neither one containing the other. Let $\tilde{X}$ be obtained by blowing up $Y \cap Z$ (defined by the ideal sheaf $\mathcal{I}_Y + \mathcal{I}_Z$). Show that the strict transforms $\tilde{Y}$ and $\tilde{Z}$ of Y and Z in $\tilde{X}$ do not meet.

[76] Hint: Let $U \subseteq X$ be an open set such that $\pi^{-1}(U) \cong U \times \mathbf{P}^n$, and let $\mathcal{L}_0$ be the invertible sheaf $\mathcal{C}(1)$ on $U \times \mathbf{P}^n$. Show that $\mathcal{L}_0$ extends to an invertible sheaf $\mathcal{L}$ on P. Then show that $\pi_* \mathcal{L} = \mathcal{E}$ is a locally free sheaf on X and that $P \cong \mathbf{P}(\mathcal{E})$.] Can you weaken the hypothesis " X regular"?

2.7.13. *II.7.13. * A Complete Nonprojective Variety.* Let k be an algebraically closed field of char $\neq 2$. Let $C \subseteq \mathbf{P}_k^2$ be the nodal cubic curve If $P_0 = (0,0,1)$ is the singular point, then $C - P_0$ is isomorphic to the multiplicative group $\mathbf{G}_m = \operatorname{Spec} k\left[t, t^{-1}\right]$ (Ex. 6.7). For each $a \in k, a \neq 0$, consider the translation of $\mathbf{G}_m$ given by $t \mapsto at$. This induces an automorphism of C which we denote by φ_a. Now consider $C \times \left(\mathbf{P}^1 - \{0\}\right)$ and $C \times \left(\mathbf{P}^1 - \{\infty\}\right)$. We glue their open subsets $C \times \left(\mathbf{P}^1 - \{0, \infty\}\right)$ by the isomorphism Thus we obtain a scheme X, which is our example. The projections to the second factor are compatible with φ, so there is a natural morphism $\pi : X \to \mathbf{P}^1$.

a. Show that π is a proper morphism, and hence that X is a complete variety over k.
b. Use the method of (Ex. 6.9) to show that[77]
c. Now show that the restriction map is of the form $\langle t, n\rangle \mapsto \langle t, 0, n\rangle$, and that the automorphism φ of $C \times \left(\mathbf{A}^1 - \{0\}\right)$ induces a map of the form $\langle t, d, n\rangle \mapsto \langle t, d+n, n\rangle$ on its Picard group.
d. Conclude that the image of the restriction map consists entirely of divisors of degree 0 on C. Hence X is not projective over k and π is not a projective morphism.

2.7.14. *II.7.14.*

a. Give an example of a noetherian scheme X and a locally free coherent sheaf $\mathcal{E}$, such that the invertible sheaf $\mathcal{O}(1)$ on $\mathbf{P}(\mathcal{E})$ is not very ample relative to X.
b. Let $f : X \to Y$ be a morphism of finite type, let $\mathcal{L}$ be an ample invertible sheaf on X, and let $\mathcal{S}$ be a sheaf of graded $\mathcal{O}_X$-algebras satisfying (†). Let $P = \operatorname{Proj} \mathcal{S}$, let $\pi : P \to X$ be the projection, and let $\mathcal{O}_P(1)$ be the associated invertible sheaf. Show that for all $n \gg 0$, the sheaf $\mathcal{O}_P(1) \otimes \pi^* \mathcal{L}^n$ is very ample on P relative to Y.[78]

2.8. **II.8: Differentials.**

2.8.1. *II.8.1.* Here we will strengthen the results of the text to include information about the sheaf of differentials at a not necessarily closed point of a scheme X.

a. Generalize (8.7) as follows. Let B be a local ring containing a field k, and assume that the residue field $k(B) = B/\mathfrak{m}$ of B is a separably generated extension of k. Then the exact sequence of (8.4A), is exact on the left also.[79]
b. Generalize (8.8) as follows. With B, k as above, assume furthermore that k is perfect, and that B is a localization of an algebra of finite type over k. Then show that B is a regular local ring if and only if $\Omega_{B/k}$ is free of rank $= \dim B +$ tr.d. $k(B)/k$.
c. Strengthen (8.15) as follows. Let X be an irreducible scheme of finite type over a perfect field k, and let $\dim X = n$. For any point $x \in X$, not necessarily closed, show that the local ring $\mathcal{O}_{x,X}$ is a regular local ring if and only if the stalk $\left(\Omega_{X/k}\right)_x$ of the sheaf of differentials at x is free of rank n.
d. Strengthen (8.16) as follows. If X is a variety over an algebraically closed field k, then $U = \left\{x \in X \,\middle|\, \mathcal{O}_x \text{ is a regular local ring }\right\}$ is an open dense subset of X.

[77]Hint: If A is a domain and if $*$ denotes the group of units, then $(A[u])^* \cong A^*$ and $\left(A\left[u, u^{-1}\right]\right)^* \cong A^* \times \mathbf{Z}$.

[78]Hint: Use (7.10) and (Ex. 5.12)

[79]Hint: In copying the proof of (8.7), first pass to B/m^2, which is a complete local ring, and then use (8.25A) to choose a field of representatives for B/m^2.

2.8.2. *II.8.2.* Let X be a variety of dimension n over k. Let $\mathcal{E}$ be a locally free sheaf of rank $> n$ on X, and let $V \subseteq \Gamma(X, \mathcal{E})$ be a vector space of global sections which generate $\mathcal{E}$. Then show that there is an element $s \in V$, such that for each $x \in X$, we have $s_x \notin \mathfrak{m}_x \mathcal{E}_x$. Conclude that there is a morphism $\mathcal{O}_X \to \mathcal{E}$ giving rise to an exact sequence where $\mathcal{E}'$ is also locally free.[80]

2.8.3. *II.8.3. Product Schemes.*

a. Let X and Y be schemes over another scheme S. Use (8.10) and (8.11) to show that
b. If X and Y are nonsingular varieties over a field k, show that
c. Let Y be a nonsingular plane cubic curve, and let X be the surface $Y \times Y$. Show that $p_g(X) = 1$ but $p_a(X) = -1$ (I, Ex. 7.2). This shows that the arithmetic genus and the geometric genus of a nonsingular projective variety may be different.

2.8.4. *II.8.4. Complete Intersections in* $\mathbf{P}^n$. A closed subscheme Y of $\mathbf{P}_k^n$ is called a **(strict, global) complete intersection** if the homogeneous ideal I of Y in $S = k[x_0, \ldots, x_n]$ can be generated by $r = \operatorname{codim}(Y, \mathbf{P}^n)$ elements (I, Ex. 2.17).

a. Let Y be a closed subscheme of codimension r in $\mathbf{P}^n$. Then Y is a complete intersection if and only if there are hypersurfaces (i.e., locally principal subschemes of codimension 1) $H_1, \ldots, H_r$, such that $Y = H_1 \cap \ldots \cap H_r$ as schemes, i.e., $\mathcal{I}_Y = \mathcal{I}_{H_1} + \ldots + \mathcal{I}_{H_r}$.[81]
b. If Y is a complete intersection of dimension $\geqslant 1$ in $\mathbf{P}^n$, and if Y is normal, then Y is projectively normal (Ex. 5.14).[82]
c. With the same hypotheses as (b), conclude that for all $l \geqslant 0$, the natural map $\Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(l)) \to \Gamma(Y, \mathcal{O}_Y(l))$ is surjective. In particular, taking $l = 0$, show that Y is connected.
d. Now suppose given integers $d_1, \ldots, d_r \geqslant 1$, with $r < n$. Use Bertini's theorem (8.18) to show that there exist nonsingular hypersurfaces $H_1, \ldots, H_r$ in $\mathbf{P}^n$, with $\deg H_i = d_i$, such that the scheme $Y = H_1 \cap \ldots \cap H_r$ is irreducible and nonsingular of codimension r in $\mathbf{P}^n$.
e. If Y is a nonsingular complete intersection as in (d), show that
f. If Y is a nonsingular hypersurface of degree d in $\mathbf{P}^n$, use (c) and (e) above to show that $p_g(Y) = \binom{d-1}{n}$. Thus $p_g(Y) = p_a(Y)$ (I, Ex. 7.2). In particular, if Y is a nonsingular plane curve of degree d, then
g. If Y is a nonsingular curve in $\mathbf{P}^3$, which is a complete intersection of nonsingular surfaces of degrees d, e, then Again the geometric genus is the same as the arithmetic genus (I, Ex. 7.2).

2.8.5. *II.8.5. Blowing up a Nonsingular Subvariety.* As in (8.24), let X be a nonsingular variety, let Y be a nonsingular subvariety of codimension $r \geqslant 2$, let $\pi : \widetilde{X} \to X$ be the blowing-up of X along Y, and let $Y' = \pi^{-1}(Y)$.

a. Show that the maps $\pi^* : \operatorname{Pic} X \to \operatorname{Pic} \tilde{X}$, and $\mathbf{Z} \to \operatorname{Pic} X$ defined by $n \mapsto$ class of nY', give rise to an isomorphism $\operatorname{Pic} \tilde{X} \cong \operatorname{Pic} X \oplus \mathbf{Z}$.
b. Show that[83]

[80]Hint: Use a method similar to the proof of Bertini's theorem (8.18).]

[81]Hint: Use the fact that the uniqueness theorem holds in S (Matsumura [2, p.107]).

[82]Hint: Apply (8.23) to the affine cone over Y.

[83]Hint: By (a) we can write in any case for some invertible sheaf $\mathcal{M}$ on X, and some integer q. By restricting to $\tilde{X} - Y' \cong X - Y$, show that $\mathcal{M} \cong \omega_X$. To determine q, proceed as follows.

- First show that $\omega_{Y'} \cong f^* \omega_X \otimes \mathcal{O}_{Y'}(-q-1)$.
- Then take a closed point $y \in Y$ and let Z be the fibre of Y' over y.
- Then show that $\omega_Z \cong \mathcal{O}_Z(-q-1)$. But since $Z \cong \mathbf{P}^{r-1}$, we have $\omega_Z \cong \mathcal{O}_Z(-r)$, so $q = r - 1$.

2.8.6. *II.8.6. The Infinitesimal Lifting Property.* The following result is very important in studying deformations of nonsingular varieties. Let k be an algebraically closed field, let A be a finitely generated k-algebra such that $\operatorname{Spec} A$ is a nonsingular variety over k. Let be an exact sequence, where B' is a k-algebra, and I is an ideal with $I^2 = 0$. Finally suppose given a k-algebra homomorphism $f : A \to B$. Then there exists a k-algebra homomorphism $g : A \to B'$ making a commutative diagram

$$\begin{tikzcd}
 & I \arrow[d, hook] \\
 & B' \arrow[d, two heads] \\
A \arrow[ur, dashed, "g"] \arrow[r, "f"] & B
\end{tikzcd}$$

Figure 4. Link to Diagram

We call this result the **infinitesimal lifting property for** A. We prove this result in several steps.

a. First suppose that $g : A \to B'$ is a given homomorphism lifting f. If $g' : A \to B'$ is another such homomorphism, show that $\theta = g - g'$ is a k-derivation of A into I, which we can consider as an element of $\operatorname{Hom}_A\left(\Omega_{A/k}, I\right)$. Note that since $I^2 = 0, I$ has a natural structure of B-module and hence also of A-module. Conversely, for any $\theta \in \operatorname{Hom}_A\left(\Omega_{A/k}, I\right), g' = g + \theta$ is another homomorphism lifting f. (For this step, you do not need the hypothesis about Spec A being nonsingular.)

b. Now let $P = k\left[x_1, \ldots, x_n\right]$ be a polynomial ring over k of which A is a quotient, and let J be the kernel. Show that there does exist a homomorphism $h : P \to B'$ making a commutative diagram,

$$\begin{tikzcd}
J \arrow[d, hook] & I \arrow[d, hook] \\
P \arrow[r, "h"'] \arrow[d, two heads] & B \arrow[d, two heads] \\
A \arrow[r, "f"'] & B'
\end{tikzcd}$$

Figure 5. Link to Diagram

and show that h induces an A-linear map $\bar{h} : J/J^2 \to I$.

c. Now use the hypothesis Spec A nonsingular and (8.17) to obtain an exact sequence Show furthermore that applying the functor $\operatorname{Hom}_A(\cdot, I)$ gives an exact sequence Let $\theta \in \operatorname{Hom}_P\left(\Omega_{P/k}, I\right)$ be an element whose image gives $\bar{h} \in \operatorname{Hom}_A\left(J/J^2, I\right)$. Consider θ as a derivation of P to B'. Then let $h' = h - \theta$, and show that h' is a homomorphism of $P \to B'$ such that $h'(J) = 0$. Thus h' induces the desired homomorphism $g : A \to B'$.

2.8.7. *II.8.7.* As an application of the infinitesimal lifting property, we consider the following general problem. Let X be a scheme of finite type over k, and let $\mathcal{F}$ be a coherent sheaf on X. We seek to classify schemes X' over k, which have a sheaf of ideals $\mathcal{I}$ such that $\mathcal{I}^2 = 0$ and $(X', \mathcal{O}_X/\mathcal{I}) \cong (X, \mathcal{O}_X)$, and such that $\mathcal{I}$ with its resulting structure of $\mathcal{O}_X$-module is isomorphic to the given sheaf $\mathcal{F}$. Such a pair $X', \mathcal{I}$ we call an **infinitesimal extension of the scheme** X **by the sheaf** $\mathcal{F}$.
One such extension, the trivial one, is obtained as follows. Take $\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{F}$ as sheaves of abelian groups, and define multiplication by Then the topological space X with the sheaf of rings $\mathcal{O}_{X'}$ is an infinitesimal extension of X by $\mathcal{F}$.
The general problem of classifying extensions of X by $\mathcal{F}$ can be quite complicated. So for now, just prove the following special case: if X is affine and nonsingular, then any extension of X by a coherent sheaf $\mathcal{F}$ is isomorphic to the trivial one. See (III, Ex. 4.10) for another case.

2.8.8. *II.8.8. Plurigenus and (some) Hodge numbers are birational invariants.* Let X be a projective nonsingular variety over k. For any $n > 0$ we define the n**th plurigenus of** X to be Thus in particular $P_1 = p_g$. Also, for any $q, 0 \leqslant q \leqslant \dim X$ we define an integer is the sheaf of regular q-forms on X. In particular, for $q = \dim X$, we recover the geometric genus again. The integers $h^{q,0}$ are called **Hodge numbers**.
Using the method of (8.19), show that P_n and $h^{q,0}$ are birational invariants of X, i.e., if X and X' are birationally equivalent nonsingular projective varieties, then $P_n(X) = P_n(X')$ and $h^{q,0}(X) = h^{q,0}(X')$.

2.9. **II.9: Formal Schemes.**

2.9.1. *II.9.1.* Let X be a noetherian scheme, Y a closed subscheme, and $\widehat{X}$ the completion of X along Y. We call the ring $\Gamma\left(\widehat{X}, \mathcal{O}_{\widehat{X}}\right)$ the ring of **formal-regular** functions on X along Y. In this exercise we show that if Y is a connected, nonsingular, positive dimensional subvariety of $X = \mathbf{P}_k^n$ over an algebraically closed field k, then $\Gamma\left(\widehat{X}, \mathcal{O}_{\widehat{X}}\right) = k$

a. Let $\mathcal{I}$ be the ideal sheaf of Y. Use (8.13) and (8.17) to show that there is an inclusion of sheaves on $Y, \mathcal{I}/\mathcal{I}^2 \hookrightarrow \mathcal{O}_Y(-1)^{n+1}$.
b. Show that for any $r \geqslant 1, \Gamma\left(Y, \mathcal{I}^r/\mathcal{I}^{r+1}\right) = 0$.
c. Use the exact sequences and induction on r to show that $\Gamma\left(Y, \mathcal{O}_X/\mathcal{I}^r\right) = k$ for all $r \geqslant 1$.[84]
d. Conclude that $\Gamma\left(\widehat{X}, \mathcal{O}_{\widehat{X}}\right) = k$.[85]

2.9.2. *II.9.2.* Use the result of (Ex. 9.1) to prove the following geometric result. Let $Y \subseteq X = \mathbf{P}_k^n$ be as above, and let $f : X \to Z$ be a morphism of k-varieties. Suppose that $f(Y)$ is a single closed point $P \in Z$. Then $f(X) = P$ also.

2.9.3. *II.9.3.* Prove the analogue of (5.6) for formal schemes, which says, if X is an affine formal scheme, and if is an exact sequence of $\mathcal{O}_x$-modules, and if F′ is coherent, then the sequence of global sections is exact. For the proof, proceed in the following steps.

a. Let I be an ideal of definition for X, and for each $n > 0$ consider the exact sequence Use (5.6), slightly modified, to show that for every open affine subset U ⊆ X, the sequence is exact.
b. Now pass to the limit, using (9.1), (9.2), and (9.6). Conclude that F ≅ lim F/J^{n}F′ and that the sequence of global sections above is exact.

[84] Use 8.21Ae.

[85] Actually, the same result holds without the hypothesis Y nonsingular, but the proof is more difficult-see Hartshorne [3, (7.3)].

2.9.4. *II.9.4.* Use (Ex. 9.3) to prove that if is an exact sequence of $\mathcal{O}_x$-modules on a noetherian formal scheme X, and if F′, F″ are coherent, then F is coherent also.

2.9.5. *II.9.5.* If F is a coherent sheaf on a noetherian formal scheme X, which can be generated by global sections, show in fact that it can be generated by a finite number of its global sections.

2.9.6. *II.9.6.* Let X be a noetherian formal scheme, let I be an ideal of definition, and for each n, let Y_n be the scheme $(\mathrm{X}, \mathcal{O}_x/\mathrm{J}^n)$. Assume that the inverse system of groups $(\Gamma(Y_n, \mathcal{O}_{Y_n}))$ satisfies the Mittag-Leffler condition. Then prove that $\operatorname{Pic}\mathrm{X} = \lim \operatorname{Pic} Y_n$.
As in the case of a scheme, we define $\operatorname{Pic}\mathrm{X}$ to be the group of locally free $\mathcal{O}_x$-modules of rank 1 under the operation $\otimes$. Proceed in the following steps.

a. Use the fact that $\ker\left(\Gamma\left(Y_{n+1}, \mathcal{O}_{Y_{n+1}}\right) \to \Gamma\left(Y_n, \mathcal{O}_{Y_n}\right)\right)$ is a nilpotent ideal to show that the inverse system $\left(\Gamma\left(Y_n, \mathcal{O}^*_{Y_n}\right)\right)$ of units in the respective rings also satisfies (ML).
b. Let F be a coherent sheaf of $\mathcal{O}_x$-modules, and assume that for each n, there is some isomorphism $\varphi_n : \tilde{F}/\mathrm{J}^n\mathrm{F} \cong \mathcal{O}_{Y_n}$. Then show that there is an isomorphism $\tilde{F} \cong \mathcal{O}_x$.[86]
   Conclude that the natural map Pic X $\to \varprojlim \operatorname{Pic} Y_n$ is injective.
c. Given an invertible sheaf $\mathcal{L}_n$ on Y_n for each n, and given isomorphisms $\mathcal{L}_{n+1}\otimes\ \mathcal{O}_{Y_n} \cong \mathcal{L}_n$, construct maps $\mathcal{L}_{n'} \to \mathcal{L}_n$ for each $n' \geqslant n$ so as to make an inverse system, and show that $\mathrm{L} = \lim \mathcal{L}_n$ is a coherent sheaf on X.
   Then show that L is locally free of rank 1, and thus conclude that the map $\operatorname{Pic}\mathrm{X} \to \lim \operatorname{Pic} Y_n$ is surjective.[87]
d. Show that the hypothesis "$(\Gamma(Y_n, \mathcal{O}_{Y_n}))$ satisfies (ML)" is satisfied if either X is affine, or each Y_n is projective over a field k.[88]

3. III: Cohomology

3.1. **III.1: Derived Functors.** Amazing! No exercises in this section.

3.2. **III.2: Cohomology of Sheaves.**

3.2.1. *V.2.1.*

a. Let $X = \mathbf{A}^1_k$ be the affine line over an infinite field k. Let P, Q be distinct closed points of X, and let $U = X - \{P, Q\}$. Show that $H^1(X, \mathbf{Z}_U) \neq 0$.
b. * More generally, let $Y \subseteq X = \mathbf{A}^n_k$ be the union of $n+1$ hyperplanes in suitably general position, and let $U = X - Y$. Show that $H^n(X, Z_U) \neq 0$. Thus the result of (2.7) is the best possible.

3.2.2. *V.2.2.* Let $X = \mathbf{P}^1_k$ be the projective line over an algebraically closed field k. Show that the exact sequence of (II, Ex. 1.21d) is a flasque resolution of $\mathcal{O}$. Conclude from (II, Ex. 1.21e) that $H^i(X, \mathcal{O}) = 0$ for all $i > 0$.

[86] Be careful, because the φ_n may not be compatible with the maps in the two inverse systems $(\mathrm{F}/\mathrm{J}^n\mathrm{F})$ and $(\mathcal{O}_{Y_n})$!
[87] Again be careful, because even though each $\mathcal{L}_n$ is locally free of rank 1, the open sets needed to make them free might get smaller and smaller with n.
[88] See (III, Ex. 11.5-11.7) for further examples and applications.

3.2.3. *V.2.3. Cohomology with Supports.* Let X be a topological space, let Y be a closed subset, and let $\mathcal{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X, \mathcal{F})$ denote the group of sections of $\mathcal{F}$ with support in Y (II, Ex. 1.20).

a. Show that $\Gamma_Y(X, \cdot)$ is a left exact functor from $\mathsf{Ab}(X)$ to Ab. We denote the right derived functors of $\Gamma_Y(X, \cdot)$ by $H_Y^i(X, \cdot)$. They are the cohomology groups of X with supports in Y, and coefficients in a given sheaf.
b. If $0 \to \mathcal{F}' \to \mathcal{F} \to \dot{\mathcal{F}}'' \to 0$ is an exact sequence of sheaves, with $\mathcal{F}'$ flasque, show that is exact.
c. Show that if $\mathcal{F}$ is flasque, then $H_Y^i(X, \mathcal{F}) = 0$ for all $i > 0$.
d. If $\mathcal{F}$ is flasque, show that the sequence is exact.
e. Let $U = X - Y$. Show that for any $\mathcal{F}$, there is a long exact sequence of cohomology groups
f. *Excision.* Let V be an open subset of X containing Y. Then there are natural functorial isomorphisms, for all i and $\mathcal{F}$,

3.2.4. *V.2.4. Mayer-Vietoris Sequence.* Let Y_1, Y_2 be two closed subsets of X. Then there is a long exact sequence of cohomology with supports

3.2.5. *V.2.5.* Let X be a Zariski space (II, Ex. 3.17). Let $P \in X$ be a closed point, and let X_P be the subset of X consisting of all points $Q \in X$ such that $P \in \{Q\}^-$. We call X_P the **local space** of X at P, and give it the induced topology.
Let $j : X_P \to X$ be the inclusion, and for any sheaf $\mathcal{F}$ on X, let $\mathcal{F}_P = j^* \mathcal{F}$. Show that for all $i, \mathcal{F}$, we have

3.2.6. *V.2.6.* Let X be a noetherian topological space, and let $\{\mathcal{I}_\alpha\}_{\alpha \in A}$ be a direct system of injective sheaves of abelian groups on X. Then $\varinjlim \mathcal{I}_\alpha$ is also injective.[89]

3.2.7. *V.2.7.* Let S^1 be the circle (with its usual topology), and let $\mathbf{Z}$ be the constant sheaf $\mathbf{Z}$.

a. Show that $H^1(S^1, \mathbf{Z}) \cong \mathbf{Z}$, using our definition of cohomology.
b. Now let $\mathcal{R}$ be the sheaf of germs of continuous real-valued functions on S^1. Show that $H^1(S^1, \mathcal{R}) = 0$.

3.3. III.3: Cohomology of a Noetherian Affine Scheme.

3.3.1. *V.3.1.* Let X be a noetherian scheme. Show that X is affine if and only if X_{red} (II, Ex. 2.3) is affine.[90]

3.3.2. *V.3.2.* Let X be a reduced noetherian scheme. Show that X is affine if and only if each irreducible component is affine.

3.3.3. *V.3.3.* Let A be a noetherian ring, and let $\mathfrak{a}$ be an ideal of A.

a. Show that $\Gamma_{\mathfrak{a}}(\cdot)$ (II, Ex. 5.6) is a left-exact functor from the category of A-modules to itself. We denote its right derived functors, calculated in $\mathsf{Mod}(A)$, by $H_{\mathfrak{a}}^i(\cdot)$.
b. Now let $X = \operatorname{Spec} A, Y = V(\mathfrak{a})$. Show that for any A-module M, where $H_Y^i(X, \cdot)$ denotes cohomology with supports in Y(Ex. 2.3).
c. For any i, show that $\Gamma_{\mathfrak{a}}\left(H_{\mathfrak{a}}^i(M)\right) = H_{\mathfrak{a}}^i(M)$.

[89]Hints: First show that a sheaf $\mathcal{I}$ is injective if and only if for every open set $U \subseteq X$, and for every subsheaf $\mathcal{R} \subseteq \mathbf{Z}_U$, and for every map $f : \mathcal{R} \to \mathcal{I}$, there exists an extension of f to a map of $\mathbf{Z}_U \to \mathcal{I}$. Secondly, show that any such sheaf $\mathcal{R}$ is finitely generated, so any map $\mathcal{R} \to \varinjlim \mathcal{I}_\alpha$ factors through one of the $\mathcal{I}_\alpha$.

[90]Hint: Use (3.7), and for any coherent sheaf $\mathcal{F}$ on X, consider the filtration $\mathcal{F} \supseteq \mathcal{N} \cdot \mathcal{F} \supseteq \mathcal{N}^2 \cdot \mathcal{F} \supseteq \ldots$, where $\mathcal{N}$ is the sheaf of nilpotent elements on X.

3.3.4. *V.3.4. Cohomological Interpretation of Depth.* If A is a ring, $\mathfrak{a}$ an ideal, and M an A module, then $\operatorname{depth}_{\mathfrak{a}} M$ is the maximum length of an M-regular sequence $x_1, \ldots, x_r$, with all $x_i \in \mathfrak{a}$. This generalizes the notion of depth introduced in $(II, \S 8)$.

a. Assume that A is noetherian. Show that if $\operatorname{depth}_{\mathfrak{a}} M \geqslant 1$, then $\Gamma_{\mathfrak{a}}(M) = 0$, and the converse is true if M is finitely generated.[91]
b. Show inductively, for M finitely generated, that for any $n \geqslant 0$, the following conditions are equivalent:
   (i) $\operatorname{depth}_{\mathfrak{a}} M \geqslant n$;
   (ii) $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n$.

3.3.5. *V.3.5.* Let X be a noetherian scheme, and let P be a closed point of X. Show that the following conditions are equivalent:

(i) $\operatorname{depth} \mathcal{O}_P \geqslant 2$;
(ii) if U is any open neighborhood of P, then every section of $\mathcal{O}_X$ over $U - P$ extends uniquely to a section of $\mathcal{O}_X$ over U.

This generalizes (I, Ex. 3.20), in view of (II, 8.22A).

3.3.6. *V.3.6.* Let X be a noetherian scheme.

a. Show that the sheaf $\mathcal{G}$ constructed in the proof of (3.6) is an injective object in the category $\mathsf{QCoh}(X)$ of quasi-coherent sheaves on X. Thus $\mathsf{QCoh}(X)$ has enough injectives.
b. * Show that any injective object of $\mathsf{QCoh}(X)$ is flasque.[92]
c. Conclude that one can compute cohomology as the derived functors of $\Gamma(X, \cdot)$, considered as a functor from $\mathsf{QCoh}(X)$ to Ab.

3.3.7. *V.3.7.* Let A be a noetherian ring, let $X = \operatorname{Spec} A$, let $\mathfrak{a} \subseteq A$ be an ideal, and let $U \subseteq X$ be the open set $X - V(\mathfrak{a})$.

a. For any A-module M, establish the following formula of Deligne:
b. Apply this in the case of an injective A-module I, to give another proof of (3.4).

3.3.8. *V.3.8.* Without the noetherian hypothesis, (3.3) and (3.4) are false. Let $A = k\,[x_0, x_1, x_2, \ldots]$ with the relations $x_0^n x_n = 0$ for $n = 1, 2, \ldots$. Let I be an injective A-module containing A. Show that $I \to I_{x_0}$ is not surjective.

3.4. **III.4: Čech Cohomology.**

3.4.1. *V.4.1.* Let $f : X \to Y$ be an affine morphism of noetherian separated schemes (II, Ex. 5.17). Show that for any quasi-coherent sheaf $\mathcal{F}$ on X, there are natural isomorphisms for all $i \geqslant 0$,[93]

[91]Hint: When M is finitely generated, both conditions are equivalent to saying that $\mathfrak{a}$ is not contained in any associated prime of M.

[92]Hints: The method of proof of (2.4) will not work, because $\mathcal{O}_U$ is not quasi-coherent on X in general. Instead, use (II, Ex. 5.15) to show that if $\mathcal{I} \in \mathbb{Q}\mathrm{co}(X)$ is injective, and if $U \subseteq X$ is an open subset, then $\mathcal{I}|_U$ is an injective object of $\mathrm{Qco}(U)$. Then cover X with open affines $\cdots$.

[93]Hint: Use (II, 5.8).

3.4.2. *V.4.2.* Prove Chevalley's theorem: Let $f : X \to Y$ be a finite surjective morphism of noetherian separated schemes, with X affine. Then Y is affine.

a. Let $f : X \to Y$ be a finite surjective morphism of integral noetherian schemes. Show that there is a coherent sheaf $\mathcal{M}$ on X, and a morphism of sheaves $\alpha : \mathcal{O}_Y^r \to f_* \mathcal{M}$ for some $r > 0$, such that α is an isomorphism at the generic point of Y.
b. For any coherent sheaf $\mathcal{F}$ on Y, show that there is a coherent sheaf $\mathcal{G}$ on X, and a morphism $\beta : f_* \mathcal{G} \to \mathcal{F}^r$ which is an isomorphism at the generic point of Y.[94]
c. Now prove Chevalley's theorem. First use (Ex. 3.1) and (Ex. 3.2) to reduce to the case X and Y integral. Then use (3.7), (Ex. 4.1), consider $\ker \beta$ and coker β, and use noetherian induction on Y.

3.4.3. *V.4.3.* Let $X = \mathbf{A}_k^2 = \operatorname{Spec} k[x, y]$, and let $U = X - \{(0, 0)\}$. Using a suitable cover of U by open affine subsets, show that $H^1(U, \mathcal{O}_U)$ is isomorphic to the k-vector space spanned by $\left\{ x^i y^j \;\middle|\; i, j < 0 \right\}$. In particular, it is infinite-dimensional.[95]

3.4.4. *V.4.4.* On an arbitrary topological space X with an arbitrary abelian sheaf $\mathcal{F}$, Čech cohomology may not give the same result as the derived functor cohomology. But here we show that for H^1, there is an isomorphism if one takes the limit over all coverings.

a. Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of the topological space X. A refinement of $\mathfrak{U}$ is a covering $\mathfrak{B} = (V_j)_{j \in J}$, together with a map $\lambda : J \to I$ of the index sets, such that for each $j \in J, V_j \subseteq U_{\lambda(j)}$. If $\mathfrak{B}$ is a refinement of $\mathfrak{X}$, show that there is a natural induced map on Čech cohomology, for any abelian sheaf $\mathcal{F}$, and for each i, The coverings of X form a partially ordered set under refinement, so we can consider the Ceech cohomology in the limit
b. For any abelian sheaf $\mathcal{F}$ on X, show that the natural maps (4.4) for each covering are compatible with the refinement maps above.
c. Now prove the following theorem. Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups. Then the natural map is an isomorphism.[96]

3.4.5. *V.4.5.* For any ringed space $(X, \mathcal{O}_X)$, let $\operatorname{Pic} X$ be the group of isomorphism classes of invertible sheaves (II, §6). Show that $\operatorname{Pic} X \cong H^1(X, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U, \mathcal{O}_X)$, with multiplication as the group operation.[97]

3.4.6. *V.4.6.* Let $(X, \mathcal{O}_X)$ be a ringed space, let $\mathcal{I}$ be a sheaf of ideals with $\mathcal{I}^2 = 0$, and let X_0 be the ringed space $(X, \mathcal{O}_X / \mathcal{I})$. Show that there is an exact sequence of sheaves of abelian groups on X, where $\mathcal{O}_X^*$ (respectively, $\mathcal{O}_{X_0}^*$) denotes the sheaf of (multiplicative) groups of units in the sheaf of rings $\mathcal{O}_X$ (respectively, $\mathcal{O}_{X_0}$) the map $\mathcal{I} \to \mathcal{O}_X^*$ is defined by $a \mapsto 1 + a$, and $\mathcal{I}$ has its usual (additive) group structure. Conclude there is an exact sequence of abelian groups

[94] Hint: Apply $\mathcal{H}om(\cdot, \mathcal{F})$ to α and use (II, Ex. 5.17e).

[95] Using (3.5), this provides another proof that U is not affine-cf. (I, Ex. 3.6).

[96] Hint: Embed $\mathcal{F}$ in a flasque sheaf $\mathcal{G}$, and let $\mathcal{R} = \mathcal{G}/\mathcal{F}$, so that we have an exact sequence Define a complex $D^{\cdot}(\mathfrak{U})$ by Then use the exact cohomology sequence of this sequence of complexes, and the natural map of complexes and see what happens under refinement.

[97] Hint: For any invertible sheaf $\mathcal{L}$ on X, cover X by open sets U_i on which $\mathcal{L}$ is free, and fix isomorphisms $\varphi_i : \mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{L}|_{U_i}$. Then on $U_i \cap U_j$, we get an isomorphism $\varphi_i^{-1} \circ \varphi_j$ of $\mathcal{O}_{U_i \cap U_j}$ with itself. These isomorphisms give an element of $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$. Now use (Ex. 4.4).

3.4.7. *V.4.7.* Let X be a subscheme of $\mathbf{P}_k^2$ defined by a single homogeneous equation $f(x_0, x_1, x_2) = 0$ of degree d. (Do not assume f is irreducible.) Assume that $(1,0,0)$ is not on X. Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Čech complex explicitly, and thus show that

3.4.8. *V.4.8. Cohomological Dimension.* Let X be a noetherian separated scheme. We define the **cohomological dimension** of X, denoted $\operatorname{cd}(X)$, to be the least integer n such that $H^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ and all $i > n$.
Thus for example, Serre's theorem (3.7) says that $\operatorname{cd}(X) = 0$ if and only if X is affine. Grothendieck's theorem (2.7) implies that $\operatorname{cd}(X) \leqslant \dim X$.

a. In the definition of $\operatorname{cd}(X)$, show that it is sufficient to consider only coherent sheaves on X. Use (II, Ex. 5.15) and (2.9).
b. If X is quasi-projective over a field k, then it is even sufficient to consider only locally free coherent sheaves on X. Use (II, 5.18).
c. Suppose X has a covering by $r+1$ open affine subsets. Use Čech cohomology to show that $\operatorname{cd}(X) \leqslant r$.
d. * If X is a quasi-projective scheme of dimension r over a field k, then X can be covered by $r+1$ open affine subsets. Conclude (independently of (2.7)) that $\operatorname{cd}(X) \leqslant \dim X$.
e. Let Y be a set-theoretic complete intersection (I, Ex. 2.17) of codimension r in $X = \mathbf{P}_k^n$. Show that $\operatorname{cd}(X - Y) \leqslant r - 1$.

3.4.9. *V.4.9.* Let $X = \operatorname{Spec} k[x_1, x_2, x_3, x_4]$ be affine four-space over a field k. Let Y_1 be the plane $x_1 = x_2 = 0$ and let Y_2 be the plane $x_3 = x_4 = 0$. Show that $Y = Y_1 \cup Y_2$ is not a set-theoretic complete intersection in X. Therefore the projective closure $\overline{Y}$ in $\mathbf{P}_k^4$ is also not a set-theoretic complete intersection.[98]

3.4.10. *V.4.10.* * Let X be a nonsingular variety over an algebraically closed field k, and let $\mathcal{F}$ be a coherent sheaf on X. Show that there is a one-to-one correspondence between the set of infinitesimal extensions of X by $\mathcal{F}$ (II, Ex. 8.7) up to isomorphism, and the group $H^1(X, \mathcal{F} \otimes \mathcal{T})$, where $\mathcal{T}$ is the tangent sheaf of X, see $(II\S8)$.[99]

3.4.11. *V.4.11.* This exercise shows that Cech cohomology will agree with the usual cohomology whenever the sheaf has no cohomology on any of the open sets. More precisely, let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups, and $\mathfrak{U} = (U_i)$ an open cover. Assume for any finite intersection $V = U_{i_0} \cap \ldots \cap U_{i_p}$ of open sets of the covering, and for any $k > 0$, that $H^k(V, \mathcal{F}|_V) = 0$. Then prove that for all $p \geqslant 0$, the natural maps of (4.4) are isomorphisms. Show also that one can recover (4.5) as a corollary of this more general result.

3.5. **III.5: The Cohomology of Projective Space.**

3.5.1. *III.5.1.* Let X be a projective scheme over a field k, and let $\mathcal{F}$ be a coherent sheaf on X. We define the Euler characteristic of $\mathcal{F}$ by If is a short exact sequence of coherent sheaves on X, show that $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$.

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[98]Hints: Use an affine analogue of (Ex. 4.8e). Then show that $H^2(X - Y, \mathcal{O}_X) \neq 0$, by using (Ex. 2.3) and (Ex. 2.4). If $P = Y_1 \cap Y_2$, imitate (Ex. 4.3) to show $H^3(X - P, \mathcal{O}_X) \neq 0$.

[99]Hint: Use (II, Ex. 8.6) and (4.5).

3.5.2. *III.5.2.*

a. Let X be a projective scheme over a field k, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over k, and let $\mathcal{F}$ be a coherent sheaf on X. Show that there is a polynomial $P(z) \in \mathbf{Q}[z]$, such that $\chi(\mathcal{F}(n)) = P(n)$ for all $n \in \mathbf{Z}$. We call P the **Hilbert polynomial** of $\mathcal{F}$ with respect to the sheaf $\mathcal{O}_X(1)$.[100]

b. Now let $X = \mathbf{P}_k^r$, and let $M = \Gamma_*(\mathcal{F})$, considered as a graded $S = k\left[x_0, \ldots, x_r\right] -$ module. Use (5.2) to show that the Hilbert polynomial of $\mathcal{F}$ just defined is the same as the Hilbert polynomial of M defined in $(I, \S 7)$.

3.5.3. *III.5.3. Arithmetic Genus.* Let X be a projective scheme of dimension r over a field k. We define the arithmetic genus p_a of X by Note that it depends only on X, not on any projective embedding.

a. If X is integral, and k algebraically closed, show that $H^0\left(X, \mathcal{O}_X\right) \cong k$, so that In particular, if X is a curve, we have[101]

b. If X is a closed subvariety of $\mathbf{P}_k^r$, show that this $p_a(X)$ coincides with the one defined in (I, Ex. 7.2), which apparently depended on the projective embedding.

c. If X is a nonsingular projective curve over an algebraically closed field k, show that $p_a(X)$ is in fact a birational invariant. Conclude that a nonsingular plane curve of degree $d \geqslant 3$ is not rational.[102]

3.5.4. *III.5.4.* Recall from (II, Ex. 6.10) the definition of the Grothendieck group $K(X)$ of a noetherian scheme X.

a. Let X be a projective scheme over a field k, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Show that there is a (unique) additive homomorphism such that for each coherent sheaf $\mathcal{F}$ on $X, P(\gamma(\mathcal{F}))$ is the Hilbert polynomial of $\mathcal{F}$ (Ex. 5.2).

b. Now let $X = \mathbf{P}_k^r$. For each $i = 0, 1, \ldots, r$, let L_i be a linear space of dimension i in X. Then show that
   (1) $K(X)$ is the free abelian group generated by $\left\{\gamma\left(\mathcal{O}_{L_i}\right) \middle| i = 0, \ldots, r\right\}$, and
   (2) the map $P : K(X) \rightarrow \mathbf{Q}[z]$ is injective.[103]

3.5.5. *III.5.5.* Let k be a field, let $X = \mathbf{P}_k^r$, and let Y be a closed subscheme of dimension $q \geqslant 1$, which is a complete intersection (II, Ex. 8.4). Then:

a. for all $n \in \mathbf{Z}$, the natural map is surjective.[104]

b. Y is connected;

c. $H^i\left(Y, \mathcal{O}_Y(n)\right) = 0$ for $0 < i < q$ and all $n \in \mathbf{Z}$;

d. $p_a(Y) = \dim_k H^q\left(Y, \mathcal{O}_Y\right)$.[105]

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[100]Hints: Use induction on dim Supp $\mathcal{F}$, general properties of numerical polynomials (I, 7.3), and suitable exact sequences

[101]Hint: Use (I, 3.4).

[102]This gives another proof of (II, 8.20.3) where we used the geometric genus.

[103]Hint: Show that (1) $\Rightarrow$ (2). Then prove (1) and (2) simultaneously, by induction on r, using (II, Ex. 6.10c).

[104]This gives a generalization and another proof of (II, Ex. 8.4c), where we assumed Y was normal.

[105]Hint: Use exact sequences and induction on the codimension, starting from the case $Y = X$ which is (5.1).

3.5.6. *III.5.6. Curves on a Nonsingular Quadric Surface.* Let Q be the nonsingular quadric surface $xy = zw$ in $X = \mathbf{P}_k^3$ over a field k. We will consider locally principal closed subschemes Y of Q. These correspond to Cartier divisors on Q by (II, 6.17.1). On the other hand, we know that $\operatorname{Pic} Q \cong \mathbf{Z} \oplus \mathbf{Z}$, so we can talk about the type (a, b) of Y(II, 6.16) and (II, 6.6.1).
Let us denote the invertible sheaf $\mathcal{L}(Y)$ by $\mathcal{O}_Q(a, b)$. Thus for any $n \in \mathbf{Z}, \mathcal{O}_Q(n) = \mathcal{O}_Q(n, n)$.

a. Use the special cases $(q, 0)$ and $(0, q)$, with $q > 0$, when Y is a disjoint union of q lines $\mathbf{P}^1$ in Q, to show:
   (1) if $|a - b| \leqslant 1$, then $H^1 (Q, \mathcal{O}_Q(a, b)) = 0$;
   (2) if $a, b < 0$, then $H^1 (Q \mathcal{O}_Q(a, b)) = 0$
   (3) If $a \leqslant -2$, then $H^1 (Q, \mathcal{O}_Q(a, 0)) \neq 0$.
b. Now use these results to show:
   (1) if Y is a locally principal closed subscheme of type (a, b), with $a, b > 0$, then Y is connected;
   (2) now assume k is algebraically closed. Then for any $a, b > 0$, there exists an irreducible nonsingular curve Y of type (a,b). Use (II, 7.6.2) and (II, 8.18).
   (3) an irreducible nonsingular curve Y of type $(a, b), a, b > 0$ on Q is projectively normal (II, Ex. 5.14) if and only if $|a - b| \leqslant 1$. In particular, this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbf{P}^3$. The simplest is the one of type $(1, 3)$, which is just the rational quartic curve (I, Ex. 3.18).
c. If Y is a locally principal subscheme of type (a, b) in Q, show that[106]

3.5.7. *III.5.7.* Let X (respectively, Y) be proper schemes over a noetherian ring A. We denote by $\mathcal{L}$ an invertible sheaf.

a. If $\mathcal{L}$ is ample on X, and Y is any closed subscheme of X, then $i^*\mathcal{L}$ is ample on Y, where $i : Y \to X$ is the inclusion.
b. $\mathcal{L}$ is ample on X if and only if $\mathcal{L}_{\text{red}} = \mathcal{L} \otimes \mathcal{O}_{X_{\text{red}}}$ is ample on X_{red} .
c. Suppose X is reduced. Then $\mathcal{L}$ is ample on X if and only if $\mathcal{L} \otimes \mathcal{O}_{X_i}$ is ample on X_i, for each irreducible component X_i of X.
d. Let $f : X \to Y$ be a finite surjective morphism, and let $\mathcal{L}$ be an invertible sheaf on Y. Then $\mathcal{L}$ is ample on Y if and only if $f^*\mathcal{L}$ is ample on X.[107]

3.5.8. *III.5.8.* Prove that every one-dimensional proper scheme X over an algebraically closed field k is projective.

a. If X is irreducible and nonsingular, then X is projective by (II, 6.7).
b. If X is integral, let $\tilde{X}$ be its normalization (II, Ex. 3.8). Show that $\tilde{X}$ is complete and nonsingular, hence projective by (a).

   Let $f : \tilde{X} \to X$ be the projection. Let $\mathcal{L}$ be a very ample invertible sheaf on $\tilde{X}$. Show there is an effective divisor $D = \sum P_i$ on $\tilde{X}$ with $\mathcal{L}(D) \cong \mathcal{L}$, and such that $f(P_i)$ is a nonsingular point of X, for each i.

   Conclude that there is an invertible sheaf $\mathcal{L}_0$ on X with $f^*\mathcal{L}_0 \cong \mathcal{L}$. Then use (Ex. 5.7d), (II, 7.6) and (II, 5.16.1) to show that X is projective.

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[106] Hint: Calculate Hilbert polynomials of suitable sheaves, and again use the special case $(q, 0)$ which is a disjoint union of q copies of $\mathbf{P}^1$. See (V, 1.5.2) for another method.

[107] Hints: Use (5.3) and compare (Ex. 3.1, Ex. 3.2, Ex. 4.1, Ex. 4.2). See also Hartshorne [5, *Ch.I*§4] for more details.

c. If X is reduced, but not necessarily irreducible, let $X_1, \ldots, X_r$ be the irreducible components of X. Use (Ex. 4.5) to show $\operatorname{Pic} X \to \bigoplus \operatorname{Pic} X_i$ is surjective. Then use (Ex. 5.7c) to show X is projective.

d. Finally, if X is any one-dimensional proper scheme over k, use (2.7) and (Ex. 4.6) to show that $\operatorname{Pic} X \to \operatorname{Pic} X_{\text{red}}$ is surjective. Then use (Ex. 5.7b) to show X is projective.

3.5.9. *III.5.9. A Nonprojective Scheme.* We show the result of (Ex. 5.8) is false in dimension 2. Let k be an algebraically closed field of characteristic 0 , and let $X = \mathbf{P}_k^2$. Let ω be the sheaf of differential 2-forms (II, §8). Define an infinitesimal extension X' of X by ω by giving the element $\xi \in H^1(X, \omega \otimes \mathcal{T})$ defined as follows (Ex. 4.10).
Let x_0, x_1, x_2 be the homogeneous coordinates of X, let U_0, U_1, U_2 be the standard open covering, and let $\xi_{ij} = (x_j/x_i)\, d\,(x_i/x_j)$. This gives a Čech 1-cocycle with values in Ω_X^1, and since $\dim X = 2$, we have $\omega \otimes \mathcal{T} \cong \Omega^1$ (II, Ex. 5.16b). Now use the exact sequence of (Ex. 4.6) and show δ is injective. We have $\omega \cong \mathcal{O}_X(-3)$ by (II, 8.20.1), so $H^2(X, \omega) \cong k$. Since char $k = 0$, you need only show that $\delta(\mathcal{O}(1)) \neq 0$, which can be done by calculating in Čech cohomology.
Since $H^1(X, \omega) = 0$, we see that $\operatorname{Pic} X' = 0$. In particular, X' has no ample invertible sheaves, so it is not projective.[108]

3.5.10. *III.5.10.* Let X be a projective scheme over a noetherian ring A, and let $\mathcal{F}^1 \to \mathcal{F}^2 \to \ldots \to \mathcal{F}^r$ be an exact sequence of coherent sheaves on X. Show that there is an integer n_0, such that for all $n \geqslant n_0$, the sequence of global sections is exact.

3.6. III.6: Ext Groups and Sheaves.

3.6.1. *III.6.1.* Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}', \mathcal{F}'' \in \mathsf{Mod}(X)$. An **extension** of $\mathcal{F}''$ by $\mathcal{F}'$ is a short exact sequence in $\mathsf{Mod}(X)$. Two extensions are isomorphic if there is an isomorphism of the short exact sequences, inducing the identity maps on $\mathcal{F}'$ and $\mathcal{F}''$. Given an extension as above consider the long exact sequence arising from $\operatorname{Hom}(\mathcal{F}'', \cdot)$, in particular the map and let $\xi \in \operatorname{Ext}^1(\mathcal{F}'', \mathcal{F}')$ be $\delta(1_{\mathcal{F}''})$. Show that this process gives a one-to-one correspondence between isomorphism classes of extensions of $\mathcal{F}''$ by $\mathcal{F}'$, and elements of the group $\operatorname{Ext}^1(\mathcal{F}'', \mathcal{F}')$.

3.6.2. *III.6.2.* Let $X = \mathbf{P}_k^1$, with k an infinite field.

a. Show that there does not exist a projective object $\mathcal{P} \in \mathsf{Mod}(X)$, together with a surjective map $\mathcal{P} \to \mathcal{O}_X \to 0$.[109]

b. Show that there does not exist a projective object $\mathcal{P}$ in either $\mathsf{QCoh}(X)$ or $\mathsf{Coh}(X)$ together with a surjection $\mathcal{P} \to \mathcal{O}_X \to 0$.[110]

3.6.3. *III.6.3.* Let X be a noetherian scheme, and let $\mathcal{F}, \mathcal{G} \in \mathsf{Mod}(X)$.

a. If $\mathcal{F}, \mathcal{G}$ are both coherent, then $\mathcal{E}\S\sqcup(\mathcal{F}, \mathcal{G})$ is coherent, for all $i \geqslant 0$.

b. If $\mathcal{F}$ is coherent and $\mathcal{G}$ is quasi-coherent, then $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ is quasi-coherent, for all $i \geqslant 0$.

[108] Note. In fact, this result can be generalized to show that for any nonsingular projective surface X over an algebraically closed field k of characteristic 0 , there is an infinitesimal extension X' of X by ω, such that X' is not projective over k.
Indeed, let D be an ample divisor on X. Then D determines an element $c_1(D) \in H^1(X, \Omega^1)$ which we use to define X', as above. Then for any divisor E on X one can show that $\delta(\mathcal{L}(E)) = (D.E)$, where $(D.E)$ is the intersection number (Chapter V), considered as an element of k. Hence if E is ample, $\delta(\mathcal{L}(E)) \neq 0$. Therefore X' has no ample divisors.
On the other hand, over a field of characteristic $p > 0$, a proper scheme X is projective if and only if X_{red} is!

[109] Hint: Consider surjections of the form $\mathcal{O}_V \to k(x) \to 0$, where $x \in X$ is a closed point, V is an open neighborhood of x, and $\mathcal{O}_V = j_!\left(\mathcal{O}_X|_V\right)$, where $j : V \to X$ is the inclusion.

[110] Hint: Consider surjections of the form $\mathcal{L} \to \mathcal{L} \otimes k(x) \to 0$, where $x \in X$ is a closed point, and $\mathcal{L}$ is an invertible sheaf on X.

3.6.4. *III.6.4.* Let X be a noetherian scheme, and suppose that every coherent sheaf on X is a quotient of a locally free sheaf. In this case we say $\mathsf{Coh}(X)$ has enough locally frees. Then for any $\mathcal{G} \in \mathsf{Mod}(X)$, show that the δ-functor $(\mathcal{E}xt^i(\cdot, \mathcal{G}))$, from $\mathsf{Coh}(X)$ to $\mathsf{Mod}(X)$ is a contravariant universal δ-functor.[111]

3.6.5. *III.6.5.* Let X be a noetherian scheme, and assume that $\mathsf{Coh}(X)$ has enough locally frees (Ex. 6.4). Then for any coherent sheaf $\mathcal{F}$ we define the **homological dimension** of $\mathcal{F}$, denoted $\mathrm{hd}(\mathcal{F})$, to be the least length of a locally free resolution of $\mathcal{F}$ (or $+\infty$ if there is no finite one). Show:

a. $\mathcal{F}$ is locally free $\Leftrightarrow \mathcal{E}xt^1(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{G} \in \mathsf{Mod}(X)$;
b. $\mathrm{hd}(\mathcal{F}) \leqslant n \Leftrightarrow \mathcal{E}\S\sqcup(\mathcal{F}, \mathcal{G}) = 0$ for all $i > n$ and all $\mathcal{G} \in \mathsf{Mod}(X)$;
c. $\mathrm{hd}(\mathcal{F}) = \sup_x \mathrm{pd}_{\mathcal{O}_x} \mathcal{F}_x$.

3.6.6. *III.6.6.* Let A be a regular local ring, and let M be a finitely generated A-module. In this case, strengthen the result (6.10 A) as follows.

a. M is projective if and only if $\mathrm{Ext}^i(M, A) = 0$ for all $i > 0$.[112]
b. Use (a) to show that for any n, pd $M \leqslant n$ if and only if $\mathrm{Ext}^i(M, A) = 0$ for all $i > n$.

3.6.7. *III.6.7.* Let $X = \operatorname{Spec} A$ be an affine noetherian scheme. Let M, N be A-modules, with M finitely generated. Then and

3.6.8. *III.6.8.* Prove the following theorem of Kleiman (see Borelli [1]): if X is a noetherian, integral, separated, locally factorial scheme, then every coherent sheaf on X is a quotient of a locally free sheaf (of finite rank).

a. First show that open sets of the form X_s, for various $s \in \Gamma(X, \mathcal{L})$, and various invertible sheaves $\mathcal{L}$ on X, form a base for the topology of X.[113]
b. Now use (II, 5.14) to show that any coherent sheaf is a quotient of a direct sum $\bigoplus \mathcal{L}_i^{n_i}$ for various invertible sheaves $\mathcal{L}_i$ and various integers n_i.

3.6.9. *III.6.9.* Let X be a noetherian, integral, separated, regular scheme. (We say a scheme is regular if all of its local rings are regular local rings.) Recall the definition of the Grothendieck group $K(X)$ from (II, Ex. 6.10).
We define similarly another group $K_1(X)$ using locally free sheaves: it is the quotient of the free abelian group generated by all locally free (coherent) sheaves, by the subgroup generated by all expressions of the form $\mathcal{E} - \mathcal{E}' - \mathcal{E}''$, whenever $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is a short exact sequence of locally free sheaves.
Clearly there is a natural group homomorphism $\varepsilon : K_1(X) \to K(X)$. Show that ε is an isomorphism (Borel and Serre [1, §4]) as follows.

a. Given a coherent sheaf $\mathcal{F}$, use (Ex. 6.8) to show that it has a locally free resolution $\mathcal{E}. \to \mathcal{F} \to 0$. Then use (6.11A) and (Ex. 6.5) to show that it has a finite locally free resolution

[111]Hint: Show $\mathcal{E}xt^i(\cdot, \mathcal{G})$ is coeffaceable for $i > 0$.

[112]Hint: Use (6.11A) and descending induction on i to show that $\mathrm{Ext}^i(M, N) = 0$ for all $i > 0$ and all finitely generated A-modules N. Then show M is a direct summand of a free A-module (Matsumura [2, p.129]).

[113]Hint: Given a closed point $x \in X$ and an open neighborhood U of x, to show there is an $\mathcal{L}, s$ such that $x \in X_s \subseteq U$, first reduce to the case that $Z = X - U$ is irreducible. Then let ζ be the generic point of Z. Let $f \in K(X)$ be a rational function with $f \in \mathcal{O}_x, f \notin \mathcal{O}_\zeta$. Let $D = (f)_\infty$, and let $\mathcal{L} = \mathcal{L}(D), s \in \Gamma(X, \mathcal{L}(D))$ correspond to D (II, §6).

b. For each $\mathcal{F}$, choose a finite locally free resolution $\mathcal{E}. \to \mathcal{F} \to 0$, and let $\delta(\mathcal{F}) = \sum(-1)^i \gamma\left(\mathcal{E}_i\right)$ in $K_1(X)$. Show that $\delta(\mathcal{F})$ is independent of the resolution chosen, that it defines a homomorphism of $K(X)$ to $K_1(X)$, and finally, that it is an inverse to ε.

3.6.10. *III.6.10. Duality for a Finite Flat Morphism.*

a. Let $f : X \to Y$ be a finite morphism of noetherian schemes. For any quasicoherent $\mathcal{O}_Y$-module $\mathcal{G}, \mathcal{H}om_Y(f_* \mathcal{O}_X, \mathcal{G})$ is a quasi-coherent $f_* \mathcal{O}_X$-module, hence corresponds to a quasi-coherent $\mathcal{O}_X$-module, which we call $f!\mathcal{G}$ (II, Ex. 5.17e).

b. Show that for any coherent $\mathcal{F}$ on X and any quasi-coherent $\mathcal{G}$ on Y, there is a natural isomorphism

c. For each $i \geqslant 0$, there is a natural map[114]

d. Now assume that X and Y are separated, $\mathsf{Coh}(X)$ has enough locally frees, and assume that $f_* \mathcal{O}_X$ is locally free on Y (this is equivalent to saying f flat-see §9). Show that φ_i is an isomorphism for all i, all $\mathcal{F}$ coherent on X, and all $\mathcal{G}$ quasi-coherent on Y.[115]

3.7. III.7: Serre Duality.

3.7.1. *III.7.1. Special case of Kodaira vanishing.* Let X be an integral projective scheme of dimension $\geqslant 1$ over a field k, and let $\mathcal{L}$ be an ample invertible sheaf on X. Then

3.7.2. *III.7.2.* Let $f : X \to Y$ be a finite morphism of projective schemes of the same dimension over a field k, and let ω_Y° be a dualizing sheaf for Y.

a. Show that $f' \omega_Y^{\circ}$ is a dualizing sheaf for X, where f' is defined as in (Ex. 6.10).

b. If X and Y are both nonsingular, and k algebraically closed, conclude that there is a natural trace map $t : f_* \omega_X \to \omega_Y$.

3.7.3. *III.7.3.* Let $X = \mathbf{P}_k^n$. Show that $H^q\left(X, \Omega_X^p\right) = 0$ for $p \neq q$, k for $p = q, 0 \leqslant p, q \leqslant n$.

3.7.4. *III.7.4. * The Cohomology Class of a Subvariety.* Let X be a nonsingular projective variety of dimension n over an algebraically closed field k. Let Y be a nonsingular subvariety of codimension p (hence dimension $n-p$). From the natural map $\Omega_X \otimes \mathcal{O}_Y \to \Omega_Y$ of (II, 8.12) we deduce a map $\Omega_X^{n-p} \to \Omega_Y^{n-p}$. This induces a map on cohomology Now $\Omega_Y^{n-p} = \omega_Y$ is a dualizing sheaf for Y, so we have the trace map Composing, we obtain a linear map $H^{n-p}\left(X, \Omega_X^{n-p}\right) \to k$. By (7.13) this corresponds to an element $\eta(Y) \in H^p\left(X, \Omega_X^p\right)$, which we call the **cohomology class of** Y.

a. If $P \in X$ is a closed point, show that $t_X(\eta(P)) = 1$, where $\eta(P) \in H^n\left(X, \Omega^n\right)$ and t_X is the trace map.

b. If $X = \mathbf{P}^n$, identify $H^p\left(X, \Omega^p\right)$ with k by (Ex. 7.3), and show that $\eta(Y) = (\deg Y) \cdot 1$, where $\deg Y$ is its degree as a projective variety (I, § 7).[116]

c. For any scheme X of finite type over k, we define a homomorphism of sheaves of abelian groups $d\log : \mathcal{O}_X^* \to \Omega_X$ by $d\log(f) = f^{-1} df$. Here $\mathcal{O}^*$ is a group under multiplication, and Ω_X is a group under addition. This induces a map on cohomology which we denote by c. See (Ex. 4.5).

d. Returning to the hypotheses above, suppose $p = 1$. Show that $\eta(Y) = c(\mathcal{L}(Y))$, where $\mathcal{L}(Y)$ is the invertible sheaf corresponding to the divisor Y.

[114]Hint: First construct a map Then compose with a suitable map from $f_* f^! \mathcal{G}$ to $\mathcal{G}$.

[115]Hints: First do $i = 0$. Then do $\mathcal{F} = \mathcal{O}_X$, using (Ex. 4.1). Then do $\mathcal{F}$ locally free. Do the general case by induction on i, writing $\mathcal{F}$ as a quotient of a locally free sheaf.

[116]Hint: Cut with a hyperplane $H \subseteq X$, and use Bertini's theorem (II, 8.18) to reduce to the case Y is a finite set of points.

3.8. **III.8: Higher Direct Images of Sheaves.**

3.8.1. *III.8.1.* Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf of abelian groups on X, and assume that $R^i f_*(\mathcal{F}) = 0$ for all $i > 0$. Show that there are natural isomorphisms, for each $i \geqslant 0$,[117]

3.8.2. *III.8.2.* Let $f : X \to Y$ be an affine morphism of schemes (II, Ex. 5.17) with X noetherian, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Show that the hypotheses of (Ex. 8.1) are satisfied, and hence that $H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F})$ for each $i \geqslant 0$.

3.8.3. *III.8.3. The Projection Formula.* Let $f : X \to Y$ be a morphism of ringed spaces, let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Prove the projection formula (cf. (II, Ex. 5.1))

3.8.4. *III.8.4.* Let Y be a noetherian scheme, and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of rank $n+1$, $n \geqslant 1$. Let $X = \mathbf{P}(\mathcal{E})$ (II, §), with the invertible sheaf $\mathcal{O}_X(1)$ and the projection morphism $\pi : X \to Y$.

a. Then
   - $\pi_*(\mathcal{O}(l)) \cong S^l(\mathcal{E})$ for $l \geqslant 0, \pi_*(\mathcal{O}(l)) = 0$ for $l < 0$ (II, 7.11);
   - $R^i\pi_*(\mathcal{O}(l)) = 0$ for $0 < i < n$ and all $l \in \mathbf{Z}$; and
   - $R^n\pi_*(\mathcal{O}(l)) = 0$ for $l > -n-1$.

b. Show there is a natural exact sequence cf. (II, 8.13), and conclude that the **relative canonical sheaf** $\omega_{X/Y} = \wedge^n \Omega_{X/Y}$ is isomorphic to $\left(\pi^* \wedge^{n+1} \mathcal{E}\right)(-n-1)$. Show furthermore that there is a natural isomorphism $R^n\pi_*\left(\omega_{X/Y}\right) \cong \mathcal{O}_Y$ (cf. (7.1.1)).

c. Now show, for any $l \in \mathbf{Z}$, that

d. Show that $p_a(X) = (-1)^n p_a(Y)$ (use (Ex. 8.1)) and $p_g(X) = 0$ (use (II, 8.11)).

e. In particular, if Y is a nonsingular projective curve of genus g, and $\mathcal{E}$ a locally free sheaf of rank 2 , then X is a projective surface with $p_a = -g, p_g = 0$, and irregularity g (7.12.3). This kind of surface is called a **geometrically ruled surface** (V, §2).

3.9. **III.9: Flat Morphisms.**

3.9.1. *III.9.1.* A flat morphism $f : X \to Y$ of finite type of noetherian schemes is open, i.e, for every open subset $U \subseteq X, f(U)$ is open in Y.[118]

3.9.2. *III.9.2.* Do the calculation of (9.8.4) for the curve of (I, Ex. 3.14). Show that you get an embedded point at the cusp of the plane cubic curve.

3.9.3. *III.9.3.* Some examples of flatness and nonflatness.

a. If $f : X \to Y$ is a finite surjective morphism of nonsingular varieties over an algebraically closed field k, then f is flat.

b. Let X be a union of two planes meeting at a point, each of which maps isomorphically to a plane Y. Show that f is not flat. For example, let $Y = \operatorname{Spec} k[x, y]$ and

c. Again let $Y = \operatorname{Spec} k[x, y]$, but take $X = \operatorname{Spec} k[x, y, z, w]/\left(z^2, zw, w^2, xz - yw\right)$. Show that $X_{\text{red}} \cong Y, X$ has no embedded points, but that f is not flat.

[117] This is a degenerate case of the Leray spectral sequence-see Godement $[1, II, 4.17.1]$.

[118] Hint: Show that $f(U)$ is constructible and stable under generization (II, Ex. 3.18) and (II, Ex. 3.19).

3.9.4. *III.9.4. Open Nature of Flatness.* Let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Then $\left\{x \in X \;\middle|\; f \text{ is flat at } x\right\}$ is an open subset of X (possibly empty).[119]

3.9.5. *III.9.5. Very Flat Families.* For any closed subscheme $X \subseteq \mathbf{P}^n$, we denote by $C(X) \subseteq \mathbf{P}^{n+1}$ the projective cone over X (I, Ex. 2.10). If $I \subseteq k\left[x_0, \ldots, x_n\right]$ is the (largest) homogeneous ideal of X, then $C(X)$ is defined by the ideal generated by I in $k\left[x_0, \ldots, x_{n+1}\right]$.

a. Give an example to show that if $\{X_t\}$ is a flat family of closed subschemes of $\mathbf{P}^n$, then $\{C\left(X_t\right)\}$ need not be a flat family in $\mathbf{P}^{n+1}$.
b. To remedy this situation, we make the following definition. Let $X \subseteq \mathbf{P}^n_T$ be a closed subscheme, where T is a noetherian integral scheme. For each $t \in T$, let $I_t \subseteq S_t = k(t)\left[x_0, \ldots, x_n\right]$ be the homogeneous ideal of X_t in $\mathbf{P}^n_{k(t)}$. We say that the family $\{X_t\}$ is **very flat** if for all $d \geqslant 0$, is independent of t. Here $(\quad)_d$ means the homogeneous part of degree d.
c. If $\{X_t\}$ is a very flat family in $\mathbf{P}^n$, show that it is flat. Show also that $\{C\left(X_t\right)\}$ is a very flat family in $\mathbf{P}^{n+1}$, and hence flat.
d. If $\left\{X_{(t)}\right\}$ is an algebraic family of projectively normal varieties in $\mathbf{P}^n_k$, parametrized by a nonsingular curve T over an algebraically closed field k, then $\left\{X_{(t)}\right\}$ is a very flat family of schemes.

3.9.6. *III.9.6.* Let $Y \subseteq \mathbf{P}^n$ be a nonsingular variety of dimension $\geqslant 2$ over an algebraically closed field k. Suppose $\mathbf{P}^{n-1}$ is a hyperplane in $\mathbf{P}^n$ which does not contain Y, and such that the scheme $Y' = Y \cap \mathbf{P}^{n-1}$ is also nonsingular. Prove that Y is a complete intersection in $\mathbf{P}^n$ if and only if Y' is a complete intersection in $\mathbf{P}^{n-1}$.[120]

3.9.7. *III.9.7.* Let $Y \subseteq X$ be a closed subscheme, where X is a scheme of finite type over a field k. Let $D = k[t]/t^2$ be the ring of dual numbers, and define an infinitesimal deformation of Y as a closed subscheme of X, to be a closed subscheme $Y' \subseteq X \underset{k}{\times} D$, which is flat over D, and whose closed fibre is Y.
Show that these Y' are classified by $H^0\left(Y, \mathcal{N}_{Y/X}\right)$, where

3.9.8. *III.9.8. *.* Let A be a finitely generated k-algebra. Write A as a quotient of a polynomial ring P over k, and let J be the kernel: Consider the exact sequence of (II, 8.4A) Apply the functor $\operatorname{Hom}_A(\cdot, A)$, and let $T^1(A)$ be the cokernel: Now use the construction of (II, Ex. 8.6) to show that $T^1(A)$ classifies infinitesimal deformations of A, i.e., algebras A' flat over $D = k[t]/t^2$, with $A' \otimes_D k \cong A$. It follows that $T^1(A)$ is independent of the given representation of A as a quotient of a polynomial ring P.

3.9.9. *III.9.9.* A k-algebra A is said to be **rigid** if it has no infinitesimal deformations, or equivalently, by (Ex. 9.8) if $T^1(A) = 0$. Let $A = k[x, y, z, w]/(x, y) \cap (z, w)$, and show that A is rigid. This corresponds to two planes in $\mathbf{A}^4$ which meet at a point.

3.9.10. *III.9.10.* A scheme X_0 over a field k is rigid if it has no infinitesimal deformations.

a. Show that $\mathbf{P}^1_k$ is rigid, using (9.13.2).
b. One might think that if X_0 is rigid over k, then every global deformation of X_0 is locally trivial. Show that this is not so, by constructing a proper, flat morphism $f : X \to \mathbf{A}^2$ over k algebraically closed, such that $X_0 \cong \mathbf{P}^1_k$, but there is no open neighborhood U of 0 in $\mathbf{A}^2$ for which $f^{-1}(U) \cong U \times \mathbf{P}^1$.

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[119]See Grothendieck EGA $IV_3, 11.1.1$.

[120]Hint: See (II, Ex. 8.4) and use (9.12) applied to the affine cones over Y and Y'.

c. * Show, however, that one can trivialize a global deformation of $\mathbf{P}^1$ after a flat base extension, in the following sense: let $f : X \to T$ be a flat projective morphism, where T is a nonsingular curve over k algebraically closed. Assume there is a closed point $t \in T$ such that $X_t \cong \mathbf{P}_k^1$. Then there exists a nonsingular curve T', and a flat morphism $g : T' \to T$, whose image contains t, such that if $X' = X \times_T T'$ is the base extension, then the new family $f' : X' \to T'$ is isomorphic to $\mathbf{P}_{T'}^1 \to T'$.

3.9.11. *III.9.11.* Let Y be a nonsingular curve of degree d in $\mathbf{P}_k^n$, over an algebraically closed field k. Show that[121]

3.10. **III.10: Smooth Morphisms.**

3.10.1. *III.10.1. Smooth $\neq$ Regular.* Over a nonperfect field, smooth and regular are not equivalent. For example, let k_0 be a field of characteristic $p > 0$, let $k = k_0(t)$, and let $X \subseteq \mathbf{A}_k^2$ be the curve defined by $y^2 = x^p - t$. Show that every local ring of X is a regular local ring, but X is not smooth over k.

3.10.2. *III.10.2.* Let $f : X \to Y$ be a proper, flat morphism of varieties over k. Suppose for some point $y \in Y$ that the fibre X_y is smooth over $k(y)$. Then show that there is an open neighborhood U of y in Y such that $f : f^{-1}(U) \to U$ is smooth.

3.10.3. *III.10.3. Tale Morphisms.* A morphism $f : X \to Y$ of schemes of finite type over k is **étale**if it is smooth of relative dimension 0 . It is **unramified** if for every $x \in X$, letting $y = f(x)$, we have $\mathrm{m}_y \cdot \mathcal{O}_x = \mathfrak{m}_x$, and $k(x)$ is a separable algebraic extension of $k(y)$.
Show that the following conditions are equivalent:

(i) f is étale;
(ii) f is flat, and $\Omega_{X/Y} = 0$;
(iii) f is flat and unramified.

3.10.4. *III.10.4.* Show that a morphism $f : X \to Y$ of schemes of finite type over k is étale if and only if the following condition is satisfied: for each $x \in X$, let $y = f(x)$. Let $\widehat{\mathcal{O}}_x$ and $\widehat{\mathcal{O}}_y$ be the completions of the local rings at x and y. Choose fields of representatives (II, 8.25A) $k(x) \subseteq \widehat{\mathcal{O}}_x$ and $k(y) \subseteq \widehat{\mathcal{O}}_y$ so that $k(y) \subseteq k(x)$ via the natural map $\widehat{\mathcal{O}}_y \to \widehat{\mathcal{O}}_x$.
Then our condition is that for every $x \in X, k(x)$ is a separable algebraic extension of $k(y)$, and the natural map is an isomorphism.

3.10.5. *III.10.5. Étale Neighborhoods.* If x is a point of a scheme X, we define an **étale neighborhood** of x to be an étale morphism $f : U \to X$, together with a point $x' \in U$ such that $f(x') = x$.
As an example of the use of étale neighborhoods, prove the following: if $\mathcal{F}$ is a coherent sheaf on X, and if every point of X has an étale neighborhood $f : U \to X$ for which $f^*\mathcal{F}$ is a free $\mathcal{O}_U$-module, then $\mathcal{F}$ is locally free on X.

3.10.6. *III.10.6.* Let Y be the plane nodal cubic curve $y^2 = x^2(x + 1)$. Show that Y has a finite étale covering X of degree 2, where X is a union of two irreducible components, each one isomorphic to the normalization of Y (Fig. 12).

[121] Hint: Compare Y to a suitable projection of Y into $\mathbf{P}^2$, as in (9.8.3) and (9.8.4).

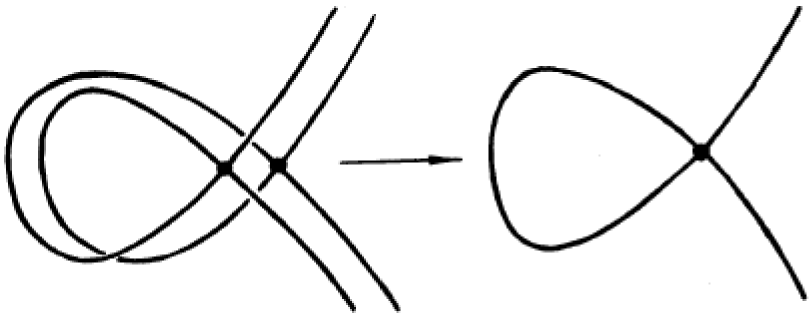

Figure 12. A finite étale covering.

3.10.7. *III.10.7. (Serre). A linear system with moving singularities.* Let k be an algebraically closed field of characteristic 2. Let $P_1, \ldots, P_7 \in \mathbf{P}_k^2$ be the seven points of the projective plane over the prime field $\mathbf{F}_2 \subseteq k$. Let D be the linear system of all cubic curves in X passing through $P_1, \ldots, P_7$.

a. D is a linear system of dimension 2 with base points $P_1, \ldots, P_7$, which determines an inseparable morphism of degree 2 from $X - \{P_i\}$ to $\mathbf{P}^2$.
b. Every curve $C \in D$ is singular.

   More precisely, either C consists of 3 lines all passing through one of the P_i, or C is an irreducible cuspidal cubic with cusp $P \neq$ any P_i.

   Furthermore, the correspondence $C \mapsto$ the singular point of C is a $1-1$ correspondence between D and $\mathbf{P}^2$. Thus the singular points of elements of D move all over.

3.10.8. *III.10.8. A linear system with moving singularities contained in the base locus (any characteristic).* In affine 3 -space with coordinates x, y, z, let C be the conic $(x-1)^2 + y^2 = 1$ in the xy-plane, and let P be the point $(0,0,t)$ on the z-axis. Let Y_t be the closure in $\mathbf{P}^3$ of the cone over C with vertex P.
Show that as t varies, the surfaces $\{Y_t\}$ form a linear system of dimension 1, with a moving singularity at P. The base locus of this linear system is the conic C plus the z-axis.

3.10.9. *III.10.9.* Let $f : X \to Y$ be a morphism of varieties over k. Assume that Y is regular, X is Cohen-Macaulay, and that every fibre of f has dimension equal to $\dim X - \dim Y$. Then f is flat.[122]

3.11. **III.11: The Theorem on Formal Functions.**

3.11.1. *III.11.1.* Show that the result of (11.2) is false without the projective hypothesis. For example, let $X = \mathbf{A}_k^n$, let $P = (0, \ldots, 0)$, let $U = X - P$, and let $f : U \to X$ be the inclusion. Then the fibres of f all have dimension 0 , but $R^{n-1} f_* \mathcal{O}_U \neq 0$.

3.11.2. *III.11.2.* Show that a projective morphism with finite fibres (= quasi-finite (II, Ex. 3.5)) is a finite morphism.

[122]Hint: Imitate the proof of (10.4), using (II, 8.21A).

3.11.3. *III.11.3. Improved Bertini's Theorem.* Let X be a normal, projective variety over an algebraically closed field k. Let D be a linear system (of effective Cartier divisors) without base points, and assume that D is **not composite with a pencil**, which means that if $f : X \to \mathbf{P}_k^n$ is the morphism determined by D, then $\dim f(X) \geqslant 2$.
Then show that every divisor in D is connected.[123]

3.11.4. *III.11.4. Principle of Connectedness.* Let $\{X_t\}$ be a flat family of closed subschemes of $\mathbf{P}_k^n$ parametrized by an irreducible curve T of finite type over k. Suppose there is a nonempty open set $U \subseteq T$, such that for all closed points $t \in U, X_t$ is connected. Then prove that X_t is connected for all $t \in T$.

3.11.5. *III.11.5. *.* Let Y be a hypersurface in $X = \mathbf{P}_k^N$ with $N \geqslant 4$. Let $\widehat{X}$ be the formal completion of X along Y (II, §). Prove that the natural map $\operatorname{Pic} \widehat{X} \to \operatorname{Pic} Y$ is an isomorphism.[124]

3.11.6. *III.11.6.* Again let Y be a hypersurface in $X = \mathbf{P}_k^N$, this time with $N \geqslant 2$.

a. If $\mathcal{F}$ is a locally free sheaf on X, show that the natural map is an isomorphism.
b. Show that the following conditions are equivalent:
   (i) For each locally free sheaf $\mathcal{F}$ on $\widehat{X}$, there exists a coherent sheaf $\mathscr{F}$ on X such that $\mathcal{F} \cong \widehat{\mathscr{F}}$ (i.e., $\mathcal{F}$ is algebraizable);
   (ii) For each locally free sheaf $\mathcal{F}$ on $\widehat{X}$, there is an integer n_0 such that $\mathcal{F}(n)$ is generated by global sections for all $n \geqslant n_0$.[125]
c. Show that the conditions (i) and (ii) of (b) imply that the natural map $\operatorname{Pic} X \to \operatorname{Pic} \widehat{X}$ is an isomorphism.[126]

3.11.7. *III.11.7.* Now let Y be a curve in $X = \mathbf{P}_k^2$.

a. Use the method of (Ex. 11.5) to show that Pic $\widehat{X} \to$ Pic Y is surjective, and its kernel is an infinite-dimensional vector space over k.
b. Conclude that there is an invertible sheaf $\mathcal{L}$ on $\widehat{X}$ which is not algebraizable.
c. Conclude also that there is a locally free sheaf $\mathcal{F}$ on $\widehat{X}$ so that no twist $\mathcal{F}(n)$ is generated by global sections. Cf. (II, 9.9.1)

3.11.8. *III.11.8.* Let $f : X \to Y$ be a projective morphism, let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y, and assume that $H^i(X_y, \mathcal{F}_y) = 0$ for some i and some $y \in Y$. Then show that $R^i f_*(\mathcal{F})$ is 0 in a neighborhood of y.

3.12. III.12: The Semicontinuity Theorem.

3.12.1. *III.12.1.* Let Y be a scheme of finite type over an algebraically closed field k. Show that the function is upper semicontinuous of the set of closed points Y.

3.12.2. *III.12.2.* Let $\{X_t\}$ be a family of hypersurfaces of the same degree in $\mathbf{P}_k^n$. Show that for each i, the function $h^i(X_t, \mathcal{O}_{X_t})$ is a constant function of t.

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[123] See (10.9.1). Hints: Use (11.5), (Ex. 5.7) and (7.9).

[124] Hint: Use (II, Ex. 9.6), and then study the maps $\operatorname{Pic} X_{n+1} \to \operatorname{Pic} X_n$ for each n using (Ex. 4.6) and (Ex. 5.5).

[125] Hint: For (ii) $\Rightarrow$ (i), show that one can find sheaves $\mathcal{E}_0, \mathcal{E}_1$ on X, which are direct sums of sheaves of the form $\mathcal{O}(-q_i)$, and an exact sequence $\widehat{\mathcal{E}}_1 \to \widehat{\mathcal{E}}_0 \to \tilde{\mathcal{F}} \to 0$ on $\widehat{X}$. Then apply (a) to the sheaf $\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_0)$.

[126] Note. In fact, (i) and (ii) always hold if $N \geqslant 3$. This fact, coupled with (Ex. 11.5) leads to Grothendieck's proof [*SGA* 2] of the Lefschetz theorem which says that if Y is a hypersurface in $\mathbf{P}_k^N$ with $N \geqslant 4$, then $\operatorname{Pic} Y \cong \mathbf{Z}$, and it. is generated by $\mathcal{O}_Y(1)$. See Hartshorne [5, *Ch.IV*] for more details.

3.12.3. *III.12.3.* Let $X_1 \subseteq \mathbf{P}_k^4$ be the **rational normal quartic curve** (which is the 4-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^4$). Let $X_0 \subseteq \mathbf{P}_k^3$ be a nonsingular rational quartic curve, such as the one in (I, Ex. 3.18b).
Use (9.8.3) to construct a flat family $\{X_t\}$ of curves in $\mathbf{P}^4$, parametrized by $T = \mathbf{A}^1$, with the given fibres X_1 and X_0 for $t = 1$ and $t = 0$.
Let $\mathcal{I} \subseteq \mathcal{O}_{\mathbf{P}^4 \times T}$ be the ideal sheaf of the total family $X \subseteq \mathbf{P}^4 \times T$. Show that $\mathcal{I}$ is flat over T.
Then show that and also This gives another example of cohomology groups jumping at a special point.

3.12.4. *III.12.4.* Let Y be an integral scheme of finite type over an algebraically closed field k. Let $f : X \to Y$ be a flat projective morphism whose fibres are all integral schemes. Let $\mathcal{L}, \mathcal{M}$ be invertible sheaves on X, and assume for each $y \in Y$ that $\mathcal{L}_y \cong \mathcal{M}_y$ on the fibre X_y.
Then show that there is an invertible sheaf $\mathcal{N}$ on Y such that $\mathcal{L} \cong \mathcal{M} \otimes f^* \mathcal{N}$.[127]

3.12.5. *III.12.5.* Let Y be an integral scheme of finite type over an algebraically closed field k. Let $\mathcal{E}$ be a locally free sheaf on Y, and let $X = \mathbf{P}(\mathcal{E})$ – see $(II, \S 7)$.
Then show that $\operatorname{Pic} X \cong (\operatorname{Pic} Y) \times \mathbf{Z}$. This strengthens (II, Ex. 7.9).

3.12.6. *III.12.6. *.* Let X be an integral projective scheme over an algebraically closed field k, and assume that $H^1(X, \mathcal{O}_X) = 0$. Let T be a connected scheme of finite type over k.

a. If $\mathcal{L}$ is an invertible sheaf on $X \times T$, show that the invertible sheaves $\mathcal{L}_t$ on $X = X \times \{t\}$ are isomorphic, for all closed points $t \in T$.
b. Show that $\operatorname{Pic}(X \times T) = \operatorname{Pic} X \times \operatorname{Pic} T$. (Do not assume that T is reduced!)[128]
   Cf. (IV, Ex. 4.10) and (V, Ex. 1.6) for examples where $\operatorname{Pic}(X \times T) \neq \operatorname{Pic} X \times$ Pic T.

4. IV: Curves

4.1. IV.1: Riemann-Roch.

4.1.1. *1.1.* Let X be a curve, and let $P \in X$ be a point. Then there exists a nonconstant rational function $f \in K(X)$, which is regular everywhere except at P.

4.1.2. *1.2.* Again let X be a curve, and let $P_1, \ldots P_r \in X$ be points. Then there is a rational function $f \in K(X)$ having poles (of some order) at each of the P_1, and regular elsewhere.

4.1.3. *1.3.* Let X be an integral, separated, regular, one-dimensional scheme of finite type over k, which is **not** proper over k. Then X is affine.[129]

4.1.4. *1.4.* Show that a separated, one-dimensional scheme of finite type over k, none of whose irreducible components is proper over k, is affine.[130]

4.1.5. *1.5.* For an effective divisor D on a curve X of genus g, show that $\dim |D| \leqslant \deg D$. Furthermore, equality holds if and only if $D = 0$ or $g = 0$.

4.1.6. *1.6.* Let X be a curve of genus g. Show that there is a finite morphism $f : X \to \mathbf{P}^1$ of degree[131] $\leqslant g + 1$.

[127]Hint: Use the results of this section to show that $f_*\left(\mathcal{L} \otimes \mathcal{M}^{-1}\right)$ is locally free of rank 1 on Y.

[128]Hint: Apply (12.11) with $i = 0, 1$ for suitable invertible sheaves on $X \times T$.

[129]Hint: Embed X in a (proper) curve $\overline{X}$ over k, and use (Ex. 1.2) to construct a morphism $f : \overline{X} \to \mathbf{P}^1$ such that $f^{-1}\left(\mathbf{A}^1\right) = X$

[130]Hint: Combine (Ex. 1.3) with (III, Ex. 3.1, Ex. 3.2, Ex. 4.2).

[131]Recall that the degree of a finite morphism of curves $f : X \to Y$ is defined as the degree of the field extension $[K(X) : K(Y)]$ (II.6).

4.1.7. *1.7.* A curve X is called **hyperelliptic** if $g \geqslant 2$ and there exists a finite morphism $f : X \to \mathbf{P}^1$ of degree 2.

a. If X is a curve of genus $g = 2$, show that the canonical divisor defines a complete linear system $|K|$ of degree 2 and dimension 1, without base points. Use (II, 7.8.1) to conclude that X is hyperelliptic.

b. Show that the curves constructed in (1.1.1) all admit a morphism of degree 2 to $\mathbf{P}^1$. Thus there exist hyperelliptic curves of any genus $g \geqslant 2$.[132]

4.1.8. *1.8. p_a of a Singular Curve.* Let X be an integral projective scheme of dimension 1 over k, and let $\tilde{X}$ be its normalization (II, Ex. 3.8). Then there is an exact sequence of sheaves on X, where $\tilde{\mathcal{O}}_P$ is the integral closure of $\mathcal{O}_P$. For each $P \in X$, let $\delta_P = \operatorname{length}(\tilde{\mathcal{O}}_P/\mathcal{O}_P)$.

a. Show that $p_a(X) = p_a(\tilde{X}) + \sum_{p\in X} \delta_p$.[133]

b. If $p_a(X) = 0$, show that X is already nonsingular and in fact isomorphic to $\mathbf{P}^1$.[134]

c. * If P is a node or an ordinary cusp (I, Ex. 5.6, Ex. 5.14), show that $\delta_P = 1$.[135]

4.1.9. *1.9. * Riemann-Roch for Singular Curves.* Let X be an integral projective scheme of dimension 1 over k. Let X_{reg} be the set of regular points of X.

a. Let $D = \sum n_i P_i$ be a divisor with support in X_{reg}, i.e., all $P_i \in X_{\text{reg}}$. Then define deg $D = \sum n_i$. Let $\mathscr{L}(D)$ be the associated invertible sheaf on X, and show that

b. Show that any Cartier divisor on X is the difference of two very ample Cartier divisors. [136]

c. Conclude that every invertible sheaf $\mathscr{L}$ on X is isomorphic to $\mathscr{L}(D)$ for some divisor D with support in X_{reg}.

d. Assume furthermore that X is a locally complete intersection in some projective space. Then by (III, 7.11) the dualizing sheaf ω_X is an invertible sheaf on X, so we can define the canonical divisor K to be a divisor with support in X_{reg} corresponding to ω_X. Then the formula of a. becomes

4.1.10. *1.10.* Let X be an integral projective scheme of dimension 1 over k, which is locally complete intersection, and has $p_a = 1$. Fix a point $P_0 \in X_{\text{reg.}}$. Imitate (1.3.7) to show that the map $P \to \mathscr{L}(P - P_0)$ gives a one-to-one correspondence between the points of X_{reg} and the elements of the group $\operatorname{Pic} X$.[137]

4.2. **IV.2: Hurwitz.**

4.2.1. *2.1.* Use (2.5.3) to show that $\mathbf{P}^n$ is simply connected.

4.2.2. *2.2 Classification of Curves of Genus 2 .* Fix an algebraically closed field k of characteristic $\neq 2$.

a. If X is a curve of genus 2 over k, the canonical linear system $|K|$ determines a finite morphism $f : X \to \mathbf{P}^1$ of degree 2 (Ex. 1.7). Show that it is ramified at exactly 6 points, with ramification index 2 at each one. Note that f is uniquely determined, up to

[132] Note: we will see later (Ex. 3.2) that there exist non-hyperelliptic curves. See also (V, Ex. 2.10).

[133] Hint: Use (III, Ex. 4.1) and (III, Ex. 5.3).

[134] This strengthens (1.3.5).

[135] Hint: Show first that δ_P depends only on the analytic isomorphism class of the singularity at P. Then compute δ_P for the node and cusp of suitable plane cubic curves. See (V, 3.9.3) for another method.

[136] Use (II, Ex. 7.5).

[137] This generalizes (II, 6.11.4) and (II, Ex. 6.7).

an automorphism of $\mathbf{P}^1$, so X determines an (unordered) set of 6 points of $\mathbf{P}^1$, up to an automorphism of $\mathbf{P}^1$.

b. Conversely, given six distinct elements $\alpha_1, \ldots, \alpha_6 \in k$, let K be the extension of $k(x)$ determined by the equation $z^2 = (x-\alpha_1)\cdots(x-\alpha_6)$. Let $f : X \to \mathbf{P}^1$ be the corresponding morphism of curves. Show that $g(X) = 2$, the map f is the same as the one determined by the canonical linear system, and f is ramified over the six points $x = \alpha_i$ of $\mathbf{P}^1$, and nowhere else. (Cf. (II, Ex. 6.4).)

c. Using (I, Ex. 6.6), show that if P_1, P_2, P_3 are three distinct points of $\mathbf{P}^1$, then there exists a unique $\varphi \in \operatorname{Aut} \mathbf{P}^1$ such that $\varphi(P_1) = 0, \varphi(P_2) = 1, \varphi(P_3) = \infty$. Thus in (a), if we order the six points of $\mathbf{P}^1$, and then normalize by sending the first three to $0, 1, x$, respectively, we may assume that X is ramified over $0, 1, \infty, \beta_1, \beta_2, \beta_3$, where $\beta_1, \beta_2, \beta_3$ are three distinct elements of $k, \neq 0, 1$.

d. Let Σ_6 be the symmetric group on 6 letters. Define an action of Σ_6 on sets of three distinct elements $\beta_1, \beta_2, \beta_3$ of $k, \neq 0, 1$, as follows: reorder the set $0, 1, \infty, \beta_1, \beta_2, \beta_3$ according to a given element $\sigma \in \Sigma_6$, then renormalise as in (c) so that the first three become $0, 1, \infty$ again. Then the last three are the new $\beta_1', \beta_2', \beta_3'$.

e. Summing up, conclude that there is a one-to-one correspondence between the set of isomorphism classes of curves of genus 2 over k, and triples of distinct elements $\beta_1, \beta_2, \beta_3$ of $k, \neq 0, 1$, modulo the action of Σ_6 described in (d). In particular, there are many nonisomorphic curves of genus 2 . We say that curves of genus 2 depend on three parameters, since they correspond to the points of an open subset of $\mathbf{A}_k^3$ modulo a finite group.

4.2.3. *2.3 Plane Curves.* Let X be a curve of degree d in $\mathbf{P}^2$. For each point $P \in X$, let $T_P(X)$ be the tangent line to X at P (I, Ex. 7.3). Considering $T_P(X)$ as a point of the dual projective plane $(\mathbf{P}^2)^*$, the map $P \to T_P(X)$ gives a morphism of X to its **dual curve** X^* in $(\mathbf{P}^2)^*$ (I, Ex. 7.3). Note that even though X is nonsingular, X^* in general will have singularities. We assume char $k = 0$ below.

a. Fix a line $L \subseteq \mathbf{P}^2$ which is not tangent to X. Define a morphism $\varphi : X \to L$ by $\varphi(P) = T_P(X) \cap L$, for each point $P \in X$. Show that φ is ramified at P if and only if either
   (1) $P \in L$, or
   (2) P is an inflection point of X, which means that the intersection multiplicity (I, Ex. 5.4) of $T_P(X)$ with X at P is $\geqslant 3$. Conclude that X has only finitely many inflection points.

b. A line of $\mathbf{P}^2$ is a **multiple tangent** of X if it is tangent to X at more than one point. It is a **bitangent** if it is tangent to X at exactly two points. If L is a multiple tangent of X, tangent to X at the points $P_1, \ldots, P_r$, and if none of the P_i is an inflection point, show that the corresponding point of the dual curve X^* is an ordinary r-fold point, which means a point of multiplicity r with distinct tangent directions (I, Ex. 5.3). Conclude that X has only finitely many multiple tangents.

c. Let $O \in \mathbf{P}^2$ be a point which is not on X, nor on any inflectional or multiple tangent of X. Let L be a line not containing O. Let $\psi : X \to L$ be the morphism defined by projection from O. Show that ψ is ramified at a point P $\in X$ if and only if the line OP is tangent to X at P, and in that case the ramification index is 2. Use Hurwitz's theorem and (I, Ex. 7.2) to conclude that there are exactly $d(d-1)$ tangents of X passing through O. Hence the degree of the dual curve (sometimes called the **class** of X) is $d(d-1)$.

d. Show that for all but a finite number of points of X, a point O of X lies on exactly $(d+1)(d-2)$ tangents of X, not counting the tangent at O.

e. Show that the degree of the morphism φ of a. is $d(d-1)$. Conclude that if $d \geqslant 2$, then X has $3d(d-2)$ inflection points, properly counted. (If $T_P(X)$ has intersection multiplicity r with X at P, then P should be counted $r-2$ times as an inflection point. If $r=3$ we call it an ordinary inflection point.) Show that an ordinary inflection point of X corresponds to an ordinary cusp of the dual curve X^*.

f. Now let X be a plane curve of degree $d \geqslant 2$, and assume that the dual curve X^* has only nodes and ordinary cusps as singularities (which should be true for sufficiently general X). Then show that X has exactly $\frac{1}{2}d(d-2)(d-3)(d+3)$ bitangents.[138]

g. For example, a plane cubic curve has exactly 9 inflection points, all ordinary. The line joining any two of them intersects the curve in a third one.

h. A plane quartic curve has exactly 28 bitangents. (This holds even if the curve has a tangent with four-fold contact, in which case the dual curve X^* has a tacnode.)

4.2.4. *2.4 A Funny Curve in Characteristic p.* Let X be the plane quartic curve $x^3y+y^3z+z^3x=0$ over a field of characteristic 3 . Show that X is nonsingular, every point of X is an inflection point, the dual curve X^* is isomorphic to X, but the natural map $X \rightarrow X^*$ is purely inseparable.

4.2.5. *2.5 Automorphisms of a Curve of Genus* $\geqslant 2$. Prove the theorem of Hurwitz that a curve X of genus $g \geqslant 2$ over a field of characteristic 0 has at most $84(g-1)$ automorphisms.
We will see later (Ex. 5.2) or (V, Ex. 1.11) that the group $G=\operatorname{Aut} X$ is finite. So let G have order n. Then G acts on the function field $K(X)$. Let L be the fixed field. Then the field extension $L \subseteq K(X)$ corresponds to a finite morphism of curves $f: X \rightarrow Y$ of degree n.

a. If $P \in X$ is a ramification point, and $e_P=r$, show that $f^{-1}f(P)$ consists of exactly n/r points, each having ramification index r. Let $P_1, \ldots, P_s$ be a maximal set of ramification points of X lying over distinct points of Y, and let $e_{P_1}=r_i$. Then show that Hurwitz's theorem implies that

b. Since $g \geqslant 2$, the left hand side of the equation is >0. Show that if $g(Y) \geqslant 0$, $s \geqslant 0, r_i \geqslant 2, i=1, \ldots, s$ are integers such that then the minimum value of this expression is $1/42$. Conclude that $n \leqslant 84(g-1)$.[139]

4.2.6. *2.6 f_* for Divisors.* Let $f: X \rightarrow Y$ be a finite morphism of curves of degree n. We define a homomorphism $f_*: \operatorname{Div} X \rightarrow \operatorname{Div} Y$ by $f_*\left(\sum n_i P_i\right)=\sum n_i f\left(P_i\right)$ for any divisor $D=\sum n_i P_i$ on X.

a. For any locally free sheaf $\mathscr{E}$ on Y, of rank r, we define $\operatorname{det} \mathscr{E}=\wedge^r \mathscr{E} \in \operatorname{Pic} Y$ (II, Ex. 6.11). In particular, for any invertible sheaf $\mathscr{M}$ on $X, f_* \mathscr{M}$ is locally free of rank n on Y, so we can consider $\operatorname{det} f_* \mathscr{M} \in \operatorname{Pic} Y$. Show that for any divisor D on X,[140] Note in particular that $\operatorname{det}\left(f_* \mathscr{L}(D)\right) \neq \mathscr{L}\left(f_* D\right)$ in general!

b. Conclude that $f_* D$ depends only on the linear equivalence class of D, so there is an induced homomorphism $f_*: \operatorname{Pic} X \rightarrow \operatorname{Pic} Y$. Show that $f_* f^*: \operatorname{Pic} Y \rightarrow \operatorname{Pic} Y$ is just multiplication by n.

[138]Hint: Show that X is the normalization of X^*. Then calculate $p_a\left(X^*\right)$ two ways: once as a plane curve of degree $d(d-1)$, and once using (Ex. 1.8).

[139]See (Ex. 5.7) for an example where this maximum is achieved. Note: It is known that this maximum is achieved for infinitely many values of g (Macbeath

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). Over a field of characteristic $p>0$, the same bound holds, provided $p>g+1$, with one exception, namely the hyperelliptic curve $y^2=x^p-x$, which has $p=2g+1$ and $2p\left(p^2-1\right)$ automorphisms (Roquette). For other bounds on the order of the group of automorphisms in characteristic p, see Singh and Stichtenoth.

[140]Hint: First consider an effective divisor D, apply f_* to the exact sequence $0 \rightarrow \mathscr{L}(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$, and use (II, Ex. 6.11).]

c. Use duality for a finite flat morphism (III, Ex. 6.10) and (III, Ex. 7.2) to show that
d. Now assume that f is separable, so we have the ramification divisor R. We define the **branch divisor** B to be the divisor $f_* R$ on Y. Show that

4.2.7. *2.7 Etale Covers of Degree 2.* Let Y be a curve over a field k of characteristic $\neq 2$. We show there is a one-to-one correspondence between finite étale morphisms $f : X \to Y$ of degree 2, and 2-torsion elements of $\operatorname{Pic} Y$, i.e., invertible sheaves $\mathscr{L}$ on Y with $\mathscr{L}^2 \cong \mathcal{O}_Y$.

a. Given an étale morphism $f : X \to Y$ of degree 2, there is a natural map $\mathcal{O}_Y \to f_* \mathcal{O}_X$. Let $\mathscr{L}$ be the cokernel. Then $\mathscr{L}$ is an invertible sheaf on $Y, \mathscr{L} \cong \det f_* \mathcal{O}_X$, and so $\mathscr{L}^2 \cong \mathcal{O}_Y$ by (Ex. 2.6). Thus an étale cover of degree 2 determines a 2-torsion element in $\operatorname{Pic} Y$.
b. Conversely, given a 2-torsion element $\mathscr{L}$ in $\operatorname{Pic} Y$, define an $\mathcal{O}_Y$-algebra structure on $\mathcal{O}_Y \oplus \mathscr{L}$ by $\langle a, b\rangle \cdot \langle a', b'\rangle = \langle aa' + \varphi(b \otimes b'), ab' + a'b\rangle$, where φ is an isomorphism of $\mathscr{L} \otimes \mathscr{L} \to \mathcal{O}_Y$. Then take $X = \operatorname{Spec}(\mathcal{O}_Y \oplus \mathscr{L})$ (II, Ex. 5.17). Show that X is an étale cover of Y.
c. Show that these two processes are inverse to each other.[141][142]

4.3. IV.3: Embeddings in Projective Space.

4.3.1. *II.3.1.* If X is a curve of genus 2 , show that a divisor D is very ample $\Leftrightarrow \deg D \geqslant 5$. This strengthens (3.3.4).

4.3.2. *II.3.2.* Let X be a plane curve of degree 4 .

a. Show that the effective canonical divisors on X are exactly the divisors $X.L$, where L is a line in $\mathbf{P}^2$.
b. If D is any effective divisor of degree 2 on X, show that $\dim |D| = 0$.
c. Conclude that X is not hyperelliptic (Ex. 1.7).

4.3.3. *II.3.3.* If X is a curve of genus $\geqslant 2$ which is a complete intersection (II, Ex. 8.4) in some $\mathbf{P}^n$, show that the canonical divisor K is very ample. Conclude that a curve of genus 2 can never be a complete intersection in any $\mathbf{P}^n$. Cf. (Ex. 5.1).

4.3.4. *II.3.4. The Rational Normal Curve.* Let X be the d-uple embedding (I, Ex. 2.12) of $\mathbf{P}^1$ in $\mathbf{P}^d$, for any $d \geqslant 1$. We call X the **rational normal curve** of degree d in $\mathbf{P}^d$.

a. Show that X is projectively normal, and that its homogeneous ideal can be generated by forms of degree 2 .
b. If X is any curve of degree d in $\mathbf{P}^n$, with $d \leqslant n$, which is not contained in any $\mathbf{P}^{n-1}$, show that in fact $d = n, g(X) = 0$, and X differs from the rational normal curve of degree d only by an automorphism of $\mathbf{P}^d$. Cf. (II. 7.8.5).
c. In particular, any curve of degree 2 in any $\mathbf{P}^n$ is a conic in some $\mathbf{P}^2$.
d. A curve of degree 3 in any $\mathbf{P}^n$ must be either a plane cubic curve, or the twisted cubic curve in $\mathbf{P}^3$.

[141]Note. This is a special case of the more general fact that for $(n, \operatorname{char} k) = 1$, the étale Galois covers of Y with group $\mathbf{Z}/n\mathbf{Z}$ are classified by the étale cohomology group $H^1_{\text{et}}(Y, \mathbf{Z}/n\mathbf{Z})$, which is equal to the group of n-torsion points of $\operatorname{Pic} Y$. See Serre

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[142]Hint: Let $\tau : X \to X$ be the involution which interchanges the points of each fibre of f. Use the trace map $a \mapsto a + \tau(a)$ from $f_* \mathcal{O}_X \to \mathcal{O}_Y$ to show that the sequence of $\mathcal{O}_{Y-}$ modules in a. is split exact.

4.3.5. *II.3.5.* Let X be a curve in $\mathbf{P}^3$, which is not contained in any plane.

a. If $O \notin X$ is a point, such that the projection from O induces a birational morphism φ from X to its image in $\mathbf{P}^2$, show that $\varphi(X)$ must be singular.[143]
b. If X has degree d and genus g, conclude that $g < \frac{1}{2}(d-1)(d-2)$. (Use (Ex. 1.8).)
c. Now let $\{X_t\}$ be the flat family of curves induced by the projection (III, 9.8.3) whose fibre over $t=1$ is X, and whose fibre X_0 over $t=0$ is a scheme with support $\varphi(X)$. Show that X_0 always has nilpotent elements. Thus the example (III, 9.8.4) is typical.

4.3.6. *II.3.6.* Curves of Degree 4.

a. If X is a curve of degree 4 in some $\mathbf{P}^n$, show that either
   (1) $g=0$, in which case X is either the rational normal quartic in $\mathbf{P}^4$ (Ex. 3.4) or the rational quartic curve in $\mathbf{P}^3$ (II, 7.8.6), or
   (2) $X \subseteq \mathbf{P}^2$, in which case $g=3$, or
   (3) $X \subseteq \mathbf{P}^3$ and $g=1$.
b. In the case $g=1$, show that X is a complete intersection of two irreducible quadric surfaces in $\mathbf{P}^3$ (I, Ex. 5.11).[144]

4.3.7. *II.3.7.* In view of (3.10), one might ask conversely, is every plane curve with nodes a projection of a nonsingular curve in $\mathbf{P}^3$? Show that the curve $xy+x^4+y^4=0$ (assume char $k \neq 2$) gives a counterexample.

4.3.8. *II.3.8.* We say a (singular) integral curve in $\mathbf{P}^n$ is **strange** if there is a point which lies on all the tangent lines at nonsingular points of the curve.

a. There are many singular strange curves, e.g., the curve given parametrically by $x=t, y=t^p, z=t^{2p}$ over a field of characteristic $p>0$.
b. Show, however, that if char $k=0$, there aren't even any singular strange curves besides $\mathbf{P}^1$.

4.3.9. *II.3.9. Bertini's Lemma.* Prove the following lemma of Bertini: if X is a curve of degree d in $\mathbf{P}^3$, not contained in any plane, then for almost all planes $H \subseteq \mathbf{P}^3$ (meaning a Zariski open subset of the dual projective space $\left(\mathbf{P}^3\right)^*$), the intersection $X \cap H$ consists of exactly d distinct points, no three of which are collinear.

4.3.10. *II.3.10. Not every secant is a multisecant.* Generalize the statement that "not every secant is a multisecant" as follows. If X is a curve in $\mathbf{P}^n$, not contained in any $\mathbf{P}^{n-1}$, and if char $k=0$, show that for almost all choices of $n-1$ points $P_1, \ldots, P_{n-1}$ on X, the linear space L^{n-2} spanned by the P_i does not contain any further points of X.

4.3.11. *II.3.11.*

a. If X is a nonsingular variety of dimension r in $\mathbf{P}^n$, and if $n>2r+1$, show that there is a point $O \notin X$, such that the projection from O induces a closed immersion of X into $\mathbf{P}^{n-1}$.
b. If X is the Veronese surface in $\mathbf{P}^5$, which is the 2-uple embedding of $\mathbf{P}^2$ (I, Ex. 2.13), show that each point of every secant line of X lies on infinitely many secant lines. Therefore, the secant variety of X has dimension 4, and so in this case there is a projection which gives a closed immersion of X into P^4 (II, Ex. 7.7).[145]

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143 Hint: Calculate $\dim H^0\left(X, \mathcal{O}_X(1)\right)$ two ways.

144 Hint: Use the exact sequence $0 \rightarrow \mathcal{I}_X \rightarrow \mathcal{O}_{\mathbf{P}^3} \rightarrow \mathcal{O}_X \rightarrow 0$ to compute $\dim H^0\left(\mathbf{P}^3, \mathcal{I}_X(2)\right)$, and thus conclude that X is contained in at least two irreducible quadric surfaces.

145 A theorem of Severi [1] states that the Veronese surface is the only surface in $\mathbf{P}^5$ for which there is a projection giving a closed immersion into $\mathbf{P}^4$. Usually one obtains a finite number of double points with transversal tangent planes.

4.3.12. *II.3.12.* For each value of $d = 2, 3, 4, 5$ and r satisfying $0 \leqslant r \leqslant \frac{1}{2}(d-1)(d-2)$, show that there exists an irreducible plane curve of degree d with r nodes and no other singularities.

4.4. IV.4: Elliptic Curves.

4.4.1. *II.4.1.* Let X be an elliptic curve over k, with char $k \neq 2$, let $P \in X$ be a point, and let R be the graded ring $R = \bigoplus_{n \geqslant 0} H^0(X, \mathcal{O}_X(nP))$. Show that for suitable choice of t, x, y as a graded ring, where $k[t, x, y]$ is graded by setting $\deg t = 1, \deg x = 2$, $\deg y = 3$.

4.4.2. *II.4.2.* If D is any divisor of degree $\geqslant 3$ on the elliptic curve X, and if we embed X in $\mathbf{P}^n$ by the complete linear system $|D|$, show that the image of X in $\mathbf{P}^n$ is projectively normal.[146]

4.4.3. *II.4.3.* Let the elliptic curve X be embedded in $\mathbf{P}^2$ so as to have the equation $y^2 = x(x-1)(x-\lambda)$. Show that any automorphism of X leaving $P_0 = (0, 1, 0)$ fixed is induced by an automorphism of $\mathbf{P}^2$ coming from the automorphism of the affine (x, y)-plane given by In each of the four cases of (4.7), describe these automorphisms of $\mathbf{P}^2$ explicitly, and hence determine the structure of the group $G = \operatorname{Aut}(X, P_0)$.

4.4.4. *II.4.4.* Let X be an elliptic curve in $\mathbf{P}^2$ given by an equation of the form Show that the j-invariant is a rational function of the a_i, with coefficients in $\mathbf{Q}$. In particular, if the a_i are all in some field $k_0 \subseteq k$, then $j \in k_0$ also. Furthermore, for every $\alpha \in k_0$, there exists an elliptic curve defined over k_0 with j-invariant equal to α.

4.4.5. *II.4.5.* Let X, P_0 be an elliptic curve having an endomorphism $f : X \to X$ of degree 2 .

a. If we represent X as a 2-1 covering of $\mathbf{P}^1$ by a morphism $\pi : X \to \mathbf{P}^1$ ramified at P_0, then as in (4.4), show that there is another morphism $\pi' : X \to \mathbf{P}^1$ and a morphism $g : \mathbf{P}^1 \to \mathbf{P}^1$, also of degree 2 , such that $\pi \circ f = g \circ \pi'$.
b. For suitable choices of coordinates in the two copies of $\mathbf{P}^1$, show that g can be taken to be the morphism $x \to x^2$.
c. Now show that g is branched over two of the branch points of π, and that g^{-1} of the other two branch points of π consists of the four branch points of π'. Deduce a relation involving the invariant λ of X.
d. Solving the above, show that there are just three values of j corresponding to elliptic curves with an endomorphism of degree 2 , and find the corresponding values of λ and j.[147]

4.4.6. *II.4.6.*

a. Let X be a curve of genus g embedded birationally in $\mathbf{P}^2$ as a curve of degree d with r nodes. Generalize the method of (Ex. 2.3) to show that X has inflection points. A node does not count as an inflection point. Assume char $k = 0$.
b. Now let X be a curve of genus g embedded as a curve of degree d in $\mathbf{P}^n, n \geqslant 3$, not contained in any $\mathbf{P}^{n-1}$. For each point $P \in X$, there is a hyperplane H containing P, such that P counts at least n times in the intersection $H \cap X$. This is called an **osculating** hyperplane at P. It generalizes the notion of tangent line for curves in $\mathbf{P}^2$.

   If P counts at least $n+1$ times in $H \cap X$, we say H is a **hyperosculating hyperplane**, and that P is a **hyperosculation point**. Use Hurwitz's theorem as above, and induction on n, to show that X has hyperosculation points.
c. If X is an elliptic curve, for any $d \geqslant 3$, embed X as a curve of degree d in $\mathbf{P}^{d-1}$, and conclude that X has exactly d^2 points of order d in its group law.

[146]Note. It is true more generally that if D is a divisor of degree $\geqslant 2g+1$ on a curve of genus g, then the embedding of X by $|D|$ is projectively normal (Mumford [4, p. 55])

[147]Answers: $j = 2^6 \cdot 3^3; j = 2^6 \cdot 5^3; j = -3^3 \cdot 5^3$.

4.4.7. *II.4.7. The Dual of a Morphism.* Let X and X' be elliptic curves over k, with base points P_0, P_0'.

a. If $f : X \to X'$ is any morphism, use (4.11) to show that $f^* : \text{Pic } X' \to \text{Pic } X$ induces a homomorphism $\widehat{f} : (X', P_0') \to (X, P_0)$. We call this the **dual** of f.

b. If $f : X \to X'$ and $g : X' \to X''$ are two morphisms, then $(g \circ f)\widehat{} = \widehat{f} \circ \widehat{g}$.

c. Assume $f(P_0) = P_0'$, and let $n = \deg f$. Show that if $Q \in X$ is any point, and $f(Q) = Q'$, then $\widehat{f}(Q') = n_X(Q)$. (Do the separable and purely inseparable cases separately, then combine.) Conclude that $f \circ \widehat{f} = n_{X'}$ and $\widehat{f} \circ f = n_X$.

d. * If $f, g : X \to X'$ are two morphisms preserving the base points P_0, P_0', then $(f+g)\widehat{} = \widehat{f} + \widehat{g}$.[148]

e. Using (d), show that for any $n \in \mathbf{Z}, \widehat{n}_X = n_X$. Conclude that $\deg n_X = n^2$.

f. Show for any f that $\deg \widehat{f} = \deg f$.

4.4.8. *II.4.8. The Algebraic Fundamental Group.* For any curve X, the **algebraic fundamental group** $\pi_1(X)$ is defined as $\varprojlim \text{Gal}(K'/K)$, where K is the function field of X, and K' runs over all Galois extensions of K such that the corresponding curve X' is étale over X (III, Ex. 10.3). Thus, for example, $\pi_1(\mathbf{P}^1) = 1$. (See 2.5.3)
Show that for an elliptic curve X, where $\mathbf{Z}_l = \lim \mathbf{Z}/l^n$ is the l-adic integers.[149][150]

4.4.9. *II.4.9. Isogenies.* We say two elliptic curves X, X' are **isogenous** if there is a finite morphism $f : X \to X'$.

a. Show that isogeny is an equivalence relation.

b. For any elliptic curve X, show that the set of elliptic curves X' isogenous to X, up to isomorphism, is countable.[151]

4.4.10. *II.4.10.* If X is an elliptic curve, show that there is an exact sequence where $R = \text{End}(X, P_0)$. In particular, we see that $\text{Pic}(X \times X)$ is bigger than the sum of the Picard groups of the factors.[152]

4.4.11. *II.4.11.* Let X be an elliptic curve over $\mathbf{C}$, defined by the elliptic functions with periods $1, \tau$. Let R be the ring of endomorphisms of X.

a. If $f \in R$ is a nonzero endomorphism corresponding to complex multiplication by α, as in (4.18), show that $\deg f = |\alpha|^2$.

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[148] Hints: It is enough to show for any $\mathcal{L} \in \text{Pic } X'$, that $(f+g)^*\mathcal{L} \cong f^*\mathcal{L} \otimes g^*\mathcal{L}$. For any f, let $\Gamma_f : X \to X \times X'$ be the graph morphism. Then it is enough to show (for $\mathcal{L}' = p_2^*\mathcal{L}$) that Let $\sigma : X \to X \times X'$ be the section $x \to (x, P_0')$. Define a subgroup of $\text{Pic}(X \times X')$ as follows: Note that this subgroup is isomorphic to the group $\text{Pic}^\circ(X'/X)$ used in the definition of the Jacobian variety. Hence there is a 1-1 correspondence between morphisms $f : X \to X'$ and elements $\mathcal{L}_f \in \text{Pic }_\sigma$ (this defines $\mathcal{L}_f$). Now compute explicitly to show that $\Gamma_g^*(\mathcal{L}_f) = \Gamma_f^*(\mathcal{L}_g)$ for any f, g.

Use the fact that $\mathcal{L}_{f+g} = \mathcal{L}_f \otimes \mathcal{L}_g$, and the fact that for any $\mathcal{L}$ on X', $p_2^*\mathcal{L} \in \text{Pic}_\sigma^\circ$ to prove the result.

[149] Hints: Any Galois étale cover X' of an elliptic curve is again an elliptic curve. If the degree of X' over X is relatively prime to p, then X' can be dominated by the cover $n_X : X \to X$ for some integer n with $(n, p) = 1$. The Galois group of the covering n_X is $\mathbf{Z}/n \times \mathbf{Z}/n$. Étale covers of degree divisible by p can occur only if the Hasse invariant of X is not zero.

[150] Note: More generally, Grothendieck has shown $[\text{SGA1}, X, 2.6, p.272]$ that the algebraic fundamental group of any curve of genus g is isomorphic to a quotient of the completion, with respect to subgroups of finite index, of the ordinary topological fundamental group of a compact Riemann surface of genus g, i.e., a group with $2g$ generators $a_1, \ldots, a_g, b_1, \ldots, b_g$ and the relation $\left(a_1 b_1 a_1^{-1} b_1^{-1}\right) \cdots \left(a_g b_g a_g^{-1} b_g^{-1}\right) = 1$.

[151] Hint: X' is uniquely determined by X and $\ker f$.

[152] Cf. (III, Ex. 12.6), (V, Ex. 1.6).

b. If $f \in R$ corresponds to $\alpha \in \mathbf{C}$ again, show that the dual $\widehat{f}$ of (Ex. 4.7) corresponds to the complex conjugate $\bar{\alpha}$ of α.

c. If $\tau \in \mathbf{Q}(\sqrt{-d})$ happens to be integral over $\mathbf{Z}$, show that $R = \mathbf{Z}[\tau]$.

4.4.12. *II.4.12.* Again let X be an elliptic curve over $\mathbf{C}$ determined by the elliptic functions with periods $1, \tau$, and assume that τ lies in the region G of (4.15B).

a. If X has any automorphisms leaving P_0 fixed other than ± 1, show that either $\tau = i$ or $\tau = \omega$, as in (4.20.1) and (4.20.2). This gives another proof of the fact (4.7) that there are only two curves, up to isomorphism, having automorphisms other than ± 1.

b. Now show that there are exactly three values of τ for which X admits an endomorphism of degree 2. Can you match these with the three values of j determined in (Ex. 4.5)?[153]

4.4.13. *II.4.13.* If $p = 13$, there is just one value of j for which the Hasse invariant of the corresponding curve is 0 . Find it.[154]

4.4.14. *II.4.14.* The Fermat curve $X : x^3 + y^3 = z^3$ gives a nonsingular curve in characteristic p for every $p \neq 3$. Determine the set $\mathfrak{P} = \left\{ p \neq 3 \;\middle|\; X_{(p)} \text{ has Hasse invariant } 0 \right\}$, and observe (modulo Dirichlet's theorem) that it is a set of primes of density $\frac{1}{2}$.

4.4.15. *II.4.15.* Let X be an elliptic curve over a field k of characteristic p. Let $F' : X_p \to X$ be the k-linear Frobenius morphism (2.4.1). Use (4.10.7) to show that the dual morphism $\widehat{F}' : X \to X_p$ is separable if and only if the Hasse invariant of X is 1 .
Now use (Ex. 4.7) to show that if the Hasse invariant is 1, then the subgroup of points of order p on X is isomorphic to $\mathbf{Z}/p$; if the Hasse invariant is 0 , it is 0 .

4.4.16. *II.4.16.* Again let X be an elliptic curve over k of characteristic p, and suppose X is defined over the field $\mathbf{F}_q$ of $q = p^r$ elements, i.e., $X \subseteq \mathbf{P}^2$ can be defined by an equation with coefficients in $\mathbf{F}_q$. Assume also that X has a rational point over $\mathbf{F}_q$. Let $F' : X_q \to X$ be the k-linear Frobenius with respect to q.

a. Show that $X_q \cong X$ as schemes over k, and that under this identification, $F' : X \to X$ is the map obtained by the q th-power map on the coordinates of points of X, embedded in $\mathbf{P}^2$.

b. Show that $1_X - F'$ is a separable morphism and its kernel is just the set $X\left(\mathbf{F}_q\right)$ of points of X with coordinates in $\mathbf{F}_q$.

c. Using (Ex. 4.7), show that $F' + \widehat{F}' = a_X$ for some integer a, and that $N = q - a + 1$, where $N = \sharp X\left(\mathbf{F}_q\right)$.

d. Use the fact that $\deg\left(m + nF'\right) > 0$ for all $m, n \in \mathbf{Z}$ to show that $|a| \leqslant 2\sqrt{q}$. This is Hasse's proof of the analogue of the Riemann hypothesis for elliptic curves (App. C, Ex. 5.6).

e. Now assume $q = p$, and show that the Hasse invariant of X is 0 if and only if $a \equiv 0 (\bmod p)$. Conclude for $p \geqslant 5$ that X has Hasse invariant 0 if and only if $N = p + 1$.

[153]Answers: $\tau = i; \tau = \sqrt{-2}; \tau = \frac{1}{2}(-1 + \sqrt{-7})$.
[154]Answer: $j = 5(\bmod 13)$.

4.4.17. *II.4.17.* Let X be the curve $y^2 + y = x^3 - x$ of (4.23.8).

a. If $Q = (a, b)$ is a point on the curve, compute the coordinates of the point $P+Q$, where $P = (0, 0)$, as a function of a, b. Use this formula to find the coordinates of $nP, n = 1, 2, \ldots, 10$.[155]
b. This equation defines a nonsingular curve over $\mathbf{F}_p$ for all $p \neq 37$.

4.4.18. *II.4.18.* Let X be the curve $y^2 = x^3 - 7x + 10$. This curve has at least 26 points with integer coordinates. Find them (use a calculator), and verify that they are all contained in the subgroup (maybe equal to all of $X(\mathbf{Q})$?) generated by $P = (1, 2)$ and $Q = (2, 2)$.

4.4.19. *II.4.19.* Let X, P_0 be an elliptic curve defined over $\mathbf{Q}$, represented as a curve in $\mathbf{P}^2$ defined by an equation with integer coefficients. Then X can be considered as the fibre over the generic point of a scheme $\overline{X}$ over $\operatorname{Spec} \mathbf{Z}$. Let $T \subseteq \operatorname{Spec} \mathbf{Z}$ be the open subset consisting of all primes $p \neq 2$ such that the fibre $X_{(p)}$ of $\overline{X}$ over p is nonsingular.

- For any n, show that $n_X : X \to X$ is defined over T, and is a flat morphism.
- Show that the kernel of n_X is also flat over T.
- Conclude that for any $p \in T$, the natural map $X(\mathbf{Q}) \to X_{(p)}(\mathbf{F}_p)$ induced on the groups of rational points, maps the n-torsion points of $X(\mathbf{Q})$ injectively into the torsion subgroup of $X_{(p)}(\mathbf{F}_p)$, for any $(n, p) = 1$.

By this method one can show easily that the groups $X(\mathbf{Q})$ in (Ex. 4.17) and (Ex. 4.18) are torsion-free.

4.4.20. *II.4.20.* Let X be an elliptic curve over a field k of characteristic $p > 0$, and let $R = \operatorname{End}(X, P_0)$ be its ring of endomorphisms.

a. Let X_p be the curve over k defined by changing the k-structure of X (2.4.1). Show that $j(X_p) = j(X)^{1/p}$. Thus $X \cong X_p$ over k if and only if $j \in \mathbf{F}_p$.
b. Show that p_X in R factors into a product $\pi\widehat{\pi}$ of two elements of degree p if and only if $X \cong X_p$. In this case, the Hasse invariant of X is 0 if and only if π and $\widehat{\pi}$ are associates in R (i.e., differ by a unit).[156]
c. If $\operatorname{Hasse}(X) = 0$ show in any case $j \in \mathbf{F}_{p^2}$.
d. For any $f \in R$, there is an induced map $f^* : H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)$. This must be multiplication by an element $\lambda_f \in k$. So we obtain a ring homomorphism $\varphi : R \to k$ by sending f to λ_f. Show that any $f \in R$ commutes with the (nonlinear) Frobenius morphism $F : X \to X$, and conclude that if Hasse $(X) \neq 0$, then the image of φ is $\mathbf{F}_p$. Therefore, R contains a prime ideal $\mathfrak{p}$ with $R/p \cong \mathbf{F}_p$.

4.4.21. *II.4.21.* Let O be the ring of integers in a quadratic number field $\mathbf{Q}(\sqrt{-d})$. Show that any subring $R \subseteq O, R \neq \mathbf{Z}$, is of the form $R = \mathbf{Z} + f \cdot O$, for a uniquely determined integer $f \geqslant 1$. This integer f is called the **conductor** of the ring R.

4.4.22. *II.4.22 *.* If $X \to \mathbf{A}^1_{\mathbf{C}}$ is a family of elliptic curves having a section, show that the family is trivial.[157]

4.5. IV.5: The Canonical Embedding.

[155] Check: $6P = (6, 14)$

[156] Use (2.5).

[157] Hints: Use the section to fix the group structure on the fibres. Show that the points of order 2 on the fibres form an étale cover of $\mathbf{A}^1_{\mathbf{C}}$, which must be trivial, since $\mathbf{A}^1_{\mathbf{C}}$ is simply connected. This implies that λ can be defined on the family, so it gives a map $\mathbf{A}^1_{\mathbf{C}} \to \mathbf{A}^1_{\mathbf{C}} - \{0, 1\}$. Any such map is constant, so λ is constant, so the family is trivial.

4.5.1. *IV.5.1.* Show that a hyperelliptic curve can never be a complete intersection in any projective space. Cf. (Ex. 3.3).

4.5.2. *IV.5.2.* If X is a curve of genus $\geqslant 2$ over a field of characteristic 0 , show that the group $\operatorname{Aut} X$ of automorphisms of X is finite.[158]

4.5.3. *IV.5.3. Moduli of Curves of Genus 4.* The hyperelliptic curves of genus 4 form an irreducible family of dimension 7 . The nonhyperelliptic ones form an irreducible family of dimension 9. The subset of those having only one g_3^1 is an irreducible family of dimension 8.[159]

4.5.4. *IV.5.4.* Another way of distinguishing curves of genus g is to ask, what is the least degree of a birational plane model with only nodes as singularities (3.11)? Let X be nonhyperelliptic of genus 4 . Then:

a. if X has two g_3^1,s, it can be represented as a plane quintic with two nodes, and conversely;
b. if X has one g_3^1, then it can be represented as a plane quintic with a tacnode (I, Ex. 5.14d), but the least degree of a plane representation with only nodes is 6 .

4.5.5. *IV.5.5. Curves of Genus 5.* Assume X is not hyperelliptic.

a. The curves of genus 5 whose canonical model in $\mathbf{P}^4$ is a complete intersection $F_2.F_2.F_2$ form a family of dimension 12 .
b. X has a g_3^1 if and only if it can be represented as a plane quintic with one node. These form an irreducible family of dimension 11.[160]
c. * In that case, the conics through the node cut out the canonical system (not counting the fixed points at the node). Mapping $\mathbf{P}^2 \rightarrow \mathbf{P}^4$ by this linear system of conics, show that the canonical curve X is contained in a cubic surface $V \subseteq \mathbf{P}^4$, with V isomorphic to $\mathbf{P}^2$ with one point blown up (II, Ex. 7.7).

   Furthermore, V is the union of all the trisecants of X corresponding to the $g_3^1(5.5.3)$, so V is contained in the intersection of all the quadric hypersurfaces containing X. Thus V and the g_3^1 are unique.[161]

4.5.6. *IV.5.6.* Show that a nonsingular plane curve of degree 5 has no g_3^1. Show that there are nonhyperelliptic curves of genus 6 which cannot be represented as a nonsingular plane quintic curve.

4.5.7. *IV.5.7.*

a. Any automorphism of a curve of genus 3 is induced by an automorphism of $\mathbf{P}^2$ via the canonical embedding.
b. * Assume char $k \neq 3$. If X is the curve given by the group $\operatorname{Aut} X$ is the simple group of order 168 , whose order is the maximum $84(g-1)$ allowed by (Ex. 2.5).[162]

[158]Hint: If X is hyperelliptic, use the unique g_2^1 and show that $\operatorname{Aut} X$ permutes the ramification points of the 2 -fold covering $X \rightarrow \mathbf{P}^1$. If X is not hyperelliptic, show that $\operatorname{Aut} X$ permutes the hyperosculation points (Ex. 4.6) of the canonical embedding. Cf. (Ex. 2.5).

[159]Hint: Use (5.2.2) to count how many complete intersections $Q \cap F_3$ there are.

[160]Hint: If $D \in g_3^1$, use $K-D$ to map $X \rightarrow \mathbf{P}^2$.

[161]Note. Conversely, if X does not have a g_3^1, then its canonical embedding is a complete intersection, as in (a). More generally, a classical theorem of Enriques and Petri shows that for any nonhyperelliptic curve of genus $g \geqslant 3$, the canonical model is projectively normal, and it is an intersection of quadric hypersurfaces unless X has a g_3^1 or $g=6$ and X has a g_5^2. See Saint-Donat [1].

[162]See Burnside [1, §232] or Klein [1].

c. * Most curves of genus 3 have no automorphisms except the identity.[163][164]

4.6. IV.6: Classification of Curves in $\mathbf{P}^3$.

4.6.1. *IV.6.1.* A rational curve of degree 4 in $\mathbf{P}^3$ is contained in a unique quadric surface Q, and Q is necessarily nonsingular.

4.6.2. *IV.6.2.* A rational curve of degree 5 in $\mathbf{P}^3$ is always contained in a cubic surface, but there are such curves which are not contained in any quadric surface.

4.6.3. *IV.6.3.* A curve of degree 5 and genus 2 in $\mathbf{P}^3$ is contained in a unique quadric surface Q. Show that for any abstract curve X of genus 2 , there exist embeddings of degree 5 in $\mathbf{P}^3$ for which Q is nonsingular, and there exist other embeddings of degree 5 for which Q is singular.

4.6.4. *IV.6.4.* There is no curve of degree 9 and genus 11 in $\mathbf{P}^3$.[165]

4.6.5. *IV.6.5.* If X is a complete intersection of surfaces of degrees a, b in $\mathbf{P}^3$, then X does not lie on any surface of degree $< \min(a, b)$.

4.6.6. *IV.6.6.* Let X be a projectively normal curve in $\mathbf{P}^3$, not contained in any plane. If $d = 6$, then $g = 3$ or 4 . If $d = 7$, then $g = 5$ or 6 . Cf. (II, Ex. 8.4) and (III, Ex. 5.6).

4.6.7. *IV.6.7.* The line, the conic, the twisted cubic curve and the elliptic quartic curve in $\mathbf{P}^3$ have no multisecants. Every other curve in $\mathbf{P}^3$ has infinitely many multisecants.[166]

4.6.8. *IV.6.8.* A curve X of genus g has a nonspecial divisor D of degree d such that $|D|$ has no base points if and only if $d \geqslant g + 1$.

4.6.9. *IV.6.9.* * Let X be an irreducible nonsingular curve in $\mathbf{P}^3$. Then for each $m \gg> 0$, there is a nonsingular surface F of degree m containing X.[167]

5. V: Surfaces

5.1. V.1: Geometry on a Surface.

5.1.1. *V.1.1.* Let C, D be any two divisors on a surface X, and let the corresponding invertible sheaves be $\mathcal{L}, \mathcal{M}$. Show that

5.1.2. *V.1.2.* Let H be a very ample divisor on the surface X, corresponding to a projective embedding $X \subseteq \mathbf{P}^N$. If we write the Hilbert polynomial of X (III, Ex. 5.2) as show that $a = H^2, b = \frac{1}{2}H^2 + 1 - \pi$, where π is the genus of a nonsingular curve representing H, and $c = 1 + p_a$. Thus the degree of X in $\mathbf{P}^N$, as defined in (I, §7), is just H^2. Show also that if C is any curve in X, then the degree of C in $\mathbf{P}^N$ is just $C.H$

[163]Hint: For each n, count the dimension of the family of curves with an automorphism T of order n. For example, if $n = 2$, then for suitable choice of coordinates, T can be written as $x \to -x, y \to y, z \to z$. Then there is an 8-dimensional family of curves fixed by T; changing coordinates there is a 4-dimensional family of such T, so the curves having an automorphism of degree 2 form a family of dimensional 12 inside the 14-dimensional family of all plane curves of degree 4.

[164]More generally it is true (at least over $\mathbf{C}$) that for any $g \geqslant 3$, a "sufficiently general" curve of genus g has no automorphisms except the identity-see Baily

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[165]Hint: Show that it would have to lie on a quadric surface, then use (6.4.1).

[166]Hint: Consider a projection from a point of the curve to $\mathbf{P}^2$.

[167]Hint: Let $\pi : \tilde{\mathbf{P}} \to \mathbf{P}^3$ be the blowing-up of X and let $Y = \pi^{-1}(X)$. Apply Bertini's theorem to the projective embedding of $\tilde{\mathbf{P}}$ corresponding to $\mathcal{I}_Y \otimes \pi^* \mathcal{O}_{\mathbf{P}^3}(m)$.

5.1.3. *V.1.3.* Recall that the arithmetic genus of a projective scheme D of dimension 1 is defined as See III, Ex. 5.3.

a. If D is an effective divisor on the surface X, use (1.6) to show that
b. $p_a(D)$ depends only on the linear equivalence class of D on X.
c. More generally, for any divisor D on X, we define the virtual arithmetic genus (which is equal to the ordinary arithmetic genus if D is effective) by the same formula: $2p_a - 2 = D.(D+K)$. Show that for any two divisors C, D we have and

5.1.4. *V.1.4.*

a. If a surface X of degree d in $\mathbf{P}^3$ contains a straight line $C = \mathbf{P}^1$, show that $C^2 = 2 - d$
b. Assume char $k = 0$, and show for every $d \geqslant 1$, there exists a nonsingular surface X of degree d in $\mathbf{P}^3$ containing the line $x = y = 0$.

5.1.5. *V.1.5.*

a. If X is a surface of degree d in $\mathbf{P}^3$, then
b. If X is a product of two nonsingular curves C, C', of genus g, g' respectively, then Cf. (II, Ex. 8.3).

5.1.6. *V.1.6.*

a. If C is a curve of genus g, show that the diagonal $\Delta \subseteq C \times C$ has self-intersection $\Delta^2 = 2 - 2g$. (Use the definition of $\Omega_{C/k}$ in (II, §8).)
b. Let $l = C \times \mathrm{pt}$ and $m = \mathrm{pt} \times C$. If $g \geqslant 1$, show that l, m, and Δ are linearly independent in $\mathrm{Num}(C \times C)$. Thus $\mathrm{Num}(C \times C)$ has rank $\geqslant 3$, and in particular, Cf. (III, Ex. 12.6), (V, Ex. 4.10).

5.1.7. *V.1.7. Algebraic Equivalence of Divisors.* Let X be a surface. Recall that we have defined an algebraic family of effective divisors on X, parametrized by a nonsingular curve T, to be an effective Cartier divisor D on $X \times T$, flat over T (III, 9.8.5). In this case, for any two closed points $0, 1 \in T$, we say the corresponding divisors D_0, D_1 on X are prealgebraically equivalent.
Two arbitrary divisors are prealgebraically equivalent if they are differences of prealgebraically equivalent effective divisors. Two divisors D, D' are algebraically equivalent if there is a finite sequence $D = D_0, D_1, \ldots, D_n = D'$ with D_i and D_{i+1} prealgebraically equivalent for each i.

a. Show that the divisors algebraically equivalent to 0 form a subgroup of $\mathrm{Div}\, X$.
b. Show that linearly equivalent divisors are algebraically equivalent.[168]
c. Show that algebraically equivalent divisors are numerically equivalent.[169][170]

[168]Hint: If (f) is a principal divisor on X, consider the principal divisor $(tf - u)$ on $X \times \mathbf{P}^1$, where t, u are the homogeneous coordinates on $\mathbf{P}^1$.

[169]Hint: Use (III, 9.9) to show that for any very ample H, if D and D' are algebraically equivalent, then $D.H = D'.H$.

[170]Note. The theorem of Néron and Severi states that the group of divisors modulo algebraic equivalence, called the Néron-Severi group, is a finitely generated abelian group. Over $\mathbf{C}$ this can be proved easily by transcendental methods (App. B, §5) or as in (Ex. 1.8) below. Over a field of arbitrary characteristic, see Lang and Néron

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for a proof, and Hartshorne

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for further discussion. Since $\mathrm{Num}\, X$ is a quotient of the Néron-Severi group, it is also finitely generated, and hence free, since it is torsion-free by construction.

5.1.8. *V.1.8. Cohomology Class of a Divisor.* For any divisor D on the surface X, we define its cohomology class $c(D) \in H^1(X, \Omega_X)$ by using the isomorphism Pic $X \cong H^1(X, \mathcal{O}_X^*)$ of (III, Ex. 4.5) and the sheaf homomorphism $d\log : \mathcal{O}^* \to \Omega_X$ (III, Ex. 7.4c). Thus we obtain a group homomorphism $c : \operatorname{Pic} X \to H^1(X, \Omega_X)$. On the other hand, $H^1(X, \Omega)$ is dual to itself by Serre duality (III, 7.13), so we have a nondegenerate bilinear map

a. Prove that this is compatible with the intersection pairing, in the following sense: for any two divisors D, E on X, we have in k.[171]

b. If char $k = 0$, use the fact that $H^1(X, \Omega_X)$ is a finite-dimensional vector space to show that $\operatorname{Num} X$ is a finitely generated free abelian group.

5.1.9. *V.1.9.*

a. If H is an ample divisor on the surface X, and if D is any divisor, show that

b. Now let X be a product of two curves $X = C \times C'$. Let $l = C \times \text{pt}$, and $m = \text{pt} \times C'$. For any divisor D on X, let $a = D.l, b = D.m$. Then we say D has type (a, b). If D has type (a, b), with $a, b \in \mathbf{Z}$, show that and equality holds if and only if $D \equiv bl + am$.[172]

5.1.10. *V.1.10. Weil's Proof of the Analogue of the Riemann Hypothesis for Curves.* Let C be a curve of genus g defined over the finite field $\mathbf{F}_q$, and let N be the number of points of C rational over $\mathbf{F}_q$. Then $N = 1 - a + q$, with $|a| \leqslant 2g\sqrt{q}$.
To prove this, we consider C as a curve over the algebraic closure k of $\mathbf{F}_q$. Let $f : C \to C$ be the k-linear Frobenius morphism obtained by taking q th powers, which makes sense since C is defined over $\mathbf{F}_q$, so $X_q \cong X$ (See $V, 2.4.1$).
Let $\Gamma \subseteq C \times C$ be the graph of f, and let $\Delta \subseteq C \times C$ be the diagonal.
Show that $\Gamma^2 = q(2 - 2g)$, and $\Gamma.\Delta = N$. Then apply (Ex. 1.9) to $D = r\Gamma + s\Delta$ for all r and s to obtain the result.[173]

5.1.11. *V.1.11.* In this problem, we assume that X is a surface for which $\operatorname{Num} X$ is finitely generated (i.e., any surface, if you accept the Néron-Severi theorem (Ex. 1.7)).

a. If H is an ample divisor on X, and $d \in \mathbf{Z}$, show that the set of effective divisors D with $D.H = d$, modulo numerical equivalence, is a finite set.[174]

b. Now let C be a curve of genus $g \geqslant 2$, and use (a) to show that the group of automorphisms of C is finite, as follows. Given an automorphism σ of C, let $\Gamma \subseteq X = C \times C$ be its graph. First show that if $\Gamma \equiv \Delta$, then $\Gamma = \Delta$, using the fact that $\Delta^2 < 0$, since $g \geqslant 2$ (Ex. 1.6). Then use (a). Cf. (V, Ex. 2.5).

5.1.12. *V.1.12.* If D is an ample divisor on the surface X, and $D' \equiv D$, then D' is also ample. Give an example to show, however, that if D is very ample, D' need not be very ample.

5.2. **V.2: Ruled Surfaces.**

5.2.1. *V.2.1.* If X is a birationally ruled surface, show that the curve C, such that X is birationally equivalent to $C \times \mathbf{P}^1$, is unique (up to isomorphism).

[171] Hint: Reduce to the case where D and E are nonsingular curves meeting transversally. Then consider the analogous map $c : \operatorname{Pic} D \to H^1(D, \Omega_D)$, and the fact (III, Ex. 7.4) that c (point) goes to 1 under the natural isomorphism of $H^1(D, \Omega_D)$ with k.

[172] Hint: Show that $H = l + m$ is ample, let $E = l - m$, let $D' = (H^2)(E^2)D - (E^2)(D \cdot H)H - (H^2)(D \cdot E)E$, and apply (1.9). This inequality is due to Castelnuovo and Severi. See Grothendieck [2].

[173] See (App. C, Ex. 5.7) for another interpretation of this result.

[174] Hint: Use the adjunction formula, the fact that p_a of an irreducible curve is $\geqslant 0$, and the fact that the intersection pairing is negative definite on $H^{\perp}$ in $\operatorname{Num} X$.

5.2.2. *V.2.2.* Let X be the ruled surface $\mathbf{P}(\mathcal{E})$ over a curve C. Show that $\mathcal{E}$ is decomposable if and only if there exist two sections C', C'' of X such that $C' \cap C'' = \varnothing$.

5.2.3. *V.2.3.*

a. If $\mathcal{E}$ is a locally free sheaf of rank r on a (nonsingular) curve C, then there is a sequence of subsheaves such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is an invertible sheaf for each $i = 1, \ldots, r$. We say that $\mathcal{E}$ is a successive extension of invertible sheaves.[175]

b. Show that this is false for varieties of dimension $\geqslant 2$. In particular, the sheaf of differentials Ω on $\mathbf{P}^2$ is not an extension of invertible sheaves.

5.2.4. *V.2.4.* Let C be a curve of genus g, and let X be the ruled surface $C \times \mathbf{P}^1$. We consider the question, for what integers $s \in \mathbf{Z}$ does there exist a section D of X with $D^2 = s$? First show that s is always an even integer, say $s = 2r$.

a. Show that $r = 0$ and any $r \geqslant g + 1$ are always possible. Cf. (V, Ex. 6.8).

b. If $g = 3$, show that $r = 1$ is not possible, and just one of the two values $r = 2, 3$ is possible, depending on whether C is hyperelliptic or not.

5.2.5. *V.2.5. Values of* e. Let C be a curve of genus $g \geqslant 1$.

a. Show that for each $0 \leqslant e \leqslant 2g - 2$ there is a ruled surface X over C with invariant e, corresponding to an indecomposable $\mathcal{E}$. Cf. (2.12).

b. Let $e < 0$, let D be any divisor of degree $d = -e$, and let $\xi \in H^1(\mathcal{L}(-D))$ be a nonzero element defining an extension Let $H \subseteq |D + K|$ be the sublinear system of codimension 1 defined by ker ξ, where ξ is considered as a linear functional on $H^0(\mathcal{L}(D + K))$. For any effective divisor E of degree $d-1$, let $L_E \subseteq |D+K|$ be the sublinear system $|D+K-E|+E$. Show that $\mathcal{E}$ is normalized if and only if for each E as above, $L_E \nsubseteq H$. Cf. proof of (2.15).

c. Now show that if $-g \leqslant e < 0$, there exists a ruled surface X over C with invariant e.[176]

d. For $g = 2$, show that $e \geqslant -2$ is also necessary for the existence of X.[177]

5.2.6. *V.2.6.* Show that every locally free sheaf of finite rank on $\mathbf{P}^1$ is isomorphic to a direct sum of invertible sheaves.[178]

5.2.7. *V.2.7.* On the elliptic ruled surface X of (2.11.6), show that the sections C_0 with $C_0^2 = 1$ form a one-dimensional algebraic family, parametrized by the points of the base curve C, and that no two are linearly equivalent.

5.2.8. *V.2.8.* A locally free sheaf $\mathcal{E}$ on a curve C is said to be **stable** if for every quotient locally free sheaf we have Replacing $>$ by $\geqslant$ defines **semistable**.

a. A decomposable $\mathcal{E}$ is never stable.

b. If $\mathcal{E}$ has rank 2 and is normalized, then $\mathcal{E}$ is stable (respectively, semistable) if and only if $\deg \mathcal{E} > 0$ (respectively, $\geqslant 0$).

c. Show that the indecomposable locally free sheaves $\mathcal{E}$ of rank 2 that are not semistable are classified, up to isomorphism, by giving
   (1) an integer $0 < e \leqslant 2g - 2$,
   (2) an element $\mathcal{L} \in \operatorname{Pic} C$ of degree $-e$, and
   (3) a nonzero $\xi \in H^1(\mathcal{L}^\vee)$, determined up to a nonzero scalar multiple.

[175] Hint: Use (II, Ex. 8.2).
[176] Hint: For any given D in (b), show that a suitable ξ exists, using an argument similar to the proof of (II, 8.18).
[177] Note. It has been shown that $e \geqslant -g$ for any ruled surface (Nagata [8]).
[178] Hint: Choose a subinvertible sheaf of maximal degree, and use induction on the rank.

5.2.9. *V.2.9.* Let Y be a nonsingular curve on a quadric cone X_0 in $\mathbf{P}^3$. Show that either

- Y is a complete intersection of X_0 with a surface of degree $a \geqslant 1$, in which case $\deg Y = 2a, g(Y) = (a-1)^2$, or,
- $\deg Y$ is odd, say $2a+1$, and $g(Y) = a^2 - a$.[179]

5.2.10. *V.2.10.* For any $n > e \geqslant 0$, let X be the rational scroll of degree $d = 2n - e$ in $\mathbf{P}^{d+1}$ given by (2.19). If $n \geqslant 2e - 2$, show that X contains a nonsingular curve Y of genus $g = d + 2$ which is a canonical curve in this embedding.
Conclude that for every $g \geqslant 4$, there exists a nonhyperelliptic curve of genus g which has a g_3^1. Cf. (V, §5).

5.2.11. *V.2.11.* Let X be a ruled surface over the curve C, defined by a normalized bundle $\mathcal{E}$, and let $\mathfrak{e}$ be the divisor on C for which $\mathcal{L}(\mathfrak{e}) \cong \bigwedge^2 \mathcal{E}$ (See 2.8 .1). Let $\mathfrak{b}$ be any divisor on C.

a. If $|\mathfrak{b}|$ and $|\mathfrak{b}+\mathfrak{e}|$ have no base points, and if $\mathfrak{b}$ is nonspecial, then there is a section $D \sim C_0 + \mathfrak{b}f$, and $|D|$ has no base points.
b. If $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ are very ample on C, and for every point $P \in C$, we have $\mathfrak{b} - P$ and $\mathfrak{b} + \mathfrak{e} - P$ nonspecial, then $C_0 + \mathfrak{b}f$ is very ample.

5.2.12. *V.2.12.* Let X be a ruled surface with invariant e over an elliptic curve C, and let $\mathfrak{b}$ be a divisor on C.

a. If $\deg \mathfrak{b} \geqslant e + 2$, then there is a section $D \sim C_0 + \mathfrak{b}f$ such that $|D|$ has no base points.
b. The linear system $|C_0 + \mathfrak{b}f|$ is very ample if and only if $\deg \mathfrak{b} \geqslant e + 3$. Note. The case $e = -1$ will require special attention.

5.2.13. *V.2.13.* For every $e \geqslant -1$ and $n \geqslant e + 3$, there is an elliptic scroll of degree $d = 2n - e$ in $\mathbf{P}^{d-1}$. In particular, there is an elliptic scroll of degree 5 in $\mathbf{P}^4$.

5.2.14. *V.2.14.* Let X be a ruled surface over a curve C of genus g, with invariant $e < 0$, and assume that char $k = p > 0$ and $g \geqslant 2$.

a. If $Y \equiv aC_0 + bf$ is an irreducible curve $\neq C_0, f$, then either
   - $a = 1, b \geqslant 0$, or
   - $2 \leqslant a \leqslant p - 1, b \geqslant \frac{1}{2}ae$, or
   - $a \geqslant p, b \geqslant \frac{1}{2}ae + 1 - g$.
b. If $a > 0$ and $b > a\left(\frac{1}{2}e + (1/p)(g-1)\right)$, then any divisor $D \equiv aC_0 + bf$ is ample. On the other hand, if D is ample, then $a > 0$ and $b > \frac{1}{2}ae$.

5.2.15. *V.2.15. Funny behavior in characteristic p.* Let C be the plane curve $x^3y + y^3z + z^3x = 0$ over a field k of characteristic 3 (V, Ex. 2.4).

a. Show that the action of the k-linear Frobenius morphism f on $H^1(C, \mathcal{O}_C)$ is identically 0 (Cf. (V, 4.21)).
b. Fix a point $P \in C$, and show that there is a nonzero $\xi \in H^1(\mathcal{L}(-P))$ such that $f^*\xi = 0$ in $H^1(\mathcal{L}(-3P))$.
c. Now let $\mathcal{E}$ be defined by ξ as an extension and let X be the corresponding ruled surface over C. Show that X contains a nonsingular curve $Y \equiv 3C_0 - 3f$, such that $\pi : Y \to C$ is purely inseparable.

   Show that the divisor $D = 2C_0$ satisfies the hypotheses of (2.21.b), but is not ample.

[179]Cf. (V, 6.4.1). Hint: Use (2.11.4).

5.2.16. *V.2.16.* Let C be a nonsingular affine curve. Show that two locally free sheaves $\mathcal{E}, \mathcal{E}'$ of the same rank are isomorphic if and only if their classes in the Grothendieck group $K(X)$ (II, Ex. 6.10) and (II, Ex. 6.11) are the same. This is false for a projective curve.

5.2.17. *V.2.17 *.*

a. Let $\varphi : \mathbf{P}_k^1 \to \mathbf{P}_k^3$ be the 3-uple embedding (I, Ex. 2.12). Let $\mathcal{I}$ be the sheaf of ideals of the twisted cubic curve C which is the image of φ. Then $\mathcal{I}/\mathcal{I}^2$ is a locally free sheaf of rank 2 on C, so $\varphi^*\left(\mathcal{I}/\mathcal{I}^2\right)$ is a locally free sheaf of rank 2 on $\mathbf{P}^1$. By (2.14), therefore, for some $l, m \in \mathbf{Z}$. Determine l and m.
b. Repeat part (a) for the embedding $\varphi : \mathbf{P}^1 \to \mathbf{P}^3$ given by $x_0 = t^4, x_1 = t^3u$, $x_2 = tu^3, x_3 = u^4$, whose image is a nonsingular rational quartic curve.[180]

5.3. V.3: Monoidal Transformations.

5.3.1. *V.3.1.* Let X be a nonsingular projective variety of any dimension, let Y be a nonsingular subvariety, and let $\pi : \tilde{X} \to X$ be obtained by blowing up Y. Show that $p_a(\tilde{X}) = p_a(X)$

5.3.2. *V.3.2.* Let C and D be curves on a surface X, meeting at a point P. Let $\pi : \tilde{X} \to X$ be the monoidal transformation with center P.
Show that Conclude that $C.D = \sum \mu_P(C) \cdot \mu_P(D)$, where the sum is taken over all intersection points of C and D, including infinitely near intersection points.

5.3.3. *V.3.3.* Let $\pi : \tilde{X} \to X$ be a monoidal transformation, and let D be a very ample divisor on X. Show that $2\pi^*D - E$ is ample on $\tilde{X}$.[181]

5.3.4. *V.3.4. Multiplicity of a Local Ring.* Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. For any $l > 0$, let $\psi(l) = \operatorname{length}\left(A/\mathfrak{m}^l\right)$. We call ψ the **Hilbert-Samuel function of** A.

a. Show that there is a polynomial $P_A(z) \in \mathbf{Q}[z]$ such that $P_A(l) = \psi(l)$ for all $l \gg 0$. This is the Hilbert-Samuel polynomial of A.[182][183]
b. Show that $\deg P_A = \dim A$.
c. Let $n = \dim A$. Then we define the multiplicity of A, denoted $\mu(A)$, to be $(n!)$. (leading coefficient of P_A). If P is a point on a noetherian scheme X, we define the multiplicity of P on $X, \mu_P(X)$, to be $\mu\left(\mathcal{O}_{P,X}\right)$.
d. Show that for a point P on a curve C on a surface X, this definition of $\mu_P(C)$ coincides with the one in the text just before (3.5.2).
e. If Y is a variety of degree d in $\mathbf{P}^n$, show that the vertex of the cone over Y is a point of multiplicity d.

5.3.5. *V.3.5.* Let $a_1, \ldots, a_r, r \geqslant 5$, be distinct elements of k, and let C be the curve in $\mathbf{P}^2$ given by the (affine) equation $y^2 = \prod_{i=1}^r (x - a_i)$. Show that the point P at infinity on the y-axis is a singular point. Compute δ_P and $g(\tilde{Y})$, where $\tilde{Y}$ is the normalization of Y. Show in this way that one obtains hyperelliptic curves of every genus $g \geqslant 2$.

5.3.6. *V.3.6.* Show that analytically isomorphic curve singularities (I, 5.6.1) are equivalent in the sense of (3.9.4), but not conversely.

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[180] Answer: If char $k \neq 2$, then $l = m = -7$; if char $k = 2$, then $l, m = -6, -8$.
[181] Hint: Use a suitable generalization of (I, Ex. 7.5) to curves in $\mathbf{P}^n$.
[182] Hint: Consider the graded ring $\operatorname{gr}_{\mathfrak{m}} A = \bigoplus_{d \geqslant 0} \mathfrak{m}^d/\mathfrak{m}^{d+1}$, and apply (I, 7.5)
[183] See Nagata [7, ChIII, §23] or Zariski-Samuel [1, vol2, Ch$VIII$, §10].

5.3.7. *V.3.7.* For each of the following singularities at $(0,0)$ in the plane, give an embedded resolution, compute δ_P, and decide which ones are equivalent.

a. $x^3 + y^5 = 0$.
b. $x^3 + x^4 + y^5 = 0$.
c. $x^3 + y^4 + y^5 = 0$.
d. $x^3 + y^5 + y^6 = 0$.
e. $x^3 + xy^3 + y^5 = 0$.

5.3.8. *V.3.8.* Show that the following two singularities have the same multiplicity, and the same configuration of infinitely near singular points with the same multiplicities, hence the same δ_P, but are not equivalent.

a. $x^4 - xy^4 = 0$.
b. $x^4 - x^2y^3 - x^2y^5 + y^8 = 0$.

5.4. **V.4: The Cubic Surface in $\mathbf{P}^3$.**

5.4.1. *V.4.1.* The linear system of conics in $\mathbf{P}^2$ with two assigned base points P_1 and P_2 (4.1) determines a morphism ψ of X' (which is $\mathbf{P}^2$ with P_1 and P_2 blown up) to a nonsingular quadric surface Y in $\mathbf{P}^3$, and furthermore X' via ψ is isomorphic to Y with one point blown up.

5.4.2. *V.4.2.* Let φ be the quadratic transformation of (4.2.3), centered at P_1, P_2, P_3. If C is an irreducible curve of degree d in $\mathbf{P}^2$, with points of multiplicity r_1, r_2, r_3 at P_1, P_2, P_3, then the strict transform C' of C by φ has degree and has points of multiplicity

- $d - r_2 - r_3$ at Q_1,
- $d - r_1 - r_3$ at Q_2 and
- $d - r_1 - r_2$ at Q_3.

The curve C may have arbitrary singularities.[184]

5.4.3. *V.4.3.* Let C be an irreducible curve in $\mathbf{P}^2$. Then there exists a finite sequence of quadratic transformations, centered at suitable triples of points, so that the strict transform C' of C has only ordinary singularities, i.e., multiple points with all distinct tangent directions (I, Ex. 5.14). Use (3.8).

5.4.4. *V.4.4.*

a. Use (4.5) to prove the following lemma on cubics: If C is an irreducible plane cubic curve, if L is a line meeting C in points P, Q, R, and L' is a line meeting C in points P', Q', R', let P'' be the third intersection of the line PP' with C, and define Q'', R'' similarly. Then P'', Q'', R'' are collinear.
b. Let P_0 be an inflection point of C, and define the group operation on the set of regular points of C by the geometric recipe "let the line PQ meet C at R, and let P_0R meet C at T, then $P + Q = T''$ as in (II, 6.10.2) and (II, 6.11.4). Use (a) to show that this operation is associative.

[184]Hint: Use (Ex. 3.2).

5.4.5. *V.4.5.* Prove Pascal's theorem: if A, B, C, A', B', C' are any six points on a conic, then the points $P = AB' \cdot A'B, Q = AC' \cdot A'C$, and $R = BC' \cdot B'C$ are collinear (Fig. 22).

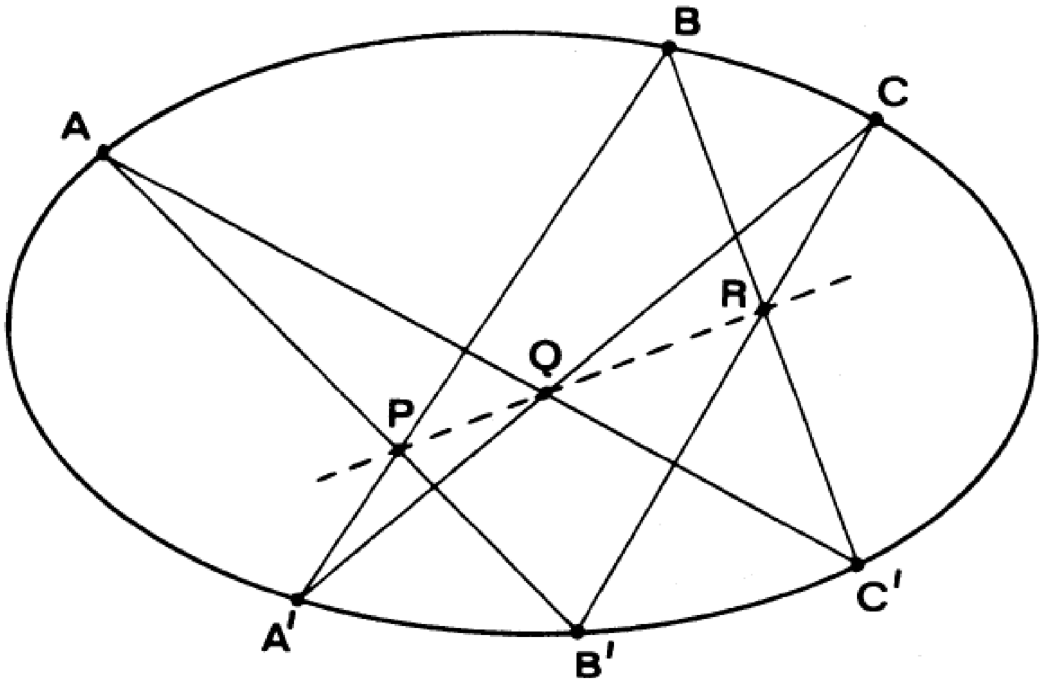

Figure 22. Pascal's theorem.

5.4.6. *V.4.6.* Generalize (4.5) as follows: given 13 points $P_1, \ldots, P_{13}$ in the plane, there are three additional determined points P_{14}, P_{15}, P_{16}, such that all quartic curves through $P_1, \ldots, P_{13}$ also pass through P_{14}, P_{15}, P_{16}. What hypotheses are necessary on $P_1, \ldots, P_{13}$ for this to be true?

5.4.7. *V.4.7.* If D is any divisor of degree d on the cubic surface (4.7.3), show that Show furthermore that for every $d > 0$, this maximum is achieved by some irreducible nonsingular curve.

5.4.8. *V.4.8.* *. Show that a divisor class D on the cubic surface contains an irreducible curve $\iff$ if it contains an irreducible nonsingular curve $\iff$ it is either

a. one of the 27 lines, or
b. a conic (meaning a curve of degree 2) with $D^2 = 0$, or
c. $D.L \geqslant 0$ for every line L, and $D^2 > 0$.[185]

5.4.9. *V.4.9.* If C is an irreducible non-singular curve of degree d on the cubic surface, and if the genus $g > 0$, then and this minimum value of $g > 0$ is achieved for each d in the range given.

5.4.10. *V.4.10.* A curious consequence of the implication (iv) $\Rightarrow$ (iii) of (4.11) is the following numerical fact: Given integers $a, b_1, \ldots, b_6$ such that $b_i > 0$ for each $i, a - b_i - b_j > 0$ for each i, j and $2a - \sum_{i \neq j} b_i > 0$ for each j, we must necessarily have $a^2 - \sum b_i^2 > 0$. Prove this directly (for $a, b_1, \ldots, b_6 \in \mathbf{R}$) using methods of freshman calculus.

[185]Hint: Generalize (4.11) to the surfaces obtained by blowing up $2, 3, 4$, or 5 points of $\mathbf{P}^2$, and combine with our earlier results about curves on $\mathbf{P}^1 \times \mathbf{P}^1$ and the rational ruled surface X_1, (2.18).

5.4.11. *V.4.11. The Weyl Groups.* Given any diagram consisting of points and line segments joining some of them, we define an abstract group, given by generators and relations, as follows:

- Each point represents a generator x_i. The relations are
- $x_i^2 = 1$ for each i;
- $(x_i x_j)^2 = 1$ if i and j are not joined by a line segment, and
- $(x_i x_j)^3 = 1$ if i and j are joined by a line segment.

a. The Weyl group $\mathbf{A}_n$ is defined using the following diagram of $n-1$ points, each joined to the next:

FIGURE 6. The $\mathbf{A}_n$ Dynkin diagram.

Show that it is isomorphic to the symmetric group Σ_n as follows:

- Map the generators of $\mathbf{A}_n$ to the elements $(12),(23),..,(n-1,n)$ of Σ_n, to get a surjective homomorphism $\mathbf{A}_n \to \Sigma_n$.
- Then estimate the number of elements of $\mathbf{A}_n$ to show in fact it is an isomorphism.

b. The Weyl group $\mathbf{E}_6$ is defined using the diagram

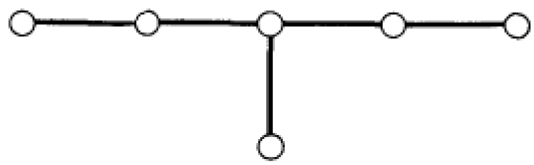

FIGURE 7. The $\mathbf{E}_6$ Dynkin diagram.

Call the generators $x_1,\ldots,x_5$ and y. Show that one obtains a surjective homomorphism $\mathbf{E}_6 \to G$, the group of automorphisms of the configuration of 27 lines (4.10.1), by sending $x_1,\ldots,x_5$ to the permutations $(12),(23),\ldots,(56)$ of the E_i, respectively, and y to the element associated with the quadratic transformation based at P_1, P_2, P_3.

c. * Estimate the number of elements in $\mathbf{E}_6$, and thus conclude that $\mathbf{E}_6 \cong G$.[186]

5.4.12. *V.4.12.* Use (4.11) to show that if D is any ample divisor on the cubic surface X, then $H^1(X, \mathcal{O}_X(-D)) = 0$. This is Kodaira's vanishing theorem for the cubic surface (III, 7.15).

5.4.13. *V.4.13.* Let X be the Del Pezzo surface of degree 4 in $\mathbf{P}^4$ obtained by blowing up 5.points of $\mathbf{P}^2$(4.7)

a. Show that X contains 16 lines.

b. Show that X is a complete intersection of two quadric hypersurfaces in $\mathbf{P}^4$ (the converse follows from (4.7.1)).

5.4.14. *V.4.14.* Using the method of (4.13.1), verify that there are nonsingular curves in $\mathbf{P}^3$ with $d=8, g=6,7; d=9, g=7,8,9; d=10, g=8,9,10,11$. Combining with (IV, §6), this completes the determination of all posible g for curves of degree $d \leqslant 10$ in $\mathbf{P}^3$.

[186]Note: See Manin [3, §25, 26] for more about Weyl groups, root systems, and exceptional curves.

5.4.15. *V.4.15.* Let $P_1, \ldots, P_r$ be a finite set of (ordinary) points of $\mathbf{P}^2$, no 3 collinear. We define an **admissible transformation** to be a quadratic transformation (4.2.3) centered at some three of the P_i (call them P_1, P_2, P_3).
This gives a new $\mathbf{P}^2$, and a new set of r points, namely Q_1, Q_2, Q_3, and the images of $P_4, \ldots, P_r$. We say that $P_1, \ldots, P_r$ are **in general position** if no three are collinear, and furthermore after any finite sequence of admissible transformations, the new set of r points also has no three collinear.

a. A set of 6 points is in general position if and only if no three are collinear and not all six lie on a conic.
b. If $P_1, \ldots, P_r$ are in general position, then the r points obtained by any finite sequence of admissible transformations are also in general position.
c. Assume the ground field k is uncountable. Then given $P_1, \ldots, P_r$ in general position, there is a dense subset $V \subseteq \mathbf{P}^2$ such that for any $P_{r+1} \in V, P_1, \ldots, P_{r+1}$ will be in general position.[187]
d. Now take $P_1, \ldots, P_r \in \mathbf{P}^2$ in general position, and let X be the surface obtained by blowing up $P_1, \ldots, P_r$. If $r = 7$, show that X has exactly 56 irreducible nonsingular curves C with $g = 0, C^2 = -1$, and that these are the only irreducible curves with negative self-intersection. Ditto for $r = 8$, the number being 240 .
e. * For $r = 9$, show that the surface X defined in (d) has infinitely many irreducible nonsingular curves C with $g = 0$ and $C^2 = -1$.[188]

5.4.16. *V.4.16.* For the Fermat cubic surface $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$, find the equations of the 27 lines explicitly, and verify their incidence relations. What is the group of automorphisms of this surface?

5.5. **V.5: Birational Transformations.**

5.5.1. *V.5.1.* Let f be a rational function on the surface X. Show that it is possible to "resolve the singularities of f'' in the following sense: there is a birational morphism $g : X' \to X$ so that f induces a morphism of X' to $\mathbf{P}^1$.[189]

5.5.2. *V.5.2.* Let $Y \cong \mathbf{P}^1$ be a curve in a surface X, with $Y^2 < 0$. Show that Y is contractible (5.7.2) to a point on a projective variety X_0 (in general singular).

5.5.3. *V.5.3.* If $\pi : \tilde{X} \to X$ is a monoidal transformation with center P, show that $H^1\left(\tilde{X}, \Omega_{\tilde{X}}\right) \cong H^1\left(X, \Omega_X\right) \oplus k$. This gives another proof of (5.8).[190]

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[187]Hint: Prove a lemma that when k is uncountable, a variety cannot be equal to the union of a countable family of proper closed subsets.

[188]Hint: Let L be the line joining P_1 and P_2. Show that there exist finite sequences of admissible transformations such that the strict transform of L becomes a plane curve of arbitrarily high degree. This example is apparently due to Kodaira-see Nagata $[5, II, p.283]$.

[189]Hints: Write the divisor of f as $(f) = \sum n_i C_i$. Then apply embedded resolution (3.9) to the curve $Y = \bigcup C_i$. Then blow up further as necessary whenever a curve of zeros meets a curve of poles until the zeros and poles of f are disjoint.

[190]Hints: Use the projection formula (III, Ex. 8.3) and (III, Ex. 8.1) to show that $H^i\left(X, \Omega_X\right) \cong H^i\left(\tilde{X}, \pi^* \Omega_X\right)$ for each i. Next use the exact sequence and a local calculation with coordinates to show that there is a natural isomorphism $\Omega_{\tilde{X}/X} \cong \Omega_E$, where E is the exceptional curve. Now use the cohomology sequence of the above sequence (you will need every term) and Serre duality to get the result.

5.5.4. *V.5.4.* Let $f : X \to X'$ be a birational morphism of nonsingular surfaces.

a. If $Y \subseteq X$ is an irreducible curve such that $f(Y)$ is a point, then $Y \cong \mathbf{P}^1$ and $Y^2 < 0$
b. Let $P' \in X'$ be a fundamental point of f^{-1}, and let $Y_1, \ldots, Y_r$ be the irreducible components of $f^{-1}(P')$. Show that the matrix $|Y_i.Y_j|$ is negative definite.

5.5.5. *V.5.5.* Let C be a curve, and let $\pi : X \to C$ and $\pi' : X' \to C$ be two geometrically ruled surfaces over C. Show that there is a finite sequence of elementary transformations (5.7.1) which transform X into X'.[191]

5.5.6. *V.5.6.* Let X be a surface with function field K. Show that every valuation ring R of K/k is one of the three kinds described in (II, Ex. 4.12).[192]

5.5.7. *V.5.7.* Let Y be an irreducible curve on a surface X, and suppose there is a morphism $f : X \to X_0$ to a projective variety X_0 of dimension 2 , such that $f(Y)$ is a point P and $f^{-1}(P) = Y$. Then show that $Y^2 < 0$.[193]

5.5.8. *V.5.8. A surface singularity.* Let k be an algebraically closed field, and let X be the surface in $\mathbf{A}_k^3$ defined by the equation $x^2 + y^3 + z^5 = 0$. It has an isolated singularity at the origin $P = (0, 0, 0)$.

a. Show that the affine ring $A = k[x, y, z]/\left(x^2 + y^3 + z^5\right)$ of X is a unique factorization domain, as follows. Let $t = z^{-1}; u = t^3 x$, and $v = t^2 y$. Show that z is irreducible in $A; t \in k[u, v]$, and $A\left[z^{-1}\right] = k\left[u, v, t^{-1}\right]$. Conclude that A is a UFD.
b. Show that the singularity at P can be resolved by eight successive blowings-up. If $\tilde{X}$ is the resulting nonsingular surface, then the inverse image of P is a union of eight projective lines, which intersect each other according to the Dynkin diagram $\mathbf{E}_8$:

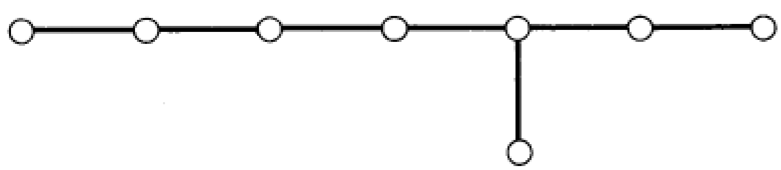

FIGURE 8. The $\mathbf{E}_8$ Dynkin Diagram

5.6. V.6: Classification of Surfaces.

5.6.1. *V.6.1.* Let X be a surface in $\mathbf{P}^n, n \geqslant 3$, defined as the complete intersection of hypersurfaces of degrees $d_1, \ldots, d_{n-2}$, with each $d_i \geqslant 2$. Show that for all but finitely many choices of $(n, d_1, \ldots, d_{n-2})$, the surface X is of general type. List the exceptional cases, and where they fit into the classification picture.

[191]Hints: First show if $D \subseteq X$ is a section of π containing a point P, and if $\tilde{D}$ is the strict transform of D by elm_P, then $\widetilde{D}^2 = D^2 - 1$ (Fig. 23).
Next show that X can be transformed into a geometrically ruled surface X'' with invariant $e \gg 0$. Then use (2.12), and study how the ruled surface $\mathbf{P}(\mathcal{E})$ with $\mathcal{E}$ decomposable behaves under elm_P.

[192]Hint: In case (3), let $f \in R$. Use (Ex. 5.1) to show that for all $i \gg 0, f \in \mathcal{O}_{X_i}$, so in fact $f \in R_0$.

[193]Hint: Let $|H|$ be a very ample (Cartier) divisor class on X_0, let $H_0 \in |H|$ be a divisor containing P, and let $H_1 \in |H|$ be a divisor not containing P. Then consider $f^* H_0, f^* H_1$ and $\tilde{H}_0 = f^*\left(H_0 - P\right)^-$.

5.6.2. *V.6.2.* Prove the following theorem of Chern and Griffiths. Let X be a nonsingular surface of degree d in $\mathbf{P}^{n+1}_{\mathbf{C}}$, which is not contained in any hyperplane. If $d < 2n$, then $p_g(X) = 0$. If $d = 2n$, then either $p_g(X) = 0$, or $p_g(X) = 1$ and X is a K3 surface.[194]

[194] Hint: Cut X with a hyperplane and use Clifford's theorem (IV, 5.4). For the last statement, use the Riemann-Roch theorem on X and the Kodaira vanishing theorem (III, 7.15).