Qual Real Analysis: Problems and Solutions

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1 Preface

I'd like to extend my gratitude to Peter Woolfitt for supplying many solutions and checking many proofs of the rest in problem sessions. Many other solutions contain input and ideas from other graduate students and faculty members at UGA, along with questions and answers posted on Math Stack Exchange or Math Overflow. However, any mistakes are absolutely my own, either in my own solutions or transcribing others'.

The first 12 or so sections are comprised strictly of actual past qual questions from UGA. The remainder of the document includes other questions (and some solutions) from other sources that I found while studying, along with a few midterms from the UGA qual course.

2 Undergraduate Analysis: Uniform Convergence

2.1 Fall 2018.1

Let $f(x) = \frac{1}{x}$. Show that f is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

Concepts Used:

- Uniform continuity:

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \quad \text{such that} \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

- Negating uniform continuity: $\exists \varepsilon > 0$ such that $\forall \delta(\varepsilon)$ there exist x, y such that $|x - y| < \delta$ *and* $|f(x) - f(y)| > \varepsilon$.
- Archimedean property: for all $x, y \in \mathbf{R}$ there exists an $n \in \mathbb{N}$ such that $nx > y$. Take $x = \varepsilon, y = 1$, so $n\varepsilon > 1$ and $\frac{1}{n} < \varepsilon$.

Strategy:
1 is the only constant around, so try to use it for uniform continuity. To negate, find a bad x: since $1/x$ blows up near zero, go hunting for small xs!

Solution:

- **Claim**: $f(x) = \frac{1}{x}$ is uniformly continuous on (c, ∞) for any $c > 0$.

- Note that

$$|x|, |y| > c > 0 \implies |xy| = |x||y| > c^2 \implies \frac{1}{|xy|} < \frac{1}{c^2}.$$

- Letting ε be arbitrary, choose $\delta < \varepsilon c^2$.
  - ♢ Note that δ does not depend on x, y.
- Then

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{x} - \frac{1}{y} \right| \\ &= \frac{|x-y|}{xy} \\ &\leq \frac{\delta}{xy} \\ &< \frac{\delta}{c^2} \\ &< \varepsilon. \end{aligned}$$

- **Claim**: f is *not* uniformly continuous when $c = 0$.
  - Take $\varepsilon < 1$, and let $\delta = \delta(\varepsilon)$ be arbitrary.
  - Let $x_n = \frac{1}{n}$ for $n \geq 1$.
  - Choose n large enough such that $\frac{1}{n} < \delta$
  - Then a computation:

$$\begin{aligned} |x_n - x_{n+1}| &= \frac{1}{n} - \frac{1}{n+1} \\ &= \frac{1}{n(n+1)} \\ &< \frac{1}{n} \\ &< \delta, \end{aligned}$$

  - Why this can be done: by the Archimedean property of $\mathbf{R}$, for any $\delta \in \mathbf{R}$, one can choose choose n such that $n\delta > 1$. We've also used that $n+1 > 1$ so $\frac{1}{n+1} < 1$
  - Note that $f(x_n) = n$, so

$$|f(x_{n+1}) - f(x_n)| = (n+1) - n = 1 > \varepsilon.$$

2.2 Fall 2017.1

Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

Concepts Used:

- $f_N \to f$ uniformly $\iff \|f_N - f\|_\infty \to 0$.
  - Applied to sums:

$$\sum_{0\leq k\leq N} f_n \xrightarrow{u} \sum_{k\geq 0} f_n \iff \left\| \sum_{k\geq N+1} f_n \right\|_\infty \to 0.$$

- An infinite sum is defined as the pointwise limit of its partial sums:

$$\sum_{n=0}^{\infty} c_n x^n := \lim_{N\to\infty} \sum_{n=0}^{N} c_n x^n.$$

- Uniformly decaying terms for uniformly convergent series: if $\sum_{n=0}^\infty f_n(x)$ converges uniformly on a set A, then

$$\|f_n\|_{\infty, A} := \sup_{x\in A} |f_n(x)| \overset{n\to\infty}{\longrightarrow} 0.$$

- M-test: if $f_n : A \to \mathbf{C}$ with $\|f_n\|_\infty < M_n$ and $\sum M_n < \infty$, then $\sum f_n$ converges uniformly and absolutely.
  - If the f_n are continuous, the uniform limit theorem implies $\sum f_n$ is also continuous.

Strategy:
No real place to start, so pick the nicest place: compact intervals. Then bounded intervals, then unbounded sets.

Solution:

- Set $f_N(x) = \sum_{n=1}^N \frac{x^n}{n!}$.
  - Then by definition, $f_N(x) \to f(x)$ pointwise on $\mathbf{R}$.
- **Claim**: f_N converges on compact intervals

- For any compact interval $[-M, M]$, we have

$$\begin{aligned}
\|f_N(x) - f(x)\|_\infty &= \sup_{x \in [-M,M]} \left| \sum_{n=N+1}^\infty \frac{x^n}{n!} \right| \\
&\le \sup_{x \in [-M,M]} \sum_{n=N+1}^\infty \left| \frac{x^n}{n!} \right| \\
&\le \sum_{n=N+1}^\infty \frac{M^n}{n!} \\
&\le \sum_{n=0}^\infty \frac{M^n}{n!} \quad \text{since all additional terms are positive} \\
&= e^M \\
&< \infty,
\end{aligned}$$

so $f_N \to f$ uniformly on $[-M, M]$ by the M-test.

◇ Note: we've used that this power series converges to e^x pointwise everywhere.

- This argument shows that f converges on any bounded set.
- **Claim**: f_N does not converge uniformly on all of $\mathbf{R}$.
  - Uniformly convergent sums have uniformly decaying terms:

$$\sum_{n \le N} g_n \overset{N\to\infty}{\longrightarrow} \sum g_n \text{ uniformly on } A \implies \|g_n\|_{\infty,A} := \sup_{x\in A} |g_n(x)| \overset{n\to\infty}{\longrightarrow} 0.$$

  - Take B_N a ball of radius N about 0, then for $N > 1$, note that $x = N$ on the boundary and so

$$\left\| \frac{x^k}{k!} \right\|_{\infty, B_N} = \frac{N^k}{k!} \overset{N\to\infty}{\longrightarrow} \infty.$$

- **Conclusion**: f_N converges on any bounded $A \subseteq \mathbf{R}$ but not on all of $\mathbf{R}$.

2.3 Spring 2017.4

Let $f(x, y)$ on $[-1, 1]^2$ be defined by

$$f(x,y) = \begin{cases} \frac{xy}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Determine if f is integrable.

Concepts Used:

- Just Calculus.
- $1/r$ is not integrable on $(0,1)$.

Solution:
Switching to polar coordinates and integrating over the quarter of the unit disc $D \cap Q_1 \subseteq I^2$ in quadrant 1, we have

$$\begin{aligned}
\int_{I^2} f\,dA &\geq \int_D f\,dA \\
&= \int_0^{\pi/2}\int_0^1 \frac{r^2\cos(\theta)\sin(\theta)}{r^4}\; r\;\,dr\,d\theta \\
&= \int_0^{\pi/2}\int_0^1 \frac{\cos(\theta)\sin(\theta)}{r}\,dr\,d\theta \\
&= \left(\int_0^1 \frac{1}{r}\,dr\right)\left(\int_0^{\pi/2}\cos(\theta)\sin(\theta)\,d\theta\right) \\
&= \left(\int_0^1 \frac{1}{r}\,dr\right)\left(\int_0^1 u\,du\right) && u = \sin(\theta) \\
&= \frac{1}{2}\left(\int_0^1 \frac{1}{r}\,dr\right) \\
&\longrightarrow \infty.
\end{aligned}$$

2.4 Fall 2014.1

Let $\{f_n\}$ be a sequence of continuous functions such that $\sum f_n$ converges uniformly.

Prove that $\sum f_n$ is also continuous.

Concepts Used:

- The uniform limit theorem.
- $\varepsilon/3$ trick.

Solution:

Claim: If $F_N \to F$ uniformly with each F_N continuous, then F is continuous.

Proof (of claim).

- Follows from an $\varepsilon/3$ argument:

$$|F(x) - F(y| \leq |F(x) - F_N(x)| + |F_N(x) - F_N(y)| + |F_N(y) - F(y)| \leq \varepsilon \to 0.$$

  - The first and last $\varepsilon/3$ come from uniform convergence of $F_N \to F$.
  - The middle $\varepsilon/3$ comes from continuity of each F_N.

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- Now setting $F_N := \sum_{n=1}^{N} f_n$ yields a finite sum of continuous functions, which is continuous.
- Each F_N is continuous and $F_N \to F$ uniformly, so F is continuous.

2.5 Spring 2015.1

Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

1. For every $\varepsilon > 0 \quad \exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
2. The sequence $\{f(x_n)\}_{n=1}^{\infty} \to f(x_0)$ for every sequence $\{x_n\} \to x_0$ in X.

Concepts Used:

- What it means for a sequence to converge.
- Trading Ns for δs.

Solution:

Proof (1 $\implies$ 2).

- Let $\{x_n\} \overset{n\to\infty}{\to} x_0$ be arbitrary; we want to show $\{f(x_n)\} \overset{n\to\infty}{\to} f(x_0)$.
  - We thus want to show that for every $\varepsilon > 0$, there exists an $N(\varepsilon)$ such that

$$n \geq N(\varepsilon) \implies \rho(f(x_n), f(x_0)) < \varepsilon.$$

- Let $\varepsilon > 0$ be arbitrary, then by (1) choose δ such that $\rho(f(x), f(x_0)) < \varepsilon$ when $d(x, x_0) < \delta$.
- Since $x_n \to x$, there is some N such that $n \geq N \implies d(x_n, x_0) < \delta$
- Then for $n \geq N$, $d(x_n, x_0) < \delta$ and thus $\rho(f(x_n), f(x_0)) < \varepsilon$, so $f(x_n) \to f(x_0)$ by definition.

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Proof (2 $\implies$ 1).

The direct implication is not a good idea here, since you need a handle on all *x in a neighborhood of x_0, not just a specific sequence.*

- By contrapositive, show that $\cancel{1} \implies \cancel{2}$.
- Need to show: if f is not ε-δ continuous at x_0, then there exists a sequence $x_n \to x_0$ where $f(x_n) \not\to f(x_0)$.
- Negating 1, we have that there exists an $\varepsilon > 0$ such that for all δ, there exists an x with $d(x, x_0) < \delta$ but $\rho(f(x), f(x_0)) > \varepsilon$
- So take a sequence of deltas $\delta_n = \frac{1}{n}$, apply this to produce a sequence x_n with $d(x_n, x_0) < \delta_n := \frac{1}{n} \longrightarrow 0$ and $\rho(f(x_n), f(x_0)) > \varepsilon$ for all n.
- This yields a sequence $x_n \to x_0$ where $f(x_n) \not\to f(x_0)$.

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2.6 Fall 2014.2

Let I be an index set and $\alpha : I \to (0, \infty)$.

a. Show that

$$\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}$$

b. Suppose $I = \mathbf{Q}$ and $\sum_{q \in \mathbb{Q}} a(q) < \infty$. Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \leq x}} a(q).$$

Show that f is continuous at $x \iff x \notin \mathbf{Q}$.

Concepts Used:

- Can always filter sets X with a function $X \to \mathbf{R}$.
- Countable union of countable sets is still countable.
- Continuity: $\lim_{y \to x} f(y) = f(x)$ from either side.
- Trick: pick enumerations of countable sets and reindex sums

Solution:

Proof (of a).

- Set $S := \sum_{i \in I} \alpha(i)$, we will show that $S < \infty \implies I$ is countable.
- Write

$$I = \bigcup_{n \geq 0} S_n, \qquad S_n := \left\{ i \in I \;\middle|\; \alpha(i) \geq \frac{1}{n} \right\}.$$

  - Note that $S_n \subseteq S$ for all n, so $\sum_{i \in I} \alpha(i) \geq \sum_{i \in S_n} \alpha(i)$ for all n.
  - It suffices to show that S_n is countable, since I is a countable union of S_n.

- There is an inequality

$$\begin{aligned} \infty > S &:= \sum_{i \in I} \alpha(i) \\ &\geq \sum_{i \in S_n} \alpha(i) \\ &\geq \sum_{i \in S_n} \frac{1}{n} \\ &= \frac{1}{n} \sum_{i \in S_n} 1 \\ &= \left(\frac{1}{n} \right) \# S_n \end{aligned}$$

$$\implies \infty > nS \geq \# S_n.$$

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Proof (of b).

- We'll prove something more general: let $Q = \{q_k\}$ be countable and $\{\alpha_k \coloneqq \alpha(q_k)\}$ be summable, and define
$$f(x) \coloneqq \sum_{q_k \leq x} \alpha_k.$$
  - f is always discontinuous precisely on the countable set Q and continuous on $\mathbf{R} \setminus Q$.
  - f is always left-continuous, is right-continuous at $x \in \mathbf{R} \setminus Q$, and *not* right-continuous at $x \in Q$
  - f has jump discontinuities at every q_m, where the jump is precisely α_m.
- This follows from computing the left and right limits:
$$\begin{aligned} f(x^+) &= \lim_{h \to 0} \sum_{q_k \leq x+h} \alpha_k = \sum_{q_k \leq x} \alpha_k = \sum_{q_k < x} \alpha_k + \sum_{q_k = x} \alpha_k \\ f(x^-) &= \lim_{h \to 0} \sum_{q_k \leq x-h} \alpha_k = \sum_{q_k < x} \alpha_k, \end{aligned}$$
where we've used that $\{q_k \leq x\} = \{q_k < x\} \coprod \{x\}$ in the first equality.
- Then if $x = q_m$ for some m,
$$\begin{aligned} f(x^+) = f(q_m^+) &= \sum_{q_k < q_m} \alpha_k + \alpha_m \\ f(x^-) = f(a_m^-) &= \sum_{q_k < q_m} \alpha_k, \end{aligned}$$
which clearly differ if $\alpha_m \neq 0$.
- Taking $x \notin Q$, we have $\{q_k \leq x\} = \{q_k < x\}$, since $\{q_k = x\} = \emptyset$, so
$$\begin{aligned} f(x^+) &= \sum_{q_k \leq x} \alpha_k = \sum_{q_k < x} \alpha_k \\ f(x^-) &= \sum_{q_k < x} \alpha_k, \end{aligned}$$
so the limits agree.
- To recover the result in the problem, let $\mathbf{Q} = \{q_k\}$ be any enumeration of the rationals.

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3 General Analysis

3.1 Fall 2021.1

Problem 3.1.1 (?)
Let $\{x_n\}_{n-1}^{\infty}$ be a sequence of real numbers such that $x_1 > 0$ and

$$x_{n+1} = 1 - (2 + x_n)^{-1} = \frac{1 + x_n}{2 + x_n}.$$

Prove that the sequence $\{x_n\}$ converges, and find its limit.

Solution:
If a limit L exists, we have $x_n \to L$ for all n, so

$$L = \frac{1+L}{2+L} \implies L^2 + L - 1 = 0 \implies L = -\frac{1}{2}\left(-1 \pm \sqrt{5}\right).$$

Noting that $\sqrt{5} > 1$, the condition $x_1 > 0$ and a small induction noting that if $x_n > 0$ then $\frac{1+x_n}{2+x_n} > 0$, the only solution can be $L = -1 + \sqrt{5}$. To see that this does converge, write $f(z) = 1 - (2+z)^{-1}$ so that $x_{n+1} = f(x_n)$. The claim is that f is a contracting map on a metric space, which implies it has a unique fixed point z_0 by the Banach fixed point theorem, and if $f(z_0) = z_0$ then $z_0 = L$. This follows from the mean value theorem, since

$$|f(z) - f(w)| = |f'(\xi)||z - w| < |z - w| \qquad \text{for some } \xi \in (z, w).$$

Since $f'(z) = (2+z)^{-2}$ satisfies $0 < f'(z) < 1$ for all z, we have

$$|f(z) - f(w)| \leq |z - w|.$$

3.2 Fall 2020.1

Problem 3.2.1 (?)
Show that if x_n is a decreasing sequence of positive real numbers such that $\sum_{n=1}^{\infty} x_n$ converges, then

$$\lim_{n \to \infty} n x_n = 0.$$

Solution:

See this MSE post for many solutions: https://math.stackexchange.com/questions/4603/if-a-n-subset0-infty-is-non-increasing-and-sum-a-n-in
Note that the "obvious" thing here is fiddly: there are bounds on the slices

$$(N - M \pm 1)x_N \leq \sum_{M \leq k \leq N} a_k \leq (N - M \pm 1)x_M,$$

but arranging it so that the constants match the indices in $(N - M \pm 1)x_N \approx Nx_N$ requires something clever.

Fix $\varepsilon > 0$, we'll find $n \gg 1$ so that $nx_n < \varepsilon$. Find n, m with $n > m$ large enough so that

$$\varepsilon > \sum_{m+1 \leq k \leq n} x_k \geq \sum_{m+1 \leq k \leq n} x_n = (m - n)x_n.$$

Then rearrange:

$$\varepsilon > (m - n)x_n \implies nx_n < \varepsilon + mx_n.$$

Now choose n large enough so that $x_n < \varepsilon$, which holds since $\sum x_n < \infty$, to obtain

$$nx_n < \varepsilon + m\varepsilon = \varepsilon(1 + m) \to 0.$$

3.3 Spring 2020.1

Prove that if $f : [0, 1] \to \mathbf{R}$ is continuous then

$$\lim_{k \to \infty} \int_0^1 kx^{k-1} f(x)\, dx = f(1).$$

Concepts Used:

- DCT
- Weierstrass Approximation Theorem
  - If $f : [a, b] \to \mathbf{R}$ is continuous, then for every $\varepsilon > 0$ there exists a polynomial $p_\varepsilon(x)$ such that $\|f - p_\varepsilon\|_\infty < \varepsilon$.

Solution:

- Suppose p is a polynomial, then integrate by parts:

$$\begin{aligned}
\lim_{k\to\infty}\int_0^1 kx^{k-1}p(x)\,dx &= \lim_{k\to\infty}\int_0^1 \left(\frac{\partial}{\partial x}x^k\right)p(x)\,dx \\
&= \lim_{k\to\infty}\left[x^k p(x)\Big|_0^1 - \int_0^1 x^k\left(\frac{\partial p}{\partial x}(x)\right)\,dx\right] \quad \text{IBP} \\
&= p(1) - \lim_{k\to\infty}\int_0^1 x^k\left(\frac{\partial p}{\partial x}(x)\right)\,dx,
\end{aligned}$$

- Thus it suffices to show that

$$\lim_{k\to\infty}\int_0^1 x^k\left(\frac{\partial p}{\partial x}(x)\right)\,dx = 0.$$

- Integrating by parts a second time yields

$$\begin{aligned}
\lim_{k\to\infty}\int_0^1 x^k\left(\frac{\partial p}{\partial x}(x)\right)\,dx &= \lim_{k\to\infty}\frac{x^{k+1}}{k+1}\frac{\partial p}{\partial x}(x)\Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1}\left(\frac{\partial^2 p}{\partial x^2}(x)\right)\,dx \\
&= \lim_{k\to\infty}\frac{p'(1)}{k+1} - \lim_{k\to\infty}\int_0^1 \frac{x^{k+1}}{k+1}\left(\frac{\partial^2 p}{\partial x^2}(x)\right)\,dx \\
&= -\lim_{k\to\infty}\int_0^1 \frac{x^{k+1}}{k+1}\left(\frac{\partial^2 p}{\partial x^2}(x)\right)\,dx \\
&= -\int_0^1 \lim_{k\to\infty}\frac{x^{k+1}}{k+1}\left(\frac{\partial^2 p}{\partial x^2}(x)\right)\,dx \quad \text{by DCT} \\
&= -\int_0^1 0\left(\frac{\partial^2 p}{\partial x^2}(x)\right)\,dx \\
&= 0.
\end{aligned}$$

  - The DCT can be applied here because polynomials are smooth and $[0,1]$ is compact, so $\frac{\partial^2 p}{\partial x^2}$ is bounded on $[0,1]$ by some constant M and

$$\int_0^1 \left|x^k\frac{\partial^2 p}{\partial x^2}(x)\right| \leq \int_0^1 1\cdot M = M < \infty.$$

- So the result holds when f is a polynomial.
- Now use the Weierstrass approximation theorem:
  - If $f : [a,b] \to \mathbf{R}$ is continuous, then for every $\varepsilon > 0$ there exists a polynomial $p_\varepsilon(x)$ such that $\|f - p_\varepsilon\|_\infty < \varepsilon$.

- Thus

$$
\begin{aligned}
\left|\int_0^1 kx^{k-1}p_\varepsilon(x)\,dx - \int_0^1 kx^{k-1}f(x)\,dx\right| &= \left|\int_0^1 kx^{k-1}\left(p_\varepsilon(x)-f(x)\right)\,dx\right| \\
&\le \left|\int_0^1 kx^{k-1}\|p_\varepsilon - f\|_\infty\,dx\right| \\
&= \|p_\varepsilon - f\|_\infty \cdot \left|\int_0^1 kx^{k-1}\,dx\right| \\
&= \|p_\varepsilon - f\|_\infty \cdot x^k\Big|_0^1 \\
&= \|p_\varepsilon - f\|_\infty \\
&\overset{\varepsilon\to 0}{\to} 0
\end{aligned}
$$

  and the integrals are equal.

- By the first argument,

$$
\int_0^1 kx^{k-1}p_\varepsilon(x)\,dx = p_\varepsilon(1) \text{ for each } \varepsilon
$$

- Since uniform convergence implies pointwise convergence, $p_\varepsilon(1) \overset{\varepsilon\to 0}{\to} f(1)$.

3.4 Fall 2019.1

Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers.

a. Prove that if $\lim_{n\to\infty} a_n = 0$, then

$$
\lim_{n\to\infty} \frac{a_1+\cdots+a_n}{n} = 0
$$

b. Prove that if $\sum_{n=1}^\infty \frac{a_n}{n}$ converges, then

$$
\lim_{n\to\infty} \frac{a_1+\cdots+a_n}{n} = 0
$$

Solution:

Concepts Used:

- Cesaro mean/summation.
- Break series apart into pieces that can be handled separately.
- Idea: once N is large enough, $a_k \approx S$, and all smaller terms will die off as $N \to \infty$.
  - See this MSE answer.

Proof (of a).

- Prove a stronger result:

$$a_k \to S \implies S_N := \frac{1}{N}\sum_{k=1}^{N} a_k \to S.$$

- For any $\varepsilon > 0$, use convergence $a_k \to S$: choose (and fix) $M = M(\varepsilon)$ large enough such that

$$k \geq M+1 \implies |a_k - S| < \varepsilon.$$

- With M fixed, choose $N = N(M, \varepsilon)$ large enough so that $\frac{1}{N}\sum_{k=1}^{M} |a_k - S| < \varepsilon$.

- Then

$$\begin{aligned}
\left| \left(\frac{1}{N}\sum_{k=1}^{N} a_k \right) - S \right| &= \frac{1}{N}\left| \left(\sum_{k=1}^{N} a_k \right) - NS \right| \\
&= \frac{1}{N}\left| \left(\sum_{k=1}^{N} a_k \right) - \sum_{k=1}^{N} S \right| \\
&= \frac{1}{N}\left| \sum_{k=1}^{N} (a_k - S) \right| \\
&\leq \frac{1}{N}\sum_{k=1}^{N} |a_k - S| \\
&= \frac{1}{N}\sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} |a_k - S| \\
&\leq \frac{1}{N}\sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} \varepsilon \quad \text{since } a_k \to S \\
&= \frac{1}{N}\sum_{k=1}^{M} |a_k - S| + (N-M)\varepsilon \\
&\leq \varepsilon + (N(M,\varepsilon) - M(\varepsilon))\varepsilon.
\end{aligned}$$

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Revisit, not so clear that the last line can be made smaller than ε, since M, N both depend on ε...

#todo

Proof (of b).

- Define
$$\Gamma_n := \sum_{k=n}^{\infty} \frac{a_k}{k}.$$
- $\Gamma_1 = \sum_{k=1}^{n} \frac{a_k}{k}$ is the original series and each Γ_n is a tail of Γ_1, so by assumption $\Gamma_n \overset{n\to\infty}{\to} 0$.
- Compute
$$\frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}(\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \mathbf{\Gamma_{n+1}})$$
.
- This comes from consider the following summation:

$$\begin{array}{r|ccccccc}
\Gamma_1 : & a_1 & +\frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & & \\
\Gamma_2 : & & \frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & & \\
\Gamma_3 : & & & \frac{a_3}{3} & +\cdots & & & \\
\hline
\sum_{i=1}^{n} \Gamma_i : & a_1 & +a_2 & +a_3 & +\cdots & a_n & +\frac{a_{n+1}}{n+1} & +\cdots
\end{array}$$

- Use part (a): since $\Gamma_n \overset{n\to\infty}{\to} 0$, we have $\frac{1}{n}\sum_{k=1}^{n} \Gamma_k \overset{n\to\infty}{\to} 0$.
- Also a minor check: $\Gamma_n \to 0 \implies \frac{1}{n}\Gamma_n \to 0$.
- Then
$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} a_k &= \frac{1}{n}(\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \mathbf{\Gamma_{n+1}}) \\
&= \left(\frac{1}{n}\sum_{k=0}^{n} \Gamma_k\right) - \left(\frac{1}{n}\Gamma_{n+1}\right) \\
&\overset{n\to\infty}{\to} 0.
\end{aligned}$$

■

3.5 Fall 2018.4

Let $f \in L^1([0,1])$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)|\sin nx|\, dx = \frac{2}{\pi}\int_0^1 f(x)\, dx$$

Hint: Begin with the case that f is the characteristic function of an interval.

Ask someone to check the last approximation part.

#todo

Solution:

Concepts Used:

- Converting floor/ceiling functions to inequalities: $x - 1 \leq \lfloor x \rfloor \leq x$.

Case of a characteristic function of an interval $[a,b]$:

- First suppose $f(x) = \chi_{[a,b]}(x)$.
- Note that $\sin(nx)$ has a period of $2\pi/n$, and thus $\left\lfloor \frac{(b-a)}{(2\pi/n)} \right\rfloor = \left\lfloor \frac{n(b-a)}{2\pi} \right\rfloor$ full periods in $[a,b]$.
- Taking the absolute value yields a new function with half the period
  - So $|\sin(nx)|$ has a period of π/n with $\left\lfloor \frac{n(b-a)}{\pi} \right\rfloor$ full periods in $[a,b]$.
- We can compute the integral over one full period (which is independent of *which* period is chosen)
  - We can use translation invariance of the integral to compute this over the period 0 to π/n.
  - Since $\sin(nx)$ is positive, it equals $|\sin(nx)|$ on its first period, so we have

$$\begin{aligned}
\int_{\text{One Period}} |\sin(nx)|\, dx &= \int_0^{\pi/n} \sin(nx)\, dx \\
&= \frac{1}{n}\int_0^{\pi} \sin(u)\, du \quad u = nx \\
&= \frac{1}{n}\left(-\cos(u) \Big|_0^{\pi} \right) \\
&= \frac{2}{n}.
\end{aligned}$$

- Then break the integral up into integrals over full periods $P_1, P_2, \cdots, P_N$ where $N := \lfloor n(b-a)/\pi \rfloor$

- Noting that each period is of length $\frac{\pi}{n}$, so letting L_n be the regions falling *outside* of a full period, we have
- Thus

$$\begin{aligned}
\int_a^b |\sin(nx)|\,dx &= \left(\sum_{j=1}^N \int_{P_j} |\sin(nx)|\,dx\right) + \int_{L_n} |\sin(nx)|\,dx \\
&= \left(\sum_{j=1}^N \frac{2}{n}\right) + \int_{L_n} |\sin(nx)|\,dx \\
&= N\left(\frac{2}{n}\right) + \int_{L_n} |\sin(nx)|\,dx \\
&:= \left\lfloor \frac{(b-a)n}{\pi} \right\rfloor \frac{2}{n} + R_n \\
&:= (b-a)C_n + R_n
\end{aligned}$$

where (claim) $C_n \overset{n\to\infty}{\to} \frac{2}{\pi}$ and $R(n) \overset{n\to\infty}{\to} 0$.

- $C_n \to \frac{2}{\pi}$:

$$\frac{n-1}{n}\left(\frac{2}{\pi}\right) = \frac{n-1}{\pi}\left(\frac{2}{n}\right) \leq \left\lfloor \frac{n}{\pi} \right\rfloor \left(\frac{2}{n}\right) \leq \frac{n}{\pi}\left(\frac{2}{n}\right) = \frac{2}{\pi},$$

  then use the fact that $\frac{n-1}{n} \to 1$.

  - Then equality follows by the Squeeze theorem.

- $R_n \to 0$:
  - We use the fact that $m(L_n) \to 0$, then $\int_{L_n} |\sin(nx)| \leq \int_{L_n} 1 = m(L_n) \to 0$.
  - This follows from the fact that L_n is the complement of $\cup_j P_j$, the set of full periods, so

$$\begin{aligned}
m(L_n) &= m(b-a) - \sum m(P_j) \\
&= (b-a) - \left\lfloor \frac{n(b-a)}{\pi} \right\rfloor \left(\frac{\pi}{n}\right) \\
&\overset{n\to\infty}{\to} (b-a) - (b-a) \\
&= 0.
\end{aligned}$$

    where we've used the fact that

$$\begin{aligned}
\left(\frac{\pi}{n}\right)\left(\frac{(b-a)n-1}{\pi}\right) &\leq \left\lfloor \frac{n(b-a)}{\pi} \right\rfloor \left(\frac{\pi}{n}\right) \\
&\leq \left(\frac{\pi}{n}\right)\left(\frac{(b-a)n}{\pi}\right) \\
&= (b-a),
\end{aligned}$$

    then taking $n \to \infty$ sends the LHS to $b-a$, forcing the middle term to be $b-a$ by the Squeeze theorem.

General case:

- By linearity of the integral, the result holds for simple functions:
  - If $f = \sum c_j \chi_{E_j}$ where $E_j = [a_j, b_j]$, we have

$$
\begin{aligned}
\int_0^1 f(x)|\sin(nx)|\,dx &= \int_0^1 \sum c_j \chi_{E_j}(x)|\sin(nx)|\,dx \\
&= \sum c_j \int_0^1 \chi_{E_j}(x)|\sin(nx)|\,dx \\
&= \sum c_j (b_j - a_j)\frac{2}{\pi} \\
&= \frac{2}{\pi}\sum c_j (b_j - a_j) \\
&= \frac{2}{\pi}\sum c_j m(E_j) \\
&:= \frac{2}{\pi}\int_0^1 f.
\end{aligned}
$$

- Since $f \in L^1$, where simple functions are dense, choose $s_n \nearrow f$ where $\|s_N - f\|_1 < \varepsilon$, then

$$
\begin{aligned}
\left|\int_0^1 f(x)|\sin(nx)|\,dx - \int_0^1 s_N(x)|\sin(nx)|\,dx\right| &= \left|\int_0^1 (f(x) - s_N(x))\,|\sin(nx)|\,dx\right| \\
&\leq \int_0^1 |f(x) - s_N(x)||\sin(nx)|\,dx \\
&= \|(f - s_N)\,|\sin(nx)|\|_1 \\
&\leq \|f - s_N\|_1 \cdot \||\sin(nx)|\|_\infty \quad \text{by Holder} \\
&\leq \varepsilon \cdot 1,
\end{aligned}
$$

- So the integrals involving s_N converge to the integral involving f, and

$$
\begin{aligned}
\lim_{n\to\infty} \int f(x)|\sin(nx)| &= \lim_{n\to\infty}\lim_{N\to\infty} \int s_N(x)|\sin(nx)| \\
&= \lim_{N\to\infty}\lim_{n\to\infty} \int s_N(x)|\sin(nx)| \quad \text{because ?} \\
&= \lim_{N\to\infty} \frac{2}{\pi}\int s_N(x) \\
&= \frac{2}{\pi}\int f,
\end{aligned}
$$

  which is the desired result.

3.6 Fall 2017.4

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

a. Show that $f_n \to 0$ pointwise but not uniformly on $[0, 1]$.

b. Show that

$$\lim_{n\to\infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

Hint for (a): Consider the maximum of f_n.

Solution:

Concepts Used:

- $\sum f_n < \infty \iff \sup f_n \to 0$.
- Negating uniform convergence: $f_n \not\to f$ uniformly iff $\exists \varepsilon$ such that $\forall N(\varepsilon)$ there exists an x_N such that $|f(x_N) - f(x)| > \varepsilon$.
- Exponential inequality: $1 + y \leq e^y$ for all $y \in \mathbf{R}$.

a.

$f_n \to 0$ pointwise:

- Finding the maximum: can check that $\frac{\partial f_n}{\partial x} = x(1-x)^{n-1}\left(1 + (n^2 - 1)x\right)$
- This has critical points $x = 0, 1, \frac{-1}{n^2+1}$, and the latter is a global max on $[0, 1]$.
- Set $x_n := \frac{-1}{n^2+1}$
- Compute

$$\lim f_n(x_n) = \lim_{n\to\infty} \frac{-n}{n^2+1}(1 + x_n)^n = 0 \cdot 1 = 0.$$

- So $\sup f_n \to 0$, forcing $f_n \to 0$ pointwise.

The convergence is not uniform:

- Let $x_n = \frac{1}{n}$ and $\varepsilon > e^{-1}$, then

$$\begin{aligned}
\|nx(1-x)^n - 0\|_\infty &\geq |nx_n(1-x_n)^n| \\
&= \left|\left(1 - \frac{1}{n}\right)^n\right| \\
&> e^{-1} \\
&> \varepsilon.
\end{aligned}$$

  - Here we've used that $(1 + \frac{x}{n})^n \leq e^x$ for all $x \in \mathbf{R}$ and all n.

  - Follows from $1 + y \leq e^y$ applied to $y = x/n$.
- Thus $\|f_n - 0\|_\infty = \|f_n\|_\infty > e^{-1} > 0$.

b. ?

Possible to use part a with $\sin(x) \leq x$ on $[0, \pi/2]$?

#todo

- Noting that $\sin(x) \leq 1$, we have

$$\begin{aligned}
\left| \int_0^1 n(1-x)^n \sin(x) \right| &\leq \int_0^1 |n(1-x)^n \sin(x)| \\
&\leq \int_0^1 |n(1-x)^n| \\
&= n \int_0^1 (1-x)^n \\
&= -\frac{n(1-x)^{n+1}}{n+1} \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}$$

3.7 Spring 2017.3

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \quad \text{where } 0 < a < b.$$

Show that

a. $\sum_{n=1}^\infty |f_n|$ is not in $L^1([0, \infty), m)$

Hint: $f_n(x)$ has a root x_n.

b.

$$\sum_{n=1}^\infty f_n \text{ is in } L^1([0, \infty), m) \quad \text{and} \quad \int_0^\infty \sum_{n=1}^\infty f_n(x)\, dm = \ln \frac{b}{a}$$

Not complete.

Add concepts.

Walk through.

Solution:

Concepts Used:

a.

- f_n has a root:

$$ae^{-nax} = be^{-nbx} \iff \frac{1}{n} = e^{-nbx}e^{nax} = e^{n(b-a)x} \iff x = \frac{\ln\left(\frac{a}{b}\right)}{n(a-b)} := x_n.$$

- Thus f_n only changes sign at x_n, and is strictly positive on one side of x_n.
- Then

$$\begin{aligned}
\int_{\mathbf{R}} \sum_n |f_n(x)|\,dx &= \sum_n \int_{\mathbf{R}} |f_n(x)|\,dx \\
&\geq \sum_n \int_{x_n}^{\infty} f_n(x)\,dx \\
&= \sum_n \frac{1}{n}\left(e^{-bnx} - e^{-anx}\Big|_{x_n}^{\infty}\right) \\
&= \sum_n \frac{1}{n}\left(e^{-bnx_n} - e^{-anx_n}\right).
\end{aligned}$$

b.

?

3.8 Fall 2016.1

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that f converges to a differentiable function on $(1, \infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^x}\right)'.$$

Hint:

$$\left(\frac{1}{n^x}\right)' = -\frac{1}{n^x}\ln n$$

Add concepts.

Solution:

Concepts Used:

- ?

- Set $f_N(x) := \sum_{n=1}^N n^{-x}$, so $f(x) = \lim_{N\to\infty} f_N(x)$.
- If an interchange of limits is justified, we have

$$\begin{aligned}
\frac{\partial}{\partial x} \lim_{N\to\infty} \sum_{n=1}^N n^{-x} &= \lim_{h\to 0} \lim_{N\to\infty} \frac{1}{h} \left[\left(\sum_{n=1}^N n^{-x} \right) - \left(\sum_{n=1}^N n^{-(x+h)} \right) \right] \\
&\underset{?}{=} \lim_{N\to\infty} \lim_{h\to 0} \frac{1}{h} \left[\left(\sum_{n=1}^N n^{-x} \right) - \left(\sum_{n=1}^N n^{-(x+h)} \right) \right] \\
&= \lim_{N\to\infty} \lim_{h\to 0} \frac{1}{h} \left[\sum_{n=1}^N n^{-x} - n^{-(x+h)} \right] \quad (1) \\
&= \lim_{N\to\infty} \sum_{n=1}^N \lim_{h\to 0} \frac{1}{h} \left[n^{-x} - n^{-(x+h)} \right] \quad \text{since this is a finite sum} \\
&:= \lim_{N\to\infty} \sum_{n=1}^N \frac{\partial}{\partial x} \left(\frac{1}{n^x} \right) \\
&= \lim_{N\to\infty} \sum_{n=1}^N -\frac{\ln(n)}{n^x},
\end{aligned}$$

where the combining of sums in (1) is valid because $\sum n^{-x}$ is absolutely convergent for $x > 1$ by the p-test.

- Thus it suffices to justify the interchange of limits and show that the last sum converges on $(1, \infty)$.
- Claim: $\sum n^{-x} \ln(n)$ converges.
  - Use the fact that for any fixed $\varepsilon > 0$,
  $$\lim_{n\to\infty} \frac{\ln(n)}{n^\varepsilon} \overset{L.H.}{=} \lim_{n\to\infty} \frac{1/n}{\varepsilon n^{\varepsilon - 1}} = \lim_{n\to\infty} \frac{1}{\varepsilon n^\varepsilon} = 0,$$
  - This implies that for a fixed $\varepsilon > 0$ and for any constant $c > 0$ there exists an N large enough such that $n \geq N$ implies $\ln(n)/n^\varepsilon < c$, i.e. $\ln(n) < cn^\varepsilon$.
  - Taking $c = 1$, we have $n \geq N \implies \ln(n) < n^\varepsilon$

  - We thus break up the sum:

$$\begin{aligned}
\sum_{n\in\mathbb{N}} \frac{\ln(n)}{n^x} &= \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{\ln(n)}{n^x} \\
&\leq \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{n^\varepsilon}{n^x} \\
&:= C_\varepsilon + \sum_{n=N}^{\infty} \frac{n^\varepsilon}{n^x} \quad \text{with } C_\varepsilon < \infty \text{ a constant} \\
&= C_\varepsilon + \sum_{n=N}^{\infty} \frac{1}{n^{x-\varepsilon}},
\end{aligned}$$

    where the last term converges by the p-test if $x - \varepsilon > 1$.
  - But ε can depend on x, and if $x \in (1, \infty)$ is fixed we can choose $\varepsilon < |x-1|$ to ensure this.
- Claim: the interchange of limits is justified.

?

3.9 Fall 2016.5

Let $\varphi \in L^\infty(\mathbf{R})$. Show that the following limit exists and satisfies the equality

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \, dx \right)^{\frac{1}{n}} = \|\varphi\|_\infty .$$

Add concepts.

Solution:

Concepts Used:

- ?

Let L be the LHS and R be the RHS.

Claim: $L \leq R$. - Since $|\varphi| \leq \|\varphi\|_\infty$ a.e., we can write

$$\begin{aligned}
L^{\frac{1}{n}} &:= \int_{\mathbf{R}} \frac{|\varphi(x)|^n}{1+x^2} \\
&\leq \int_{\mathbf{R}} \frac{\|\varphi\|_\infty^n}{1+x^2} \\
&= \|\varphi\|_\infty^n \int_{\mathbf{R}} \frac{1}{1+x^2} \\
&= \|\varphi\|_\infty^n \arctan(x)\Big|_{-\infty}^{\infty} \\
&= \|\varphi\|_\infty^n \left(\frac{\pi}{2} - \frac{-\pi}{2}\right) \\
&= \pi \|\varphi\|_\infty^n
\end{aligned}$$

$$\begin{aligned}
\implies L^{\frac{1}{n}} &\leq \sqrt[n]{\pi \|\varphi\|_\infty^n} \\
\implies L &\leq \pi^{\frac{1}{n}} \|\varphi\|_\infty \\
&\stackrel{n\to\infty}{\to} \|\varphi\|_\infty,
\end{aligned}$$

where we've used the fact that $c^{\frac{1}{n}} \stackrel{n\to\infty}{\to} 1$ for any constant c.

Actually true? Need conditions?

Claim: $R \leq L$.

- We will show that $R \leq L + \varepsilon$ for every $\varepsilon > 0$.
- Set

$$S_\varepsilon := \left\{ x \in \mathbf{R}^n \;\middle|\; |\varphi(x)| \geq \|\varphi\|_\infty - \varepsilon \right\}.$$

- Then we have

$$\begin{aligned}
\int_{\mathbf{R}} \frac{|\varphi(x)|^n}{1+x^2}\,dx &\geq \int_{S_\varepsilon} \frac{|\varphi(x)|^n}{1+x^2}\,dx \quad && S_\varepsilon \subset \mathbf{R} \\
&\geq \int_{S_\varepsilon} \frac{(\|\varphi\|_\infty - \varepsilon)^n}{1+x^2}\,dx && \text{by definition of } S_\varepsilon \\
&= (\|\varphi\|_\infty - \varepsilon)^n \int_{S_\varepsilon} \frac{1}{1+x^2}\,dx \\
&= (\|\varphi\|_\infty - \varepsilon)^n \, C_\varepsilon && \text{where } C_\varepsilon \text{ is some constant}
\end{aligned}$$

$$\begin{aligned}
\implies \left(\int_{\mathbf{R}} \frac{|\varphi(x)|^n}{1+x^2}\,dx \right)^{\frac{1}{n}} &\geq (\|\varphi\|_\infty - \varepsilon)\, C_\varepsilon^{\frac{1}{n}} \\
&\stackrel{n\to\infty}{\to} (\|\varphi\|_\infty - \varepsilon) \cdot 1 \\
&\stackrel{\varepsilon\to 0}{\to} \|\varphi\|_\infty,
\end{aligned}$$

where we've again used the fact that $c^{\frac{1}{n}} \to 1$ for any constant.

3.10 Fall 2016.6

Let $f, g \in L^2(\mathbf{R})$. Show that

$$\lim_{n\to\infty} \int_{\mathbf{R}} f(x)g(x+n)\,dx = 0$$

Rewrite solution.

Concepts Used:

- Cauchy Schwarz: $\|fg\|_1 \leq \|f\|_1 \|g\|_1$.
- Small tails in L^p.

Solution:

- Use the fact that L^p has small tails: if $h \in L^2(\mathbf{R})$, then for any $\varepsilon > 0$,

$$\forall \varepsilon,\ \exists N \in \mathbb{N} \quad \text{suchthat} \quad \int_{|x|\geq N} |h(x)|^2\,dx < \varepsilon.$$

- So choose N large enough so that

$$\int_{\|x\|\geq N} |g(x)|^2 < \varepsilon$$
$$\int_{\|x\|\geq N} |f(x)|^2 < \varepsilon$$
$$.$$

- Then write

$$\int_{\mathbf{R}^d} f(x)g(x+n)\,dx = \int_{\|x\|\leq N} f(x)g(x+n)\,dx + \int_{\|x\|\geq N} f(x)g(x+n)\,dx.$$

- Bounding the second term: apply Cauchy-Schwarz

$$\int_{\|x\|\geq N} f(x)g(x+n)\,dx \leq \left(\int_{\|x\|\geq N} |f(x)|^2\right)^{\frac{1}{2}} \cdot \left(\int_{\|x\|\geq N} |g(x)|^2\right)^{\frac{1}{2}} \leq \varepsilon^{\frac{1}{2}} \cdot \|g\|_2.$$

- Bounding the first term: also Cauchy-Schwarz, after variable changes

$$\begin{aligned}
\int_{\|x\| \leq N} f(x)g(x+n)\,dx &= \int_{-N}^{N} f(x)g(x+n)\,dx \\
&= \int_{-N+n}^{N+n} f(x-n)g(x)\,dx \\
&\leq \int_{-N+n}^{\infty} f(x-n)g(x)\,dx \\
&\leq \left(\int_{-N+n}^{\infty} |f(x-n)|^2\right)^{\frac{1}{2}} \cdot \left(\int_{-N+n}^{\infty} |g(x)|^2\right)^{\frac{1}{2}} \\
&\leq \|f\|_2 \cdot \varepsilon^{\frac{1}{2}}.
\end{aligned}$$

- Then as long as $n \geq 2N$, we have

$$\int |f(x)g(x+n)| \leq (\|f\|_2 + \|g\|_2) \cdot \varepsilon^{\frac{1}{2}}.$$

3.11 Spring 2016.1

For $n \in \mathbb{N}$, define

$$e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that $e_n < E_n$, and prove Bernoulli's inequality:

$$(1+x)^n \geq 1 + nx \qquad\qquad -1 < x < \infty,\ n \in \mathbb{N}.$$

Use this to show the following:

1. The sequence e_n is increasing.
2. The sequence E_n is decreasing.
3. $2 < e_n < E_n < 4$.
4. $\lim_{n\to\infty} e_n = \lim_{n\to\infty} E_n$.

3.12 Fall 2015.1

Define

$$f(x) = c_0 + c_1 x^1 + c_2 x^2 + \ldots + c_n x^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number x_m such that $f(x_m) \leq f(x)$ for all $x \in \mathbf{R}$.

3.13 Spring 2014.2

Let $\{a_n\}$ be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that $\sum a_n^2 < \infty$.

Note: Assume a_n, b_n are all non-negative.

Have someone check!

Solution:

- Define a sequence of operators

$$\begin{aligned} T_N : \ell^2 &\to \ell^1 \\ \{b_n\} &\mapsto \sum_{n=1}^N a_n b_n. \end{aligned}$$

- By assumption, these are well defined: the image is ℓ^1 since $|T_N(\{b_n\})| < \infty$ for all N and all $\{b_n\} \in \ell^2$.
- So each $T_N \in (\ell^2)^\vee$ is a linear functional on ℓ^2.
- For each $x \in \ell^2$, we have $\|T_N(x)\|_{\mathbf{R}} = \sum_{n=1}^N a_n b_n < \infty$ by assumption, so each T_N is pointwise bounded.
- By the Uniform Boundedness Principle, $\sup_N \|T_N\|_{\text{op}} < \infty$.
- Define $T = \lim_{N\to\infty} T_N$, then $\|T\|_{\text{op}} < \infty$.
- By the Riesz Representation theorem,

$$\sqrt{\sum a_n^2} \coloneqq \|\{a_n\}\|_{\ell^2} = \|T\|_{(\ell^2)^\vee} = \|T\|_{\text{op}} < \infty.$$

- So $\sum a_n^2 < \infty$.

4 | Measure Theory: Sets

4.1 Fall 2021.3

Recall that a set $E \subset \mathbb{R}^d$ is measurable if for every $c > 0$ there is an open set $U \subseteq \mathbf{R}^d$ such that $m^*(U \setminus E) < \epsilon$.

a. Prove that if E is measurable then for all $\epsilon > 0$ there exists an elementary set F, such that $m(E\Delta F) < \epsilon$.

   Here $m(E)$ denotes the Lebesgue measure of E, a set F is called elementary if it is a finite union of rectangles and $E\Delta F$ denotes the symmetric difference of the sets E and F.

b. Let $E \subset \mathbb{R}$ be a measurable set, such that $0 < m(E) < \infty$. Use part (a) to show that

$$\lim_{n\to\infty} \int_E \sin(nt)dt = 0$$

4.2 Spring 2020.2

Let m_* denote the Lebesgue outer measure on $\mathbf{R}$.

a.. Prove that for every $E \subseteq \mathbf{R}$ there exists a Borel set B containing E such that

$$m_*(B) = m_*(E).$$

b.. Prove that if $E \subseteq \mathbf{R}$ has the property that

$$m_*(A) = m_*(A \bigcap E) + m_*(A \bigcap E^c)$$

for every set $A \subseteq \mathbf{R}$, then there exists a Borel set $B \subseteq \mathbf{R}$ such that $E = B \setminus N$ with $m_*(N) = 0$.

Be sure to address the case when $m_*(E) = \infty$.

Concepts Used:

- Definition of outer measure:

$$m_*(E) = \inf_{\{Q_j\} \rightrightarrows E} \sum |Q_j|$$

  where $\{Q_j\}$ is a countable collection of closed cubes.
- Break $\mathbf{R}$ into $\coprod_{n\in\mathbf{Z}}[n, n+1)$, each with finite measure.
- Theorem: $m_*(Q) = |Q|$ for Q a closed cube (i.e. the outer measure equals the volume).

Solution:

Proof .

- $m_*(Q) \leq |Q|$:
- Since $Q \subseteq Q$, $Q \rightrightarrows Q$ and $m_*(Q) \leq |Q|$ since m_* is an infimum over such coverings.
- $|Q| \leq m_*(Q)$:
- Fix $\varepsilon > 0$.
- Let $\{Q_i\}_{i=1}^{\infty} \rightrightarrows Q$ be arbitrary, it suffices to show that

$$|Q| \leq \left(\sum_{i=1}^{\infty} |Q_i| \right) + \varepsilon.$$

- Pick open cubes S_i such that $Q_i \subseteq S_i$ and $|Q_i| \leq |S_i| \leq (1+\varepsilon)|Q_i|$.
- Then $\{S_i\} \rightrightarrows Q$, so by compactness of Q pick a finite subcover with N elements.
- Note

$$Q \subseteq \bigcup_{i=1}^{N} S_i \implies |Q| \leq \sum_{i=1}^{N} |S_i| \leq \sum_{i=1}^{N} (1+\varepsilon)|Q_j| \leq (1+\varepsilon) \sum_{i=1}^{\infty} |Q_i|.$$

- Taking an infimum over coverings on the RHS preserves the inequality, so

$$|Q| \leq (1+\varepsilon) m_*(Q)$$

- Take $\varepsilon \to 0$ to obtain final inequality.

■

a.

- If $m_*(E) = \infty$, then take $B = \mathbf{R}^n$ since $m(\mathbf{R}^n) = \infty$.
- Suppose $N := m_*(E) < \infty$.
- Since $m_*(E)$ is an infimum, by definition, for every $\varepsilon > 0$ there exists a covering by closed cubes $\{Q_i(\varepsilon)\}_{i=1}^{\infty} \rightrightarrows E$ depending on ε such that

$$\sum_{i=1}^{\infty} |Q_i(\varepsilon)| < N + \varepsilon.$$

- For each fixed n, set $\varepsilon_n = \frac{1}{n}$ to produce such a covering $\{Q_i(\varepsilon_n)\}_{i=1}^{\infty}$ and set $B_n := \bigcup_{i=1}^{\infty} Q_i(\varepsilon_n)$.

- The outer measure of cubes is *equal* to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^{\infty} |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

- Now set $B \coloneqq \bigcap_{n=1}^{\infty} B_n$.
  - Since $E \subseteq B_n$ for every n, $E \subseteq B$
  - Since B is a countable intersection of countable unions of closed sets, B is Borel.
  - Since $B_n \subseteq B$ for every n, we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \leq m_*(B) \leq m_*(B_n) < N + \frac{1}{n} \quad \text{forall} \quad n \in \mathbf{Z}^{\geq 1}.$$

- This forces $m_*(E) = m_*(B)$.

b.

Suppose $m_*(E) < \infty$.

- By (a), find a Borel set $B \supseteq E$ such that $m_*(B) = m_*(E)$
- Note that $E \subseteq B \implies B \bigcap E = E$ and $B \bigcap E^c = B \setminus E$.
- By assumption,

$$\begin{aligned} m_*(B) &= m_*(B \bigcap E) + m_*(B \bigcap E^c) \\ m_*(E) &= m_*(E) + m_*(B \setminus E) \\ m_*(E) - m_*(E) &= m_*(B \setminus E) \qquad \text{since } m_*(E) < \infty \\ \implies m_*(B \setminus E) &= 0. \end{aligned}$$

- So take $N = B \setminus E$; this shows $m_*(N) = 0$ and $E = B \setminus (B \setminus E) = B \setminus N$.

If $m_*(E) = \infty$:

- Apply result to $E_R \coloneqq E \bigcap [R, R+1)^n \subset \mathbf{R}^n$ for $R \in \mathbf{Z}$, so $E = \coprod_R E_R$
- Obtain B_R, N_R such that $E_R = B_R \setminus N_R$, $m_*(E_R) = m_*(B_R)$, and $m_*(N_R) = 0$.
- Note that
  - $B \coloneqq \bigcup_R B_R$ is a union of Borel sets and thus still Borel
  - $E = \bigcup_R E_R$
  - $N \coloneqq B \setminus E$
  - $N' \coloneqq \bigcup_R N_R$ is a union of null sets and thus still null
- Since $E_R \subset B_R$ for every R, we have $E \subset B$
- We can compute

$$N = B \setminus E = \left(\bigcup_R B_R \right) \setminus \left(\bigcup_R E_R \right) \subseteq \bigcup_R (B_R \setminus E_R) = \bigcup_R N_R \coloneqq N'$$

where $m_*(N') = 0$ since N' is null, and thus subadditivity forces $m_*(N) = 0$.

4.3 Fall 2019.3.

Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) = 1$ and $\{B_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{B}$-measurable subsets of X, and

$$B := \left\{ x \in X \;\middle|\; x \in B_n \text{ for infinitely many } n \right\}.$$

a. Argue that B is also a $\mathcal{B}$-measurable subset of X.

b. Prove that if $\sum_{n=1}^{\infty} \mu(B_n) < \infty$ then $\mu(B) = 0$.

c. Prove that if $\sum_{n=1}^{\infty} \mu(B_n) = \infty$ **and** the sequence of set complements $\{B_n^c\}_{n=1}^{\infty}$ satisfies

$$\mu\left(\bigcap_{n=k}^{K} B_n^c \right) = \prod_{n=k}^{K} (1 - \mu(B_n))$$

for all positive integers k and K with $k < K$, then $\mu(B) = 1$.

Hint: Use the fact that $1 - x \leq e^{-x}$ for all x.

Concepts Used:

- Borel-Cantelli: for a sequence of sets X_n,

$$\left\{ x \;\middle|\; x \in X_n \text{ for infinitely many } n \right\} = \bigcap_{N \geq 1} \bigcup_{n \geq N} X_n = \limsup_n X_n$$
$$\left\{ x \;\middle|\; x \in X_n \text{ for all but finitely many } n \right\} = \bigcup_{N \geq 1} \bigcap_{n \geq N} X_n = \liminf X_n.$$

- Properties of logs and exponentials:

$$\prod_n e^{x_n} = e^{\sum_n x_n} \quad \text{and} \quad \sum_n \log(x_n) = \log\left(\prod_n x_n \right).$$

- Tails of convergent sums vanish.
- Continuity of measure: $B_n \searrow B$ and $\mu(B_0) < \infty$ implies $\lim_n \mu(B_n) = \mu(B)$, and $B_n \nearrow B \implies \lim_n \mu(B_n) = \mu(B)$.

Solution:

Proof (of a).

- The Borel σ-algebra is closed under countable unions/intersections/complements,
- $B = \limsup_n B_n = \cap_{N\geq 1} \cup_{n\geq N} B_n$ is an intersection of unions of measurable sets.

$\blacksquare$

Proof (of b).

- Tails of convergent sums vanish, so

$$\sum_{n\geq N} \mu(B_n) \xrightarrow{N\to\infty} 0.$$

- Also,

$$B_M \coloneqq \bigcap_{N=1}^{M} \bigcup_{n\geq N} B_n \searrow B.$$

- A computation:

$$\begin{aligned}
\mu(B) &\coloneqq \mu\left(\bigcap_{N\geq 1} \bigcup_{n\geq N} B_n \right) \\
&\leq \mu\left(\bigcup_{n\geq N} B_n \right) && \forall N \\
&\leq \sum_{n\geq N} \mu(B_n) && \forall N \\
&\xrightarrow{N\to\infty} 0,
\end{aligned}$$

where we've used that we're intersecting over fewer sets and this can only increase measure.

$\blacksquare$

Proof (of c).

- Since $\mu(X) = 1$, in order to show $\mu(B) = 1$ it suffices to show $\mu(X \setminus B) = 0$.
- A computation:

$$\begin{aligned}
\mu(B^c) &= \mu\left(\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_n\right)^c\right) \\
&= \mu\left(\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_n^c\right) \\
&\leq \sum_{N=1}^{\infty} \mu\left(\bigcap_{n=N}^{\infty} B_n^c\right) \\
&= \sum_{N=1}^{\infty} \lim_{K\to\infty} \mu\left(\bigcap_{n=N}^{K} B_n^c\right) && \text{continuity of measure from above} \\
&= \sum_{N=1}^{\infty} \lim_{K\to\infty} \prod_{n=N}^{K} (1 - \mu(B_n)) && \text{by assumption} \\
&\leq \sum_{N=1}^{\infty} \lim_{K\to\infty} \prod_{n=N}^{K} e^{-\mu(B_n)} && \text{by hint} \\
&= \sum_{N=1}^{\infty} \lim_{K\to\infty} e^{-\sum_{n=N}^{K} \mu(B_n)} \\
&= \sum_{N=1}^{\infty} e^{-\lim_{K\to\infty} \sum_{n=N}^{K} \mu(B_n)} && \text{by continuity of } f(x) = e^x \\
&= \sum_{N=1}^{\infty} e^{-\sum_{n=N}^{\infty} \mu(B_n)} \\
&= \sum_{N=1}^{\infty} 0 \\
&= 0.
\end{aligned}$$

- Here we've used that every tail of a divergent sum is divergent: if $\sum_{n=1}^{\infty} a_n \to \infty$ then for every N, the tail $\sum_{n=N}^{\infty} a_n \to \infty$ as well.
- We've also use that if $b_n \to \infty$ then $e^{-b_n} \to 0$.

■

4.4 Spring 2019.2

Let $\mathcal{B}$ denote the set of all Borel subsets of $\mathbf{R}$ and $\mu : \mathcal{B} \to [0, \infty)$ denote a finite Borel measure on $\mathbf{R}$.

a. Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k \supseteq F_{k+1}$ for all k, then

$$\lim_{k \to \infty} \mu(F_k) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

b. Suppose μ has the property that $\mu(E) = 0$ for every $E \in \mathcal{B}$ with Lebesgue measure $m(E) = 0$. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ so that if $E \in \mathcal{B}$ with $m(E) < \delta$, then $\mu(E) < \varepsilon$.

Concepts Used:

- Proof of continuity of measure.
- Using limsup/liminf sets (intersections of unions and vice-versa) and (sub)additivity to bound measures.
  - Control over lower bound: use tails of convergent sums
  - Control over upper bound: use rapidly converging coefficients like $\sum 1/2^n$
- Convergent sums have vanishing tails.
- Intersecting over *more* sets can only lose measure, taking a union over *more* can only gain measure.
- Similarly intersecting over *fewer* sets can only *gain* measure, and taking a union over *fewer* sets can only *lose* measure.

Strategy:
Use a limsup or liminf of sets and continuity of measure. Note that choosing a limsup vs a liminf is fiddly – for one choice, you can only get one of the bounds you need, for the other choice you can get both.

Proof (of a).

- Observation: μ finite means $\mu(E) < \infty$ for all $E \in \mathcal{B}$, which we'll need in several places.
- Prove a more general statement: for any measure μ,

$$\mu(F_1) < \infty,\ F_k \searrow F \implies \lim_{k \to \infty} \mu(F_k) = \mu(F),$$

  where $F_k \searrow F$ means $F_1 \supseteq F_2 \supseteq \cdots$ with $\bigcap_{k=1}^{\infty} F_k = F$.
  - Note that $\mu(F)$ makes sense: each $F_k \in \mathcal{B}$, which is a σ-algebra and closed under countable intersections.
- Take disjoint annuli by setting $E_k := F_k \setminus F_{k+1}$

- Funny step: write

$$F_1 = F\coprod \coprod_{k=1}^{\infty} E_k.$$

  - This is because $x \in F_1$ iff x is in every F_k, so in F, **or**
  - $x \notin F_1$ but $x \in F_2$, noting incidentally $x \in F_3, F_4, \cdots$, **or**,
  - $x \notin F_2$ but $x \in F_3$, and so on.

- Now take measures, and note that we get a telescoping sum:

$$\begin{aligned}
\mu(F_1) &= \mu(F) + \sum_{k=1}^{\infty} \mu(E_k) \\
&= \mu(F) + \lim_{N\to\infty} \sum_{k=1}^{N} \mu(E_k) \\
&:= \mu(F) + \lim_{N\to\infty} \sum_{k=1}^{N} \mu(F_k \setminus F_{k+1}) \\
&:= \mu(F) + \lim_{N\to\infty} \sum_{k=1}^{N} \mu(F_k) - \mu(F_{k+1}) && \text{to be justified} \\
&= \mu(F) + \lim_{N\to\infty} [(\mu(F_1) - \mu(F_2)) + (\mu(F_2) - \mu(F_3)) + \cdots \\
&\qquad\qquad + (\mu(F_{N-1}) - \mu(F_N)) + (\mu(F_N) - \mu(F_{N+1}))] \\
&= \mu(F) + \lim_{N\to\infty} \mu(F_1) - \mu(F_{N+1}) \\
&= \mu(F) + \mu(F_1) - \lim_{N\to\infty} \mu(F_{N+1}).
\end{aligned}$$

- Justifying the measure subtraction: the general statement is that for any pair of sets $A \subseteq X$, $\mu(X \setminus A) = \mu(X) - \mu(A)$ when $\mu(A) < \infty$:

$$\begin{aligned}
X &= A\coprod(X \setminus A) \\
\implies \mu(X) &= \mu(A) + \mu(X \setminus A) && \text{countable additivity} \\
\implies \mu(X) - \mu(A) &= \mu(X \setminus A) && \text{if } \mu(A) < \infty.
\end{aligned}$$

- Now use that $\mu(F_1) < \infty$ to justify subtracting it from both sides:

$$\begin{aligned}
\mu(F_1) &= \mu(F) + \mu(F_1) - \lim_{N\to\infty} \mu(F_{N+1}) \\
\implies 0 &= \mu(F_1) - \lim_{N\to\infty} \mu(F_{N+1}) \\
\lim_{N\to\infty} \mu(F_{N+1}) &= \mu(F_1).
\end{aligned}$$

- Now use that $\lim_{N\to\infty} \mu(F_{N+1}) = \lim_{N\to\infty} \mu(F_N)$ to conclude.

$\blacksquare$

Proof (of b).

- Toward a contradiction, negate the implication: there exists an $\varepsilon > 0$ such that for all δ, there exists an $E \in \mathcal{B}$
$$m(E) < \delta \qquad \text{but} \qquad \mu(E) > \varepsilon.$$
  - **Goal**: produce a set A with $m(A) = 0$ **but** $\mu(A) \neq 0$.
- Take a sequence $\delta_n = \alpha(n)$, some function to be determined later, produce sets E_n with
$$m(E_n) < \delta_n \qquad \text{but} \qquad \mu(E_n) > \varepsilon \quad \forall n.$$
- Set
$$A_M \coloneqq \bigcap_{N=1}^{M} \bigcup_{n=N}^{\infty} E_n \coloneqq \bigcap_{N=1}^{M} F_N \qquad F_N \coloneqq \bigcup_{n=N}^{\infty} E_n.$$
  - Observation: $F_N \supseteq F_{N+1}$ for all N, since the right-hand side involves taking a union over *fewer* sets.
  - Notation: define
$$A_\infty \coloneqq \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n.$$
- Bounding the Lebesgue measure m from above:
$$\begin{aligned}
m(A_\infty) \coloneqq m\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n\right) & \\
\leq m\left(\bigcup_{n=N}^{\infty} E_n\right) & \qquad \forall N \\
\leq \sum_{n=N}^{\infty} m(E_n) & \qquad \forall N \quad \text{by countable subadditivity} \\
\leq \sum_{n=N}^{\infty} \alpha(n) & \\
\overset{N\to\infty}{\longrightarrow} 0, &
\end{aligned}$$
where we've used that intersecting over *fewer* sets (i.e. none) can only increase measure in the first bound.
  - We have control over the sequence $\alpha(n)$, so we can choose it to be summable so that the tails converge to zero as rapidly as we'd like.
  - So e.g. for any $\varepsilon_1 > 0$, we can choose $\alpha(n) \coloneqq \varepsilon_1/2^n$, then
$$\sum_{n=N}^{\infty} \alpha(n) \leq \sum_{n=1}^{\infty} \frac{\varepsilon_1}{2^n} = \varepsilon_1 \to 0.$$

- Bounding the μ measure from below:

$$
\begin{aligned}
\mu(A_\infty) &\coloneqq \mu\left(\bigcap_{N=1}^\infty F_N\right) \\
&= \lim_{N\to\infty} \mu(F_N) && \text{by part (1)} \\
&= \lim_{N\to\infty} \mu\left(\bigcup_{n=N}^\infty E_n\right) \\
&\geq \lim_{N\to\infty} \mu(E_N) \\
&\geq \lim_{N\to\infty} \varepsilon \\
&= \varepsilon \\
&> 0,
\end{aligned}
$$

where we've used that taking a union over *fewer* sets can only make the measure smaller.

■

4.5 Fall 2018.2

Let $E \subset \mathbf{R}$ be a Lebesgue measurable set. Show that there is a Borel set $B \subset E$ such that $m(E \setminus B) = 0$.

Move this to review notes to clean things up.

What a mess, redo!!

Concepts Used:

- Definition of measurability: there exists an open $O \supset E$ such that $m_*(O \setminus E) < \varepsilon$ for all $\varepsilon > 0$.
- Theorem: E is Lebesgue measurable iff there exists a closed set $F \subseteq E$ such that $m_*(E \setminus F) < \varepsilon$ for all $\varepsilon > 0$.
- Every F_σ, G_δ is Borel.
- Claim: E is measurable $\iff$ for every ε there exist $F_\varepsilon \subset E \subset G_\varepsilon$ with F_ε closed and G_ε open and $m(G_\varepsilon \setminus E) < \varepsilon$ and $m(E \setminus F_\varepsilon) < \varepsilon$.
  - Proof: existence of G_ε is the definition of measurability.
  - Existence of F_ε: ?
- Claim: E is measurable $\implies$ there exists an open $O \supseteq E$ such that $m(O \setminus E) = 0$.
  - Since E is measurable, for each $n \in \mathbb{N}$ choose $G_n \supseteq E$ such that $m_*(G_n \setminus E) < \frac{1}{n}$.
  - Set $O_N \coloneqq \bigcap_{n=1}^N G_n$ and $O \coloneqq \bigcap_{n=1}^\infty G_n$.

- Suppose E is bounded.
  - ◇ Note $O_N \searrow O$ and $m_*(O_1) < \infty$ if E is bounded, since in this case

$$m_*(G_n \setminus E) = m_*(G_1) - m_*(E) < 1 \iff m_*(G_1) < m_*(E) + \frac{1}{n} < \infty.$$

  - ◇ Note $O_N \setminus E \searrow O \setminus E$ since $O_N \setminus E := O_N \bigcap E^c \supseteq O_{N+1} \bigcap E^c$ for all N, and again $m_*(O_1 \setminus E) < \infty$.
  - ◇ So it's valid to apply continuity of measure from above:

$$\begin{aligned} m_*(O \setminus E) &= \lim_{N\to\infty} m_*(O_N \setminus E) \\ &\leq \lim_{N\to\infty} m_*(G_N \setminus E) \\ &= \lim_{N\to\infty} \frac{1}{N} = 0, \end{aligned}$$

    where the inequality uses subadditivity on $\bigcap_{n=1}^N G_n \subseteq G_N$
- Suppose E is unbounded.
  - ◇ Write $E^k = E \bigcap [k, k+1]^d \subset \mathbf{R}^d$ as the intersection of E with an annulus, and note that $E = \coprod_{k\in\mathbb{N}} E_k$.
  - ◇ Each E_k is bounded, so apply the previous case to obtain $O_k \supseteq E_k$ with $m(O_k \setminus E_k) = 0$.
  - ◇ So write $O_k = E_k \coprod N_k$ where $N_k := O_k \setminus E_k$ is a null set.
  - ◇ Define $O = \bigcup_{k\in\mathbb{N}} O_k$, note that $E \subseteq O$.
  - ◇ Now note

$$\begin{aligned} O \setminus E &= (\textstyle\coprod_k O_k) \setminus (\coprod_K E_k) \\ &\subseteq \textstyle\coprod_k (O_k \setminus E_k) \\ \implies m_*(O \setminus E) &\leq m_*\left(\textstyle\coprod (O_k \setminus E_k)\right) = 0, \end{aligned}$$

    since any countable union of null sets is again null.
- So $O \supseteq E$ with $m(O \setminus E) = 0$.

• Theorem: since E is measurable, E^c is measurable

- Proof: It suffices to write E^c as the union of two measurable sets, $E^c = S \bigcup (E^c - S)$, where S is to be determined.
- We'll produce an S such that $m_*(E^c - S) = 0$ and use the fact that any subset of a null set is measurable.
- Since E is measurable, for every $\varepsilon > 0$ there exists an open $\mathcal{O}_\varepsilon \supseteq E$ such that $m_*(\mathcal{O}_\varepsilon \setminus E) < \varepsilon$.
- Take the sequence $\left\{ \varepsilon_n := \frac{1}{n} \right\}$ to produce a sequence of sets $\mathcal{O}_n$.
- Note that each $\mathcal{O}_n^c$ is closed and

$$\mathcal{O}_n \supseteq E \iff \mathcal{O}_n^c \subseteq E^c.$$

- Set $S := \bigcup_n \mathcal{O}_n^c$, which is a union of closed sets, thus an F_σ set, thus Borel, thus measurable.
- Note that $S \subseteq E^c$ since each $\mathcal{O}_n \subseteq E^c$.

- Note that

$$
\begin{aligned}
E^c \setminus S &:= E^c \setminus \left(\bigcup_{n=1}^\infty \mathcal{O}_n^c \right) \\
&:= E^c \bigcap \left(\bigcup_{n=1}^\infty \mathcal{O}_n^c \right)^c && \text{definition of set minus} \\
&= E^c \bigcap \left(\bigcap_{n=1}^\infty \mathcal{O}_n \right)^c && \text{De Morgan's law} \\
&= E^c \bigcup \left(\bigcap_{n=1}^\infty \mathcal{O}_n \right) \\
&:= \left(\bigcap_{n=1}^\infty \mathcal{O}_n \right) \setminus E \\
&\subseteq \mathcal{O}_N \setminus E \quad \text{for every } N \in \mathbb{N}.
\end{aligned}
$$

- Then by subadditivity,

$$
m_*(E^c \setminus S) \leq m_*(\mathcal{O}_N \setminus E) \leq \frac{1}{N} \quad \forall N \implies m_*(E^c \setminus S) = 0.
$$

- Thus $E^c \setminus S$ is measurable.

Solution:

- Since E is measurable, E^c is measurable.
- Since E^c is measurable exists an open $O \supseteq E^c$ such that $m(O \setminus E^c) = 0$.
- Set $B := O^c$, then $O \supseteq E^c \iff \mathcal{O}^c \subseteq E \iff B \subseteq E$.
- Computing measures yields

$$
E \setminus B := E \setminus \mathcal{O}^c := E \bigcap (\mathcal{O}^c)^c = E \bigcap \mathcal{O} = \mathcal{O} \bigcap (E^c)^c := \mathcal{O} \setminus E^c,
$$

  thus $m(E \setminus B) = m(\mathcal{O} \setminus E^c) = 0$.
- Since $\mathcal{O}$ is open, B is closed and thus Borel.

d.irect Proof (Todo)

Try to construct the set.

4.6 Spring 2018.1

Define

$$
E := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.
$$

Prove that $m(E) = 0$.

Concepts Used:

- Borel-Cantelli: If $\{E_k\}_{k\in\mathbf{Z}} \subset 2^{\mathbf{R}}$ is a countable collection of Lebesgue measurable sets with $\sum_{k\in\mathbf{Z}} m(E_k) < \infty$, then almost every $x \in \mathbf{R}$ is in *at most finitely* many E_k.
  - Equivalently (?), $m(\limsup_{k\to\infty} E_k) = 0$, where $\limsup_{k\to\infty} E_k = \bigcap_{k=1}^\infty \bigcup_{j\geq k} E_j$, the elements which are in E_k for infinitely many k.

Solution:

- Strategy: Borel-Cantelli.
- We'll show that $m(E)\bigcap[n, n+1] = 0$ for all $n \in \mathbf{Z}$; then the result follows from

$$m(E) = m\left(\bigcup_{n\in\mathbf{Z}} E\bigcap[n, n+1]\right) \leq \sum_{n=1}^{\infty} m(E\bigcap[n, n+1]) = 0.$$

- By translation invariance of measure, it suffices to show $m(E\bigcap[0, 1]) = 0$.
  - So WLOG, replace E with $E\bigcap[0, 1]$.
- Define

$$E_j := \left\{x \in [0,1] \;\middle|\; \exists p \in \mathbf{Z}^{\geq 0} \text{ s.t. } \left|x - \frac{p}{j}\right| < \frac{1}{j^3}\right\}.$$

  - Note that $E_j \subseteq \coprod_{p\in\mathbf{Z}^{\geq 0}} B_{j^{-3}}\left(\frac{p}{j}\right)$, i.e. a union over integers p of intervals of radius $1/j^3$ around the points p/j. Since $1/j^3 < 1/j$, this union is in fact disjoint.
- Importantly, note that

$$\limsup_{j\to\infty} E_j := \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j = E$$

  since

$$\begin{aligned}
x \in \limsup_j E_j &\iff x \in E_j \text{ for infinitely many } j \\
&\iff \text{there are infinitely many } j \text{ for which there exist a } p \text{ such that } \left|x - \frac{p}{j}\right| < j^{-3} \\
&\iff \text{there are infinitely many such pairs } p, j \\
&\iff x \in E.
\end{aligned}$$

- Intersecting with $[0, 1]$, we can write E_j as a union of intervals:

$$E_j = \left(0, j^{-3}\right) \quad \coprod \quad B_{j^{-3}}\left(\frac{1}{j}\right) \coprod B_{j^{-3}}\left(\frac{2}{j}\right) \coprod \cdots \coprod B_{j^{-3}}\left(\frac{j-1}{j}\right) \quad \coprod \quad (1 - j^{-3}, 1),$$

  where we've separated out the "boundary" terms to emphasize that they are balls about 0 and 1 intersected with $[0, 1]$.

- Since E_j is a union of open sets, it is Borel and thus Lebesgue measurable.
- Computing the measure of E_j:
  - For a fixed j, there are exactly $j+1$ possible choices for a numerator $(0, 1, \cdots, j)$, thus there are exactly $j+1$ sets appearing in the above decomposition.
  - The first and last intervals are length $\frac{1}{j^3}$
  - The remaining $(j+1)-2 = j-1$ intervals are twice this length, $\frac{2}{j^3}$
  - Thus

$$m(E_j) = 2\left(\frac{1}{j^3}\right) + (j-1)\left(\frac{2}{j^3}\right) = \frac{2}{j^2}$$

- Note that

$$\sum_{j \in \mathbb{N}} m(E_j) = 2 \sum_{j \in \mathbb{N}} \frac{1}{j^2} < \infty,$$

  which converges by the p-test for sums.
- But then

$$\begin{aligned} m(E) &= m(\limsup_j E_j) \\ &= m\Big(\bigcap_{n \in \mathbb{N}} \bigcup_{j \geq n} E_j\Big) \\ &\leq m\Big(\bigcup_{j \geq N} E_j\Big) \quad \text{for every } N \\ &\leq \sum_{j \geq N} m(E_j) \\ &\overset{N \to \infty}{\to} 0 \quad . \end{aligned}$$

- Thus E is measurable as a subset of a null set and $m(E) = 0$.

4.7 Fall 2017.2

Let $f(x) = x^2$ and $E \subset [0, \infty) := \mathbf{R}^+$.

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

$$\begin{aligned}\varphi: \mathcal{L}(\mathbf{R}^+) &\to \mathcal{L}(\mathbf{R}^+) \\ E &\mapsto f(E)\end{aligned}$$

is a bijection from the class of Lebesgue measurable sets of $[0, \infty)$ to itself.

Walk through.

Solution:

a.

It suffices to consider the bounded case, i.e. $E \subseteq B_M(0)$ for some M. Then write $E_n = B_n(0) \bigcap E$ and apply the theorem to E_n, and by subadditivity, $m^*(E) = m^*(\bigcup_n E_n) \leq \sum_n m^*(E_n) = 0$.
Lemma: $f(x) = x^2, f^{-1}(x) = \sqrt{x}$ are Lipschitz on any compact subset of $[0, \infty)$.
Proof: Let $g = f$ or f^{-1}. Then $g \in C^1([0, M])$ for any M, so g is differentiable and g' is continuous. Since g' is continuous on a compact interval, it is bounded, so $|g'(x)| \leq L$ for all x. Applying the MVT,

$$|f(x) - f(y)| = f'(c)|x - y| \leq L|x - y|.$$

Lemma: If g is Lipschitz on $\mathbf{R}^n$, then $m(E) = 0 \implies m(g(E)) = 0$.
Proof: If g is Lipschitz, then

$$g(B_r(x)) \subseteq B_{Lr}(x),$$

which is a dilated ball/cube, and so

$$m^*(B_{Lr}(x)) \leq L^n \cdot m^*(B_r(x)).$$

Now choose $\{Q_j\} \rightrightarrows E$; then $\{g(Q_j)\} \rightrightarrows g(E)$.
By the above observation,

$$|g(Q_j)| \leq L^n |Q_j|,$$

and so

$$m^*(g(E)) \leq \sum_j |g(Q_j)| \leq \sum_j L^n |Q_j| = L^n \sum_j |Q_j| \to 0.$$

Now just take $g(x) = x^2$ for one direction, and $g(x) = f^{-1}(x) = \sqrt{x}$ for the other.

b.

Lemma: E is measurable iff $E = K \coprod N$ for some K compact, N null.

Write $E = K \coprod N$ where K is compact and N is null.
Then $\varphi^{-1}(E) = \varphi^{-1}(K \coprod N) = \varphi^{-1}(K) \coprod \varphi^{-1}(N)$.
Since $\varphi^{-1}(N)$ is null by part (a) and $\varphi^{-1}(K)$ is the preimage of a compact set under a continuous map and thus compact, $\varphi^{-1}(E) = K' \coprod N'$ where K' is compact and N' is null, so $\varphi^{-1}(E)$ is measurable.
So φ is a measurable function, and thus yields a well-defined map $\mathcal{L}(\mathbf{R}) \to \mathcal{L}(\mathbf{R})$ since it preserves measurable sets. Restricting to $[0, \infty)$, f is bijection, and thus so is φ.

4.8 Spring 2017.1

Let K be the set of numbers in $[0,1]$ whose decimal expansions do not use the digit 4.

We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 *with* $399\cdots$. *For example,* $0.8754 = 0.8753999\cdots$.

Show that K is a compact, nowhere dense set without isolated points, and find the Lebesgue measure $m(K)$.

Concepts Used:

- Definition: A is *nowhere dense* $\iff$ every interval I contains a subinterval $S \subseteq A^c$.
  - Equivalently, the interior of the closure is empty, $\left(\overline{K}\right)^\circ = \emptyset$.

Solution:
Claim: K **is compact**.

- It suffices to show that $K^c := [0,1] \setminus K$ is open; Then K will be a closed and bounded subset of $\mathbf{R}$ and thus compact by Heine-Borel.
- Strategy: write K^c as the union of open balls (since these form a basis for the Euclidean topology on $\mathbf{R}$).
  - Do this by showing every point $x \in K^c$ is an interior point, i.e. x admits a neighborhood N_x such that $N_x \subseteq K^c$.
- Identify K^c as the set of real numbers in $[0,1]$ whose decimal expansion **does** contain a 4.
  - We will show that there exists a neighborhood small enough such that all points in it contain a 4 in their decimal expansions.
- Let $x \in K^c$, suppose a 4 occurs as the kth digit, and write

$$x = 0.d_1 d_2 \cdots d_{k-1}\ 4\ d_{k+1} \cdots = \left(\sum_{j=1}^{k} d_j 10^{-j} \right) + \left(4 \cdot 10^{-k} \right) + \left(\sum_{j=k+1}^{\infty} d_j 10^{-j} \right).$$

- Set $r_x < 10^{-k}$ and let $y \in [0,1] \bigcap B_{r_x}(x)$ be arbitrary and write

$$y = \sum_{j=1}^{\infty} c_j 10^{-j}.$$

- Thus $|x-y| < r_x < 10^{-k}$, and the first k digits of x and y must agree:

- We first compute the difference:

$$x - y = \sum_{i=1}^{\infty} d_j 10^{-j} - \sum_{i=1}^{\infty} c_j 10^{-j} = \sum_{i=1}^{\infty} (d_j - c_j)\, 10^{-j}$$

- Thus (claim)

$$|x - y| \leq \sum_{j=1}^{\infty} |d_j - c_j| 10^{j} < 10^{-k} \iff |d_j - c_j| = 0 \quad \forall j \leq k.$$

- Otherwise we can note that any term $|d_j - c_j| \geq 1$ and there is a contribution to $|x - y|$ of at least $1 \cdot 10^{-j}$ for some $j < k$, whereas

$$j < k \iff 10^{-j} > 10^{-k},$$

a contradiction.

- This means that for all $j \leq k$ we have $d_j = c_j$, and in particular $d_k = 4 = c_k$, so y has a 4 in its decimal expansion.
- But then $K^c = \bigcup_x B_{r_x}(x)$ is a union of open sets and thus open.

Claim: K **is nowhere dense and** $m(K) = 0$**:**

- Strategy: Show $\left(\overline{K}\right)^{\circ} = \emptyset$.
- Since K is closed, $\overline{K} = K$, so it suffices to show that K does not properly contain any interval.
- It suffices to show $m(K^c) = 1$, since this implies $m(K) = 0$ and since any interval has strictly positive measure, this will mean K can not contain an interval.
- As in the construction of the Cantor set, let
  - K_0 denote $[0, 1]$ with 1 interval $\left(\frac{4}{10}, \frac{5}{10}\right)$ of length $\frac{1}{10}$ deleted, so

$$m(K_0^c) = \frac{1}{10}.$$

  - K_1 denote K_0 with 9 intervals $\left(\frac{1}{100}, \frac{5}{100}\right), \left(\frac{14}{100}, \frac{15}{100}\right), \cdots \left(\frac{94}{100}, \frac{95}{100}\right)$ of length $\frac{1}{100}$ deleted, so

$$m(K_1^c) = \frac{1}{10} + \frac{9}{100}.$$

  - K_n denote K_{n-1} with 9^n such intervals of length $\frac{1}{10^{n+1}}$ deleted, so

$$m(K_n^c) = \frac{1}{10} + \frac{9}{100} + \cdots + \frac{9^n}{10^{n+1}}.$$

- Then compute

$$m(K^c) = \sum_{j=0}^{\infty} \frac{9^n}{10^{n+1}} = \frac{1}{10} \sum_{j=0}^{\infty} \left(\frac{9}{10}\right)^n = \frac{1}{10} \left(\frac{1}{1 - \frac{9}{10}}\right) = 1.$$

Claim: K **has no isolated points**:

- A point $x \in K$ is isolated iff there there is an open ball $B_r(x)$ containing x such that $B_r(x) \subsetneq K^c$.
  - So every point in this ball **should** have a 4 in its decimal expansion.
- Strategy: show that if $x \in K$, every neighborhood of x intersects K.
- Note that $m(K_n) = \left(\frac{9}{10}\right)^n \overset{n\to\infty}{\to} 0$
- Also note that we deleted open intervals, and the endpoints of these intervals are never deleted.
  - Thus endpoints of deleted intervals are elements of K.
- Fix x. Then for every ε, by the Archimedean property of $\mathbf{R}$, choose n such that $\left(\frac{9}{10}\right)^n < \varepsilon$.
- Then there is an endpoint x_n of some deleted interval I_n satisfying

$$|x - x_n| \leq \left(\frac{9}{10}\right)^n < \varepsilon.$$

- So every ball containing x contains some endpoint of a removed interval, and thus an element of K.

4.9 Spring 2017.2

a. Let μ be a measure on a measurable space $(X, \mathcal{M})$ and f a positive measurable function.

Define a measure λ by

$$\lambda(E) := \int_E f \, d\mu, \quad E \in \mathcal{M}$$

Show that for g any positive measurable function,

$$\int_X g \, d\lambda = \int_X fg \, d\mu$$

b. Let $E \subset \mathbf{R}$ be a measurable set such that

$$\int_E x^2 \, dm = 0.$$

Show that $m(E) = 0$.

Concepts Used:

- Absolute continuity of measures: $\lambda \ll \mu \iff E \in \mathcal{M}, \mu(E) = 0 \implies \lambda(E) = 0$.
- Radon-Nikodym: if $\lambda \ll \mu$, then there exists a measurable function $\frac{\partial \lambda}{\partial \mu} := f$ where $\lambda(E) = \int_E f \, d\mu$.
- Chebyshev's inequality:

$$A_c := \left\{ x \in X \;\middle|\; |f(x)| \geq c \right\} \implies \mu(A_c) \leq c^{-p} \int_{A_c} |f|^p \, d\mu \quad \forall 0 < p < \infty.$$

Solution:

a.

- Strategy: use approximation by simple functions to show absolute continuity and apply Radon-Nikodym
- Claim: $\lambda \ll \mu$, i.e. $\mu(E) = 0 \implies \lambda(E) = 0$.
  - Note that if this holds, by Radon-Nikodym, $f = \frac{\partial \lambda}{\partial \mu} \implies d\lambda = f d\mu$, which would yield

$$\int g \, d\lambda = \int gf \, d\mu.$$

- So let E be measurable and suppose $\mu(E) = 0$.
- Then

$$\lambda(E) := \int_E f \, d\mu = \lim_{n\to\infty} \left\{ \int_E s_n \, d\mu \;\middle|\; s_n := \sum_{j=1}^{\infty} c_j \mu(E_j),\ s_n \nearrow f \right\}$$

  where we take a sequence of simple functions increasing to f.
- But since each $E_j \subseteq E$, we must have $\mu(E_j) = 0$ for any such E_j, so every such s_n must be zero and thus $\lambda(E) = 0$.

What is the final step in this approximation?

b.

- Set $g(x) = x^2$, note that g is positive and measurable.
- By part (a), there exists a positive f such that for any $E \subseteq \mathbf{R}$,

$$\int_E g \, dm = \int_E gf \, d\mu$$

  - The LHS is zero by assumption and thus so is the RHS.

- $m \ll \mu$ by construction.
- Note that gf is positive.

- Define $A_k = \left\{ x \in X \;\middle|\; gf \cdot \chi_E > \frac{1}{k} \right\}$, for $k \in \mathbf{Z}^{\geq 0}$
- Then by Chebyshev with $p = 1$, for every k we have

$$\mu(A_k) \leq k \int_E gf \; d\mu = 0$$

- Then noting that $A_k \searrow A := \left\{ x \in X \;\middle|\; gf \cdot \chi_E(x) > 0 \right\}$, we have $\mu(A) = 0$.
- Since gf is positive, we have

$$x \in E \iff gf\chi_E(x) > 0 \iff x \in A$$

so $E = A$ and $\mu(E) = \mu(A)$.
- But $m \ll \mu$ and $\mu(E) = 0$, so we can conclude that $m(E) = 0$.

4.10 Fall 2016.4

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{E_n\} \subset \mathcal{M}$ satisfies

$$\lim_{n \to \infty} \mu\left(X \backslash E_n\right) = 0.$$

Define

$$G := \left\{ x \in X \;\middle|\; x \in E_n \text{ for only finitely many } n \right\}.$$

Show that $G \in \mathcal{M}$ and $\mu(G) = 0$.

Add concepts.

Solution:

- Claim: $G \in \mathcal{M}$.
  - Claim:

$$G = \left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n \right)^c = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n^c.$$

- ◇ This follows because x is in the RHS $\iff x \in E_n^c$ for all but finitely many n $\iff x \in E_n$ for at most finitely many n.
- – But $\mathcal{M}$ is a σ-algebra, and this shows G is obtained by countable unions/intersections/complements of measurable sets, so $G \in \mathcal{M}$.

- Claim: $\mu(G) = 0$.
  - We have

$$\begin{aligned}
\mu(G) &= \mu\left(\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n^c\right) \\
&\leq \sum_{N=1}^{\infty} \mu\left(\bigcap_{n=N}^{\infty} E_n^c\right) \\
&\leq \sum_{N=1}^{\infty} \mu(E_M^c) \\
&:= \sum_{N=1}^{\infty} \mu(X \setminus E_N) \\
&\overset{N\to\infty}{\to} 0.
\end{aligned}$$

Last step seems wrong!

4.11 Spring 2016.3

Let f be Lebesgue measurable on $\mathbf{R}$ and $E \subset \mathbf{R}$ be measurable such that

$$0 < A = \int_E f(x)dx < \infty.$$

Show that for every $0 < t < 1$, there exists a measurable set $E_t \subset E$ such that

$$\int_{E_t} f(x)dx = tA.$$

4.12 Spring 2016.5

Let $(X, \mathcal{M}, \mu)$ be a measure space. For $f \in L^1(\mu)$ and $\lambda > 0$, define

$$\varphi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that φ, ψ are Borel measurable and

$$\int_X |f| \, d\mu = \int_0^\infty [\varphi(\lambda) + \psi(\lambda)] \, d\lambda$$

4.13 Spring 2016.2

Let $0 < \lambda < 1$ and construct a Cantor set C_λ by successively removing middle intervals of length λ.

Prove that $m(C_\lambda) = 0$.

4.14 Fall 2015.2

Let $f : \mathbf{R} \to \mathbf{R}$ be Lebesgue measurable.

1. Show that there is a sequence of simple functions $s_n(x)$ such that $s_n(x) \to f(x)$ for all $x \in \mathbf{R}$.
2. Show that there is a Borel measurable function g such that $g = f$ almost everywhere.

4.15 Spring 2015.3

Let μ be a finite Borel measure on $\mathbf{R}$ and $E \subset \mathbf{R}$ Borel. Prove that the following statements are equivalent:

1. $\forall \varepsilon > 0$ there exists G open and F closed such that
$$F \subseteq E \subseteq G \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$
2. There exists a $V \in G_\delta$ and $H \in F_\sigma$ such that
$$H \subseteq E \subseteq V \quad \text{and} \quad \mu(V \setminus H) = 0$$

4.16 Spring 2014.3

Let $f : \mathbf{R} \to \mathbf{R}$ and suppose

$$\forall x \in \mathbf{R}, \quad f(x) \geq \limsup_{y \to x} f(y)$$

Prove that f is Borel measurable.

4.17 Spring 2014.4

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose f is a measurable function on X. Show that

$$\lim_{n\to\infty} \int_X f^n \, d\mu = \begin{cases} \infty & \text{or} \\ \mu(f^{-1}(1)), \end{cases}$$

and characterize the collection of functions of each type.

5 | Measure Theory: Functions

5.1 Spring 2021.1

Problem 5.1.1 (Spring 2021, 1)
Let $(X, \mathcal{M}, \mu)$ be a measure space and let $E_n \in \mathcal{M}$ be a measurable set for $n \geq 1$. Let $f_n := \chi_{E_n}$ be the indicator function of the set E and show that

a. $f_n \overset{n\to\infty}{\rightrightarrows} 1$ uniformly $\iff$ there exists $N \in \mathbb{N}$ such that $E_n = X$ for all $n \geq N$.

b. $f_n(x) \overset{n\to\infty}{\rightrightarrows} 1$ for almost every $x \iff$

$$\mu\left(\bigcap_{n\geq 0} \bigcup_{k\geq n} (X \setminus E_k)\right) = 0.$$

Solution:
Part a:
$\implies$:

- Suppose $\chi_{E_n} \to 1$ uniformly, we want to produce an N such that $n \geq N \implies x \in E_n$ for all $x \in X$.
- Take $\varepsilon := 1/2$. By uniform convergence, for N large enough,

$$\begin{aligned}
&\forall n \geq N \quad |\chi_{E_n}(x) - 1| < 1/2 && \forall x \in X \\
&\iff \forall n \geq N \quad \chi_{E_n}(x) = 1 && \forall x \in X \\
&\iff \forall n \geq N \quad x \in E_n && \forall x \in X && \iff \forall n \geq N \quad E_n = X,
\end{aligned}$$

where we've used that $E_n \subseteq X$ by definition and this shows $X \subseteq E_n$. So this N suffices.

$\impliedby$:

- Let $\varepsilon > 0$ be arbitrary.

- Choose N such that $n \geq N \implies X = E_n$. Then

$$\begin{aligned} &\forall n \geq N \quad x \in E_n && \forall x \in X \\ &\forall n \geq N \quad \chi_{E_n}(x) = 1 && \forall x \in X \\ &\forall n \geq N \quad |\chi_{E_n}(x) - 1| = 0 < \varepsilon && \forall x \in X, \end{aligned}$$

so $\chi_{E_n} \to 1$ uniformly.

Part b:

- Define

$$\begin{aligned} S &:= \left\{ x \in X \;\middle|\; \chi_{E_k}(x) \to 1 \right\} \\ &:= \left\{ x \in X \;\middle|\; \forall \varepsilon, \exists N \text{ s.t. } |\chi_{E_k}(x) - 1| < \varepsilon, \forall k \geq N \right\} \\ L &:= \bigcap_{n \geq 0} \bigcup_{k \geq n} (X \setminus E_k), \end{aligned}$$

so S is the set where $f_n \to f$ and $X \setminus S$ is the exceptional set where $f_n \not\to f$ doesn't converge pointwise.

- **Claim**: $L = X \setminus S$, so if $x \in S \iff x \in X \setminus L$.

- Proof of claim: Suppose there exists an N such that the first line below is true. Then for a fixed x, there are equivalent statements:

$$\begin{aligned} & x \in S \\ \iff & \exists N \text{ s.t. } \forall \varepsilon > 0, \quad |\chi_{E_k}(x) - 1| < \varepsilon && \forall k \geq N \\ \iff & \exists N \text{ s.t. } |\chi_{E_k}(x) - 1| = 0 && \forall k \geq N \\ \iff & \exists N \text{ s.t. } \chi_{E_k}(x) = 1 && \forall k \geq N \\ \iff & \exists N \text{ s.t. } x \in E_k && \forall k \geq N \\ \iff & \exists N \text{ s.t. } x \notin X \setminus E_k && \forall k \geq N \\ \iff & \exists N \text{ s.t. } x \notin \bigcup_{k \geq N} X \setminus E_k \\ \iff & x \notin \bigcap_{n \geq 0} \bigcup_{k \geq n} X \setminus E_k \\ \iff & x \notin L \\ \iff & x \in X \setminus L. \end{aligned}$$

- Proving the iff: $f_n \to f$ almost everywhere $\iff \mu(X \setminus S) = 0 \iff \mu(L) = 0$.

5.2 Spring 2021.3

Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $\{f_n\}_{n=1}^{\infty} \subseteq L^1(X, \mu)$. Suppose $f \in L^1(X, \mu)$ such that $f_n(x) \overset{n\to\infty}{\to} f(x)$ for almost every $x \in X$. Prove that for every $\varepsilon > 0$ there exists $M > 0$ and a set $E \subseteq X$ such that $\mu(E) \leq \varepsilon$ and $|f_n(x)| \leq M$ for all $x \in X \setminus E$ and all $n \in \mathbb{N}$.

5.3 Fall 2020.2

a. Let $f : \mathbf{R} \to \mathbf{R}$. Prove that

$$f(x) \leq \liminf_{y\to x} f(y) \text{ for each } x \in \mathbf{R} \iff \left\{x \in \mathbf{R} \,\middle|\, f(x) > a\right\} \text{ is open for all } a \in \mathbf{R}$$

b. Recall that a function $f : \mathbf{R} \to \mathbf{R}$ is called *lower semi-continuous* iff it satisfies either condition in part (a) above.

Prove that if $\mathcal{F}$ is any family of lower semi-continuous functions, then

$$g(x) = \sup\left\{f(x) \,\middle|\, f \in \mathcal{F}\right\}$$

is Borel measurable.

Note that $\mathcal{F}$ need not be a countable family.

5.4 Fall 2016.2

Let $f, g : [a, b] \to \mathbf{R}$ be measurable with

$$\int_a^b f(x)\; dx = \int_a^b g(x)\; dx.$$

Show that either

1. $f(x) = g(x)$ almost everywhere, or
2. There exists a measurable set $E \subset [a, b]$ such that

$$\int_E f(x)\, dx > \int_E g(x)\, dx$$

Concepts Used:

- Monotonicity of the Lebesgue integral: $f \leq g$ on $A \implies \int_A f \leq \int_A g$

Strategy:
Take the assumption and the negation of (1) and show (2). The obvious move: define the set A where they differ. The non-obvious move: split A itself up to get a strict inequality.

Solution:

- Write $X := [a, b]$,
- Suppose it is *not* the case that $f = g$ almost everywhere; then letting $A := \left\{ x \in X \;\middle|\; f(x) \neq g(x) \right\}$, we have $m(A) > 0$.
- Write

$$A = A_1 \coprod A_2 := \{f > g\} \coprod \{f < g\}.$$

- Both A_i are measurable:
  - Since f, g are measurable functions, so is $h := f - g$.
  - We can write

$$\begin{aligned} A_1 &:= \left\{ x \in X \;\middle|\; h > 0 \right\} = h^{-1}((0, \infty)) \\ A_2 &:= \left\{ x \in X \;\middle|\; h < 0 \right\} = h^{-1}((-\infty, 0)), \end{aligned}$$

  and pullbacks of Borel sets by measurable functions are measurable.

- Then on E, we have $f(x) > g(x)$ pointwise. This is preserved by monotonicity of the integral, thus

$$f(x) > g(x) \text{ on } E \implies \int_E f(x)\,dx > \int_E g(x)\,dx.$$

5.5 Spring 2016.4

Let $E \subset \mathbf{R}$ be measurable with $m(E) < \infty$. Define

$$f(x) = m(E \cap (E + x)).$$

Show that

1. $f \in L^1(\mathbf{R})$.
2. f is uniformly continuous.
3. $\lim_{|x| \to \infty} f(x) = 0$.

Hint:

$$\chi_{E\cap(E+x)}(y) = \chi_E(y)\chi_E(y-x)$$

6 | Integrals: Convergence

6.1 Fall 2020.3

Problem 6.1.1 (?)
Let f be a non-negative Lebesgue measurable function on $[1, \infty)$.

a. Prove that

$$1 \leq \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b \frac{1}{f(x)}\,dx\right)$$

for any $1 \leq a < b < \infty$.

b. Prove that if f satisfies

$$\int_1^t f(x)\,dx \leq t^2\log(t)$$

for all $t \in [1, \infty)$, then

$$\int_1^\infty \frac{1}{f(x)}\,dx = \infty.$$

Hint: write

$$\int_1^\infty \frac{1}{f(x)}\,dx = \sum_{k=0}^\infty \int_{2^k}^{2^{k+1}} \frac{1}{f(x)}\,dx.$$

Solution:
Part 1: By Holder with $p = q = 2$ on $L_1[a, b]$,

$$(b-a)^2 = \|\mathrm{id}\|_1^2 = \left\|f^{\frac12}f^{-\frac12}\right\|_1^2 \leq \left\|f^{\frac12}\right\|_2^2 \cdot \left\|f^{-\frac12}\right\|_2^2 = \int_a^b f(x)\,dx \cdot \int_a^b \frac{1}{f(x)}\,dx.$$

Part 2: It suffices to show

$$\int_{2^k}^{2^{k+1}} \frac{1}{f} > c_k \text{ where } \sum_{k\geq 0} c_k = \infty.$$

Manipulate the given inequality a bit:

$$\int_a^b f \leq \int_1^b f \leq b^2 \log(b) \implies \left(\int_a^b f\right)^{-1} \geq \frac{1}{b^2 \log(b)}$$
$$\implies .$$

Rewrite the bound in part 1:

$$\int_a^b \frac{1}{f} \geq \left(\int_a^b f\right)^{-1} (b-a)^2 \geq \frac{(b-a)^2}{b^2 \log(b)}.$$

Now set $a = 2^k, b = 2^{k+1}$:

$$\int_{2^k}^{2^{k+1}} \frac{1}{f(x)}\, dx \geq \frac{(2^{k+1} - 2^k)^2}{2^{2(k+1)}(k+1)\log(2)} = \frac{2^{2k}}{2^{2k} \cdot 4(k+1)\log(2)} = \mathsf{O}(1/k),$$

and $\sum 1/k = \infty$.

6.2 Spring 2021.2

Problem 6.2.1 (?)
Calculate the following limit, justifying each step of your calculation:

$$L := \lim_{n\to\infty} \int_0^n \frac{\cos\left(\frac{x}{n}\right)}{x^2 + \cos\left(\frac{x}{n}\right)}\, dx.$$

Solution:

- If interchanging a limit and integral is justified, we have

$$\begin{aligned}
L &:= \lim_{n\to\infty} \int_{(0,n)} \frac{\cos\left(\frac{x}{n}\right)}{x^2+\cos\left(\frac{x}{n}\right)}\,dx \\
&= \lim_{n\to\infty} \int_{(0,\infty)} \chi_{(0,n)}(x) \frac{\cos\left(\frac{x}{n}\right)}{x^2+\cos\left(\frac{x}{n}\right)}\,dx \\
&\overset{\text{DCT}}{=} \int_{(0,\infty)} \lim_{n\to\infty} \chi_{(0,n)}(x) \frac{\cos\left(\frac{x}{n}\right)}{x^2+\cos\left(\frac{x}{n}\right)}\,dx \\
&= \int_{(0,\infty)} \chi_{(0,\infty)}(x) \lim_{n\to\infty} \frac{\cos\left(\frac{x}{n}\right)}{x^2+\cos\left(\frac{x}{n}\right)}\,dx \\
&= \int_{(0,\infty)} \frac{\lim_{n\to\infty}\cos\left(\frac{x}{n}\right)}{\lim_{n\to\infty} x^2+\cos\left(\frac{x}{n}\right)}\,dx \\
&= \int_{(0,\infty)} \frac{\cos\left(\lim_{n\to\infty}\frac{x}{n}\right)}{x^2+\cos\left(\lim_{n\to\infty}\frac{x}{n}\right)}\,dx \\
&= \int_{(0,\infty)} \frac{1}{x^2+1}\,dx \\
&= \arctan(x)\Big|_0^\infty \\
&= \frac{\pi}{2},
\end{aligned}$$

  where we've used that $\cos(\theta)$ is continuous on $\mathbf{R}$ to pass a limit inside, noting that x is fixed in the integrand.

- Justifying the interchange: DCT. Write $f_n(x) := \cos(x/n)/(x^2+\cos(x/n))$.
- On (α,∞) for any $\alpha>1$:
  - We have

$$|f_n(x)| \leq \left|\frac{1}{x^2+\cos(x/n)}\right| \leq \frac{1}{x^2-1},$$

    where we've used that $-1\leq\cos(x/n)\leq 1$ for every x, and so the denominator is minimized when $\cos(x/n)=-1$, and this maximizes the quantity.
  - Setting $g(x) := 1/(x^2-1)$, we have $g\in L^1(\alpha,\infty)$ by the limit comparison test with $h(x) := x^2$:

$$\frac{g(x)}{h(x)} := \frac{x^2-1}{x^2} = 1-\frac{1}{x^2} \overset{x\to\infty}{\longrightarrow} 1 < \infty,$$

    and so g, h either both converge or both diverge. But $\int_\alpha^\infty \frac{1}{x^2}\,dx < \infty$ by the p-test for integrals since $\alpha > 1$.

- On $(0,\alpha)$:
  - Just use that $f_n(x)$ is bounded by a constant:

$$|f_n(x)| = \left|\frac{\cos(x/n)}{x^2+\cos(x/n)}\right| \leq \left|\frac{\cos(x/n)}{\cos(x/n)}\right| = 1,$$

    where we've used that x^2 is positive, and removing it from the denominator only makes the quantity larger.

– Then check that $\int_0^\alpha 1\,dx = \alpha < \infty$, so $1 \in L^1(0,\alpha)$.

6.3 Spring 2021.5

Problem 6.3.1 (?)
Let $f_n \in L^2([0,1])$ for $n \in \mathbb{N}$, and assume that

- $\|f_n\|_2 \leq n^{\frac{-51}{100}}$ for all $n \in \mathbb{N}$,
- $\widehat{f}_n$ is supported in the interval $[2^n, 2^{n+1}]$, so

$$\widehat{f}_n(\xi) := \int_0^1 f_n(x) e^{2\pi i \xi \cdot x}\,dx = 0 \qquad\qquad \text{for } \xi \notin [2^n, 2^{n+1}].$$

Prove that $\sum_{n\in\mathbb{N}} f_n$ converges in the Hilbert space $L^2([0,1])$.

Hint: Plancherel's identity may be helpful.

Warning 6.3.1
Although this mentions Plancherel, probably what is needed is Parseval's identity:

$$\sum_{k\in\mathbf{Z}} \left|\widehat{f}(k)\right|^2 = \int_0^1 |f(x)|^2\,dx.$$

6.4 Fall 2019.2

Prove that

$$\left|\frac{d^n}{dx^n}\frac{\sin x}{x}\right| \leq \frac{1}{n}$$

for all $x \neq 0$ and positive integers n.

Hint: Consider $\int_0^1 \cos(tx)dt$

Solution:

Concepts Used:

- DCT
- Bounding in the right place. Don't evaluate the actual integral!

- By induction on the number of limits we can pass through the integral.
- For $n = 1$ we first pass one derivative into the integral: let $x_n \to x$ be any sequence converging to x, then

$$\begin{aligned}
\frac{\partial}{\partial x} \frac{\sin(x)}{x} &= \frac{\partial}{\partial x} \int_0^1 \cos(tx)\,dt \\
&= \lim_{x_n \to x} \frac{1}{x_n - x} \left(\int_0^1 \cos(tx_n)\,dt - \int_0^1 \cos(tx)\,dt \right) \\
&= \lim_{x_n \to x} \left(\int_0^1 \frac{\cos(tx_n) - \cos(tx)}{x_n - x}\,dt \right) \\
&= \lim_{x_n \to x} \left(\int_0^1 \left(t\sin(tx) \Big|_{x=\xi_n} \right) dt \right) \quad \text{where} \quad \xi_n \in [x_n, x] \text{ by MVT}, \xi_n \to x \\
&= \lim_{\xi_n \to x} \left(\int_0^1 t\sin(t\xi_n)\,dt \right) \\
&=_{\text{DCT}} \int_0^1 \lim_{\xi_n \to x} t\sin(t\xi_n)\,dt \\
&= \int_0^1 t\sin(tx)\,dt
\end{aligned}$$

.

- Taking absolute values we obtain an upper bound

$$\begin{aligned}
\left| \frac{\partial}{\partial x} \frac{\sin(x)}{x} \right| &= \left| \int_0^1 t\sin(tx)\,dt \right| \\
&\leq \int_0^1 |t\sin(tx)|\,dt \\
&\leq \int_0^1 1\,dt = 1,
\end{aligned}$$

since $t \in [0,1] \implies |t| < 1$, and $|\sin(xt)| \leq 1$ for any x and t.

- Note that this bound also justifies the DCT, since the functions $f_n(t) = t\sin(t\xi_n)$ are uniformly dominated by $g(t) = 1$ on $L^1([0,1])$.

Note: integrating by parts here yields the actual formula:

$$\begin{aligned}\int_0^1 t\sin(tx)\,dt &=_{IBP} \left(\frac{-t\cos(tx)}{x}\right)\Big|_{t=0}^{t=1} - \int_0^1 \frac{\cos(tx)}{x}\,dt \\ &= \frac{-\cos(x)}{x} - \frac{\sin(x)}{x^2} \\ &= \frac{x\cos(x)-\sin(x)}{x^2}.\end{aligned}$$

- For the inductive step, we assume that we can pass $n-1$ limits through the integral and show we can pass the nth through as well.

$$\begin{aligned}\frac{\partial^n}{\partial x^n}\frac{\sin(x)}{x} &= \frac{\partial^n}{\partial x^n}\int_0^1 \cos(tx)\,dt \\ &= \frac{\partial}{\partial x}\int_0^1 \frac{\partial^{n-1}}{\partial x^{n-1}}\cos(tx)\,dt \\ &= \frac{\partial}{\partial x}\int_0^1 t^{n-1} f_{n-1}(x,t)\,dt\end{aligned}$$

  - Note that $f_n(x,t) = \pm\sin(tx)$ when n is odd and $f_n(x,t) = \pm\cos(tx)$ when n is even, and a constant factor of t is multiplied when each derivative is taken.

- We continue as in the base case:

$$\begin{aligned}\frac{\partial}{\partial x}\int_0^1 t^{n-1} f_{n-1}(x,t)\,dt &= \lim_{x_k\to x}\int_0^1 t^{n-1}\left(\frac{f_{n-1}(x_n,t)-f_{n-1}(x,t)}{x_n - x}\right)dt \\ &=_{\text{IVT}} \lim_{x_k\to x}\int_0^1 t^{n-1}\frac{\partial f_{n-1}}{\partial x}(\xi_k,t)\,dt \quad \text{where } \xi_k\in[x_k,x],\ \xi_k\to x \\ &=_{\text{DCT}} \int_0^1 \lim_{x_k\to x} t^{n-1}\frac{\partial f_{n-1}}{\partial x}(\xi_k,t)\,dt \\ &:= \int_0^1 \lim_{x_k\to x} t^n f_n(\xi_k,t)\,dt \\ &:= \int_0^1 t^n f_n(x,t)\,dt.\end{aligned}$$

  - We've used the fact that $f_0(x) = \cos(tx)$ is smooth as a function of x, and in particular continuous
  - The DCT is justified because the functions $h_{n,k}(x,t) = t^n f_n(\xi_k,t)$ are again uniformly (in k) bounded by 1 since $t \leq 1 \implies t^n \leq 1$ and each f_n is a sin or cosine.

- Now take absolute values

$$
\begin{aligned}
\left| \frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} \right| &= \left| \int_0^1 -t^n f_n(x,t)\ dt \right| \\
&\leq \int_0^1 |t^n f_n(x,t)|\ dt \\
&\leq \int_0^1 |t^n||f_n(x,t)|\ dt \\
&\leq \int_0^1 |t^n| \cdot 1\ dt \\
&\leq \int_0^1 t^n\ dt \quad \text{since } t \text{ is positive} \\
&= \frac{1}{n+1} \\
&< \frac{1}{n}.
\end{aligned}
$$

  - We've again used the fact that $f_n(x,t)$ is of the form $\pm\cos(tx)$ or $\pm\sin(tx)$, both of which are bounded by 1.

6.5 Spring 2020.5

Compute the following limit and justify your calculations:

$$
\lim_{n\to\infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)}\ dx.
$$

Not finished, flesh out.

Walk through.

Solution:

Concepts Used:

- DCT
- Passing limits through products and quotients

Note that

$$\begin{aligned}\lim_n \left(1+\frac{x^2}{n}\right)^{-(n+1)} &= \frac{1}{\lim_n \left(1+\frac{x^2}{n}\right)^1 \left(1+\frac{x^2}{n}\right)^n} \\ &= \frac{1}{1\cdot e^{x^2}} \\ &= e^{-x^2}.\end{aligned}$$

If passing the limit through the integral is justified, we will have

$$\begin{aligned}\lim_{n\to\infty} \int_0^n \left(1+\frac{x^2}{n}\right)^{-(n+1)} dx &= \lim_{n\to\infty} \int_{\mathbf{R}} \chi_{[0,n]} \left(1+\frac{x^2}{n}\right)^{-(n+1)} dx \\ &= \int_{\mathbf{R}} \lim_{n\to\infty} \chi_{[0,n]} \left(1+\frac{x^2}{n}\right)^{-(n+1)} dx \quad \text{bytheDCT} \\ &= \int_{\mathbf{R}} \lim_{n\to\infty} \left(1+\frac{x^2}{n}\right)^{-(n+1)} dx \\ &= \int_0^\infty e^{-x^2} \\ &= \frac{\sqrt{\pi}}{2}.\end{aligned}$$

Computing the last integral:

$$\begin{aligned}\left(\int_{\mathbf{R}} e^{-x^2}\,dx\right)^2 &= \left(\int_{\mathbf{R}} e^{-x^2}\,dx\right)\left(\int_{\mathbf{R}} e^{-y^2}\,dx\right) \\ &= \int_{\mathbf{R}}\int_{\mathbf{R}} e^{-(x+y)^2}\,dx \\ &= \int_0^{2\pi}\int_0^\infty e^{-r^2} r\,dr\,d\theta \qquad u=r^2 \\ &= \frac{1}{2}\int_0^{2\pi}\int_0^\infty e^{-u}\,du\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi} 1 \\ &= \pi,\end{aligned}$$

and now use the fact that the function is even so $\int_0^\infty f = \frac{1}{2}\int_{\mathbf{R}} f$.
Justifying the DCT:

- Apply Bernoulli's inequality:

$$1+\frac{x^2}{n}^{n+1} \geq 1+\frac{x^2}{n}\left(1+x^2\right) \geq 1+x^2,$$

where the last inequality follows from the fact that $1+\frac{x^2}{n} \geq 1$

6.6 Spring 2019.3

Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2 \leq M$ for all $k \in \mathbb{N}$.

Prove that if $f_k \to f$ almost everywhere, then $f \in L^2([0,1])$ with $\|f\|_2 \leq M$ and

$$\lim_{k\to\infty} \int_0^1 f_k(x)dx = \int_0^1 f(x)dx$$

Hint: Try using Fatou's Lemma to show that $\|f\|_2 \leq M$ and then try applying Egorov's Theorem.

Solution:

Concepts Used:

- Definition of L^+: space of measurable function $X \to [0, \infty]$.
- Fatou: For any sequence of L^+ functions, $\int \liminf f_n \leq \liminf \int f_n$.
- Egorov's Theorem: If $E \subseteq \mathbf{R}^n$ is measurable, $m(E) > 0$, $f_k : E \to \mathbf{R}$ a sequence of measurable functions where $\lim_{n\to\infty} f_n(x)$ exists and is finite a.e., then $f_n \to f$ *almost uniformly*: for every $\varepsilon > 0$ there exists a closed subset $F_\varepsilon \subseteq E$ with $m(E \setminus F) < \varepsilon$ and $f_n \to f$ uniformly on F.

L^2 bound:

- Since $f_k \to f$ almost everywhere, $\liminf_n f_n(x) = f(x)$ a.e.
- $\|f_n\|_2 < \infty$ implies each f_n is measurable and thus $|f_n|^2 \in L^+$, so we can apply Fatou:

$$\begin{aligned}
\|f\|_2^2 &= \int |f(x)|^2 \\
&= \int \liminf_n |f_n(x)|^2 \\
&\underset{\text{Fatou}}{\leq} \liminf_n \int |f_n(x)|^2 \\
&\leq \liminf_n M \\
&= M.
\end{aligned}$$

- Thus $\|f\|_2 \leq \sqrt{M} < \infty$ implying $f \in L^2$.

What is the "right" proof here that uses the first part?

Equality of Integrals:

- Take the sequence $\varepsilon_n = \frac{1}{n}$

- Apply Egorov's theorem: obtain a set F_ε such that $f_n \to f$ uniformly on F_ε and $m(I \setminus F_\varepsilon) < \varepsilon$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \int_0^1 f_n - f \right| &\leq \lim_{n\to\infty} \int_0^1 |f_n - f| \\
&= \lim_{n\to\infty} \left(\int_{F_\varepsilon} |f_n - f| + \int_{I\setminus F_\varepsilon} |f_n - f| \right) \\
&= \int_{F_\varepsilon} \lim_{n\to\infty} |f_n - f| + \lim_{n\to\infty} \int_{I\setminus F_\varepsilon} |f_n - f| \quad \text{by uniform convergence} \\
&= 0 + \lim_{n\to\infty} \int_{I\setminus F_\varepsilon} |f_n - f|,
\end{aligned}$$

so it suffices to show $\int_{I\setminus F_\varepsilon} |f_n - f| \overset{n\to\infty}{\to} 0$.

- We can obtain a bound using Holder's inequality with $p = q = 2$:

$$\begin{aligned}
\int_{I\setminus F_\varepsilon} |f_n - f| &\leq \left(\int_{I\setminus F_\varepsilon} |f_n - f|^2 \right) \left(\int_{I\setminus F_\varepsilon} |1|^2 \right) \\
&= \left(\int_{I\setminus F_\varepsilon} |f_n - f|^2 \right) \mu(F_\varepsilon) \\
&\leq \|f_n - f\|_2 \mu(F_\varepsilon) \\
&\leq (\|f_n\|_2 + \|f\|_2)\, \mu(F_\varepsilon) \\
&\leq 2M \cdot \mu(F_\varepsilon)
\end{aligned}$$

where M is now a constant not depending on ε or n.

- Now take a nested sequence of sets F_ε with $\mu(F_\varepsilon) \to 0$ and applying continuity of measure yields the desired statement.

6.7 Fall 2018.6

Compute the following limit and justify your calculations:

$$\lim_{n\to\infty} \int_1^n \frac{dx}{(1 + \frac{x}{n})^n \sqrt[n]{x}}$$

Add concepts.

Solution:

Concepts Used:

- ?

- Note that $x^{\frac{1}{n}} \overset{n\to\infty}{\to} 1$ for any $0 < x < \infty$.
- Thus the integrand converges to $\frac{1}{e^x}$, which is integrable on $(0, \infty)$ and integrates to 1.
- Break the integrand up:

$$\int_0^\infty \frac{1}{(1+\frac{x}{n})^n x^{\frac{1}{n}}} dx = \int_0^1 \frac{1}{(1+\frac{x}{n})^n x^{\frac{1}{n}}} dx = \int_1^\infty \frac{1}{(1+\frac{x}{n})^n x^{\frac{1}{n}}} dx.$$

6.8 Fall 2018.3

Suppose $f(x)$ and $xf(x)$ are integrable on $\mathbf{R}$. Define F by

$$F(t) := \int_{-\infty}^\infty f(x)\cos(xt) dx$$

Show that

$$F'(t) = -\int_{-\infty}^\infty xf(x)\sin(xt) dx.$$

Walk through.

Solution:

Concepts Used:

- Mean Value Theorem
- DCT

$$\begin{aligned}
\frac{\partial}{\partial t} F(t) &= \frac{\partial}{\partial t} \int_{\mathbf{R}} f(x)\cos(xt)\ dx \\
&\overset{DCT}{=} \int_{\mathbf{R}} f(x) \frac{\partial}{\partial t} \cos(xt)\ dx \\
&= \int_{\mathbf{R}} xf(x)\cos(xt)\ dx,
\end{aligned}$$

so it only remains to justify the DCT.

- Fix t, then let $t_n \to t$ be arbitrary.

- Define

$$h_n(x,t) = f(x)\left(\frac{\cos(tx) - \cos(t_n x)}{t_n - t}\right) \overset{n\to\infty}{\to} \frac{\partial}{\partial t}\left(f(x)\cos(xt)\right)$$

since $\cos(tx)$ is differentiable in t and this is the limit definition of differentiability.

- Note that

$$\begin{aligned}\frac{\partial}{\partial t}\cos(tx) &:= \lim_{t_n \to t} \frac{\cos(tx) - \cos(t_n x)}{t_n - t} \\ &\overset{MVT}{=} \frac{\partial}{\partial t}\cos(tx)\Big|_{t=\xi_n} && \text{for some } \xi_n \in [t, t_n] \text{ or } [t_n, t] \\ &= x\sin(\xi_n x)\end{aligned}$$

where $\xi_n \overset{n\to\infty}{\to} t$ since wlog $t_n \leq \xi_n \leq t$ and $t_n \nearrow t$.

- We then have

$$|h_n(x)| = |f(x)x\sin(\xi_n x)| \leq |xf(x)| \quad \text{since } |\sin(\xi_n x)| \leq 1$$

for every x and every n.

- Since $xf(x) \in L^1(\mathbf{R})$ by assumption, the DCT applies.

6.9 Spring 2018.5

Suppose that

- $f_n, f \in L^1$,
- $f_n \to f$ almost everywhere, and
- $\int |f_n| \to \int |f|$.

Show that $\int f_n \to \int f$.

Solution:

Concepts Used:

- $\int |f_n - f| \to \iff \int f_n = \int f$.
- Fatou:

$$\int \liminf f_n \leq \liminf \int f_n$$
$$\int \limsup f_n \geq \limsup \int f_n.$$

- Since $\int |f_n| \overset{n\to\infty}{\to} \int |f|$, define

$$h_n = |f_n - f| \qquad \overset{n\to\infty}{\to} 0 \ a.e.$$
$$g_n = |f_n| + |f| \qquad \overset{n\to\infty}{\to} 2|f| \ a.e.$$

  - Note that $g_n - h_n \overset{n\to\infty}{\to} 2|f| - 0 = 2|f|$.

- Then

$$\begin{aligned}
\int 2|f| &= \int \liminf_n (g_n - h_n) \\
&= \int \liminf_n (g_n) + \int \liminf_n (-h_n) \\
&= \int \liminf_n (g_n) - \int \limsup_n (h_n) \\
&= \int 2|f| - \int \limsup_n (h_n) \\
&\leq \int 2|f| - \limsup_n \int h_n \quad \text{by Fatou,}
\end{aligned}$$

- Since $f \in L^1$, $\int 2|f| = 2\|f\|_1 < \infty$ and it makes sense to subtract it from both sides, thus

$$\begin{aligned}
0 &\leq -\limsup_n \int h_n \\
&:= -\limsup_n \int |f_n - f|.
\end{aligned}$$

  which forces $\limsup_n \int |f_n - f| = 0$, since
  - The integral of a nonnegative function is nonnegative, so $\int |f_n - f| \geq 0$.
  - So $(-\int |f_n - f|) \leq 0$.
  - But the above inequality shows $(-\int |f_n - f|) \geq 0$ as well.

- Since $\liminf_n \int h_n \leq \limsup_n \int h_n = 0$, $\lim_n \int h_n$ exists and is equal to zero.

- But then

$$\left| \int f_n - \int f \right| = \left| \int f_n - f \right| \leq \int |f_n - f|,$$

and taking $\lim_{n\to\infty}$ on both sides yields

$$\lim_{n\to\infty} \left| \int f_n - \int f \right| \leq \lim_{n\to\infty} \int |f_n - f| = 0,$$

so $\lim_{n\to\infty} \int f_n = \int f$.

6.10 Spring 2018.2

Let

$$f_n(x) := \frac{x}{1+x^n}, \quad x \geq 0.$$

a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on $[0, \infty)$?

b. Compute

$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx$$

Add concepts.

Solution:

Concepts Used:

- ?

Part a Claim: f_n does not converge uniformly to its limit.

- Note each $f_n(x)$ is clearly continuous on $(0, \infty)$, since it is a quotient of continuous functions where the denominator is never zero.
- Note

$$x < 1 \implies x^n \overset{n\to\infty}{\to} 0 \quad \text{and} \quad x > 1 \implies x^n \overset{n\to\infty}{\to} \infty.$$

- Thus

$$f_n(x) = \frac{x}{1+x^n} \overset{n\to\infty}{\longrightarrow} f(x) := \begin{cases} x, & 0 \leq x < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & x > 1 \end{cases}.$$

- If $f_n \to f$ uniformly on $[0,\infty)$, it would converge uniformly on every subset and thus uniformly on $(0,\infty)$.
  - Then f would be a uniform limit of continuous functions on $(0,\infty)$ and thus continuous on $(0,\infty)$.
  - By uniqueness of limits, f_n would converge to the pointwise limit f above, which is not continuous at $x = 1$, a contradiction.

Part b - If the DCT applies, interchange the limit and integral:

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx &= \int_0^\infty \lim_{n\to\infty} f_n(x)\,dx \quad \text{DCT} \\
&= \int_0^\infty f(x)\,dx \\
&= \int_0^1 x\,dx + \int_1^\infty 0\,dx \\
&= \frac{1}{2}x^2 \Big|_0^1 \\
&= \frac{1}{2}.
\end{aligned}$$

- To justify the DCT, write
$$\int_0^\infty f_n(x) = \int_0^1 f_n(x) + \int_1^\infty f_n(x).$$
- f_n restricted to $(0,1)$ is uniformly bounded by $g_0(x) = 1$ in the first integral, since
$$x \in [0,1] \implies \frac{x}{1+x^n} < \frac{1}{1+x^n} < 1 := g(x)$$
so
$$\int_0^1 f_n(x)\,dx \le \int_0^1 1\,dx = 1 < \infty.$$
Also note that $g_0 \cdot \chi_{(0,1)} \in L^1((0,\infty))$.
- The f_n restricted to $(1,\infty)$ are uniformly bounded by $g_1(x) = \frac{1}{x^2}$ on $[1,\infty)$, since
$$x \in (1,\infty) \implies \frac{x}{1+x^n} \le \frac{x}{x^n} = \frac{1}{x^{n-1}} \le \frac{1}{x^2} \in L^1([1,\infty) \text{ when } n \ge 3,$$
by the p-test for integrals.
- So set
$$g := g_0 \cdot \chi_{(0,1)} + g_1 \cdot \chi_{[1,\infty)},$$
then by the above arguments $g \in L^1((0,\infty))$ and $f_n \le g$ everywhere, so the DCT applies.

6.11 Fall 2016.3

Let $f \in L^1(\mathbf{R})$. Show that

$$\lim_{x \to 0} \int_{\mathbf{R}} |f(y-x) - f(y)| \, dy = 0$$

Missing some stuff.

Solution:

Concepts Used:

- $C_c^\infty \hookrightarrow L^p$ is dense.
- If $f \ldots$?

- Fixing notation, set $\tau_x f(y) := f(y-x)$; we then want to show

$$\|\tau_x f - f\|_{L^1} \overset{x \to 0}{\to} 0.$$

- Claim: by an $\varepsilon/3$ argument, it suffices to show this for compactly supported functions:
  - Since $f \in L^1$, choose $g_n \subset C_c^\infty(\mathbf{R}^1)$ smooth and compactly supported so that

$$\|f - g\|_{L^1} < \varepsilon.$$

  - Claim: $\|\tau_x f - \tau_x g\| < \varepsilon$.
    - ♢ Proof 1: translation invariance of the integral.
    - ♢ Proof 2: Apply a change of variables:

$$\begin{aligned}
\|\tau_x f - \tau_x g\|_1 &:= \int_{\mathbf{R}} |\tau_x f(y) - \tau_x g(y)| \, dy \\
&= \int_{\mathbf{R}} |f(y-x) - g(y-x)| \, dy \\
&= \int_{\mathbf{R}} |f(u) - g(u)| \, du \qquad (u = y - x, \, du = dy) \\
&= \|f - g\|_1 \\
&< \varepsilon.
\end{aligned}$$

  - Then

$$\begin{aligned}
\|\tau_x f - f\|_1 &= \|\tau_x f - \tau_x g + \tau_x g - g + g - f\|_1 \\
&\leq \|\tau_x f - \tau_x g\|_1 + \|\tau_x g - g\|_1 + \|g - f\|_1 \\
&\leq 2\varepsilon + \|\tau_x g - g\|_1.
\end{aligned}$$

- To show this for compactly supported functions:

– Let $g \in C_c^\infty(\mathbf{R}^1)$, let $E = \operatorname{supp}(g)$, and write

$$\begin{aligned}
\|\tau_x g - g\|_1 &= \int_{\mathbf{R}} |g(y-x) - g(y)|\, dy \\
&= \int_E |g(y-x) - g(y)|\, dy + \int_{E^c} |g(y-x) - g(y)|\, dy \\
&= \int_E |g(y-x) - g(y)|\, dy.
\end{aligned}$$

– But g is smooth and compactly supported on E, and thus uniformly continuous on E, so

$$\begin{aligned}
\lim_{x\to 0} \int_E |g(y-x) - g(y)|\, dy &= \int_E \lim_{x\to 0} |g(y-x) - g(y)|\, dy \\
&= \int_E 0\, dy \\
&= 0.
\end{aligned}$$

6.12 Fall 2015.3

Problem 6.12.1 (?)
Compute the following limit:

$$\lim_{n\to\infty} \int_1^n \frac{ne^{-x}}{1+nx^2} \sin\left(\frac{x}{n}\right) dx$$

Solution:

$$I = \lim_{n\to\infty} \int_1^\infty \frac{e^{-x}}{\frac{1}{n} + x^2} \sin\left(\frac{x}{n}\right) \chi_{[1,n]}\, dx = \int_1^\infty \frac{e^{-x}}{x^2} \lim_{n\to\infty} \sin\left(\frac{x}{n}\right) \chi_{[1,n]}\, dx = 0,$$

since $\sin(x/n) \to 0$. Passing the limit through the integral is justified by the DCT: write

$$f_n(x) := \frac{ne^{-x}}{1+nx^2} \sin\left(\frac{x}{n}\right) \chi_{[1,n]}.$$

Then

$$|f_n(x)| \le g(x) := \frac{e^{-x}}{x^2} \in L^1(1,\infty),$$

since

$$\|f\|_{L^1(1,\infty)} = \int_1^\infty \left|\frac{1}{x^2 e^x}\right| dx \le \int_1^\infty \left|\frac{1}{x^2}\right| dx = 1 < \infty.$$

6.13 Fall 2015.4

Let $f : [1, \infty) \to \mathbf{R}$ such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x\to\infty} f(x) \leq 1 + \frac{\pi}{4}$$

7 Integrals: Approximation

7.1 Fall 2021.2

Problem 7.1.1 (?) a. Let $F \subset \mathbb{R}$ be closed, and define

$$\delta_F(y) := \inf_{x\in F} |x-y|.$$

For $y \notin F$, show that

$$\int_F |x-y|^{-2} dx \leq \frac{2}{\delta_F(y)},$$

b. Let $F \subset \mathbb{R}$ be a closed set whose complement has finite measure, i.e. $m(\mathbf{R} \setminus F) < \infty$. Define the function

$$I(x) := \int_{\mathbb{R}} \frac{\delta_F(y)}{|x-y|^2} dy$$

Prove that $I(x) = \infty$ if $x \notin F$, however $I(x) < \infty$ for almost every $x \in F$.

Hint: investigate $\int_F I(x)dx$.

Solution (Part a):
Let $y \in F^c$ which is open, then one can find an epsilon ball about y avoiding F. We can take $\varepsilon := \delta_F(y)$ to define $A := B_\varepsilon(y)$, and we still have $A \subseteq F^c$ and $F \subseteq A^c$. Note that

$|x-y|^2 = (x-y)^2$ since this is always positive, then

$$\begin{aligned}
\int_F |x-y|^{-2}\,dx &\leq \int_{A^c} |x-y|^{-2}\,dx \\
&= \int_{-\infty}^{-\varepsilon} (x-y)^{-2}\,\,dx + \int_{\varepsilon}^{\infty} (x-y)^{-2}\,\,dx \\
&= \int_{-\infty}^{-\varepsilon} u^{-2}\,dx + \int_{\varepsilon}^{\infty} u^{-2}\,dx \\
&= -u^{-1}\Big|_{u=-\varepsilon}^{u=-\infty} - u^{-1}\Big|_{u=\infty}^{u=\varepsilon} \\
&= \frac{2}{\varepsilon} \\
&:= \frac{2}{\delta_F(y)}.
\end{aligned}$$

Solution (Part b):
Estimate:

$$\begin{aligned}
\int_F I(x)\,dx &:= \int_F \int_{\mathbf{R}} \frac{\delta_F(y)}{(x-y)^2}\,dy\,dx \\
&= \int_{\mathbf{R}} \delta_F(y) \int_F \frac{1}{(x-y)^2}\,dx\,dy \\
&= \int_F \delta_F(y) \int_F \frac{1}{(x-y)^2}\,dx\,dy + \int_{F^c} \delta_F(y) \int_F \frac{1}{(x-y)^2}\,dx\,dy \\
&= 0 + \int_{F^c} \delta_F(y) \int_F \frac{1}{(x-y)^2}\,dx\,dy \\
&\leq \int_{F^c} 2\,dy \\
&= 2\mu(F^c) \\
&< \infty,
\end{aligned}$$

where we've used that $y \in F \implies \delta_F(y) = 0$ and applied the bound from the first part. We've also implicitly used Fubini-Tonelli to change the order of integration, justified by positivity of the integrand and the finite iterated integral. This forces $I(x) < \infty$ for almost every $x \in F$, since if $I(x)$ is unbounded on any positive measure set then this integral would diverge.
If $x \notin F$, pick an ε-ball A about x avoiding F so that $|x-y| > \varepsilon$ for any $y \in A^c$ and thus $(x-y)^{-2} \leq \varepsilon^{-2}$. Note that $\delta_F(y) \geq \varepsilon$, so

$$\begin{aligned}
I(x) &= \int_{\mathbf{R}} \delta_F(y)(x-y)^{-2}\,dy \\
&\geq \int_{A^c} \delta_F(y)(x-y)^{-2}\,dy \\
&\geq \int_{A^c} \delta_F(y)\varepsilon^{-2}\,dy \\
&\geq \int_{A^c} \varepsilon^{-1}\,dy \\
&= \mu(A^c)\varepsilon^{-1},
\end{aligned}$$

which can be made arbitrarily large by taking $\varepsilon \to 0$.
#todo: Not great, A^c depends on ε so this ratio has a competing numerator...

7.2 Spring 2018.3

Let f be a non-negative measurable function on $[0,1]$.

Show that

$$\lim_{p\to\infty}\left(\int_{[0,1]} f(x)^p dx\right)^{\frac{1}{p}} = \|f\|_\infty.$$

Concepts Used:

- $\|f\|_\infty \coloneqq \inf_t \left\{t \;\middle|\; m\left(\left\{x\in\mathbf{R}^n \;\middle|\; f(x) > t\right\}\right) = 0\right\}$, i.e. this is the lowest upper bound that holds almost everywhere.

Solution:

- $\|f\|_p \le \|f\|_\infty$:
  - Note $|f(x)| \le \|f\|_\infty$ almost everywhere and taking pth powers preserves this inequality.
  - Thus

$$\begin{aligned}
& |f(x)| \le \|f\|_\infty && \text{a.e. by definition} \\
\implies & |f(x)|^p \le \|f\|_\infty^p && \text{for } p \ge 0 \\
\implies & \|f\|_p^p = \int_X |f(x)|^p \, dx \\
& \le \int_X \|f\|_\infty^p \, dx \\
& = \|f\|_\infty^p \int_X 1\, dx \\
& = \|f\|_\infty^p \cdot m(X) && \text{since the norm doesn't depend on } x \\
& = \|f\|_\infty^p && \text{since } m(X) = 1.
\end{aligned}$$

  $\diamondsuit$ Thus $\|f\|_p \le \|f\|_\infty$ for all p and taking $\lim_{p\to\infty}$ preserves this inequality.

- $\|f\|_p \ge \|f\|_\infty$:
  - Fix $\varepsilon > 0$.
  - Define

$$S_\varepsilon \coloneqq \left\{x\in\mathbf{R}^n \;\middle|\; |f(x)| \ge \|f\|_\infty - \varepsilon\right\}.$$

◇ Note that $m(S_\varepsilon) > 0$; otherwise if $m(S_\varepsilon) = 0$, then $t := \|f\|_\infty - \varepsilon < \|f\|_\varepsilon$. But this produces a *smaller* upper bound almost everywhere than $\|f\|_\varepsilon$, contradicting the definition of $\|f\|_\varepsilon$ as an infimum over such bounds.

– Then

$$\begin{aligned}
\|f\|_p^p &= \int_X |f(x)|^p \, dx \\
&\geq \int_{S_\varepsilon} |f(x)|^p \, dx \quad \text{since } S_\varepsilon \subseteq X \\
&\geq \int_{S_\varepsilon} |\|f\|_\infty - \varepsilon|^p \, dx \quad \text{since on } S_\varepsilon, |f| \geq \|f\|_\infty - \varepsilon \\
&= |\|f\|_\infty - \varepsilon|^p \cdot m(S_\varepsilon) \quad \text{since the integrand is independent of } x \\
&\geq 0 \quad \text{since } m(S_\varepsilon) > 0
\end{aligned}$$

– Taking pth roots for $p \geq 1$ preserves the inequality, so

$$\implies \|f\|_p \geq |\|f\|_\infty - \varepsilon| \cdot m(S_\varepsilon)^{\frac{1}{p}} \xrightarrow{p\to\infty} |\|f\|_\infty - \varepsilon| \xrightarrow{\varepsilon\to 0} \|f\|_\infty$$

where we've used the fact that above arguments work

– Thus $\|f\|_p \geq \|f\|_\infty$.

7.3 Spring 2018.4

Let $f \in L^2([0,1])$ and suppose

$$\int_{[0,1]} f(x) x^n dx = 0 \text{ for all integers } n \geq 0.$$

Show that $f = 0$ almost everywhere.

Concepts Used:

- Weierstrass Approximation: A continuous function on a compact set can be uniformly approximated by polynomials.
- $C^1([0,1])$ is dense in $L^2([0,1])$
- Polynomials are dense in $L^p(X, \mathcal{M}, \mu)$ for any $X \subseteq \mathbf{R}^n$ compact and μ a finite measure, for all $1 \leq p < \infty$.
  - Use Weierstrass Approximation, then uniform convergence implies $L^p(\mu)$ convergence by DCT.

Solution:

Proof (using Fourier transforms).

- Fix $k \in \mathbf{Z}$.
- Since $e^{2\pi ikx}$ is continuous on the compact interval $[0,1]$, it is uniformly continuous.
- Thus there is a sequence of polynomials P_ℓ such that

$$P_{\ell,k} \overset{\ell\to\infty}{\longrightarrow} e^{2\pi ikx} \text{ uniformly on } [0,1].$$

- Note applying linearity to the assumption $\int f(x)\, x^n$, we have

$$\int f(x)x^n \, dx = 0 \ \forall n \implies \int f(x)p(x)\, dx = 0$$

  for any polynomial $p(x)$, and in particular for $P_{\ell,k}(x)$ for every ℓ and every k.

- But then

$$\begin{aligned}
\langle f,\ e_k\rangle &= \int_0^1 f(x)e^{-2\pi ikx}\ dx \\
&= \int_0^1 f(x) \lim_{\ell\to\infty} P_\ell(x) \\
&= \lim_{\ell\to\infty} \int_0^1 f(x)P_\ell(x) \qquad \text{by uniform convergence on a compact interval} \\
&= \lim_{\ell\to\infty} 0 \quad \text{by assumption} \\
&= 0 \quad \forall k \in \mathbf{Z},
\end{aligned}$$

  so f is orthogonal to every e_k.

- Thus $f \in S^\perp := \operatorname{span}_{\mathbf{C}} \{e_k\}_{k\in\mathbf{Z}}^\perp \subseteq L^2([0,1])$, but since this is a basis, S is dense and thus $S^\perp = \{0\}$ in $L^2([0,1])$.
- Thus $f \equiv 0$ in $L^2([0,1])$, which implies that f is zero almost everywhere.

■

Proof (Alternative).

- By density of polynomials, for $f \in L^2([0,1])$ choose $p_\varepsilon(x)$ such that $\|f - p_\varepsilon\| < \varepsilon$ by Weierstrass approximation.
- Then on one hand,

$$\begin{aligned} \|f(f-p_\varepsilon)\|_1 &= \left\|f^2\right\|_1 - \|f \cdot p_\varepsilon\|_1 \\ &= \left\|f^2\right\|_1 - 0 \quad \text{by assumption} \\ &= \|f\|_2^2. \end{aligned}$$

  - Where we've used that $\|f^2\|_1 = \int |f^2| = \int |f|^2 = \|f\|_2^2$.
- On the other hand

$$\begin{aligned} \|f(f-p_\varepsilon)\| &\leq \|f\|_1 \|f - p_\varepsilon\|_\infty \quad \text{by Holder} \\ &\leq \varepsilon \|f\|_1 \\ &\leq \varepsilon \|f\|_2 \sqrt{m(X)} \\ &= \varepsilon \|f\|_2 \quad \text{since } m(X) = 1. \end{aligned}$$

  - Where we've used that $\|fg\|_1 = \int |fg| = \int |f||g| \leq \int \|f\|_\infty |g| = \|f\|_\infty \|g\|_1$.
- Combining these,

$$\|f\|_2^2 \leq \|f\|_2 \varepsilon \implies \|f\|_2 < \varepsilon \to 0, .$$

  so $\|f\|_2 = 0$, which implies $f = 0$ almost everywhere.

■

7.4 Spring 2015.2

Let $f : \mathbf{R} \to \mathbf{C}$ be continuous with period 1. Prove that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N f(n\alpha) = \int_0^1 f(t)\,dt \quad \forall \alpha \in \mathbf{R} \setminus \mathbf{Q}.$$

Hint: show this first for the functions $f(t) = e^{2\pi i k t}$ for $k \in \mathbf{Z}$.

7.5 Fall 2014.4

Problem 7.5.1 (?)
Let $g \in L^\infty([0,1])$ Prove that

$$\int_{[0,1]} f(x)g(x)\,dx = 0 \quad \text{for all continuous } f:[0,1] \to \mathbf{R} \implies g(x) = 0 \text{ almost everywhere.}$$

Concepts Used:

- Polar decomposition: $f = \operatorname{sign}(f) \cdot |f|$.
- $L^\infty[0,1] \subseteq L^1[0,1]$.

Solution:
Use that $L^\infty[0,1] \subseteq L^1[0,1]$, so fixing g, choose a sequence of compactly supported continuous functions f_k converging to $\operatorname{sign}(g)$ in L^1. We can arrange so that $|g_k| \leq 1$. Then

$$\begin{aligned}
\int |g| &= \int \operatorname{sign}(g) \cdot g \\
&= \int \lim_k g_k \cdot g \\
&\overset{\text{DCT}}{=} \lim_k \int g_k \cdot g \\
&= \lim_k 0 \\
&= 0,
\end{aligned}$$

where the DCT applies since defining $h_k \coloneqq g_k \cdot g$ we have $|h_k| \leq g \in L^1[0,1]$, and each integral is zero since g_k is continuous (and we use the hypothesis).

8 | L^1

8.1 Spring 2021.4

Let f, g be Lebesgue integrable on $\mathbf{R}$ and let $g_n(x) \coloneqq g(x-n)$. Prove that

$$\lim_{n\to\infty} \|f + g_n\|_1 = \|f\|_1 + \|g\|_1.$$

Concepts Used:

- For $f \in L^1(X)$, $\|f\|_1 := \int_X |f(x)|\,dx < \infty$.
- Small tails in L_1: if $f \in L^1(\mathbf{R}^n)$, then for every $\varepsilon > 0$ exists some radius R such that

$$\|f\|_{L^1(B_R^c)} < \varepsilon.$$

- Shift g to the right far enough so that the two densities are mostly disjoint:

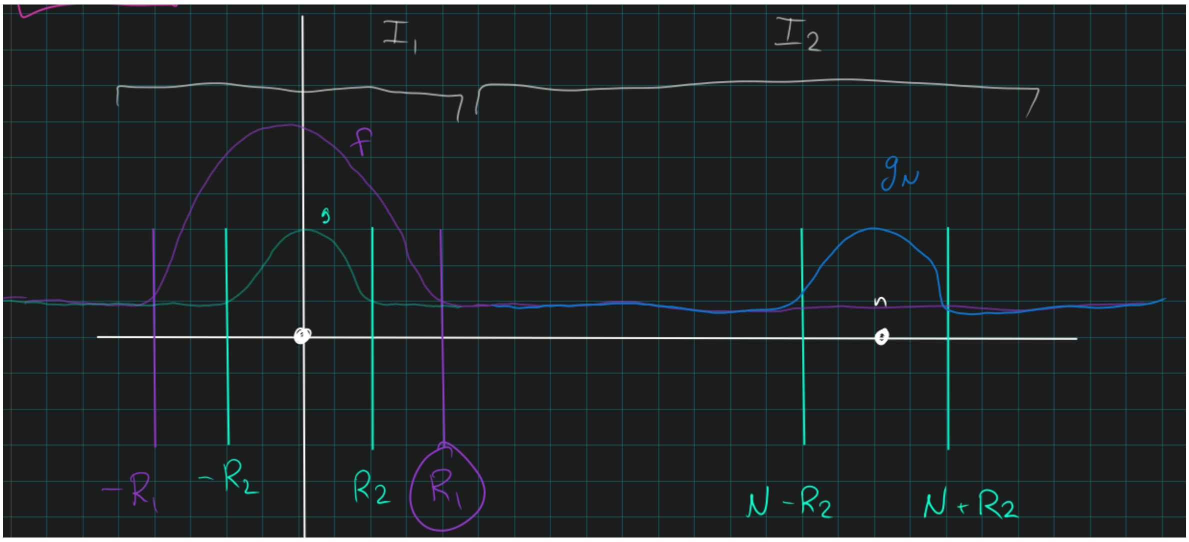

Figure 1: Shifting density

- Any integral $\int_a^b f$ can be written as $\|f\|_1 - O(\text{err})$.
- Bounding technique:

$$a - \varepsilon \leq b \leq a + \varepsilon \implies b = a.$$

Solution:

- Fix ε.
- Using small tails for $f, g \in L^1$, choose $R_1, R_2 \gg 0$ so that

$$\int_{B_{R_1}(0)^c} |f| < \varepsilon$$
$$\int_{B_{R_2}(0)^c} |g| < \varepsilon.$$

- Note that this implies

$$\int_{-R_1}^{R_1} |f| = \|f\|_1 - 2\varepsilon$$
$$\int_{-R_2}^{R_2} |g_N| = \|g_N\| - 2\varepsilon.$$

- Also note that by translation invariance of the Lebesgue integral, $\|g\|_1 = \|g_N\|_1$.

- Now use N to make the densities almost disjoint: choose $N \gg 1$ so that $N - R_2 > R_1$:

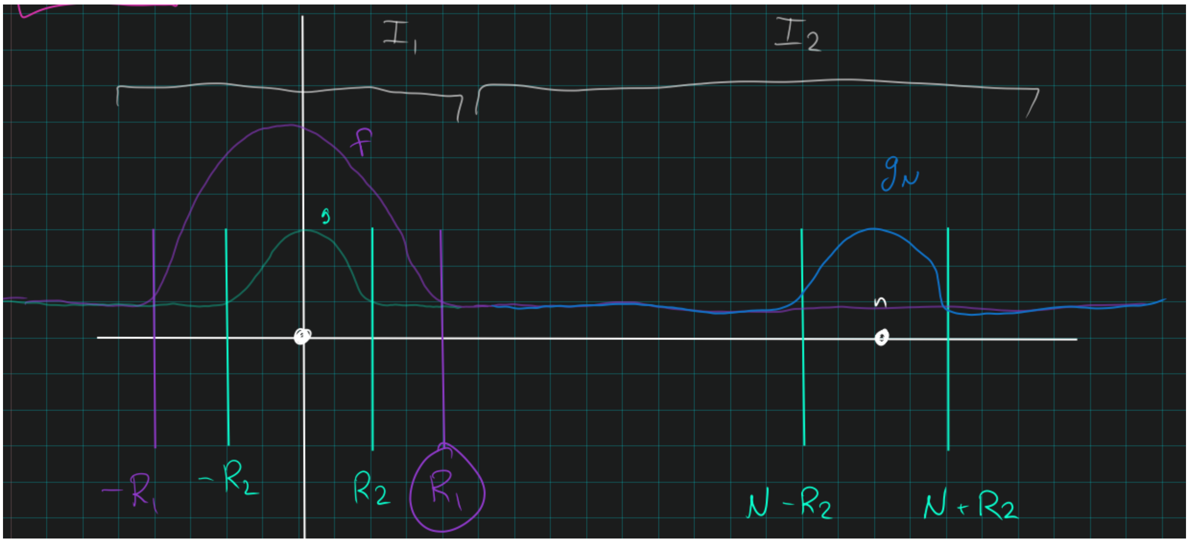

Figure 2: Shifting density

- Consider the change of variables $x \mapsto x - N$:

$$\int_{-R_2}^{R_2} |g(x)|\, dx = \int_{N-R_2}^{N+R_2} |g(x-N)|\, dx := \int_{N-R_2}^{N+R_2} |g_N(x)|\, dx.$$

  - Use this to conclude that

$$\int_{N-R_2}^{N+R_2} |g_N| = \|g_N\| - 2\varepsilon.$$

- Now split the integral in the problem statement at R_1:

$$\|f + g_N\|_1 = \int_{\mathbf{R}} |f + g_N| = \int_{-\infty}^{R_1} |f + g_N| + \int_{R_1}^{\infty} |f + g_N| := I_1 + I_2.$$

- **Idea**: from the picture,
  - On I_1, f is big and g_N is small

- On I_2, f is small and g_N is big

- Casework: estimate I_1, I_2 separately, bounding from above and below.

- I_1 upper bound:

$$\begin{aligned}
I_1 &:= \int_{-\infty}^{R_1} |f + g_N| \\
&\leq \int_{-\infty}^{R_1} |f| + |g_N| \\
&= \int_{-\infty}^{R_1} |f| + \int_{-\infty}^{R_1} |g_N| \\
&\leq \int_{-\infty}^{R_1} |f| + \int_{-\infty}^{N-R_2} |g_N| && R_1 < N - R_2 \\
&= \|f\|_1 - \int_{R_1}^{\infty} |f| + \int_{-\infty}^{N-R_2} |g_N| \\
&\leq \|f\|_1 - \int_{R_1}^{\infty} |f| + \varepsilon \\
&\leq \|f\|_1 + \varepsilon.
\end{aligned}$$

  - In the last step we've used that we're subtracting off a positive number, so forgetting it only makes things larger.
  - We've also used monotonicity of the Lebesgue integral: if $A \leq B$, then $(c, A) \subseteq (c, B)$ and $\int_c^A |f| \leq \int_c^B |f|$ since $|f|$ is positive.

- I_1 lower bound:

$$\begin{aligned}
I_1 &:= \int_{-\infty}^{R_1} |f + g_N| \\
&\geq \int_{-\infty}^{R_1} |f| - |g_N| \\
&= \int_{-\infty}^{R_1} |f| - \int_{-\infty}^{R_1} |g_N| \\
&\geq \int_{-\infty}^{R_1} |f| - \int_{-\infty}^{N-R_2} |g_N| && R_1 < N - R_2 \\
&= \|f\|_1 - \int_{R_1}^{\infty} |f| - \int_{-\infty}^{N-R_2} |g_N| \\
&\geq \|f\|_1 - \varepsilon - \varepsilon \\
&= \|f\|_1 - 2\varepsilon.
\end{aligned}$$

  - Now we've used that the integral with g_N comes in with a negative sign, so extending the range of integration only makes things *smaller*. We've also used the ε bound on both f and g_N here, and both are tail estimates.

- Taken together we conclude

$$\|f\|_1 - 2\varepsilon \leq I_1 \leq \|f\|_1 \qquad \varepsilon \to 0 \implies I_1 = \|f\|_1.$$

- I_2 lower bound:

$$\begin{aligned}
I_2 &:= \int_{R_1}^{\infty} |f + g_N| \\
&\leq \int_{R_1}^{\infty} |f| + \int_{R_1}^{\infty} g_N \\
&\leq \int_{R_1}^{\infty} |f| + \|g_N\|_1 - \int_{-\infty}^{R_1} |g_N| \\
&\leq \varepsilon + \|g_N\|_1 - \int_{-\infty}^{R_1} |g_N| \\
&\leq \varepsilon + \|g_N\|_1 \\
&= \varepsilon + \|g\|_1.
\end{aligned}$$

  - Here we've again thrown away negative terms, only increasing the bound, and used the tail estimate on f.

- I_2 upper bound:

$$\begin{aligned}
I_2 &:= \int_{R_1}^{\infty} |f + g_N| \\
&= \int_{R_1}^{\infty} |g_N + f| \\
&\geq \int_{R_1}^{\infty} |g_N| - \int_{R_1}^{\infty} |f| \\
&= \|g_N\| - \int_{-\infty}^{R_1} |g_N| - \int_{R_1}^{\infty} |f| \\
&\geq \|g_N\| - 2\varepsilon.
\end{aligned}$$

- Here we've swapped the order under the absolute value, and used the tail estimates on both g and f.
- Taken together:

$$\|g\|_1 - \varepsilon \leq I_2 \leq \|g\|_1 + 2\varepsilon.$$

- Note that we have two inequalities:

$$\begin{aligned}
\|f\|_1 - 2\varepsilon &\leq \int_{-\infty}^{R_1} |f - g_N| \leq \|f\|_1 + \varepsilon \\
\|g\|_1 - 2\varepsilon &\leq \int_{R_1}^{\infty} |f - g_N| \leq \|g\|_1 + \varepsilon.
\end{aligned}$$

- Add these to obtain

$$\|f\|_1 + \|g\|_1 - 4\varepsilon \leq I_1 + I_2 := \|f - g_N\|_1 \leq \|f\| + \|g\|_1 + 2\varepsilon.$$

- Check that as $N \to \infty$ as $\varepsilon \to 0$ to yield the result.

8.2 Fall 2020.4

Problem 8.2.1 (?)
Prove that if $xf(x) \in L^1(\mathbf{R})$, then

$$F(y) := \int f(x)\cos(yx)\,dx$$

defines a C^1 function.

Solution:

- Fix y_0, we'll show F' exists and is continuous at y_0.
- Fix a sequence $y_n \searrow y_0$ and define

$$h_n(x) := \frac{h(x, y_n) - h(x, y_0)}{y_n - y_0} \qquad h(x, y) := f(x)\cos(yx).$$

- We can then write

$$\frac{\partial h}{\partial y}(x, y_0) = \lim_{n\to\infty} h_n(x).$$

- Apply the MVT:

$$h_n(x) := \frac{h(x, y_n) - h(x, y_0)}{y_n - y_0} = \frac{\partial h}{\partial y}(x, \tilde{y}) \qquad \text{for some } \tilde{y} \in [y_0, y_n].$$

- Use this to get a bound for DCT:

$$\begin{aligned} |h_n(x)| &:= \left| \frac{h(x, y_n) - h(x, y_0)}{y_n - y_0} \right| \\ &= \left| \frac{\partial h}{\partial y}(x, \tilde{y}) \right| \\ &\le \sup_{y \in [y_0, y_n]} \left| \frac{\partial h}{\partial y}(x, y) \right| \\ &\le \sup_{y \in [y_0, y_n]} |xf(x)\sin(yx)| \\ &\le |xf(x)|, \end{aligned}$$

  and by assumption $xf(x) \in L^1$.

- So this justifies commuting an integral and a limit:

$$\begin{aligned} F'(y_0) &:= \lim_{y_n \to y_0} \frac{F(y_n) - F(y_0)}{y_n - y_0} \\ &= \lim_{n \to 0} \int h_n(x)\, dx \\ &\overset{\text{DCT}}{=} \int \lim_{n \to \infty} h_n(x)\, dx \\ &:= \int \frac{\partial h}{\partial y}(x, y_0)\, dx \\ &:= -\int x f(x) \sin(yx)\, dx, \end{aligned}$$

and since this limit exists and is finite, F is differentiable at y_0.

- That F is continuous:

$$\begin{aligned} \lim_{y_n \to y_0} F'(y_n) &= \lim_{y_n \to y_0} \int \frac{\partial h}{\partial y}(x, y_n)\, dx \\ &\overset{\text{DCT}}{=} \int \lim_{y_n \to y_0} \frac{\partial h}{\partial y}(x, y_n)\, dx \\ &= -\int \lim_{y_n \to y_0} x f(x) \sin(y_n x)\, dx \\ &= -\int x f(x) \sin(y_0 x)\, dx, \end{aligned}$$

where we've used that $y \mapsto \sin(yx)$ is clearly continuous.

8.3 Spring 2020.3

Problem 8.3.1 (?)

a. Prove that if $g \in L^1(\mathbf{R})$ then

$$\lim_{N \to \infty} \int_{|x| \geq N} |f(x)|\, dx = 0,$$

and demonstrate that it is not necessarily the case that $f(x) \to 0$ as $|x| \to \infty$.

b. Prove that if $f \in L^1([1, \infty])$ and is decreasing, then $\lim_{x \to \infty} f(x) = 0$ and in fact $\lim_{x \to \infty} x f(x) = 0$.

c. If $f : [1, \infty) \to [0, \infty)$ is decreasing with $\lim_{x \to \infty} x f(x) = 0$, does this ensure that $f \in L^1([1, \infty))$?

Concepts Used:

- Limits
- Cauchy Criterion for Integrals: $\int_a^\infty f(x)\,dx$ converges iff for every $\varepsilon > 0$ there exists an M_0 such that $A, B \geq M_0$ implies $\left|\int_A^B f\right| < \varepsilon$, i.e. $\left|\int_A^B f\right| \xrightarrow{A\to\infty} 0$.
- Integrals of L^1 functions have vanishing tails: $\int_N^\infty |f| \xrightarrow{N\to\infty} 0$.
- Mean Value Theorem for Integrals: $\int_a^b f(t)\,dt = (b-a)f(c)$ for some $c \in [a,b]$.

Proof (of a).
Stated integral equality:

- Let $\varepsilon > 0$
- $C_c(\mathbf{R}^n) \hookrightarrow L^1(\mathbf{R}^n)$ is dense so choose $\{f_n\} \to f$ with $\|f_n - f\|_1 \to 0$.
- Since $\{f_n\}$ are compactly supported, choose $N_0 \gg 1$ such that f_n is zero outside of $B_{N_0}(\mathbf{0})$.
- Then

$$\begin{aligned}
N \geq N_0 \implies \int_{|x|>N} |f| &= \int_{|x|>N} |f - f_n + f_n| \\
&\leq \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n| \\
&= \int_{|x|>N} |f - f_n| \\
&\leq \int_{|x|>N} \|f - f_n\|_1 \\
&= \|f_n - f\|_1 \left(\int_{|x|>N} 1\right) \\
&\xrightarrow{n\to\infty} 0 \left(\int_{|x|>N} 1\right) \\
&= 0 \\
&\xrightarrow{N\to\infty} 0.
\end{aligned}$$

To see that this doesn't force $f(x) \to 0$ as $|x| \to \infty$:

- Take $f(x)$ to be a train of rectangles of height 1 and area $1/2^j$ centered on even integers.
- Then

$$\int_{|x|>N} |f| = \sum_{j=N}^\infty 1/2^j \xrightarrow{N\to\infty} 0$$

  as the tail of a convergent sum.
- However $f(x) = 1$ for infinitely many even integers $x > N$, so $f(x) \not\to 0$ as $|x| \to \infty$.

■

Proof (of b, Solution 1, a slight trick).

- Since f is decreasing on $[1,\infty)$, for any $t \in [x-n, x]$ we have

$$x - n \le t \le x \implies f(x) \le f(t) \le f(x-n).$$

- Integrate over $[x, 2x]$, using monotonicity of the integral:

$$\begin{aligned} \int_x^{2x} f(x)\,dt &\le \int_x^{2x} f(t)\,dt \le \int_x^{2x} f(x-n)\,dt \\ \implies f(x)\int_x^{2x} dt &\le \int_x^{2x} f(t)\,dt \le f(x-n)\int_x^{2x} dt \\ \implies xf(x) &\le \int_x^{2x} f(t)\,dt \le xf(x-n). \end{aligned}$$

- By the Cauchy Criterion for integrals, $\lim_{x\to\infty} \int_x^{2x} f(t)\ dt = 0$.
- So the LHS term $xf(x) \overset{x\to\infty}{\longrightarrow} 0$.
- Since $x > 1$, $|f(x)| \le |xf(x)|$
- Thus $f(x) \overset{x\to\infty}{\longrightarrow} 0$ as well.

■

Proof (of b, Solution 2: Variation on the trick).

- Use mean value theorem for integrals:

$$\int_x^{2x} f(t)\,dt = xf(c_x) \quad \text{for some } c_x \in [x, 2x] \text{ depending on } x.$$

- Since f is decreasing,

$$\begin{aligned} x \le c_x \le 2x &\implies f(2x) \le f(c_x) \le f(x) \\ &\implies 2xf(2x) \le 2xf(c_x) \le 2xf(x) \\ &\implies 2xf(2x) \le 2x\int_x^{2x} f(t)\,dt \le 2xf(x) \end{aligned}$$

.

- By Cauchy Criterion, $\int_x^{2x} f \to 0$.
- So $2xf(2x) \to 0$, which by a change of variables gives $uf(u) \to 0$.
- Since $u \ge 1$, $f(u) \le uf(u)$ so $f(u) \to 0$ as well.

■

Proof (of b, Solution 3: Contradiction).
Just showing $f(x) \overset{x\to\infty}{\longrightarrow} 0$:

- Toward a contradiction, suppose not.
- Since f is decreasing, it can not diverge to $+\infty$
- If $f(x) \to -\infty$, then $f \notin L^1(\mathbf{R})$: choose $x_0 \gg 1$ so that $t \geq x_0 \implies f(t) < -1$, then
- Then $t \geq x_0 \implies |f(t)| \geq 1$, so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f(t)|\, dt \geq \int_{x_0}^\infty 1 = \infty.$$

- Otherwise $f(x) \to L \neq 0$, some finite limit.
- If $L > 0$:
  - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L - \varepsilon \leq f(t) \leq L$
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L - \varepsilon)\, dt = \infty$$

- If $L < 0$:
  - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L \leq f(t) \leq L + \varepsilon$.
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L)\, dt = \infty$$

Showing $xf(x) \overset{x\to\infty}{\longrightarrow} 0$.

- Toward a contradiction, suppose not.
- (How to show that $xf(x) \not\to +\infty$?)
- If $xf(x) \to -\infty$
  - Choose a sequence $\Gamma = \{\widehat{x}_i\}$ such that $x_i \to \infty$ and $x_i f(x_i) \to -\infty$.
  - Choose a subsequence $\Gamma' = \{x_i\}$ such that $x_i f(x_i) \leq -1$ for all i and $x_i \leq x_{i+1}$.
  - Choose a further subsequence $S = \left\{x_i \in \Gamma' \,\middle|\, 2x_i < x_{i+1}\right\}$.
  - Then since f is always decreasing, for $t \geq x_0$, $|f|$ is increasing, and $|f(x_i)| \leq |f(2x_i)|$, so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f| \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(t)|\, dt \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(x_i)| = \sum_{x_i \in S} x_i f(x_i) \to \infty.$$

- If $xf(x) \to L \neq 0$ for $0 < L < \infty$:
  - Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L - \varepsilon \leq x_i f(x_i) \leq L$ for all i.

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)|\, dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i)\, dt = \sum_S x_i f(x_i) \geq \sum_S (L - \varepsilon) \to \infty.$$

- If $xf(x) \to L \neq 0$ for $-\infty < L < 0$:
  - Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L \leq x_i f(x_i) \leq L + \varepsilon$ for all i.

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)|\, dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i)\, dt = \sum_S x_i f(x_i) \geq \sum_S (L) \to \infty.$$

■

Proof (of b, Solution 4: Akos' suggestion).
For $x \geq 1$,

$$|xf(x)| = \left| \int_x^{2x} f(x)\, dt \right| \leq \int_x^{2x} |f(x)|\, dt \leq \int_x^{2x} |f(t)|\, dt \leq \int_x^\infty |f(t)|\, dt \overset{x\to\infty}{\longrightarrow} 0$$

where we've used

- Since f is decreasing and $\lim_{x\to\infty} f(x) = 0$ from part (a), f is non-negative.
- Since f is positive and decreasing, for every $t \in [a, b]$ we have $|f(a)| \leq |f(t)|$.
- By part (a), the last integral goes to zero.

■

Proof (of b, Solution 5: Peter's).

- Toward a contradiction, produce a sequence $x_i \to \infty$ with $x_i f(x_i) \to \infty$ and $x_i f(x_i) > \varepsilon > 0$, then

$$\begin{aligned}
\int f(x)\, dx &\geq \sum_{i=1}^\infty \int_{x_i}^{x_{i+1}} f(x)\, dx \\
&\geq \sum_{i=1}^\infty \int_{x_i}^{x_{i+1}} f(x_{i+1})\, dx \\
&= \sum_{i=1}^\infty f(x_{i+1}) \int_{x_i}^{x_{i+1}} dx \\
&\geq \sum_{i=1}^\infty (x_{i+1} - x_i) f(x_{i+1}) \\
&\geq \sum_{i=1}^\infty (x_{i+1} - x_i) \frac{\varepsilon}{x_{i+1}} \\
&= \varepsilon \sum_{i=1}^\infty \left(1 - \frac{x_{i-1}}{x_i}\right) \to \infty
\end{aligned}$$

  which can be ensured by passing to a subsequence where $\sum \frac{x_{i-1}}{x_i} < \infty$.

■

Proof (of c).

- No: take $f(x) = \frac{1}{x \ln x}$
- Then by a u-substitution,

$$\int_0^x f = \ln(\ln(x)) \overset{x\to\infty}{\longrightarrow} \infty$$

  is unbounded, so $f \not\in L^1([1,\infty))$.
- But

$$xf(x) = \frac{1}{\ln(x)} \overset{x\to\infty}{\longrightarrow} 0.$$

■

8.4 Fall 2019.5

a. Show that if f is continuous with compact support on $\mathbf{R}$, then

$$\lim_{y\to 0} \int_{\mathbb{R}} |f(x-y) - f(x)|dx = 0$$

b. Let $f \in L^1(\mathbf{R})$ and for each $h > 0$ let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y|\leq h} f(x-y)dy$$

- Prove that $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$ for all $h > 0$.
- Prove that $\mathcal{A}_h f \to f$ in $L^1(\mathbf{R})$ as $h \to 0^+$.

Walk through.

Concepts Used:

- Continuity in L^1 (recall that DCT won't work! Notes 19.4, prove it for a dense subset first).
- Lebesgue differentiation in 1-dimensional case. See HW 5.6.

Solution:

Proof (of a).

- Fix $\varepsilon > 0$. If we can find a set A such that the following calculation holds for h small enough, we're done:

$$\begin{aligned}\int_{\mathbf{R}} |f(x-h) - f(x)|\, dx &= \int_A |f(x-h) - f(x)|\, dx \\ &\leq \int_A \varepsilon \\ &= \varepsilon \mu(A) \longrightarrow 0,\end{aligned}$$

  provided $h \to 0$ as $\varepsilon \to 0$, which we can arrange if $|h| < \varepsilon$.

- Choose $A \supseteq \operatorname{supp} f$ compact such that $\operatorname{supp} f \pm 1 \subseteq A$
  - Why this can be done: $\operatorname{supp} f$ is compact, so closed and bounded, and contained in some compact interval $[-M, M]$. So e.g. $A := [-M-1, M+1]$ suffices.

- Note that f is still continuous on A, since it is zero on $A \setminus \operatorname{supp} f$, and thus uniformly continuous (by Heine-Cantor, for example).

- We can rephrase the usual definition of uniform continuity:

$$\forall \varepsilon \exists \delta = \delta(\varepsilon) \text{ such that } |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon \quad \forall x, y \in A$$

  as

$$\forall \varepsilon \exists \delta = \delta(\varepsilon) \text{ such that } |h| < \delta \implies |f(x-h) - f(x)| < \varepsilon \quad \forall x \text{ such that } x, x \pm h \in A$$

- So fix ε and choose such a δ for A, and choose h such that $|h| < \min(1, \delta)$. Then the desired computation goes through by uniform continuity of f on A.

■

Proof (of b).
We have

$$\begin{aligned}
\int_{\mathbf{R}} |A_h(f)(x)|\, dx &= \int_{\mathbf{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y)\, dy \right| dx \\
&\leq \frac{1}{2h} \int_{\mathbf{R}} \int_{x-h}^{x+h} |f(y)|\, dy\, dx \\
&=_{FT} \frac{1}{2h} \int_{\mathbf{R}} \int_{y-h}^{y+h} |f(y)|\, \mathbf{dx}\, \mathbf{dy} \\
&= \int_{\mathbf{R}} |f(y)|\, dy \\
&= \|f\|_1,
\end{aligned}$$

and (rough sketch)

$$\begin{aligned}
\int_{\mathbf{R}} |A_h(f)(x) - f(x)|\, dx &= \int_{\mathbf{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y)\, dy \right) - f(x) \right| dx \\
&= \int_{\mathbf{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y)\, dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x)\, dy \right| dx \\
&\leq_{FT} \frac{1}{2h} \int_{\mathbf{R}} \int_{B(h,x)} |f(y-x) - f(x)|\, \mathbf{dx}\, \mathbf{dy} \\
&\leq \frac{1}{2h} \int_{\mathbf{R}} \|\tau_x f - f\|_1\, dy \\
&\to 0 \quad \text{by (a).}
\end{aligned}$$

■

Remark 8.4.1: This works for arbitrary $f \in L^1$, using approximation by continuous functions with compact support:

- Choose $g \in C_c^0$ such that $\|f - g\|_1 \to 0$.
- By translation invariance, $\|\tau_h f - \tau_h g\|_1 \to 0$.
- Write

$$\begin{aligned}
\|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\
&\leq \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\
&\to \|\tau_h g - g\|,
\end{aligned}$$

so it suffices to show that $\|\tau_h g - g\| \to 0$.

8.5 Fall 2017.3

Let

$$S = \operatorname*{span}_{\mathbf{C}} \left\{ \chi_{(a,b)} \;\middle|\; a, b \in \mathbf{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form (a, b).

Show that for every $f \in L^1(\mathbf{R})$, there exists a sequence of functions $\{f_n\} \subset S$ such that

$$\lim_{n\to\infty} \|f_n - f\|_1 = 0$$

Concepts Used:

- From homework: E is Lebesgue measurable iff there exists a finite union of closed cubes A such that $m(E\Delta A) < \varepsilon$.

Solution:

- Idea: first show this for characteristic functions, then simple functions, then for arbitrary f.
- For characteristic functions:
  - Consider χ_A for A a measurable set.
  - By regularity of the Lebesgue measure, for every $\varepsilon > 0$ we can find an I_ε such that $m(A\Delta I_\varepsilon) < \varepsilon$ where I_ε is a finite disjoint union of intervals.
  - Then use

    $$\varepsilon > m(A\Delta I\varepsilon) = \int_X |\chi_A - \chi_{I_\varepsilon}|,$$

    so the χ_{I_ε} converge to χ_A in L_1.
  - Then just note that $\chi_{I_\varepsilon} = \sum_{j\leq N} \chi_{I_j}$ where $I_\varepsilon = \coprod_{j\leq N} I_j$, so $\chi_{I_\varepsilon} \in S$.
- For simple functions:
  - Let $\psi = \sum_{k\leq N} c_k \chi_{E_k}$.
  - By the argument above, for each k we can find $I_{\varepsilon,k}$ such that $\chi_{I_{\varepsilon,k}}$ converges to χ_{E_k} in L^1.
  - So defining $\psi_\varepsilon = \sum_{k\leq N} c_k \chi_{I_{\varepsilon,k}}$, the claim is that this will converge to φ in L_1.
  - Note that

    $$\psi_\varepsilon = \sum_k c_k \chi_{I_{\varepsilon,k}} = \sum_k c_k \sum_j \chi_{I_{j,k}} = \sum_{k,j} c_k \chi_{I_{j,k}} \in S$$

    since now the $I_{j,k}$ are indicators of intervals.

- Moreover

$$\|\psi_\varepsilon - \psi\| = \left\|\sum_k c_k \left(\chi_{E_k} - \chi_{I_{\varepsilon,k}}\right)\right\| \leq \sum_k c_k \left\|\chi_{E_k} - \chi_{I_{\varepsilon,k}}\right\|,$$

where the last norm can be bounded by the proof for characteristic functions.

- For arbitrary functions:
  - Now just use that every $f \in L^1$ can be approximated by simple functions φ_n so that $\|f - \varphi_n\|_1 < \varepsilon$ for $n \gg 1$.
  - So find $\varphi_n \to f$, and for each n, find $g_{n,k} \in S$ with $\|g_{n,k} - \varphi_n\|_1 \overset{k\to\infty}{\longrightarrow} 0$, an approximation by functions in S.
  - Then

$$\|f - g_{n,k}\| \leq \|f - \varphi_n\| + \|\varphi_n - g_{n,k}\|,$$

which can be made arbitrarily small.

8.6 Spring 2015.4

Problem 8.6.1 (?)
Define

$$f(x,y) := \begin{cases} \frac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that $f \in L^1(\mathbf{R}^2)$.

Solution:
Note that

$$\begin{aligned}
\int_{\mathbf{R}^2} |f|\, d\mu &= \int_0^\infty \int_x^\infty x^{\frac{1}{3}} (1+xy)^{\frac{-3}{2}}\, dy\, dx \\
&= \int_0^\infty -2x^{-\frac{2}{3}} (1+xy)^{-\frac{1}{2}} \Big|_{y=x}^{y=\infty} dx \\
&= \int_0^\infty \frac{2}{x^{\frac{2}{3}} (1+x^2)^{\frac{1}{2}}} \\
&= \int_0^1 \frac{2}{x^{\frac{2}{3}} (1+x^2)^{\frac{1}{2}}} + \int_1^\infty \frac{2}{x^{\frac{2}{3}} (1+x^2)^{\frac{1}{2}}} \\
&= \int_0^1 \frac{2}{x^{\frac{2}{3}}} + \int_1^\infty \frac{2}{x^{\frac{5}{3}}} \\
&< \infty,
\end{aligned}$$

where

- For the first term: We've entirely neglected the $1+x^2$ factor, since neglecting to divide by a positive number can only make the integrand larger,
- For the second term:
$$1+x^2 \geq 0 \implies \frac{1}{\sqrt{1+x^2}} \leq \frac{1}{\sqrt{x^2}} = \frac{1}{x}$$
- Both terms converge by the p-tests.

The use of iterated integration is justified by Tonelli's theorem on $|f| = f$, since f is non-negative and clearly measurable on $\mathbf{R}^2$, and if any iterated integral is finite then it is equal to $\int |f|$.

8.7 Fall 2014.3

Problem 8.7.1 (?)
Let $f \in L^1(\mathbf{R})$. Show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } \qquad m(E) < \delta \implies \int_E |f(x)|\,dx < \varepsilon$$

Solution (by contradiction):

- Note that if $m(E) = 0$ then $\int_E f = 0$ for any f.
- Toward a contradiction, suppose there exists an $\varepsilon > 0$ such that for all $\delta > 0$ there exists a set $E_\delta \subseteq \mathbf{R}$ with $m(E) < \delta$ but $\int_{E_\delta} |f| > \varepsilon$.
- Let $\delta_n \searrow 0$ be any sequence converging to zero and choose E_n with $\int_{E_n} |f| > \varepsilon$ for every n.
- Define $E := \limsup_n E_n := \bigcap_{N \geq 1} \bigcup_{n \geq N} E_n$, then $m(E) = 0$ by Borel-Cantelli.
- Now estimate using Fatou:
$$\begin{aligned}
\int_E |f| &= \int_X \chi_E |f| \\
&= \int_X \limsup_n \chi_{E_n} |f| \\
&\geq \limsup_n \int_X \chi_{E_n} |f| \\
&\geq \limsup_n \int_{E_n} |f| \\
&\geq \limsup_n \varepsilon \\
&= \varepsilon,
\end{aligned}$$
however $\displaystyle\int_E |f|\,dm = 0$ since $m(E) = 0$, a contradiction. $\lightning$.

Solution (direct):
Note that this is clear for simple functions: let $\varphi = \sum_{k\leq n} c_k m(A_k) < \infty$ be simple function. then φ is necessarily bounded on $\mathbf{R}$, so let $M := \sup_{\mathbf{R}} \varphi$ and estimate

$$\begin{aligned}\int_E \varphi &:= \sum_k c_k m(A_k \cap E) \\ &\leq \sum_k M \cdot m(E) \\ &= CMm(E),\end{aligned}$$

for some constant C, so choosing $\delta < \frac{\varepsilon}{CM}$ (and its corresponding E with $m(E) < \delta$) bounds this above by ε.
For arbitrary $f \in L^1$, there is a sequence of simple functions φ_n with $\int \varphi_n \nearrow \int f$ and $\|\varphi_n - f\|_{L_1} \overset{n\to\infty}{\longrightarrow} 0$. Choose δ and E as above, and use the triangle inequality to estimate

$$\begin{aligned}\int_E |f| &= \int_E |f - \varphi_n + \varphi_n| \\ &\leq \int_E |f - \varphi_n| + \int_E |\varphi_n|,\end{aligned}$$

choose $n \gg 1$ to bound the first term by ε, noting that the second term is bounded by ε by the case for simple functions.

8.8 Spring 2014.1

Problem 8.8.1 (?)

1. Give an example of a continuous $f \in L^1(\mathbf{R})$ such that $f(x) \not\to 0$ as$|x| \to \infty$.
2. Show that if f is *uniformly* continuous, then

$$\lim_{|x|\to\infty} f(x) = 0.$$

Solution:
Part 1: Take a train of triangles with base points at k and $k+1$, each of area 2^{-k}. Then $\int |f| \approx \sum_{k\geq 0} 2^{-k} < \infty$, but $f(x) \not\to 0$ since $f(x) > 0$ infinitely often.
Part 2:

- Idea: use contradiction to produce a sequence with arbitrarily large terms, and bound below an integral in a ball about each point.
- Suppose $\lim_{|x|\to\infty} f(x) = L > 0$.
  - Then for any ε there exists an M such that $x \geq M \implies |f(x) - L| < \varepsilon$, so $L - \varepsilon \leq f(x) \leq L + \varepsilon$

- Choosing $\varepsilon = L/2$ yields $L/2 \leq f(x) \leq 3L/2$, and so

$$\int_{\mathbf{R}} |f| \geq \int_{|x| \geq M} |f| \geq \int_{|x| \geq M} L/2 \to \infty,$$

contradicting $f \in L^1(\mathbf{R})$. ↯.

- So it must be that this limit does not exist. Fix $\varepsilon > 0$, then there are infinitely many x such that $f(x) > \varepsilon$, so choose a sequence $x_n \to \infty$ with $f(x_n) > \varepsilon$ for each n.
- Now use uniform continuity: pick a uniform $\delta = \delta(\varepsilon)$ such that $x \in B_\delta(x_n) \implies |f(x) - f(x_n)| < \varepsilon/4$.
- Now use that $f(x_n) - \varepsilon/4 \leq f(x) \leq f(x_n) + \varepsilon/4$ implies that $f(x) \geq 3\varepsilon/4$ whenever $x \in B_\delta(x_n)$ for any n to estimate

$$\int_{B_\delta(x_n)} |f(x)|\, dx \geq 2\delta \cdot 3\varepsilon/4 := C = C_{\delta,\varepsilon} > 0,$$

where C is a constant.

- But now we've contradicted $f \in L^1$:

$$\int_{\mathbf{R}} |f| \geq \sum_{n \geq 1} \int_{B_\delta(x_n)} |f| \geq \sum_{n \geq 1} C \to \infty,$$

provided we pass to a further subsequence of x_n such that the balls $B_\delta(x_n)$ are disjoint. ↯

9 | Fubini-Tonelli

9.1 Spring 2021.6

Warning 9.1.1
This problem may be much harder than expected. Recommended skip.

Let $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a measurable function and for $x \in \mathbf{R}$ define the set

$$E_x := \left\{ y \in \mathbf{R} \;\middle|\; \mu\left(z \in \mathbf{R} \;\middle|\; f(x,z) = f(x,y) \right) > 0 \right\}.$$

Show that the following set is a measurable subset of $\mathbf{R} \times \mathbf{R}$:

$$E := \bigcup_{x \in \mathbf{R}} \{x\} \times E_x.$$

Hint: consider the measurable function $h(x,y,z) \coloneqq f(x,y) - f(x,z)$.

9.2 Fall 2021.4

Problem 9.2.1 (?)
Let f be a measurable function on $\mathbb{R}$. Show that the graph of f has measure zero in $\mathbb{R}^2$.

Solution:
Write

$$\Gamma \coloneqq \left\{ (x, f(x)) \;\middle|\; x \in \mathbf{R} \right\} \subseteq \mathbf{R}^d.$$

Then

$$\begin{aligned} \mu(\Gamma) &= \int_{\mathbf{R}^d} \chi_\Gamma \, d\mu \\ &= \int_{\mathbf{R}^{d-1}} \int_{\mathbf{R}} \chi_\Gamma(x,y) \, dy \, dx \\ &= \int_{\mathbf{R}^{d-1}} 0 \, dx \\ &= 0, \end{aligned}$$

using that $\int_{\mathbf{R}} \chi_\Gamma(x,y)\,dy = 0$ since if x is fixed then $\chi_\Gamma(x,y) = \{f(x)\}$ is a point with measure zero. Since f is measurable, Γ is a measurable set and χ_Γ is measurable. Since the iterated integral was finite, the equalities are justified by Fubini-Tonelli.

9.3 Spring 2020.4

Let $f, g \in L^1(\mathbf{R})$. Argue that $H(x,y) \coloneqq f(y)g(x-y)$ defines a function in $L^1(\mathbf{R}^2)$ and deduce from this fact that

$$(f * g)(x) \coloneqq \int_{\mathbf{R}} f(y) g(x-y) \, dy$$

defines a function in $L^1(\mathbf{R})$ that satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Strategy:
Just do it! Sort out the justification afterward. Use Tonelli.

Concepts Used:

- Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
- Fubini: $f(x, y) \in L^1$ yields *integrable* slices and equality of iterated integrals
- F/T: apply Tonelli to $|f|$; if finite, $f \in L^1$ and apply Fubini to f
- See Folland's Real Analysis II, p. 68 for a discussion of using Fubini *and* Tonelli.

Solution:

- If these norms can be computed via iterated integrals, we have

$$
\begin{aligned}
\|f * g\|_1 &:= \int_{\mathbf{R}} |(f * g)(x)|\, dx \\
&:= \int_{\mathbf{R}} \left| \int_{\mathbf{R}} H(x, y)\, dy \right| dx \\
&:= \int_{\mathbf{R}} \left| \int_{\mathbf{R}} f(y) g(x - y)\, dy \right| dx \\
&\leq \int_{\mathbf{R}} \int_{\mathbf{R}} |f(y) g(x - y)|\, dx\, dy \\
&:= \int_{\mathbf{R}} \int_{\mathbf{R}} |H(x, y)|\, dx\, dy \\
&:= \int_{\mathbf{R}^2} |H|\, d\mu_{\mathbf{R}^2} \\
&:= \|H\|_{L^1(\mathbf{R}^2)}.
\end{aligned}
$$

  So it suffices to show $\|H\|_1 < \infty$.

- A preliminary computation, the validity of which we will show afterward:

$$
\begin{aligned}
\|H\|_1 &:= \|H\|_{L^1(\mathbf{R}^2)} \\
&= \int_{\mathbf{R}} \left(\int_{\mathbf{R}} |f(y) g(x - y)|\, dy \right) dx && \text{Tonelli} \\
&= \int_{\mathbf{R}} \left(\int_{\mathbf{R}} |f(y) g(x - y)|\, dx \right) dy && \text{Tonelli} \\
&= \int_{\mathbf{R}} \left(\int_{\mathbf{R}} |f(y) g(t)|\, dt \right) dy && \text{setting } t = x - y,\ dt = -dx \\
&= \int_{\mathbf{R}} \left(\int_{\mathbf{R}} |f(y)| \cdot |g(t)|\, dt \right) dy \\
&= \int_{\mathbf{R}} |f(y)| \cdot \left(\int_{\mathbf{R}} |g(t)|\, dt \right) dy \\
&:= \int_{\mathbf{R}} |f(y)| \cdot \|g\|_1\, dy \\
&= \|g\|_1 \int_{\mathbf{R}} |f(y)|\, dy && \text{the norm is a constant} \\
&:= \|g\|_1 \|f\|_1 \\
&< \infty && \text{by assumption.}
\end{aligned}
$$

- We've used Tonelli twice: to equate the integral to the iterated integral, and to switch the order of integration, so it remains to show the hypothesis of Tonelli are fulfilled.

Claim: H is measurable on $\mathbf{R}^2$:

Proof (?).

- It suffices to show $\tilde{f}(x,y) := f(y)$ and $\tilde{g}(x,y) := g(x-y)$ are both measurable on $\mathbf{R}^2$.
  - Then use that products of measurable functions are measurable.
- $f \in L^1$ by assumption, and L^1 functions are measurable by definition.
- The function $(x,y) \mapsto g(x-y)$ is measurable on $\mathbf{R}^2$:
  - g is measurable on $\mathbf{R}$ by assumption, so the cylinder function $G(x,y) := g(x)$ on $\mathbf{R}^2$ is measurable (result from course).
  - Define a linear transformation
  $$T := \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2(\mathbf{R}) \qquad \implies \quad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ y \end{bmatrix},$$
  and linear functions are measurable.
  - Write
  $$\tilde{g}(x-y) := G(x-y,y) := (G \circ T)(x,y),$$
  and compositions of measurable functions are measurable.

■

- Apply **Tonelli** to $|H|$
  - H measurable implies $|H|$ is measurable.
  - $|H|$ is non-negative.
  - So the iterated integrals are equal in the extended sense
  - The calculation shows the iterated integral is finite, so $\int |H|$ is finite and H is thus integrable on $\mathbf{R}^2$.

Note: Fubini is not needed, since we're not calculating the actual integral, just showing H is integrable.

9.4 Spring 2019.4

Let f be a non-negative function on $\mathbf{R}^n$ and $\mathcal{A} = \{(x,t) \in \mathbf{R}^n \times \mathbf{R} : 0 \leq t \leq f(x)\}$.

Prove the validity of the following two statements:

a. f is a Lebesgue measurable function on $\mathbf{R}^n$ $\iff$ $\mathcal{A}$ is a Lebesgue measurable subset of $\mathbf{R}^{n+1}$

b. If f is a Lebesgue measurable function on $\mathbf{R}^n$, then

$$m(\mathcal{A}) = \int_{\mathbf{R}^n} f(x)dx = \int_0^\infty m\left(\{x \in \mathbf{R}^n : f(x) \geq t\}\right) dt$$

Concepts Used:

- See Stein and Shakarchi p.82 corollary 3.3.
- Tonelli
- Important trick! $\left\{(x,t) \;\middle|\; 0 \leq t \leq f(x)\right\} = \{f(x) \geq t\} \cap \{t \geq 0\}$

Solution:

Proof (a, $\implies$).
$\implies$:

- Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is a measurable function.
- Rewrite A:

$$\begin{aligned}
A &= \left\{ (x,t) \in \mathbf{R}^d \times \mathbf{R} \;\middle|\; 0 \leq t \leq f(x) \right\} \\
&= \left\{ (x,t) \in \mathbf{R}^d \times \mathbf{R} \;\middle|\; 0 \leq t < \infty \right\} \cap \left\{ (x,t) \in \mathbf{R}^d \times \mathbf{R} \;\middle|\; t \leq f(x) \right\} \\
&= \left(\mathbf{R}^d \times [0,\infty) \right) \cap \left\{ (x,t) \in \mathbf{R}^d \times \mathbf{R} \;\middle|\; f(x) - t \geq 0 \right\} \\
&\coloneqq \left(\mathbf{R}^d \times [0,\infty) \right) \cap H^{-1}\left([0,\infty) \right),
\end{aligned}$$

  where we define

$$\begin{aligned}
H : \mathbf{R}^d \times \mathbf{R} &\to \mathbf{R} \\
(x,t) &\mapsto f(x) - t.
\end{aligned}$$

  - Note: this is "clearly" measurable!

- If we can show both sets are measurable, we're done, since σ-algebras are closed under countable intersections.
- The first set is measurable since it is a Borel set in $\mathbf{R}^{d+1}$.
- For the same reason, it suffices to show H is a measurable function.
- Define cylinder functions

$$\begin{aligned}
F : \mathbf{R}^d \times \mathbf{R} &\to \mathbf{R} \\
(x,t) &\mapsto f(x)
\end{aligned}$$

  and

$$\begin{aligned}
G : \mathbf{R}^d \times \mathbf{R} &\to \mathbf{R} \\
(x,t) &\mapsto t
\end{aligned}$$

  - F is a cylinder of f, and since f is measurable by assumption, F is measurable.
  - G is a cylinder on the identity for $\mathbf{R}$, which is measurable, so G is measurable.

- Define

$$\begin{aligned}
H : \mathbf{R}^d &\to \mathbf{R} \\
(x,t) &\mapsto F(x,t) - G(x,t) \coloneqq f(x) - t,
\end{aligned}$$

  which are linear combinations of measurable functions and thus measurable.

■

Proof (a, $\Longleftarrow$).
$\Longleftarrow$:

- Suppose $\mathcal{A}$ is a measurable set.
- A corollary of Tonelli applied to χ_X: if E is measurable, then for a.e. t the following slice is measurable:

$$\begin{aligned} \mathcal{A}_t \coloneqq \left\{ x \in \mathbf{R}^d \;\middle|\; (x,t) \in \mathcal{A} \right\} &= \left\{ x \in \mathbf{R}^d \;\middle|\; f(x) \geq t \geq 0 \right\} \\ &= f^{-1}\left([t,\infty)\right). \end{aligned}$$

  - But maybe this isn't enough, because we need $f^{-1}\left([\alpha,\infty)\right)$ for *all* α
- But the other slice is also measurable for a.e. x:

$$\begin{aligned} \mathcal{A}_x &\coloneqq \left\{ t \in \mathbf{R} \;\middle|\; (x,t) \in \mathcal{A} \right\} \\ &= \left\{ t \in \mathbf{R} \;\middle|\; 0 \leq t \leq f(x) \right\} \\ &= \left\{ t \in \mathbf{R} \;\middle|\; t \in [0, f(x)] \right\} \\ &= [0, f(x)]. \end{aligned}$$

- Moreover the function $x \mapsto m(\mathcal{A}_x)$ is a measurable function of x
- Now note $m(\mathcal{A}_x) = f(x) - 0 = f(x)$, so f must be measurable.

■

Proof (of b).

- Writing down what the slices are

$$\mathcal{A} = \left\{(x,t) \in \mathbf{R}^n \times \mathbf{R} \;\middle|\; 0 \leq t \leq f(x)\right\}$$
$$\mathcal{A}_t = \left\{x \in \mathbf{R}^n \;\middle|\; t \leq f(x)\right\}.$$

- Then

$$\begin{aligned}
\int_{\mathbf{R}^n} f(x)\, dx &= \int_{\mathbf{R}^n} \int_0^{f(x)} 1\, dt\, dx \\
&= \int_{\mathbf{R}^n} \int_0^{\infty} \chi_{\mathcal{A}}\, dt\, dx \\
&\overset{F.T.}{=} \int_0^{\infty} \int_{\mathbf{R}^n} \chi_{\mathcal{A}}\, dx\, dt \\
&= \int_0^{\infty} m(\mathcal{A}_t)\, dt,
\end{aligned}$$

  where we just use that $\int\int \chi_{\mathcal{A}} = m(\mathcal{A})$

- By Tonelli, all of these integrals are equal.
  - This is justified because f was assumed measurable on $\mathbf{R}^n$, thus by (a) $\mathcal{A}$ is a measurable set and thus χ_A is a measurable function on $\mathbf{R}^n \times \mathbf{R}$.

■

9.5 Fall 2018.5

Let $f \geq 0$ be a measurable function on $\mathbf{R}$. Show that

$$\int_{\mathbf{R}} f = \int_0^{\infty} m(\{x : f(x) > t\})dt$$

Concepts Used:

- Claim: If $E \subseteq \mathbf{R}^a \times \mathbf{R}^b$ is a measurable set, then for almost every $y \in \mathbf{R}^b$, the slice E^y is measurable and

$$m(E) = \int_{\mathbf{R}^b} m(E^y)\, dy.$$

  - Set $g = \chi_E$, which is non-negative and measurable, so apply Tonelli.
  - Conclude that $g^y = \chi_{E^y}$ is measurable, the function $y \mapsto \int g^y(x)\, dx$ is measurable, and $\int\int g^y(x)\, dx\, dy = \int g$.
  - But $\int g = m(E)$ and $\int\int g^y(x)\, dx\, dy = \int m(E^y)\, dy$.

Solution:

Note: f is a function $\mathbf{R} \to \mathbf{R}$ in the original problem, but here I've assumed $f : \mathbf{R}^n \to \mathbf{R}$.

- Since $f \geq 0$, set
$$E := \left\{ (x,t) \in \mathbf{R}^n \times \mathbf{R} \;\middle|\; f(x) > t \right\} = \left\{ (x,t) \in \mathbf{R}^n \times \mathbf{R} \;\middle|\; 0 \leq t < f(x) \right\}.$$
- Claim: since f is measurable, E is measurable and thus $m(E)$ makes sense.
  - Since f is measurable, $F(x,t) := t - f(x)$ is measurable on $\mathbf{R}^n \times \mathbf{R}$.
  - Then write $E = \{F < 0\} \cap \{t \geq 0\}$ as an intersection of measurable sets.
- We have slices
$$\begin{aligned} E^t &:= \left\{ x \in \mathbf{R}^n \;\middle|\; (x,t) \in E \right\} = \left\{ x \in \mathbf{R}^n \;\middle|\; 0 \leq t < f(x) \right\} \\ E^x &:= \left\{ t \in \mathbf{R} \;\middle|\; (x,t) \in E \right\} = \left\{ t \in \mathbf{R} \;\middle|\; 0 \leq t \leq f(x) \right\} = [0, f(x)]. \end{aligned}$$
  - E_t is precisely the set that appears in the original RHS integrand.
  - $m(E^x) = f(x)$.
- Claim: χ_E satisfies the conditions of Tonelli, and thus $m(E) = \int \chi_E$ is equal to any iterated integral.
  - Non-negative: clear since $0 \leq \chi_E \leq 1$
  - Measurable: characteristic functions of measurable sets are measurable.
- Conclude:
  1. For almost every x, E^x is a measurable set, $x \mapsto m(E^x)$ is a measurable function, and $m(E) = \int_{\mathbf{R}^n} m(E^x)\, dx$
  2. For almost every t, E^t is a measurable set, $t \mapsto m(E^t)$ is a measurable function, and $m(E) = \int_{\mathbf{R}} m(E^t)\, dt$
- On one hand,
$$\begin{aligned} m(E) &= \int_{\mathbf{R}^{n+1}} \chi_E(x,t) \\ &= \int_{\mathbf{R}} \int_{\mathbf{R}^n} \chi_E(x,t)\, dt\, dx \quad \text{by Tonelli} \\ &= \int_{\mathbf{R}^n} m(E^x)\, dx \quad \text{first conclusion} \\ &= \int_{\mathbf{R}^n} f(x)\, dx. \end{aligned}$$
- On the other hand,
$$\begin{aligned} m(E) &= \int_{\mathbf{R}^{n+1}} \chi_E(x,t) \\ &= \int_{\mathbf{R}} \int_{\mathbf{R}^n} \chi_E(x,t)\, dx\, dt \quad \text{by Tonelli} \\ &= \int_{\mathbf{R}} m(E^t)\, dt \quad \text{second conclusion.} \end{aligned}$$

- Thus

$$\int_{\mathbf{R}^n} f\,dx = m(E) = \int_{\mathbf{R}} m(E^t)\,dt = \int_{\mathbf{R}} m\left(\left\{x \;\middle|\; f(x) > t\right\}\right).$$

9.6 Fall 2015.5

Problem 9.6.1 (?)
Let $f, g \in L^1(\mathbf{R})$ be Borel measurable.

- Show that
  - The function

$$F(x, y) := f(x-y)g(y)$$

    is Borel measurable on $\mathbf{R}^2$, and
  - For almost every $x \in \mathbf{R}$, the function $f(x-y)g(y)$ is integrable with respect to y on $\mathbf{R}$.
- Show that $f * g \in L^1(\mathbf{R})$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

Solution:

- $F \in \mathcal{B}(\mathbf{R}^2)$:
  - Write a function $\tilde{f}(x, y) := f(x)$
  - Write a linear transformation $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathrm{GL}_2$, so $T[x, y] = [x-y, 0]$
  - Write $f(x-y) := (\tilde{f} \circ T)(x, y)$, which is a composition of measurable functions and thus measurable.
  - A product of measurable functions is measurable.
- $f * g \in L^1(\mathbf{R})$: estimate

$$\begin{aligned}
\int |f * g| d\mu &= \int_{\mathbf{R}} \int_{\mathbf{R}} |f(x-y)g(y)|\,dx\,dy \\
&= \int_{\mathbf{R}} \int_{\mathbf{R}} |f(x-y)||g(y)|\,dx\,dy \\
&= \int_{\mathbf{R}} |g(y)| \int_{\mathbf{R}} |f(x-y)|\,dx\,dy \\
&= \|g\|_1 \|f\|_1,
\end{aligned}$$

  where we've used translation invariance of the L^1 norm and Fubini-Tonelli justified by the finite result.

- $F_x(y) := f(x-y)g(y)$ is integrable with respect to y for almost every x:
  - This follows from Fubini-Tonelli, which says that if $F(x,y)$ is integrable, the slices $F^x(y)$ are integrable for almost every x. Here take $F(x,y) := f(x-y)g(y)$.

9.7 Spring 2014.5

Problem 9.7.1 (?)
Let $f, g \in L^1([0,1])$ and for all $x \in [0,1]$ define

$$F(x) := \int_0^x f(y)\,dy \quad \text{and} \quad G(x) := \int_0^x g(y)\,dy.$$

Prove that

$$\int_0^1 F(x)g(x)\,dx = F(1)G(1) - \int_0^1 f(x)G(x)\,dx$$

10 | L^2 and Fourier Analysis

10.1 Fall 2020.5

Problem 10.1.1 (?)
Suppose $\varphi \in L^1(\mathbf{R})$ with

$$\int \varphi(x)\,dx = \alpha.$$

For each $\delta > 0$ and $f \in L^1(\mathbf{R})$, define

$$A_\delta f(x) := \int f(x-y)\delta^{-1}\varphi\left(\delta^{-1}y\right)\,dy.$$

a. Prove that for all $\delta > 0$,

$$\|A_\delta f\|_1 \le \|\varphi\|_1 \|f\|_1.$$

b. Prove that

$$A_\delta f \to \alpha f \text{ in } L^1(\mathbf{R}) \quad \text{as} \quad \delta \to 0^+.$$

Hint: you may use without proof the fact that for all $f \in L^1(\mathbf{R})$,

$$\lim_{y \to 0} \int_{\mathbf{R}} |f(x-y) - f(x)|\, dx = 0.$$

Remark 10.1.1: See Folland 8.14.

Solution (Part 1):
This is a direct application of Fubini-Tonelli:

$$\begin{aligned}
\|A_\delta f\| &:= \int \left| \int f(x-y)\delta^{-1}\varphi(\delta^{-1}y)\, dy \right| dx \\
&\leq \int \int \left| f(x-y)\delta^{-1}\varphi(\delta^{-1}y) \right| dy\, dx \\
&\overset{FT}{=} \int \int |f(x-y)| \cdot \left| \delta^{-1}\varphi(\delta^{-1}y) \right| dx\, dy \\
&= \int \left| \delta^{-1}\varphi(\delta^{-1}y) \right| \left(\int |f(x-y)|\, dx \right) dy \\
&= \int \left| \delta^{-1}\varphi(\delta^{-1}y) \right| \cdot \|f\|\, dy \\
&= \|f\| \cdot \int \left| \delta^{-1}\varphi(\delta^{-1}y) \right| dy \\
&= \|f\| \cdot \|\varphi\|.
\end{aligned}$$

Here we've used translation and dilation invariance of the Lebesgue integral.

Solution (Part 2):
Write $\varphi_\delta(y) := \delta^{-1}\varphi(\delta^{-1}y)$, then

$$\begin{aligned}
\|A_\delta f - \alpha f\|_1 &:= \int |A_\delta f(x) - \alpha f(x)|\, dx \\
&= \int \left| \int f(x-y)\varphi_\delta(y)\, dy - \alpha f(x) \right| dx \\
&= \int \left| \int \tau_y f(x)\varphi_\delta(y)\, dy - \int f(x)\varphi_\delta(y)\, dy \right| dx \\
&\leq \int \int |\tau_y f(x) - f(x)| \cdot |\varphi_\delta(y)|\, dy\, dx \\
&= \int \int |\tau_y f(x) - f(x)| \cdot |\varphi_\delta(y)|\, dx\, dy \\
&= \int |\varphi_\delta(y)| \cdot \|\tau_y f - f\|_1\, dy,
\end{aligned}$$

where the interchange of integration order is justified by Tonelli since the integrands are positive. The goal is to now make this small when δ is small.

One way to do this immediately: make a change of variables $y = tz$ to get

$$\|A_\delta f - \alpha f\|_1 \leq \int |\varphi(z)| \|\tau_{tz} f - f\|_1 \, dz,$$

use that $\|\tau_{tz} f - f\|_1 \leq 2\|f\|_1 < \infty$ by the triangle inequality and apply the DCT:

$$\lim_{t \to 0} \int |\varphi(z)| \cdot \|\tau_{tz} f - f\|_1 \, dz = \int |\varphi(z)| \lim_{t \to 0} \|\tau_{tz} f - f\|_1 \, dz = 0.$$

More directly, use continuity in L^1 (as per the hint) to pick a $h > 0$ such that

$$\|\tau_y f - f\| < \varepsilon \quad \text{for } y \in A := \left\{ y \;\middle|\; |y| \leq h \right\}.$$

Now choose $\delta_0 \gg 1$ large enough so that

$$\int_{A^c} |\varphi_\delta(y)| \, dy < \varepsilon \quad \text{for all } \delta > \delta_0.$$

Now

$$\begin{aligned}
\int_{\mathbf{R}} |\varphi_\delta(y)| \cdot \|\tau_y f - f\|_1 \, dy &= \int_A |\varphi_\delta(y)| \cdot \|\tau_y f - f\|_1 \, dy + \int_{A^c} |\varphi_\delta(y)| \cdot \|\tau_y f - f\|_1 \, dy \\
&\leq \int_A |\varphi_\delta(y)| \cdot \varepsilon \, dy + \int_{A^c} |\varphi_\delta(y)| \cdot 2\|f\|_1 \, dy \\
&\leq \varepsilon \|\varphi_\delta\|_1 + 2\varepsilon \|f\|_1 \\
&\longrightarrow 0.
\end{aligned}$$

10.2 Spring 2020.6

Problem 10.2.1 (?)

a. Show that

$$L^2([0,1]) \subseteq L^1([0,1]) \quad \text{and} \quad \ell^1(\mathbf{Z}) \subseteq \ell^2(\mathbf{Z}).$$

b. For $f \in L^1([0,1])$ define

$$\widehat{f}(n) := \int_0^1 f(x) e^{-2\pi i n x} \, dx.$$

Prove that if $f \in L^1([0,1])$ and $\left\{ \widehat{f}(n) \right\} \in \ell^1(\mathbf{Z})$ then

$$S_N f(x) := \sum_{|n| \leq N} \widehat{f}(n) e^{2\pi i n x}.$$

converges uniformly on $[0,1]$ to a continuous function g such that $g = f$ almost everywhere.

Hint: One approach is to argue that if $f \in L^1([0,1])$ with $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbf{Z})$ then $f \in L^2([0,1])$.

Concepts Used:
From Neil:

1. $\widehat{f}$ in ℓ^1 ensures that S_N converges uniformly to something, call it g.
2. $\widehat{f} \in \ell^1$ Implies $\widehat{f} \in \ell^2$ which (by characterization of an o.n.b.) implies f is in L^2 (Parseval) and (again by characterization of an o.n.b.) that S_N converges to f in L^2 (and hence a subsequence converges to f a.e.)
3. By uniqueness of limits $f = g$.

Other stuff:

- For $e_n(x) := e^{2\pi i n x}$, the set $\{e_n\}$ is an orthonormal basis for $L^2([0,1])$.
- For any orthonormal sequence in a Hilbert space, we have Bessel's inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

- When $\{e_n\}$ is a basis, the above is an *equality* (Parseval)
- Arguing uniform convergence: since $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbf{Z})$, we should be able to apply the M test.

Solution (From Neil):
Claim: if $f \in L^1[0,1]$ and $\widehat{f} \in \ell^1(\mathbf{Z})$, then $S_N f \to f$ uniformly.

- Since $\widehat{f} \in \ell^1(\mathbf{Z})$, we have $S_N f \to g$ uniformly for some continuous g by the M-test.
- Now consider $\widehat{g}$. We have

$$\widehat{g}(n) = \int_0^1 \sum_m \left(\widehat{f}(m) e_m(x)\right) e_{-n}(x)\, dx = \widehat{f}(n),$$

  using that $\int_0^1 e_n(x)\, dx = \chi_{n=0}$.

- We'll now show $f - g = 0$ a.e. by mollifying against an approximate identity $\varphi \in L^1$, setting

$$\varphi_\varepsilon(x) := \varepsilon^{-1} \varphi(\varepsilon^{-1} x) \in L^1[0,1].$$

- A computation:

$$\begin{aligned}
\widehat{f * \varphi_\varepsilon}(n) &= \widehat{f} \cdot \widehat{\varphi_\varepsilon}(n) \\
&= \widehat{g} \cdot \widehat{\varphi_\varepsilon}(n) \\
&= \widehat{g * \varphi_\varepsilon}(n),
\end{aligned}$$

so

$$\widehat{(f-g)*\varphi_\varepsilon} = 0 \quad \forall n \implies (f-g)*\varphi_\varepsilon \equiv 0,$$

using that $(f-g)*\varphi_\varepsilon \in L^2$ and $\{e_n\}$ for a complete orthonormal basis of L^2.

- Now use that $(f-g)*\varphi_\varepsilon \to (f-g)$ in L^1 and $(f-g)*\varphi_\varepsilon \equiv 0$ to conclude $f-g=0$ a.e.

Solution (Part 1):

Claim: $\ell^1(\mathbf{Z}) \subseteq \ell^2(\mathbf{Z})$.

Proof (?).

- Set $\mathbf{c} = \left\{ c_k \;\middle|\; k \in \mathbf{Z} \right\} \in \ell^1(\mathbf{Z})$.
- It suffices to show that if $\sum_{k\in\mathbf{Z}} |c_k| < \infty$ then $\sum_{k\in\mathbf{Z}} |c_k|^2 < \infty$.
- Let $S = \left\{ c_k \;\middle|\; |c_k| \leq 1 \right\}$, then $c_k \in S \implies |c_k|^2 \leq |c_k|$
- Claim: S^c can only contain finitely many elements, all of which are finite.
  - If not, either $S^c := \{c_j\}_{j=1}^\infty$ is infinite with every $|c_j| > 1$, which forces

$$\sum_{c_k \in S^c} |c_k| = \sum_{j=1}^\infty |c_j| > \sum_{j=1}^\infty 1 = \infty.$$

  - If any $c_j = \infty$, then $\sum_{k\in\mathbf{Z}} |c_k| \geq c_j = \infty$.

- So S^c is a finite set of finite integers, let $N = \max\left\{ |c_j|^2 \;\middle|\; c_j \in S^c \right\} < \infty$.
- Rewrite the sum

$$\begin{aligned}
\sum_{k\in\mathbf{Z}} |c_k|^2 &= \sum_{c_k \in S} |c_k|^2 + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \sum_{c_k \in S} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \sum_{k\in\mathbf{Z}} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \quad \text{since the } |c_k| \text{ are all positive} \\
&= \|\mathbf{c}\|_{\ell^1} + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \|\mathbf{c}\|_{\ell^1} + |S^c| \cdot N \\
&< \infty.
\end{aligned}$$

■

Claim: $L^2([0,1]) \subseteq L^1([0,1])$.

Proof (?).

- It suffices to show that $\int |f|^2 < \infty \implies \int |f| < \infty$.
- Define $S = \left\{ x \in [0,1] \;\middle|\; |f(x)| \leq 1 \right\}$, then $x \in S^c \implies |f(x)|^2 \geq |f(x)|$.
- Break up the integral:

$$\begin{aligned}
\int_{\mathbf{R}} |f| &= \int_S |f| + \int_{S^c} |f| \\
&\leq \int_S |f| + \int_{S^c} |f|^2 \\
&\leq \int_S |f| + \|f\|_2 \\
&\leq \sup_{x \in S} \{|f(x)|\} \cdot \mu(S) + \|f\|_2 \\
&= 1 \cdot \mu(S) + \|f\|_2 \quad \text{by definition of } S \\
&\leq 1 \cdot \mu([0,1]) + \|f\|_2 \quad \text{since } S \subseteq [0,1] \\
&= 1 + \|f\|_2 \\
&< \infty.
\end{aligned}$$

■

Note: this proof shows $L^2(X) \subseteq L^1(X)$ whenever $\mu(X) < \infty$.

Solution (Part 2):

- First, $S_N f$ converges in $\mathcal{H}$ to something, say $g := \lim_{n\to\infty} S_n f$, since

$$\|g - S_N f\| = \left\| \sum_{|n| \geq N} \widehat{f}(n) e_n(x) \right\| \leq \sum_{|n| \geq N} \left| \widehat{f}(n) \right| \overset{N\to\infty}{\longrightarrow} 0,$$

  where the last term is the tail of a convergent sum since $\left\{ \widehat{f}(n) \right\} \in \ell^1$.
- This also shows that $S_N \to g$ uniformly.
- g is continuous, as the uniform limit of continuous functions.
- Showing that $g = f$ a.e.: it suffices to show that S_N converges to f in L^p for some p, since this will provide a subsequence that converges to f a.e..
- Claim: $\widehat{f} \in \ell^1 \subseteq \ell^2$ implies that $f \in L^2$. This follows from Parseval:

$$\infty > \left\| \widehat{f} \right\|_{\ell^2}^2 = \sum_{n \in \mathbf{Z}} \left| \widehat{f}(n) \right|^2 = \int_0^1 |f(z)|^2 \, dz = \|f\|_{L^2}^2.$$

- Claim: $S_N \to f$ in L^2.

- This follows from the fact that $\{e_n\}_{n\in\mathbf{Z}}$ is a complete orthonormal basis, so $f = \sum \langle f,\ e_n\rangle e_n$ uniquely, recognizing $\widehat{f}(n) = \langle f,\ e_n\rangle$, and writing

$$f = \sum_n \langle f,\ e_n\rangle e_n = \sum_n \widehat{f}(n) e_n := \lim_{N\to\infty} S_N f.$$

- So a subsequence $\{S_{N_k}\}_{k\geq 0}$ converges to f a.e.. Since $S_N \to g$ a.e., $f = g$ a.e. by uniqueness of limits.

10.3 Fall 2017.5

Let φ be a compactly supported smooth function that vanishes outside of an interval $[-N, N]$ such that $\int_{\mathbf{R}} \varphi(x)\, dx = 1$.

For $f \in L^1(\mathbf{R})$, define

$$K_j(x) := j\varphi(jx), \qquad f * K_j(x) := \int_{\mathbf{R}} f(x-y) K_j(y)\, dy$$

and prove the following:

1. Each $f * K_j$ is smooth and compactly supported.

2. $$\lim_{j\to\infty} \|f * K_j - f\|_1 = 0$$

Hint:

$$\lim_{y\to 0} \int_{\mathbf{R}} |f(x-y) - f(x)| dy = 0$$

Add concepts.

Solution:

Concepts Used:

- ?

Part a
Lemma: If $\varphi \in C_c^1$, then $(f * \varphi)' = f * \varphi'$ almost everywhere.

Silly Proof:

$$\begin{aligned}\mathcal{F}((f*\varphi)') &= 2\pi i\xi\ \mathcal{F}(f*\varphi)\\ &= 2\pi i\xi\ \mathcal{F}(f)\ \mathcal{F}(\varphi)\\ &= \mathcal{F}(f)\cdot(2\pi i\xi\ \mathcal{F}(\varphi))\\ &= \mathcal{F}(f)\cdot\mathcal{F}(\varphi')\\ &= \mathcal{F}(f*\varphi').\end{aligned}$$

Actual proof:

$$\begin{aligned}(f*\varphi)'(x) &= (\varphi*f)'(x)\\ &= \lim_{h\to 0}\frac{(\varphi*f)'(x+h)-(\varphi*f)'(x)}{h}\\ &= \lim_{h\to 0}\int\frac{\varphi(x+h-y)-\varphi(x-y)}{h}f(y)\\ &\overset{DCT}{=}\int\lim_{h\to 0}\frac{\varphi(x+h-y)-\varphi(x-y)}{h}f(y)\\ &= \int\varphi'(x-y)f(y)\\ &= (\varphi'*f)(x)\\ &= (f*\varphi')(x).\end{aligned}$$

To see that the DCT is justified, we can apply the MVT on the interval $[0,h]$ to f to obtain

$$\frac{\varphi(x+h-y)-\varphi(x-y)}{h}=\varphi'(c)\quad c\in[0,h],$$

and since φ' is continuous and compactly supported, φ' is bounded by some $M<\infty$ by the extreme value theorem and thus

$$\begin{aligned}\int\left|\frac{\varphi(x+h-y)-\varphi(x-y)}{h}f(y)\right| &= \int|\varphi'(c)f(y)|\\ &\leq\int|M||f|\\ &= |M|\int|f|<\infty,\end{aligned}$$

since $f\in L^1$ by assumption, so we can take $g:=|M||f|$ as the dominating function.
Applying this theorem infinitely many times shows that $f*\varphi$ is smooth.
To see that $f*\varphi$ is compactly supported, approximate f by a *continuous* compactly supported function h, so $\|h-f\|_1\overset{L^1}{\to}0$.
Now let $g_x(y)=\varphi(x-y)$, and note that $\mathrm{supp}(g)=x-\mathrm{supp}(\varphi)$ which is still compact.
But since $\mathrm{supp}(h)$ is bounded, there is some N such that

$$|x|>N\implies A_x:=\mathrm{supp}(h)\cap\mathrm{supp}(g_x)=\emptyset$$

and thus

$$\begin{aligned}(h * \varphi)(x) &= \int_{\mathbf{R}} \varphi(x-y)h(y)\ dy \\ &= \int_{A_x} g_x(y)h(y) \\ &= 0,\end{aligned}$$

so $\left\{x \,\middle|\, f * g(x) = 0\right\}$ is open, and its complement is closed and bounded and thus compact.

Part b

$$\begin{aligned}\|f * K_j - f\|_1 &= \int \left| \int f(x-y)K_j(y)\ dy - f(x) \right| dx \\ &= \int \left| \int f(x-y)K_j(y)\ dy - \int f(x)K_j(y)\ dy \right| dx \\ &= \int \left| \int (f(x-y) - f(x))K_j(y)\ dy \right| dx \\ &\leq \int \int |(f(x-y) - f(x))| \cdot |K_j(y)|\ dy\ dx \\ &\overset{FT}{=} \int \int |(f(x-y) - f(x))| \cdot |K_j(y)|\ \mathbf{dx\ dy} \\ &= \int |K_j(y)| \left(\int |(f(x-y) - f(x))|\ dx \right)\ dy \\ &= \int |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy.\end{aligned}$$

We now split the integral up into pieces.

1. Chose δ small enough such that $|y| < \delta \implies \|f - \tau_y f\|_1 < \varepsilon$ by continuity of translation in L^1, and
2. Since φ is compactly supported, choose J large enough such that

$$j > J \implies \int_{|y| \geq \delta} |K_j(y)|\ dy = \int_{|y| \geq \delta} |j\varphi(jy)| = 0$$

Then

$$\begin{aligned}\|f * K_j - f\|_1 &\leq \int |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy \\ &= \int_{|y|<\delta} |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy + \int_{|y|\geq\delta} |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy \\ &= \varepsilon \int_{|y|\geq\delta} |K_j(y)| + 0 \\ &\leq \varepsilon(1) \to 0.\end{aligned}$$

10.4 Spring 2017.5

Let $f, g \in L^2(\mathbf{R})$. Prove that the formula

$$h(x) := \int_{-\infty}^{\infty} f(t)g(x-t)\,dt$$

defines a uniformly continuous function h on $\mathbf{R}$.

10.5 Spring 2015.6

Let $f \in L^1(\mathbf{R})$ and g be a bounded measurable function on $\mathbf{R}$.

1. Show that the convolution $f * g$ is well-defined, bounded, and uniformly continuous on $\mathbf{R}$.
2. Prove that one further assumes that $g \in C^1(\mathbf{R})$ with bounded derivative, then $f * g \in C^1(\mathbf{R})$ and

$$\frac{d}{dx}(f * g) = f * \left(\frac{d}{dx} g\right)$$

10.6 Fall 2014.5

1. Let $f \in C_c^0(\mathbf{R}^n)$, and show

$$\lim_{t \to 0} \int_{\mathbf{R}^n} |f(x+t) - f(x)|\,dx = 0.$$

2. Extend the above result to $f \in L^1(\mathbf{R}^n)$ and show that

$$f \in L^1(\mathbf{R}^n), \quad g \in L^\infty(\mathbf{R}^n) \quad \implies f * g \text{ is bounded and uniformly continuous.}$$

11 | Functional Analysis: General

11.1 Fall 2019.4

Problem 11.1.1 (?)
Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$.

a. Prove that for every $x \in \mathcal{H}$ one has

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

b. Prove that for any sequence $\{a_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$ there exists an element $x \in \mathcal{H}$ such that

$$a_n = \langle x,\ u_n \rangle \text{ for all } n \in \mathbb{N}$$

and

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$$

Concepts Used:

- Bessel's Inequality
- Pythagoras
- Surjectivity of the Riesz map
- Parseval's Identity
- Trick – remember to write out finite sum S_N, and consider $\|x - S_N\|$.

Proof (of a).

- Equivalently, we can show

$$\|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 \geq 0.$$

- Claim: the LHS is the norm of an element in H, and thus non-negative. More precisely, set $S_N := \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n$, then the above is equal to

$$\left\| x - \lim_{N \to \infty} S_N \right\|^2.$$

Note that if this is true, we're done.

- To see this, expand the norm in terms of inner products:

$$\begin{aligned}
\|x - S_N\|^2 &= \langle x - S_N,\ x - S_N\rangle \\
&= \langle x,\ x\rangle - \langle x,\ S_N\rangle - \langle S_N,\ x\rangle + \langle S_N,\ S_N\rangle \\
&= \|x\|^2 + \|S_N\|^2 - \left(\langle x,\ S_N\rangle + \overline{\langle x,\ S_N\rangle}\right) \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\left(\langle x,\ S_N\rangle\right) \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\left(\left\langle x,\ \sum_{n=1}^{N} \langle x,\ u_n\rangle u_n\right\rangle\right) \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\left(\sum_{n=1}^{N} \langle x,\ \langle x,\ u_n\rangle u_n\rangle\right) \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\left(\sum_{n=1}^{N} \overline{\langle x,\ u_n\rangle}\langle x,\ u_n\rangle\right) \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\sum_{n=1}^{N} |\langle x,\ u_n\rangle|^2 \\
&= \|x\|^2 + \|S_N\|^2 - 2\sum_{n=1}^{N} |\langle x,\ u_n\rangle|^2 \\
&= \|x\|^2 + \left\|\sum_{n=1}^{N} \langle x,\ u_n\rangle u_n\right\|^2 - 2\sum_{n=1}^{N} |\langle x,\ u_n\rangle|^2 \\
&= \|x\|^2 + \left\langle \sum_{n=1}^{N} \langle x,\ u_n\rangle u_n,\ \sum_{m=1}^{N} \langle x,\ u_m\rangle u_m \right\rangle - 2\sum_{n=1}^{N} |\langle x,\ u_n\rangle|^2 \\
&= \|x\|^2 + \sum_{n,m\leq N} \langle x,\ u_n\rangle\overline{\langle x,\ u_m\rangle}\langle u_n,\ u_m\rangle - 2\sum_{n=1}^{N} |\langle x,\ u_n\rangle|^2 \\
&= \|x\|^2 + \sum_{n,m\leq N} \langle x,\ u_n\rangle\overline{\langle x,\ u_m\rangle}\delta_{mn} - 2\sum_{n=1}^{N} |\langle x,\ u_n\rangle|^2 \\
&= \|x\|^2 + \sum_{n\leq N} |\langle x,\ u_n\rangle|^2 - 2\sum_{n=1}^{N} |\langle x,\ u_n\rangle|^2 \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n\rangle|^2.
\end{aligned}$$

- Now take $\lim_{N\to\infty}$ and use that $\|-\|$ is continuous.

■

Proof (of b).

- Set

$$x := \sum_{n \in \mathbb{N}} a_n u_n.$$

- Checking the first desired property:

$$\begin{aligned}
\langle x,\ u_m \rangle &= \left\langle \sum_{n \geq 1} a_n u_n,\ u_m \right\rangle \\
&= \sum_{n \geq 1} a_n \langle u_n,\ u_m \rangle \\
&= \sum_{n \geq 1} a_n \delta_{mn} \\
&= a_m.
\end{aligned}$$

- That $x \in H$: this would follow from

$$\|x\|^2 = \sum_n |\langle x,\ u_n \rangle|^2 = \sum_n |a_n|^2 < \infty.$$

  The inequality holds by assumption since $\{a_n\} \in \ell^2$, so it suffices to show the first equality:

$$\begin{aligned}
\|x\|^2 &:= \langle x,\ x \rangle \\
&= \left\langle \sum_n a_n u_n,\ \sum_m a_m u_m \right\rangle \\
&= \sum_{n,m} a_n \overline{a_m} \langle u_n,\ u_m \rangle \\
&= \sum_{n,m} a_n \overline{a_m} \delta_{mn} \\
&= \sum_n a_n \overline{a_n} \\
&= \sum_n |a_n|^2 \\
&= \sum_n |\langle x,\ u_n \rangle|^2.
\end{aligned}$$

■

11.2 Spring 2019.5

a. Show that $L^2([0,1]) \subseteq L^1([0,1])$ and argue that $L^2([0,1])$ in fact forms a dense subset of $L^1([0,1])$.

b. Let Λ be a continuous linear functional on $L^1([0,1])$.

Prove the Riesz Representation Theorem for $L^1([0,1])$ by following the steps below:

i. Establish the existence of a function $g \in L^2([0,1])$ which represents Λ in the sense that

$$\Lambda(f) = f(x)g(x)dx \text{ for all } f \in L^2([0,1]).$$

Hint: You may use, without proof, the Riesz Representation Theorem for $L^2([0,1])$.

ii. Argue that the g obtained above must in fact belong to $L^\infty([0,1])$ and represent Λ in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{for all } f \in L^1([0,1])$$

with

$$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^\vee}$$

Concepts Used:

- Holders' inequality: $\|fg\|_1 \leq \|f\|_p \|f\|_q$
- Riesz Representation for L^2: If $\Lambda \in (L^2)^\vee$ then there exists a unique $g \in L^2$ such that $\Lambda(f) = \int fg$.
- $\|f\|_{L^\infty(X)} \coloneqq \inf \left\{ t \geq 0 \;\middle|\; |f(x)| \leq t \text{ almost everywhere} \right\}$.
- **Lemma**: $m(X) < \infty \implies L^p(X) \subset L^2(X)$.

Proof .

- Write Holder's inequality as $\|fg\|_1 \leq \|f\|_a \|g\|_b$ where $\frac{1}{a} + \frac{1}{b} = 1$, then

$$\|f\|_p^p = \||f|^p\|_1 \leq \||f|^p\|_a \ \|1\|_b.$$

- Now take $a = \frac{2}{p}$ and this reduces to

$$\begin{aligned} &\|f\|_p^p \leq \|f\|_2^p \ m(X)^{\frac{1}{b}} \\ \implies &\|f\|_p \leq \|f\|_2 \cdot O(m(X)) < \infty. \end{aligned}$$

■

Proof (of a).

- Note $X = [0,1] \implies m(X) = 1$.
- By Holder's inequality with $p = q = 2$,

$$\|f\|_1 = \|f \cdot 1\|_1 \leq \|f\|_2 \cdot \|1\|_2 = \|f\|_2 \cdot m(X)^{\frac{1}{2}} = \|f\|_2,$$

- Thus $L^2(X) \subseteq L^1(X)$
- Since they share a common dense subset (simple functions), L^2 is dense in L^1

■

Proof (of b, Existence of g representing Λ).
For all of part b, let $\Lambda \in L^1(X)^\vee$ be arbitrary. Let $f \in L^2 \subseteq L^1$ be arbitrary.
Claim: $\Lambda \in L^1(X)^\vee \implies \Lambda \in L^2(X)^\vee$.

- Suffices to show that $\|\Gamma\|_{L^2(X)^\vee} \coloneqq \sup_{\|f\|_2 = 1} |\Gamma(f)| < \infty$, since bounded implies continuous.
- By the lemma, $\|f\|_1 \leq C\|f\|_2$ for some constant $C \approx m(X)$.
- Note

$$\|\Lambda\|_{L^1(X)^\vee} \coloneqq \sup_{\|f\|_1 = 1} |\Lambda(f)|$$

- Define $\widehat{f} = \frac{f}{\|f\|_1}$ so $\left\|\widehat{f}\right\|_1 = 1$
- Since $\|\Lambda\|_{1^\vee}$ is a supremum over *all* $f \in L^1(X)$ with $\|f\|_1 = 1$,

$$\left|\Lambda(\widehat{f})\right| \leq \|\Lambda\|_{(L^1(X))^\vee},$$

- Then

$$\begin{aligned}\frac{|\Lambda(f)|}{\|f\|_1} &= \left|\Lambda(\widehat{f})\right| \leq \|\Lambda\|_{L^1(X)^\vee} \\ \implies |\Lambda(f)| &\leq \|\Lambda\|_{1^\vee} \cdot \|f\|_1 \\ &\leq \|\Lambda\|_{1^\vee} \cdot C\|f\|_2 < \infty \quad \text{by assumption,}\end{aligned}$$

- So $\Lambda \in (L^2)^\vee$.

Now apply Riesz Representation for L^2: there is a $g \in L^2$ such that

$$f \in L^2 \implies \Lambda(f) = \langle f, \ g \rangle := \int_0^1 f(x)\overline{g(x)}\,dx.$$

■

Proof (of b, g is in L^∞).

- It suffices to show $\|g\|_{L^\infty(X)} < \infty$.
- Since we're assuming $\|\Gamma\|_{L^1(X)^\vee} < \infty$, it suffices to show the stated equality.

Is this assumed..? Or did we show it..?

- Claim: $\|\Lambda\|_{L^1(X)^\vee} = \|g\|_{L^\infty(X)}$
  - The result will follow since Λ was assumed to be in $L^1(X)^\vee$, so $\|\Lambda\|_{L^1(X)^\vee} < \infty$.
  - $\leq$:

$$\begin{aligned}\|\Lambda\|_{L^1(X)^\vee} &= \sup_{\|f\|_1=1} |\Lambda(f)| \\ &= \sup_{\|f\|_1=1} \left|\int_X f\overline{g}\right| \quad \text{by (i)} \\ &= \sup_{\|f\|_1=1} \int_X |f\overline{g}| \\ &:= \sup_{\|f\|_1=1} \|fg\|_1 \\ &\leq \sup_{\|f\|_1=1} \|f\|_1\|g\|_\infty \quad \text{by Holder with } p=1, q=\infty \\ &= \|g\|_\infty,\end{aligned}$$

  - $\geq$:
    - ◇ Suppose toward a contradiction that $\|g\|_\infty > \|\Lambda\|_{1^\vee}$.
    - ◇ Then there exists some $E \subseteq X$ with $m(E) > 0$ such that

$$x \in E \implies |g(x)| > \|\Lambda\|_{L^1(X)^\vee}.$$

◇ Define

$$h = \frac{1}{m(E)} \frac{\overline{g}}{|g|} \chi_E.$$

◇ Note $\|h\|_{L^1(X)} = 1$.

◇ Then

$$\begin{aligned}
\Lambda(h) &= \int_X hg \\
&:= \int_X \frac{1}{m(E)} \frac{g\overline{g}}{|g|} \chi_E \\
&= \frac{1}{m(E)} \int_E |g| \\
&\geq \frac{1}{m(E)} \|g\|_\infty m(E) \\
&= \|g\|_\infty \\
&> \|\Lambda\|_{L^1(X)^\vee},
\end{aligned}$$

a contradiction since $\|\Lambda\|_{L^1(X)^\vee}$ is the supremum over all h_α with $\|h_\alpha\|_{L^1(X)} = 1$.

■

11.3 Spring 2016.6

Without using the Riesz Representation Theorem, compute

$$\sup \left\{ \left| \int_0^1 f(x) e^x \, dx \right| \;\middle|\; f \in L^2([0,1], m), \;\; \|f\|_2 \leq 1 \right\}$$

11.4 Spring 2015.5

Let $\mathcal{H}$ be a Hilbert space.

1. Let $x \in \mathcal{H}$ and $\{u_n\}_{n=1}^N$ be an orthonormal set. Prove that the best approximation to x in $\mathcal{H}$ by an element in $\operatorname{span}_{\mathbf{C}} \{u_n\}$ is given by

$$\widehat{x} := \sum_{n=1}^N \langle x, \; u_n \rangle u_n.$$

2. Conclude that finite dimensional subspaces of $\mathcal{H}$ are always closed.

11.5 Fall 2015.6

Let $f : [0,1] \to \mathbf{R}$ be continuous. Show that

$$\sup\left\{\|fg\|_1 \;\middle|\; g \in L^1[0,1],\ \|g\|_1 \le 1\right\} = \|f\|_\infty$$

11.6 Fall 2014.6

Let $1 \le p, q \le \infty$ be conjugate exponents, and show that

$$f \in L^p(\mathbf{R}^n) \implies \|f\|_p = \sup_{\|g\|_q = 1} \left|\int f(x)g(x)dx\right|$$

12 | Banach and Hilbert Spaces

12.1 Fall 2021.5

Consider the Hilbert space $\mathcal{H} = L^2([0,1])$.

a. Prove that of $E \subset \mathcal{H}$ is closed and convex then E contains an element of smallest norm.

Hint: Show that if $\|f_n\|_2 \to \min\{f \in E : \|f\|_2\}$ then $\{f_n\}$ is a Cauchy sequence.

b. Construct a non-empty closed subset $E \subset \mathcal{H}$ which does not contain an element of smallest norm.

12.2 Spring 2019.1

Let $C([0,1])$ denote the space of all continuous real-valued functions on $[0,1]$.

a. Prove that $C([0,1])$ is complete under the uniform norm $\|f\|_u := \sup_{x\in[0,1]} |f(x)|$.

b. Prove that $C([0,1])$ is not complete under the L^1-norm $\|f\|_1 = \int_0^1 |f(x)|\, dx$.

Add concepts.

Solution:

Proof (of a).

- Let $\{f_n\}$ be a Cauchy sequence in $C(I, \|-\|_\infty)$, so $\lim_n \lim_m \|f_m - f_n\|_\infty = 0$, we will show it converges to some f in this space.
- For each fixed $x_0 \in [0,1]$, the sequence of real numbers $\{f_n(x_0)\}$ is Cauchy in $\mathbf{R}$ since

$$x_0 \in I \implies |f_m(x_0) - f_n(x_0)| \leq \sup_{x \in I} |f_m(x) - f_n(x)| := \|f_m - f_n\|_\infty \overset{m>n\to\infty}{\to} 0,$$

- Since $\mathbf{R}$ is complete, this sequence converges and we can define $f(x) := \lim_{k\to\infty} f_n(x)$.
- Thus $f_n \to f$ pointwise by construction
- Claim: $\|f - f_n\| \overset{n\to\infty}{\to} 0$, so f_n converges to f in $C([0,1], \|-\|_\infty)$.
  - Proof:
    - ♢ Fix $\varepsilon > 0$; we will show there exists an N such that $n \geq N \implies \|f_n - f\| < \varepsilon$
    - ♢ Fix an $x_0 \in I$. Since $f_n \to f$ pointwise, choose N_1 large enough so that

$$n \geq N_1 \implies |f_n(x_0) - f(x_0)| < \varepsilon/2.$$

    - ♢ Since $\|f_n - f_m\|_\infty \to 0$, choose and N_2 large enough so that

$$n, m \geq N_2 \implies \|f_n - f_m\|_\infty < \varepsilon/2.$$

    - ♢ Then for $n, m \geq \max(N_1, N_2)$, we have

$$\begin{aligned}
|f_n(x_0) - f(x_0)| &= |f_n(x_0) - f(x_0) + f_m(x_0) - f_m(x_0)| \\
&= |f_n(x_0) - f_m(x_0) + f_m(x_0) - f(x_0)| \\
&\leq |f_n(x_0) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\
&< |f_n(x_0) - f_m(x_0)| + \frac{\varepsilon}{2} \\
&\leq \sup_{x\in I} |f_n(x) - f_m(x)| + \frac{\varepsilon}{2} \\
&< \|f_n - f_m\|_\infty + \frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
\implies |f_n(x_0) - f(x_0)| &< \varepsilon \\
\implies \sup_{x\in I} |f_n(x_0) - f(x_0)| &\leq \sup_{x \in I} \varepsilon \quad \text{by order limit laws} \\
\implies \|f_n - f\| &\leq \varepsilon
\end{aligned}$$

.

- f is the uniform limit of continuous functions and thus continuous, so $f \in C([0,1])$.

■

Proof (of b).

- It suffices to produce a Cauchy sequence that does not converge to a continuous function.
- Take the following sequence of functions:
  - f_1 increases linearly from 0 to 1 on $[0, 1/2]$ and is 1 on $[1/2, 1]$
  - f_2 is 0 on $[0, 1/4]$ increases linearly from 0 to 1 on $[1/4, 1/2]$ and is 1 on $[1/2, 1]$
  - f_3 is 0 on $[0, 3/8]$ increases linearly from 0 to 1 on $[3/8, 1/2]$ and is 1 on $[1/2, 1]$
  - f_3 is 0 on $[0, (1/2 - 3/8)/2]$ increases linearly from 0 to 1 on $[(1/2 - 3/8)/2, 1/2]$ and is 1 on $[1/2, 1]$

Idea: take sequence starting points for the triangles: $0, 0 + \frac{1}{4}, 0 + \frac{1}{4} + \frac{1}{8}, \cdots$ *which converges to* $1/2$ *since* $\sum_{k=1}^{\infty} \frac{1}{2^k} = -\frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{2^k}$.

- Then each f_n is clearly integrable, since its graph is contained in the unit square.
- $\{f_n\}$ is Cauchy: geometrically subtracting areas yields a single triangle whose area tends to 0.
- But f_n converges to $\chi_{[\frac{1}{2}, 1]}$ which is discontinuous.

show that $\int_0^1 |f_n(x) - f_m(x)|\, dx \to 0$ rigorously, show that no $g \in L^1([0, 1])$ can converge to this indicator function.

■

12.3 Spring 2017.6

Show that the space $C^1([a, b])$ is a Banach space when equipped with the norm

$$\|f\| := \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |f'(x)|.$$

Add concepts.

Concepts Used:

- See https://math.stackexchange.com/questions/507263/prove-that-c1a-b-with-the-c1-norm-is-a-banach-space/

Solution:

- Denote this norm $\|-\|_u$
- Let f_n be a Cauchy sequence in this space, so $\|f_n\|_u < \infty$ for every n and $\|f_j - f_k\|_u \overset{j,k\to\infty}{\to} 0$.

and define a candidate limit: for each $x \in I$, set

$$f(x) \coloneqq \lim_{n\to\infty} f_n(x).$$

- Note that

$$\begin{aligned}\|f_n\|_\infty &\leq \|f_n\|_u < \infty \\ \|f_n'\|_\infty &\leq \|f_n\|_u < \infty.\end{aligned}$$

  - Thus both f_n, f_n' are Cauchy sequences in $C^0([a,b], \|-\|_\infty)$, which is a Banach space, so they converge.

- So
  - $f_n \to f$ uniformly (by uniqueness of limits),
  - $f_n' \to g$ uniformly for some g, and
  - $f, g \in C^0([a,b])$.

- Claim: $g = f'$
  - For any fixed $a \in I$, we have

$$\begin{aligned}f_n(x) - f_n(a) &\overset{u}{\to} f(x) - f(a) \\ \int_a^x f_n' &\overset{u}{\to} \int_a^x g.\end{aligned}$$

  - By the FTC, the left-hand sides are equal.
  - By uniqueness of limits so are the right-hand sides, so $f' = g$.

- Claim: the limit f is an element in this space.
  - Since $f, f' \in C^0([a,b])$, they are bounded, and so $\|f\|_u < \infty$.

- Claim: $\|f_n - f\|_u \overset{n\to\infty}{\to} 0$
- Thus the Cauchy sequence $\{f_n\}$ converges to a function f in the u-norm where f is an element of this space, making it complete.

12.4 Fall 2017.6

Let X be a complete metric space and define a norm

$$\|f\| := \max\{|f(x)| : x \in X\}.$$

Show that $(C^0(\mathbf{R}), \|{-}\|)$ (the space of continuous functions $f : X \to \mathbf{R}$) is complete.

Add concepts.

Shouldn't this be a supremum? The max may not exist?

Review and clean up.

Solution:

Let $\{f_k\}$ be a Cauchy sequence, so $\|f_k\| < \infty$ for all k. Then for a fixed x, the sequence $f_k(x)$ is Cauchy in $\mathbf{R}$ and thus converges to some $f(x)$, so define f by $f(x) := \lim_{k\to\infty} f_k(x)$.
Then $\|f_k - f\| = \max_{x\in X} |f_k(x) - f(x)| \overset{k\to\infty}{\to} 0$, and thus $f_k \to f$ uniformly and thus f is continuous. It just remains to show that f has bounded norm.
Choose N large enough so that $\|f - f_N\| < \varepsilon$, and write $\|f_N\| := M < \infty$

$$\|f\| \leq \|f - f_N\| + \|f_N\| < \varepsilon + M < \infty.$$

13 | Extras

Exercise 13.0.1 (?)
Compute the following limits:

- $\lim_{n\to\infty} \sum_{k\geq 1} \frac{1}{k^2} \sin^n(k)$
- $\lim_{n\to\infty} \sum_{k\geq 1} \frac{1}{k} e^{-k/n}$

Solution:
For the first, use that

$$\left| \sum_{k\geq 1} \frac{1}{k^2} \sin^n(k) \right| \leq \sum_{k\geq 1} \left| \frac{1}{k^2} \sin^n(k) \right| \sum_{k\geq 1} \left| \frac{1}{k^2} \right| < \infty,$$

since $|\sin(x)| \leq 1$ and $x^n < x$ for $|x| \leq 1$. By the dominated convergence theorem, we can pass the limit inside. Using the same fact as above, $\lim_{n\to\infty} \sin^n(x) = 0$, For the second, the claim is that it diverges (very slowly). Note that $\lim_{n\to\infty} e^{-k/n} = 1$ for any k. By Fatou, we have

$$\liminf_{n\to\infty} \sum_{k\geq 1} \frac{e^{-k/n}}{k} \geq \sum_{k\geq 1} \liminf_{n\to\infty} \frac{e^{-k/n}}{k} = \sum_{k\geq 1} \frac{1}{k} = \infty.$$

Exercise 13.0.2 (?)
Let $(\Omega, \mathcal{B})$ be a measurable space with a Borel σ-algebra and $\mu_n : \mathcal{B} \to [0, \infty]$ be a σ-additive measure for each n. Show that the following map is again a σ-additive measure on $\mathcal{B}$:

$$\mu(B) := \sum_{n\geq 1} \mu_n(B).$$

Solution:
Apply Fubini-Tonelli to commute two sums:

$$\begin{aligned}
\mu\left(\bigcup_{1\leq k\leq M} E_k\right) := &= \sum_{n\geq 1} \mu_n \left(\bigcup_{1\leq k\leq M} E_k\right) \\
&= \sum_{n\geq 1} \sum_{1\leq k\leq M} \mu_n\left(E_k\right) \\
&= \sum_{1\leq k\leq M} \sum_{n\geq 1} \mu_n\left(E_k\right) \text{FT} \\
&:= \sum_{1\leq k\leq M} \mu(E_k).
\end{aligned}$$

14 | Extra Problems: Measure Theory

14.1 Greatest Hits

- $\star$: Show that for $E \subseteq \mathbf{R}^n$, TFAE:

  1. E is measurable
  2. $E = H \cup Z$ here H is F_σ and Z is null
  3. $E = V \setminus Z'$ where $V \in G_\delta$ and Z' is null.

- $\star$: Show that if $E \subseteq \mathbf{R}^n$ is measurable then $m(E) = \sup\left\{m(K) \;\middle|\; K \subset E \text{ compact}\right\}$ iff for all $\varepsilon > 0$ there exists a compact $K \subseteq E$ such that $m(K) \geq m(E) - \varepsilon$.

- $\star$: Show that cylinder functions are measurable, i.e. if f is measurable on $\mathbf{R}^s$, then $F(x,y) := f(x)$ is measurable on $\mathbf{R}^s \times \mathbf{R}^t$ for any t.

- $\star$: Prove that the Lebesgue integral is translation invariant, i.e. if $\tau_h(x) = x + h$ then $\int \tau_h f = \int f$.

- $\star$: Prove that the Lebesgue integral is dilation invariant, i.e. if $f_\delta(x) = \frac{f(\frac{x}{\delta})}{\delta^n}$ then $\int f_\delta = \int f$.

- $\star$: Prove continuity in L^1, i.e.

$$f \in L^1 \implies \lim_{h\to 0} \int |f(x+h) - f(x)| = 0.$$

- $\star$: Show that

$$f, g \in L^1 \implies f * g \in L^1 \quad \text{and} \quad \|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

- $\star$: Show that if $X \subseteq \mathbf{R}$ with $\mu(X) < \infty$ then

$$\|f\|_p \overset{p\to\infty}{\to} \|f\|_\infty.$$

14.2 Topology

- Show that every compact set is closed and bounded.
- Show that if a subset of a metric space is complete and totally bounded, then it is compact.
- Show that if K is compact and F is closed with K, F disjoint then $\operatorname{dist}(K, F) > 0$.

14.3 Continuity

- Show that a continuous function on a compact set is uniformly continuous.

14.4 Differentiation

- Show that if $f \in C^1(\mathbf{R})$ and both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ exist, then $\lim_{x\to\infty} f'(x)$ must be zero.

14.5 Advanced Limitology

- If f is continuous, is it necessarily the case that f' is continuous?
- If $f_n \to f$, is it necessarily the case that f_n' converges to f' (or at all)?
- Is it true that the sum of differentiable functions is differentiable?
- Is it true that the limit of integrals equals the integral of the limit?
- Is it true that a limit of continuous functions is continuous?
- Show that a subset of a metric space is closed iff it is complete.
- Show that if $m(E) < \infty$ and $f_n \to f$ uniformly, then $\lim \int_E f_n = \int_E f$.

14.6 Uniform Convergence

- Show that a uniform limit of bounded functions is bounded.
- Show that a uniform limit of continuous function is continuous.
  - I.e. if $f_n \to f$ uniformly with each f_n continuous then f is continuous.
- Show that
  - $f_n : [a, b] \to \mathbf{R}$ are continuously differentiable with derivatives f_n'
  - The sequence of derivatives f_n' converges uniformly to some function g
  - There exists *at least one* point x_0 such that $\lim_n f_n(x_0)$ exists,
  - Then $f_n \to f$ uniformly to some differentiable f, and $f' = g$.
- Prove that uniform convergence implies pointwise convergence implies a.e. convergence, but none of the implications may be reversed.
- Show that $\sum \frac{x^n}{n!}$ converges uniformly on any compact subset of $\mathbf{R}$.

14.7 Measure Theory

- Show that continuity of measure from above/below holds for outer measures.
- Show that a countable union of null sets is null.

Measurability

- Show that $f = 0$ a.e. iff $\int_E f = 0$ for every measurable set E.

Integrability

- Show that if f is a measurable function, then $f = 0$ a.e. iff $\int f = 0$.

- Show that a bounded function is Lebesgue integrable iff it is measurable.
- Show that simple functions are dense in L^1.
- Show that step functions are dense in L^1.
- Show that smooth compactly supported functions are dense in L^1.

14.8 Convergence

- Prove Fatou's lemma using the Monotone Convergence Theorem.
- Show that if $\{f_n\}$ is in L^1 and $\sum \int |f_n| < \infty$ then $\sum f_n$ converges to an L^1 function and

$$\int \sum f_n = \sum \int f_n.$$

14.9 Convolution

- Show that if f, g are continuous and compactly supported, then so is $f * g$.
- Show that if $f \in L^1$ and g is bounded, then $f * g$ is bounded and uniformly continuous.
- If f, g are compactly supported, is it necessarily the case that $f * g$ is compactly supported?
- Show that under any of the following assumptions, $f * g$ vanishes at infinity:
  - $f, g \in L^1$ are both bounded.
  - $f, g \in L^1$ with just g bounded.
  - f, g smooth and compactly supported (and in fact $f * g$ is smooth)
  - $f \in L^1$ and g smooth and compactly supported (and in fact $f * g$ is smooth)
- Show that if $f \in L^1$ and g' exists with $\frac{\partial g}{\partial x_i}$ all bounded, then

$$\frac{\partial}{\partial x_i}(f * g) = f * \frac{\partial g}{\partial x_i}$$

14.10 Fourier Analysis

- Show that if $f \in L^1$ then $\widehat{f}$ is bounded and uniformly continuous.
- Is it the case that $f \in L^1$ implies $\widehat{f} \in L^1$?
- Show that if $f, \widehat{f} \in L^1$ then f is bounded, uniformly continuous, and vanishes at infinity.
  - Show that this is not true for arbitrary L^1 functions.
- Show that if $f \in L^1$ and $\widehat{f} = 0$ almost everywhere then $f = 0$ almost everywhere.
  - Prove that $\widehat{f} = \widehat{g}$ implies that $f = g$ a.e.

- Show that if $f, g \in L^1$ then
$$\int \widehat{f} g = \int f \widehat{g}.$$
  - Give an example showing that this fails if g is not bounded.
- Show that if $f \in C^1$ then f is equal to its Fourier *series*.

14.11 Approximate Identities

- Show that if φ is an approximate identity, then
$$\|f * \varphi_t - f\|_1 \overset{t\to 0}{\to} 0.$$
  - Show that if additionally $|\varphi(x)| \leq c(1+|x|)^{-n-\varepsilon}$ for some $c, \varepsilon > 0$, then this converges is almost everywhere.
- Show that is f is bounded and uniformly continuous and φ_t is an approximation to the identity, then $f * \varphi_t$ uniformly converges to f.

L^p Spaces

- Show that if $E \subseteq \mathbf{R}^n$ is measurable with $\mu(E) < \infty$ and $f \in L^p(X)$ then
$$\|f\|_{L^p(X)} \overset{p\to\infty}{\to} \|f\|_\infty.$$
- Is it true that the converse to the DCT holds? I.e. if $\int f_n \to \int f$, is there a $g \in L^p$ such that $f_n < g$ a.e. for every n?
- Prove continuity in L^p: If f is uniformly continuous then for all p,
$$\|\tau_h f - f\|_p \overset{h\to 0}{\to} 0.$$
- Prove the following inclusions of L^p spaces for $m(X) < \infty$:
$$L^\infty(X) \subset L^2(X) \subset L^1(X)$$
$$\ell^2(\mathbf{Z}) \subset \ell^1(\mathbf{Z}) \subset \ell^\infty(\mathbf{Z}).$$

14.12 Unsorted

Proposition 14.12.1 ***(Volumes of Rectangles).***

If $\{R_j\} \rightrightarrows R$ is a covering of R by rectangles,

$$R = \coprod_j^{\circ} R_j \implies |R| = \sum |R|_j$$
$$R \subseteq \bigcup_j R_j \implies |R| \leq \sum |R|_j.$$

- Show that any disjoint intervals is countable.
- Show that every open $U \subseteq \mathbf{R}$ is a countable union of disjoint open intervals.
- Show that every open $U \subseteq \mathbf{R}^n$ is a countable union of *almost* disjoint closed cubes.
- Show that that Cantor middle-thirds set is compact, totally disconnected, and perfect, with outer measure zero.
- Prove the Borel-Cantelli lemma.

15 | Extra Problems from Problem Sets

15.1 Continuous on compact implies uniformly continuous

Problem 15.1.1 (?)
Show that a continuous function on a compact set is uniformly continuous.

Solution:
Use a stronger result: a continuous function on a compact metric is uniformly continuous. Fix ε. Suppose f is continuous, then for each $z \in X$ choose $\delta_z = \delta(\varepsilon, z)$ to ensure $B_\delta(z) \hookrightarrow B_\varepsilon(f(z))$ and form the cover $\{B_{\delta_z}(z)\}_{x \in X} \rightrightarrows X$. By compactness, choose a finite subcover corresponding to $\{z_1, \cdots, z_m\}$ and choose $\delta = \min\{\delta_1, \cdots, \delta_m\}$. The claim is that this δ works for uniform continuity: if $|x - y| < \delta$ then $|x - y| < \delta_i$ for all i. Note that $x \in B_{\delta_z}(z)$ for one of the finitely many z above, and if we adjust δ to $\delta/2$, we can arrange so that both $x, y \in B_{\delta_z}(z)$ for some z, since

$$|x - y| = |x - z + z - y| \leq |x - z| + |z - y| < \frac{\delta}{2} + \frac{\delta}{2} = \delta < \delta_z,$$

and similarly

$$|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| < \varepsilon + \varepsilon,$$

so just adjust the original ε chosen by the continuity of f to $\varepsilon/2$.

15.2 2010 6.1

Problem 15.2.1 (?)
Show that

$$\int_{\mathbb{B}^n} \frac{1}{|x|^p}\, dx < \infty \iff p < n$$

$$\int_{\mathbf{R}^n \setminus \mathbb{B}^n} \frac{1}{|x|^p}\, dx < \infty \iff p > n.$$

Solution:
Todo

15.3 2010 6.2

Show that

$$\int_{\mathbf{R}^n} |f| = \int_0^\infty m(A_t)\, dt \qquad\qquad A_t := \left\{ x \in \mathbf{R}^n \;\middle|\; |f(x)| > t \right\}.$$

Solution:
Todo

15.4 2010 6.5

Suppose $F \subseteq \mathbf{R}$ with $m(F^c) < \infty$ and let $\delta(x) := d(x, F)$ and

$$I_F(x) := \int_{\mathbf{R}} \frac{\delta(y)}{|x-y|^2}\, dy.$$

a. Show that δ is continuous.

b. Show that if $x \in F^c$ then $I_F(x) = \infty$.

c. Show that $I_F(x) < \infty$ for almost every x

Solution:
Todo

15.5 2010 7.1

Let $(X, \mathcal{M}, \mu)$ be a measure space and prove the following properties of $L^\infty(X, \mathcal{M}, \mu)$:

- If f, g are measurable on X then

$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty.$$

- $\|-\|_\infty$ is a norm on L^∞ making it a Banach space.
- $\|f_n - f\|_\infty \overset{n\to\infty}{\to} 0 \iff$ there exists an $E \in \mathcal{M}$ such that $\mu(X \setminus E) = 0$ and $f_n \to f$ uniformly on E.
- Simple functions are dense in L^∞.

15.6 2010 7.2

Show that for $0 < p < q \leq \infty$, $\|a\|_{\ell^q} \leq \|a\|_{\ell^p}$ over $\mathbf{C}$, where $\|a\|_\infty := \sup_j |a_j|$.

15.7 2010 7.3

Let f, g be non-negative measurable functions on $[0, \infty)$ with

$$A := \int_0^\infty f(y) y^{-1/2}\, dy < \infty$$
$$B := \left(\int_0^\infty |g(y)| \right)^2 dy < \infty.$$

Show that

$$\int_0^\infty \left(\int_0^\infty f(y)\, dy \right) \frac{g(x)}{x}\, dx \leq AB.$$

15.8 2010 7.4

Let $(X, \mathcal{M}, \mu)$ be a measure space and $0 < p < q < \infty$. Prove that if $L^q(X) \subseteq L^p(X)$, then X does not contain sets of arbitrarily large finite measure.

15.9 2010 7.5

Suppose $0 < a < b \leq \infty$, and find examples of functions $f \in L^p((0, \infty))$ if and only if:

- $a < p < b$
- $a \leq p \leq b$
- $p = a$

Hint: consider functions of the following form:

$$f(x) := x^{-\alpha} |\log(x)|^{\beta}.$$

15.10 2010 7.6

Define

$$F(x) := \left(\frac{\sin(\pi x)}{\pi x} \right)^2$$
$$G(x) := \begin{cases} 1 - |x| & |x| \leq 1 \\ 0 & \text{else.} \end{cases}$$

a. Show that $\widehat{G}(\xi) = F(\xi)$

b. Compute $\widehat{F}$.

c. Give an example of a function $g \notin L^1(\mathbf{R})$ which is the Fourier transform of an L^1 function.

Hint: write $\widehat{G}(\xi) = H(\xi) + H(-\xi)$ *where*

$$H(\xi) := e^{2\pi i \xi} \int_0^1 y e^{2\pi i y \xi} \, dy.$$

15.11 2010 7.7

Show that for each $\epsilon > 0$ the following function is the Fourier transform of an $L^1(\mathbf{R}^n)$ function:

$$F(\xi) := \left(\frac{1}{1+|\xi|^2}\right)^{\epsilon}.$$

Hint: show that

$$\begin{aligned}
K_\delta(x) &:= \delta^{-n/2} e^{\frac{-\pi|x|^2}{\delta}} \\
f(x) &:= \int_0^\infty K_\delta(x) e^{-\pi\delta} \delta^{\epsilon-1}\, d\delta \\
\Gamma(s) &:= \int_0^\infty e^{-t} t^{s-1}\, dt \\
\implies \widehat{f}(\xi) &= \int_0^\infty e^{-\pi\delta|\xi|^2} e^{-\pi\delta} \delta^{\epsilon-1} = \pi^{-s}\Gamma(\epsilon) F(\xi).
\end{aligned}$$

15.12 2010 7 Challenge 1: Generalized Holder

Suppose that

$$1 \le p_j \le \infty, \qquad \sum_{j=1}^n \frac{1}{p_j} = \frac{1}{r} \le 1.$$

Show that if $f_j \in L^{p_j}$ for each $1 \le j \le n$, then

$$\prod f_j \in L^r, \qquad \left\|\prod f_j\right\|_r \le \prod \|f_j\|_{p_j}.$$

15.13 2010 7 Challenge 2: Young's Inequality

Suppose $1 \le p, q, r \le \infty$ with

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Prove that

$$f \in L^p, g \in L^q \implies f * g \in L^r \text{ and } \|f * g\|_r \le \|f\|_p \|g\|_q.$$

15.14 2010 9.1

Show that the set $\{u_k(j) \coloneqq \delta_{ij}\} \subseteq \ell^2(\mathbf{Z})$ and forms an orthonormal system.

15.15 2010 9.2

Consider $L^2([0,1])$ and define

$$
\begin{aligned}
e_0(x) &= 1 \\
e_1(x) &= \sqrt{3}(2x-1).
\end{aligned}
$$

a. Show that $\{e_0, e_1\}$ is an orthonormal system.

b. Show that the polynomial $p(x)$ where $\deg(p) = 1$ which is closest to $f(x) = x^2$ in $L^2([0,1])$ is given by

$$
h(x) = x - \frac{1}{6}.
$$

Compute $\|f - g\|_2$.

15.16 2010 9.3

Let $E \subseteq H$ a Hilbert space.

a. Show that $E \perp\subseteq H$ is a closed subspace.

b. Show that $(E^\perp)^\perp = \mathrm{cl}_H(E)$.

15.17 2010 9.5b

Let $f \in L^1((0, 2\pi))$.

i. Show that for an $\epsilon > 0$ one can write $f = g + h$ where $g \in L^2((0, 2\pi))$ and $\|H\|_1 < \epsilon$.

15.18 2010 9.6

Prove that every closed convex $K \subset H$ a Hilbert space has a unique element of minimal norm.

15.19 2010 9 Challenge

Let U be a unitary operator on H a Hilbert space, let $M := \left\{ x \in H \;\middle|\; Ux = x \right\}$, let P be the orthogonal projection onto M, and define

$$S_N := \frac{1}{N} \sum_{n=0}^{N-1} U^n.$$

Show that for all $x \in H$,

$$\|S_N x - Px\|_H \overset{N\to\infty}{\to} 0.$$

15.20 2010 10.1

Let ν, μ be signed measures, and show that

$$\nu \perp \mu \text{ and } \nu \ll |\mu| \implies \nu = 0.$$

15.21 2010 10.2

Let $f \in L^1(\mathbf{R}^n)$ with $f \neq 0$.

a. Prove that there exists a $c > 0$ such that

$$Hf(x) \geq \frac{c}{(1+|x|)^n}.$$

15.22 2010 10.3

Consider the function

$$f(x) := \begin{cases} \frac{1}{|x|\left(\log\left(\frac{1}{x}\right)\right)^2} & |x| \leq \frac{1}{2} \\ 0 & \text{else.} \end{cases}$$

a. Show that $f \in L^1(\mathbf{R})$.

b. Show that there exists a $c > 0$ such that for all $|x| \leq 1/2$,

$$Hf(x) \geq \frac{c}{|x| \log\left(\frac{1}{|x|}\right)}.$$

Conclude that Hf is not locally integrable.

15.23 2010 10.4

Let $f \in L^1(\mathbf{R})$ and let $\mathcal{U} := \left\{(x, y) \in \mathbf{R}^2 \;\middle|\; y > 0\right\}$ denote the upper half plane. For $(x, y) \in \mathcal{U}$ define

$$u(x, y) := f * P_y(x) \qquad \text{where } P_y(x) := \frac{1}{\pi}\left(\frac{y}{t^2 + y^2}\right).$$

a. Prove that there exists a constant C independent of f such that for all $x \in \mathbf{R}$,

$$\sup_{y>0} |u(x, y)| \leq C \cdot Hf(x).$$

Hint: write the following and try to estimate each term:

$$u(x, y) = \int_{|t|<y} f(x - t)P_y(t)\,dt + \sum_{k=0}^{\infty} \int_{A_k} f(x - t)P_y(t)\,dt \qquad A_k := \left\{2^k y \leq |t| < 2^{k+1} y\right\}.$$

b. Following the proof of the Lebesgue differentiation theorem, show that for $f \in L^1(\mathbf{R})$ and for almost every $x \in \mathbf{R}$,

$$u(x, y) \overset{y \to 0}{\to} f(x).$$

16 | Midterm Exam 2 (December 2014)

16.1 Fall 2014 Midterm 1.1

Note: (a) is a repeat.

- Let $\Lambda \in L^2(X)^\vee$.
  - Show that $M := \left\{f \in L^2(X) \;\middle|\; \Lambda(f) = 0\right\} \subseteq L^2(X)$ is a closed subspace, and $L^2(X) = M \oplus M\perp$.
  - Prove that there exists a unique $g \in L^2(X)$ such that $\Lambda(f) = \int_X g\overline{f}$.

16.2 Fall 2014 Midterm 1.2

a. In parts:

- Given a definition of $L^\infty(\mathbf{R}^n)$.
- Verify that $\|-\|_\infty$ defines a norm on $L^\infty(\mathbf{R}^n)$.
- Carefully proved that $(L^\infty(\mathbf{R}^n), \|-\|_\infty)$ is a Banach space.

b. Prove that for any measurable $f : \mathbf{R}^n \to \mathbf{C}$,

$$L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) \subset L^2(\mathbf{R}^n) \quad \text{and} \quad \|f\|_2 \leq \|f\|_1^{\frac{1}{2}} \cdot \|f\|_\infty^{\frac{1}{2}}.$$

16.3 Fall 2014 Midterm 1.3

a. Prove that if $f, g : \mathbf{R}^n \to \mathbf{C}$ is both measurable then $F(x, y) \coloneqq f(x)$ and $h(x, y) \coloneqq f(x-y)g(y)$ is measurable on $\mathbf{R}^n \times \mathbf{R}^n$.

b. Show that if $f \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ and $g \in L^1(\mathbf{R}^n)$, then $f * g \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ is well defined, and carefully show that it satisfies the following properties:

$$\|f * g\|_\infty \leq \|g\|_1 \|f\|_\infty \|f * g\|_1 \qquad \leq \|g\|_1 \|f\|_1 \|f * g\|_2 \leq \|g\|_1 \|f\|_2.$$

Hint: first show $|f * g|^2 \leq \|g\|_1 \left(|f|^2 * |g|\right)$.

16.4 Fall 2014 Midterm 1.4

Note: (a) is a repeat.

Let $f : [0, 1] \to \mathbf{R}$ be continuous, and prove the Weierstrass approximation theorem: for any $\varepsilon > 0$ there exists a polynomial P such that $\|f - P\|_\infty < \varepsilon$.

17 | Midterm Exam 1 (October 2018)

17.1 Fall 2018 Midterm 1.1

Let $E \subseteq \mathbf{R}^n$ be bounded. Prove the following are equivalent:

1. For any $\epsilon > 0$ there exists and open set G and a closed set F such that
$$F \subseteq E \subseteq G \qquad\qquad m(G \setminus F) < \epsilon.$$
2. There exists a G_δ set V and an F_σ set H such that
$$m(V \setminus H) = 0.$$

17.2 Fall 2018 Midterm 1.2

Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of extended real-valued Lebesgue measurable functions.

a. Prove that $\sup_k f_k$ is a Lebesgue measurable function.

b. Prove that if $\lim_{k\to\infty} f_k(x)$ exists for every $x \in \mathbf{R}^n$ then $\lim_{k\to\infty} f_k$ is also a measurable function.

17.3 Fall 2018 Midterm 1.3

a. Prove that if $E \subseteq \mathbf{R}^n$ is a Lebesgue measurable set, then for any $h \in \mathbf{R}$ the set
$$E + h \coloneqq \left\{ x + h \;\middle|\; x \in E \right\}$$
is also Lebesgue measurable and satisfies $m(E + h) = m(E)$.

b. Prove that if f is a non-negative measurable function on $\mathbf{R}^n$ and $h \in \mathbf{R}^n$ then the function
$$\tau_h d(x) \coloneqq f(x - h)$$
is a non-negative measurable function and
$$\int f(x)\, dx = \int f(x - h)\, dx.$$

17.4 Fall 2018 Midterm 1.4

Let $f : \mathbf{R}^n \to \mathbf{R}$ be a Lebesgue measurable function.

a. Prove that for all $\alpha > 0$,

$$A_\alpha := \left\{ x \in \mathbf{R}^n \;\middle|\; |f(x)| > \alpha \right\} \implies m(A_\alpha) \leq \frac{1}{\alpha} \int |f(x)| \, dx.$$

b. Prove that

$$\int |f(x)| \, dx = 0 \iff f = 0 \text{ almost everywhere.}$$

17.5 Fall 2018 Midterm 1.5

Let $\{f_k\}_{k=1}^\infty \subseteq L^2([0,1])$ be a sequence which *converges in* L^1 to a function f.

a. Prove that $f \in L^1([0,1])$.

b. Give an example illustrating that f_k may not converge to f almost everywhere.

c. Prove that $\{f_k\}$ must contain a subsequence that converges to f almost everywhere.

18 | Midterm Exam 2 (November 2018)

18.1 Fall 2018 Midterm 2.1

Let $f, g \in L^1([0,1])$, define $F(x) = \int_0^x f(y)\, dy$ and $G(x) = \int_0^x g(y)\, dy$, and show

$$\int_0^1 F(x)g(x)\, dx = F(1)G(1) - \int_0^1 f(x)G(x)\, dx.$$

18.2 Fall 2018 Midterm 2.2

Let $\varphi \in L^1(\mathbf{R}^n)$ such that $\int \varphi = 1$ and define $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$. Show that if f is bounded and uniformly continuous then $f * \varphi_t \overset{t\to 0}{\rightrightarrows} f$ uniformly.

18.3 Fall 2018 Midterm 2.3

Let $g \in L^\infty([0,1])$.

a. Prove

$$\|g\|_{L^p([0,1])} \overset{p\to\infty}{\to} \|g\|_{L^\infty([0,1])}.$$

b. Prove that the map

$$\begin{aligned} \Lambda_g : L^1([0,1]) &\to \mathbf{C} \\ f &\mapsto \int_0^1 fg \end{aligned}$$

defines an element of $L^1([0,1])^\vee$ with $\|\Lambda_g\|_{L^1([0,1])^\vee} = \|g\|_{L^\infty([0,1])}$.

18.4 Fall 2018 Midterm 2.4

See section 20.3

19 | Practice Exam (November 2014)

19.1 Fall 2018 Practice Midterm 1.1

Let $m_*(E)$ denote the Lebesgue outer measure of a set $E \subseteq \mathbf{R}^n$.

a. Prove using the definition of Lebesgue outer measure that

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m_*(E_j).$$

b. Prove that for any $E \subseteq \mathbf{R}^n$ and any $\epsilon > 0$ there exists an open set G with $E \subseteq G$ and

$$m_*(E) \leq m_*(G) \leq m_*(E) + \epsilon.$$

19.2 Fall 2018 Practice Midterm 1.2

a. See section 17.1

b. Let f_k be a sequence of extended real-valued Lebesgue measurable function.

i. Prove that $\inf_k f_k, \sup_k f_k$ are both Lebesgue measurable function.

Hint: argue that

$$\left\{x \;\middle|\; \inf_k f_k(x) < a\right\} = \bigcup_k \left\{x \;\middle|\; f_k(x) < a\right\}.$$

ii. Carefully state Fatou's Lemma and deduce the Monotone Converge Theorem from it.

19.3 Fall 2018 Practice Midterm 1.3

a. Prove that if $f, g \in L^+(\mathbf{R})$ then

$$\int (f+g) = \int f + \int g.$$

Extend this to establish that if $\{f_k\} \subseteq L^+(\mathbf{R}^n)$ then

$$\int \sum_k f_k = \sum_k \int f_k.$$

b. Let $\{E_j\}_{j\in\mathbb{N}} \subseteq \mathcal{M}(\mathbf{R}^n)$ with $E_j \nearrow E$. Use the countable additivity of μ_f on $\mathcal{M}(\mathbf{R}^n)$ established above to show that

$$\mu_f(E) = \lim_{j\to\infty} \mu_f(E_j).$$

19.4 Fall 2018 Practice Midterm 1.4

a. Show that $f \in L^1(\mathbf{R}^n) \implies |f(x)| < \infty$ almost everywhere.

b. Show that if $\{f_k\} \subseteq L^1(\mathbf{R}^n)$ with $\sum \|f_k\|_1 < \infty$ then $\sum f_k$ converges almost everywhere and in L^1.

c. Use the Dominated Convergence Theorem to evaluate

$$\lim_{t\to 0} \int_0^1 \frac{e^{tx^2} - 1}{t}\, dx.$$

20 | Practice Exam (November 2014)

20.1 Fall 2018 Practice Midterm 2.1

a. Carefully state Tonelli's theorem for a nonnegative function $F(x,t)$ on $\mathbf{R}^n \times \mathbf{R}$.

b. Let $f : \mathbf{R}^n \to [0, \infty]$ and define

$$\mathcal{A} := \left\{ (x,t) \in \mathbf{R}^n \times \mathbf{R} \;\middle|\; 0 \le t \le f(x) \right\}.$$

Prove the validity of the following two statements:

1. f is Lebesgue measurable on $\mathbf{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of $\mathbf{R}^{n+1}$.
2. If f is Lebesgue measurable on $\mathbf{R}^n$ then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)dx = \int_0^\infty m\left(\left\{ x \in \mathbb{R}^n \;\middle|\; f(x) \ge t \right\} \right) dt.$$

20.2 Fall 2018 Practice Midterm 2.2

a. Let $f, g \in L^1(\mathbf{R}^n)$ and give a definition of $f * g$.

b. Prove that if f, g are integrable and bounded, then

$$(f * g)(x) \overset{|x| \to \infty}{\to} 0.$$

c. In parts:

1. Define the *Fourier transform* of an integrable function f on $\mathbf{R}^n$.
2. Give an outline of the proof of the Fourier inversion formula.
3. Give an example of a function $f \in L^1(\mathbf{R}^n)$ such that $\widehat{f}$ is not in $L^1(\mathbf{R}^n)$.

20.3 Fall 2018 Practice Midterm 2.3

Let $\{u_n\}_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space H.

a. Let $x \in H$ and verify that

$$\left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle \, u_n \right\|_H^2 = \|x\|_H^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2 .$$

for any $N \in \mathbb{N}$ and deduce that

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|_H^2 .$$

b. Let $\{a_n\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ and prove that there exists an $x \in H$ such that $a_n = \langle x,\ u_n \rangle$ for all $n \in \mathbb{N}$, and moreover x may be chosen such that

$$\|x\|_H = \left(\sum_{n\in\mathbb{N}} |a_n|^2 \right)^{\frac{1}{2}} .$$

c. Prove that if $\{u_n\}$ is *complete*, Bessel's inequality becomes an equality.

Solution (part b):

- Take $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
- Define $x := \lim\limits_{N\to\infty} S_N$ where $S_N = \sum_{k=1}^N a_k u_k$
- $\{S_N\}$ is Cauchy and H is complete, so $x \in H$.
- By construction,

$$\langle x,\ u_n \rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n \rangle = a_n$$

  since the u_k are all orthogonal.
- By Pythagoras since the u_k are normal,

$$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2 .$$

Solution (part c):
Let x and u_n be arbitrary.

$$\left\langle x - \sum_{k=1}^\infty \langle x,\ u_k\rangle u_k,\ u_n \right\rangle = \langle x,\ u_n\rangle - \left\langle \sum_{k=1}^\infty \langle x,\ u_k\rangle u_k,\ u_n \right\rangle$$
$$= \langle x,\ u_n\rangle - \sum_{k=1}^\infty \langle\langle x,\ u_k\rangle u_k,\ u_n\rangle$$
$$= \langle x,\ u_n\rangle - \sum_{k=1}^\infty \langle x,\ u_k\rangle\langle u_k,\ u_n\rangle$$
$$= \langle x,\ u_n\rangle - \langle x,\ u_n\rangle = 0$$
$$\implies x - \sum_{k=1}^\infty \langle x,\ u_k\rangle u_k = 0 \quad \text{by completeness.}$$

So

$$x = \sum_{k=1}^\infty \langle x,\ u_k\rangle u_k \implies \|x\|^2 = \sum_{k=1}^\infty |\langle x,\ u_k\rangle|^2.\blacksquare.$$

20.4 Fall 2018 Practice Midterm 2.4

a. Prove Holder's inequality: let $f \in L^p, g \in L^q$ with p, q conjugate, and show that

$$\|fg\|_p \le \|f\|_p \cdot \|g\|_q.$$

b. Prove Minkowski's Inequality:

$$1 \le p < \infty \implies \|f+g\|_p \le \|f\|_p + \|g\|_p.$$

Conclude that if $f, g \in L^p(\mathbf{R}^n)$ then so is $f+g$.

c. Let $X = [0,1] \subset \mathbf{R}$.

1. Give a definition of the Banach space $L^\infty(X)$ of essentially bounded functions of X.
2. Let f be non-negative and measurable on X, prove that

$$\int_X f(x)^p \, dx \overset{p\to\infty}{\to} \begin{cases} \infty & \text{or} \\ m\left(\left\{f^{-1}(1)\right\}\right) \end{cases},$$

and characterize the functions of each type

Solution:

$$\begin{aligned}
\int f^p &= \int_{x<1} f^p + \int_{x=1} f^p + \int_{x>1} f^p \\
&= \int_{x<1} f^p + \int_{x=1} 1 + \int_{x>1} f^p \\
&= \int_{x<1} f^p + m(\{f=1\}) + \int_{x>1} f^p \\
&\overset{p\to\infty}{\to} 0 + m(\{f=1\}) + \begin{cases} 0 & m(\{x \geq 1\}) = 0 \\ \infty & m(\{x \geq 1\}) > 0. \end{cases}
\end{aligned}$$

20.5 Fall 2018 Practice Midterm 2.5

Let X be a normed vector space.

a. Give the definition of what it means for a map $L : X \to \mathbf{C}$ to be a *linear functional.*

b. Define what it means for L to be *bounded* and show L is bounded $\iff$ L is continuous.

c. Prove that $(X^\vee, \|-\|_{\mathrm{op}})$ is a Banach space.

DZG: this comes from some tex file that I found when studying for quals, so is definitely not my own content! I've just copied it here for extra practice.

21 | May 2016 Qual

21.1 May 2016, 1

Consider the function $f(x) = \frac{x}{1-x^2}$, $x \in (0,1)$.

1. By using the $\epsilon - \delta$ definition of the limit only, prove that f is continuous on $(0,1)$. (Note: You are not allowed to trivialize the problem by using properties of limits).

2. Is f uniformly continuous on $(0,1)$? Justify your answer.

Proof .

Fix $x \in (0,1)$ and let $\epsilon > 0$. Then we have

$$\begin{aligned}
|f(x) - f(y)| &= \left| \frac{x}{1-x^2} - \frac{y}{1-y^2} \right| \\
&= \left| \frac{x(1-y^2) - y(1-x^2)}{(1-x^2)(1-y^2)} \right| \\
&= \left| \frac{x-y}{(1-x)(1+x)(1-y)(1+y)} \right|.
\end{aligned}$$

Now, choose $\delta > 0$ such that

$$\delta < \min\{\frac{1}{2}(1-x)^2\epsilon, \frac{1}{2}(1-x)\}.$$

When $x - \delta < y < x + \delta$,

$$\begin{aligned}
|f(x) - f(y)| &= \left| \frac{x-y}{(1-x)(1+x)(1-y)(1+y)} \right| \\
&\leq \left| \frac{x-y}{(1-x)(1-y)} \right| \\
&\leq \left| \frac{x-y}{(1-x)(1-(x+\frac{1}{2}(1-x)))} \right| \\
&= \left| \frac{x-y}{(1-x)(1-(x+\frac{1}{2}(1-x)))} \right| \\
&= \left| \frac{2}{(1-x)^2} \right| |x-y| \\
&< \epsilon.
\end{aligned}$$

As our choice of $x \in (0,1)$ was arbitrary, we conclude that f is continuous on all of $(0,1)$. ■

Proof .

Proof. We will show that the function f is not uniformly continuous. Consider the sequence $(x_n)_{n=1}^{\infty}$ in $(0,1)$ defined by $x_n = \frac{n}{n+1}$. Observe that

$$\begin{aligned}
f(x_n) &= \frac{\frac{n}{n+1}}{1 - \left(\frac{n}{n+1}\right)^2} \\
&= \frac{n(n+1)}{(n+1)^2 - n^2} \\
&= \frac{n(n+1)}{[(n+1)-n][(n+1)+n]} \\
&= \frac{n(n+1)}{2n+1}.
\end{aligned}$$

Written as $x_n = 1 - \frac{1}{n+1}$, one can more easily see that $(x_n)_{n=1}^{\infty}$ converges to 1 in $\mathbb{R}$, hence is Cauchy in $(0,1)$. Now, let $\delta > 0$ and choose $N \in \mathbb{N}$ such that $|x_n - x_m| < \delta$ when $n, m \geq N$.

For $\epsilon < \frac{1}{8}$ we have

$$\begin{aligned}
|f(x_n) - f(x_{n+1})| &= \left| \frac{n(n+1)}{2n+1} - \frac{(n+1)(n+2)}{2n+3} \right| \\
&= \left| \frac{n(n+1)(2n+3) - (n+1)(n+2)(2n+1)}{(2n+1)(2n+3)} \right| \\
&= \left| \frac{(2n^3+5n^2+3n) - (2n^3+7n^2+7n+2)}{(2n+1)(2n+3)} \right| \\
&= \left| \frac{2n^2+4n+2}{4n^2+8n+3} \right| \\
&\geq \left| \frac{2n^2}{16n^2} \right| \\
&= \frac{1}{8}.
\end{aligned}$$

So for any $\delta > 0$, we see that there exists two points $x_n, x_{n+1} \in (0,1)$ such that $|x_n - x_{n+1}| < \delta$ when n is sufficiently large but $f(x_n) - f(x_{n+1})| \not< \epsilon$. Therefore $f(x)$ is not uniformly continuous.

■

21.2 (May 2016, 2)

Let $\{a_k\}_{k=1}^{\infty}$ be a bounded sequence of real numbers and E given by:

$$E := \Big\{ s \in \mathbb{R} : \text{ the set } \{k \in \mathbb{N} : a_k \geq s\} \text{ has at most finitely many elements} \Big\}.$$

Prove that $\limsup_{k\to\infty} a_k = \inf E$.

Proof .
Proof. Let $e \in E$. As there are only finitely many $a_k \geq s$, there exists some $N \in \mathbb{N}$ such that $a_k < e$ for all $k \geq N$. Define $T_k := \{a_k : k \geq n\}$. It is clear that e is thus an upper bound for T_N. So,

$$e \geq \sup T_N \geq \limsup a_k.$$

Thus, $\limsup a_k$ is a lower bound for E, meaning $\inf E \geq \limsup a_n$.
Conversely, suppose $k \in \mathbb{N}$.

$$T_k = \{a_n : n \geq k\}.$$

So, $\sup T_k \geq a_n$ for all $a_n \in T_k$. Then, $\{a_k : a_k \geq \sup T_k\}$ must be finite, so $\{k \in \mathbb{N} : a_k \geq \sup T_k\}$ is finite. So, $\sup T_k \in E$ for all $k \in \mathbb{N}$. Since $\inf E$ is a lower bound for E, $\inf E \leq \sup T_k$ for all $k \in \mathbb{N}$. Thus,

$$\inf E \leq \lim(\sup T_k) = \limsup a_k.$$

We have both inequalities, therefore $\limsup a_k = \inf E$. ∎

21.3 (May 2016, 3)

Assume (X, d) is a compact metric space.

1. Prove that X is both complete and separable.

2. Suppose $\{x_k\}_{k=1}^{\infty} \subseteq X$ is a sequence such that the series $\sum_{k=1}^{\infty} d(x_k, x_{k+1})$ converges. Prove that the sequence $\{x_k\}_{k=1}^{\infty}$ converges in X.

21.4 (May 2016, 4)

Suppose that $f \colon [0, 2] \to \mathbb{R}$ is continuous on $[0, 2]$, differentiable on $(0, 2)$, and such that $f(0) = f(2) = 0$, $f(c) = 1$ for some $c \in (0, 2)$. Prove that there exists $x \in (0, 2)$ such that $|f'(x)| > 1$.

Proof .
Proof. We will consider three cases. First, suppose $c < 1$. Then, by the mean value theorem, there exists $x \in (0, c)$ such that $f'(x)(c - 0) = f(c) - f(0)$ so $f'(x) = \frac{f(c)}{c} = \frac{1}{c} > 1$ since $c < 1$. Similarly, if $c > 1$ then by the mean value theorem there exits $y \in (c, 2)$ such that

$$|f'(y)| = \left|\frac{f(2) - f(c)}{2 - c}\right| = \left|\frac{-f(c)}{2 - c}\right| = \left|\frac{-1}{2 - c}\right| > 1$$

since $1 < c < 2$.
Now, suppose $c = 1$. If there exists $x \in (0, 1)$ such that $x < f(x)$ then by the mean value theorem on the interval $(0, x)$ there exists $s \in (0, x)$ such that $f'(s) = \frac{f(x)}{x} > 1$ since $f(x) > x$. Likewise, if there exists $x \in (0, 1)$ such that $x > f(x)$ then the mean value theorem on $(x, 1)$ gives a point $t \in (x, 1)$ such that $|f'(t)| = \left|\frac{f(1)-f(x)}{1-x}\right| = \left|\frac{1-f(x)}{1-x}\right| > 1$ since $x > f(x)$. So, on $(0, 1)$, if the proposition does not hold then $f(x) = x$. Similarly, if there exists $x \in (1, 2)$ such that $f(x) > 2 - x$ then the mean value theorem yields a point $u \in (x, 2)$ such that

$$|f'(u)| = \left|\frac{f(2) - f(x)}{2 - x}\right| = \left|\frac{-f(x)}{2 - x}\right| > 1.$$

since $f(x) > 2 - x$. If there exists $y \in (1, 2)$ such that $f(y) < 2 - y$ then again by the mean value theorem there exists $v \in (1, y)$ such that

$$|f'(v)| = \left|\frac{f(y) - f(1)}{y - 1}\right| = \left|\frac{f(y) - 1}{y - 1}\right| > 1.$$

since $f(y) < 2 - y$ so $|f(y) - 1| > |y - 1|$. So, on $(1, 2)$ if the proposition does not hold then $f(x) = 2 - x$. However, notice that since $f(x)$ is differentiable at $x = 1$ we cannot have $f(x) = x$ on $(0, 1)$ and $f(x) = 2 - x$ on $(1, 2)$.

■

21.5 (May 2016, 5)

Let $f_n(x) = n^\beta x(1 - x^2)^n$, $x \in [0, 1]$, $n \in \mathbb{N}$.

1. Prove that $\{f_n\}_{n=1}^\infty$ converges pointwise on $[0, 1]$ for every $\beta \in \mathbb{R}$.
2. Show that the convergence in part (a) is uniform for all $\beta < \frac{1}{2}$, but not uniform for any $\beta \geq \frac{1}{2}$.

21.6 (May 2016, 6)

1. Suppose $f\colon [-1, 1] \to \mathbb{R}$ is a bounded function that is continuous at 0. Let $\alpha(x) = -1$ for $x \in [-1, 0]$ and $\alpha(x) = 1$ for $x \in (0, 1]$. Prove that $f \in \mathcal{R}(\alpha)[-1, 1]$, i.e., f is Riemann integrable with respect to α on $[-1, 1]$, and $\int_{-1}^{1} f d\alpha = 2f(0)$.
2. Let $g\colon [0, 1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 g(x)x^{3k+2}dx = 0$ for all $k = 0, 1, 2, \ldots$. Prove that $g(x) = 0$ for all $x \in [0, 1]$.

Proof .
Proof. Let $\epsilon > 0$. Choose $\delta > 0$ so that if $|x| < \delta$, then $|f(x) - f(0)| < \epsilon$. Let P be a partition of $[-1, 1]$ with $0 \in P$ and mesh$(P) < \delta$. Then

$$|U(f, P, \alpha) - L(f, P, \alpha)| = |\sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i| = (|\sup_{x\in[0,x_k]} f(x) - \inf_{x\in[0,x_k]} f(x)|)2 < 4\epsilon.$$

Thus f is integrable with respect to α. Additionally, we have $L(f, P, \alpha) \leq 2f(0) \leq U(f, P, \alpha)$ for all partitions P of the form described above, and so $\int_{-1}^{1} f d\alpha = 2f(0)$.

■

Proof .
Proof. Since $g(x)$ is continuous, so is $g(x^{1/3})$. Thus by the Weierstrauss Approximation Theorem, we can find a sequence of polynomials $(p_n(x)) \to g(x^{1/3})$ uniformly. Since this holds for all values $x \in [0, 1]$, we have that $(p_n(x^3))$ converges to $g(x)$ uniformly. Then we have $(x^2 p_n(x^3))$ converges to $x^2 g(x)$ uniformly. Note that by assumption, $\int_0^1 g(x)x^2 p_n(x^3)dx = 0$,

and so

$$0 = \lim_{n\to\infty} \int_0^1 g(x)x^2 p_n(x^3)dx = \int_0^1 \lim_{n\to\infty} g(x)x^2 p_n(x^3)dx = \int_0^1 x^2 g^2(x)dx.$$

Since $x^2g^2(x)$ is non-negative, and its integral is zero, we conclude that $x^2g^2(x) = 0$ for all x. Therefore, we have $g(x) = 0$.

■

22 | Metric Spaces and Topology

22.1 (May 2019, 1)

Let (M, d_M), (N, d_N) be metric spaces. Define

$$d_{M\times N} \colon (M \times N) \times (M \times N) \to \mathbb{R}$$

by

$$d_{M\times N}((x_1, y_1), (x_2, y_2)) := d_M(x_1, x_2) + d_N(y_1, y_2)..$$

1. Prove that $(M \times N, d_{M\times N})$ is a metric space.

2. Let $S \subseteq M$ and $T \subseteq N$ be compact sets in (M, d_M) and (N, d_N), respectively. Prove that $S \times T$ is a compact set in $(M \times N, d_{M\times N})$.

Proof .
Proof. To prove that $(M \times N, d_{M\times N})$ is a metric space we must prove that $d_{M\times N}$ is a metric on $M \times N$.

- Positive Definite-

Let $(x_1, y_1), (x_2, y_2) \in M \times N$. Then since d_M is a metric on M, then $d_M(x_1, x_2) \geq 0$ for all $x_i, x_j \in M$ and d_N is a metric on N and likewise $d_N(y_1, y_2) \geq 0$ for any $y_i, y_j \in N$.
Then by definition

$$d_{M\times N}((x_1, y_1), (x_2, y_2)) = d_M(x_1, x_2) + d_N(y_1, y_2) \geq 0 + 0 = 0..$$

Hence since $(x_1, y_1), (x_2, y_2)$ are arbitrary, $d_{M\times N}((x_1, y_1), (x_2, y_2)) \geq 0$ for all $(x_i, y_i), (x_j, y_j) \in M \times N$.
Suppose that $d_{M\times N}((x_1, y_1), (x_2, y_2)) = 0$. By definition

$$d_{M\times N}((x_1, y_1), (x_2, y_2)) = d_M(x_1, x_2) + d_N(y_1, y_2).$$

Therefore $d_M(x_1,x_2) + d_N(y_1,y_2) = 0$, since d_M, d_N are metrics, then $d_M(x_1,x_2) \geq 0, d_N(y_1,y_2) \geq 0$, which implies that $d_M(x_1,x_2) = d_N(y_1,y_2) = 0$ and also since they are metrics we have that $x_1 = x_2, y_1 = y_2$. Hence, $(x_1,y_1) = (x_2,y_2)$.
Now suppose that $(x_1,y_1) = (x_2,y_2)$. Then $x_1 = x_2, y_1 = y_2$ and for the metrics d_M, d_N we would have $d_M(x_1,x_2) = 0, d_N(y_1,y_2) = 0$. Thus

$$d_{M\times N}((x_1,y_1),(x_2,y_2)) = d_M(x_1,x_2) + d_N(y_1,y_2) = 0 + 0 = 0.$$

Therefore $d_{M\times N}((x_1,y_1),(x_2,y_2)) = 0$ if and only if $(x_1,y_1) = (x_2,y_2)$.

- Symmetric

Let $(x_1,y_1),(x_2,y_2) \in M \times N$. Then since d_M is a metric on M, then $d_M(x_1,x_2) = d_M(x_2,x_1)$ for all $x_i, x_j \in M$ and d_N is a metric on N and likewise $d_N(y_1,y_2) = d_N(y_2,y_1)$ for any $y_i, y_j \in N$. Therefore,

$$\begin{aligned} d_{M\times N}((x_1,y_1),(x_2,y_2)) &= d_M(x_1,x_2) + d_N(y_1,y_2) \\ &= d_M(x_2,x_1) + d_N(y_2,y_1) \\ &= d_{M\times N}((x_2,y_2),(x_1,y_1)). \end{aligned}$$

- Triangle Inequality

Since d_M, d_N are metrics then for all $x_1,x_2,x_3 \in M, y_1,y_2,y_3 \in N$ we have that

$$d_M(x_1,x_2) \leq d_M(x_1,x_3) + d_M(x_3,x_2)$$

and that

$$d_N(y_1,y_2) \leq d_N(y_1,y_3) + d_N(y_3,y_2)..$$

Therefore,

$$\begin{aligned} d_{M\times N}((x_1,y_1),(x_2,y_2)) &= d_M(x_1,x_2) + d_N(y_1,y_2) \\ d_M(x_1,x_2) + d_N(y_1,y_2) &\leq d_M(x_1,x_3) + d_M(x_3,x_2) + d_N(y_1,y_3) + d_N(y_3,y_2) \\ d_M(x_1,x_3) + d_M(x_3,x_2) + d_N(y_1,y_3) + d_N(y_3,y_2) &= d_M(x_1,x_3) + d_N(y_1,y_3) + d_M(x_3,x_2) + d_N(y_3,y_2) \\ d_M(x_1,x_3) + d_N(y_1,y_3) + d_M(x_3,x_2) + d_N(y_3,y_2) &= d_M((x_1,y_1),(x_3,y_3)) + d_M((x_3,y_3),(x_2,y_2)). \end{aligned}$$

Hence

$$d_{M\times N}((x_1,y_1),(x_2,y_2)) \leq d_M((x_1,y_1),(x_3,y_3)) + d_M((x_3,y_3),(x_2,y_2))..$$

Therefore $d_{M\times N}$ is a metric on $M \times N$ and $(M \times N, d_{M\times N})$ is a metric space.

■

Proof .
Proof. By part a we showed that $(M \times N, d_{M\times N})$ is a metric space. Let $\{s_n, t_n\}$ be a sequence

in $S \times T$. Since $\{s_n\}$ is a sequence on a compact set S in a metric space (M, d_M) then it has a convergent subsequence s_{n_k}. Let $\lim_{k\to\infty} s_{n_k} = s_0$.
Since $\{t_{n_k}\}$ is a sequence on a compact set T in a metric space. Thus $\{t_{n_k}\}$ has a convergent subsequence $\{t_{n_{k_j}}\}$. Let $\lim_{j\to\infty} t_{n_{k_j}} = t_0$. Thus $\{s_{n_{k_j}}\}$ is a subsequence of $\{s_{n_k}\}$. And since $\{s_{n_k}\}$ converges to s_0, then any subsequence also converges to s_0.
Let $\epsilon > 0$ be given. Then for $\epsilon/2$ there exists $N_1, N_2 \in \mathbb{N}$ such that for all $n_{k_j} \geq N_1, d_M(s_{n_{k_j}}, s_0) < \epsilon/2$, and for all $n_{k_j} \geq N_2, d_N(t_{n_{k_j}}, t_0) < \epsilon/2$. Choose $N = \text{Max}(\{N_1, N_2\})$. Then

$$d_{M\times N}((s_{n_{k_j}}, t_{n_{k_j}}), (s_0, t_0)) = d_M(s_{n_{k_j}}, s_0) + d_N(t_{n_{k_j}}, t_0) < \epsilon/2 + \epsilon/2 = \epsilon..$$

Therefore

$$d_{M\times N}((s_{n_{k_j}}, t_{n_{k_j}}), (s_0, t_0)) < \epsilon..$$

Hence $\{(s_{n_{k_j}}, t_{n_{k_j}})$ converges to (s_0, t_0). Therefore $S \times T$ is sequentially compact and $S \times T$ is therefore compact.

■

22.2 (June 2003, 1b,c)

(b) Show by example that the union of infinitely many compact subsets of a metric space need not be compact. (c) If (X, d) is a metric space and $K \subset X$ is compact, define $d(x_0, K) = \inf_{y\in K} d(x_0, y)$. Prove that there exists a point $y_0 \in K$ such that $d(x_0, K) = d(x_0, y_0)$.

22.3 (January 2009, 4a)

Consider the metric space $(\mathbb{Q}, d)$ where $\mathbb{Q}$ denotes the rational numbers and $d(x, y) = |x - y|$. Let $E = \{x \in \mathbb{Q} : x > 0,\ 2 < x^2 < 3\}$. Is E closed and bounded in $\mathbb{Q}$? Is E compact in $\mathbb{Q}$?

22.4 (January 2011 3a)

Let (X, d) be a metric space, $K \subset X$ be compact, and $F \subset X$ be closed. If $K \cap F = \emptyset$, prove that there exists an $\epsilon > 0$ so that $d(k, f) \geq \epsilon$ for all $k \in K$ and $f \in F$.

Proof .
Proof. We prove this by contrapositive. Suppose for all $\epsilon > 0$, there exists $k \in K$, $f \in F$ such that $d(k, f) < \epsilon$. Then for all $n \in \mathbb{N}$, we can choose $k_n \in K$, $f_n \in F$ such that $d(k_n, f_n) < \frac{1}{n}$.

Since k_n is a sequence in K, which is compact (and therefore sequentially compact), there exists a subsequence $k_{n_j} \subseteq k_n$ with the property that k_{n_j} converges to some $k_0 \in K$. Find $N \in \mathbb{N}$ such that for $n \geq N$, $d(k_{n_j}, k_0) < \frac{\epsilon}{2}$ and $\frac{1}{n} < \frac{\epsilon}{2}$. Then

$$d(f_{n_j}, k_0) \leq d(f_{n_j}, k_{n_j}) + d(k_{n_j}, k_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, f_{n_j} also converges to k_0, and since F is closed, $k_0 \in F$. So $K \cap F \neq \emptyset$.

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22.5 5?

Let (X, d) be an unbounded and connected metric space. Prove that for each $x_0 \in X$, the set $\{x \in X : d(x, x_0) = r\}$ is nonempty.

23 | Sequences and Series

23.1 (June 2013 1a)

Let $a_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$. Prove that $\lim_{n\to\infty} a_n = 1/2$.

23.2 (January 2014 2)

1. Produce sequences $\{a_n\}$, $\{b_n\}$ of positive real numbers such that
$$\liminf_{n\to\infty}(a_n b_n) > \left(\liminf_{n\to\infty} a_n\right)\left(\liminf_{n\to\infty} b_n\right)..$$

2. If $\{a_n\}$, $\{b_n\}$ are sequences of positive real numbers and $\{a_n\}$ converges, prove that
$$\liminf_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\liminf_{n\to\infty} b_n\right)..$$

23.3 (May 2011 4a)

Determine the values of $x \in \mathbb{R}$ for which $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1 + n|x|^n}$ converges, justifying your answer carefully.

23.4 (June 2005 3b)

If the series $\sum_{n=0}^{\infty} a_n$ converges conditionally, show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is 1.

23.5 (January 2011 5)

Suppose $\{a_n\}$ is a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$ and $\sum a_n$ diverges. Prove that for all $x > 0$ there exist integers $n(1) < n(2) < \ldots$ such that $\sum_{k=1}^{\infty} a_{n(k)} = x$.

(Note: Many variations on this problem are possible including more general rearrangements. You may also wish to show that if $\sum a_n$ converges conditionally then given any $x \in \mathbb{R}$ there is a rearrangement of $\{b_n\}$ of $\{a_n\}$ such that $\sum b_n = r$. See Rudin Thm. 3.54 for a further generalization.)

23.6 (June 2008 # 4b)

Assume $\beta > 0$, $a_n > 0$, $n = 1, 2, \ldots$, and the series $\sum a_n$ is divergent. Show that $\sum \dfrac{a_n}{\beta + a_n}$ is also divergent.

24 | Continuity of Functions

25 | Differential Calculus

25.1 (June 2005 1a)

Use the definition of the derivative to prove that if f and g are differentiable at x, then fg is differentiable at x.

25.2 (January 2006 2b)

Assume that f is differentiable at a. Evaluate

$$\lim_{x\to a}\frac{a^n f(x)-x^n f(a)}{x-a},\quad n\in\mathbb{N}..$$

25.3 (June 2007 3a)

Suppose that $f,g:\mathbb{R}\to\mathbb{R}$ are differentiable, that $f(x)\le g(x)$ for all $x\in\mathbb{R}$, and that $f(x_0)=g(x_0)$ for some x_0. Prove that $f'(x_0)=g'(x_0)$.

25.4 (June 2008 3a)

Prove that if f' exists and is bounded on $(a,b]$, then $\lim_{x\to a^+}f(x)$ exists.

25.5 (January 2012 4b, extended)

Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function with $f'\in C(\mathbb{R})$. Assume that there are $a,b\in\mathbb{R}$ with $\lim_{x\to\infty}f(x)=a$ and $\lim_{x\to\infty}f'(x)=b$. Prove that $b=0$. Then, find a counterexample to show that the assumption $\lim_{x\to\infty}f'(x)$ exists is necessary.

25.6 (June 2012 1a)

Suppose that $f:\mathbb{R}\to\mathbb{R}$ satisfies $f(0)=0$. Prove that f is differentiable at $x=0$ if and only if there is a function $g:\mathbb{R}\to\mathbb{R}$ which is continuous at $x=0$ and satisfies $f(x)=xg(x)$ for all $x\in\mathbb{R}$.

26 Integral Calculus

26.1 (January 2006 4b)

Suppose that f is continuous and $f(x) \geq 0$ on $[0,1]$. If $f(0) > 0$, prove that $\int_0^1 f(x)dx > 0$.

26.2 (June 2005 1b)

Use the definition of the Riemann integral to prove that if f is bounded on $[a,b]$ and is continuous everywhere except for finitely many points in (a,b), then $f \in \mathscr{R}$ on $[a,b]$.

26.3 (January 2010 5)

Suppose that $f : [a,b] \to \mathbb{R}$ is continuous, $f \geq 0$ on $[a,b]$, and put $M = \sup\{f(x) : x \in [a,b]\}$. Prove that

$$\lim_{p\to\infty} \left(\int_a^b f(x)^p \, dx \right)^{1/p} = M..$$

26.4 (January 2009 4b)

Let f be a continuous real-valued function on $[0,1]$. Prove that there exists at least one point $\xi \in [0,1]$ such that $\int_0^1 x^4 f(x) \, dx = \frac{1}{5} f(\xi)$.

Proof .
Proof. Assume that f is a continuous real-valued function on $[0,1]$. Then, by the Intermediate Value Theorem we have that f attains its maximum and minimum on $[0,1]$. That is, for some $a,b \in [0,1]$,

$$f(a) = \min_{[0,1]} f(x) \qquad \text{and} \qquad f(b) = \max_{[0,1]} f(x)..$$

We now have $f(a) \leq f(x) \leq f(b)$ for all $x \in [0,1]$. This gives

$$f(a) \int_0^1 x^4 dx \leq \int_0^1 x^4 f(x) dx \leq f(b) \int_0^1 x^4 dx..$$

By the Fundamental Theorem of Calculus we know that

$$\int_0^1 x^4 dx = \frac{1}{5}..$$

Thus, it follows that

$$\frac{1}{5}f(a) \leq \int_0^1 x^4 f(x)dx \leq \frac{1}{5}f(b).$$

giving

$$f(a) \leq 5\int_0^1 x^4 f(x)dx \leq f(b)..$$

By the Intermediate Value Theorem, there exists $\xi \in [0,1]$ such that

$$f(\xi) = 5\int_0^1 x^4 f(x)dx..$$

Therefore, we have that there exists $\xi \in [0,1]$ such that

$$\int_0^1 x^4 f(x)dx = \frac{1}{5}f(\xi).$$

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26.5 (June 2009 5b)

Let φ be a real-valued function defined on $[0,1]$ such that φ, φ', and φ'' are continuous on $[0,1]$. Prove that

$$\int_0^1 \cos x \frac{x\varphi'(x) - \varphi(x) + \varphi(0)}{x^2}\, dx < \frac{3}{2}||\varphi''||_\infty,$$

where $||\varphi''||_\infty = \sup_{[0,1]} |\varphi''(x)|$. Note that $3/2$ may not be the smallest possible constant.

27 | Sequences and Series of Functions

27.1 (June 2010 6a)

Let $f : [0,1] \to \mathbb{R}$ be continuous with $f(0) \neq f(1)$ and define $f_n(x) = f(x^n)$. Prove that f_n does not converge uniformly on $[0,1]$.

27.2 (January 2008 5a)

Let $f_n(x) = \frac{x}{1+nx^2}$ for $n \in \mathbb{N}$. Let $\mathcal{F} := \{f_n : n = 1,2,3,\ldots\}$ and $[a,b]$ be any compact subset of $\mathbb{R}$. Is $\mathcal{F}$ equicontinuous? Justify your answer.

27.3 (January 2005 4, June 2010 6b)

If $f : [0,1] \to \mathbb{R}$ is continuous, prove that

$$\lim_{n\to\infty} \int_0^1 f(x^n)\, dx = f(0)..$$

27.4 (January 2020 4a)

Let $M < \infty$ and $\mathcal{F} \subseteq C[a,b]$. Assume that each $f \in \mathcal{F}$ is differentiable on (a,b) and satisfies $|f(a)| \leq M$ and $|f'(x)| \leq M$ for all $x \in (a,b)$. Prove that $\mathcal{F}$ is equicontinuous on $[a,b]$.

27.5 (June 2005 5)

Suppose that $f \in C([0,1])$ and that $\int_0^1 f(x)x^n\, dx = 0$ for all $n = 99, 100, 101, \ldots$. Show that $f \equiv 0$.

Note: Many variations on this problem exist. See June 2012 6b and others.

27.6 (January 2005 3b)

Suppose $f_n : [0,1] \to \mathbb{R}$ are continuous functions converging uniformly to $f : [0,1] \to \mathbb{R}$. Either prove that

$$\lim_{n\to\infty} \int_{1/n}^{1} f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

or give a counterexample.

28 | Miscellaneous Topics

Bounded Variation

28.1 (January 2018)

Let $f\colon [a,b] \to \mathbb{R}$. Suppose $f \in \mathrm{BV}[a,b]$. Prove f is the difference of two increasing functions.

28.2 (January 2007, 6a)

Let f be a function of bounded variation on $[a,b]$. Furthermore, assume that for some $c > 0$, $|f(x)| \geq c$ on $[a,b]$. Show that $g(x) = 1/f(x)$ is of bounded variation on $[a,b]$.

28.3 (January 2017, 2a)

Define $f\colon [0,1] \to [-1,1]$ by

$$f(x) := \begin{cases} x\sin\left(\frac{1}{x}\right) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}.$$

Determine, with justification, whether f is if bounded variation on the interval $[0,1]$.

28.4 (January 2020, 6a)

Let $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ and a strictly increasing sequence $\{x_n\}_{n=1}^{\infty} \subseteq (0,1)$ be given. Assume that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and define $\alpha \colon [0,1] \to \mathbb{R}$ by

$$\alpha(x) := \begin{cases} a_n & x = x_n \\ 0 & \text{otherwise} \end{cases}..$$

Prove or disprove: α has bounded variation on $[0,1]$.

Metric Spaces and Topology

1. Find an example of a metric space X and a subset $E \subseteq X$ such that E is closed and bounded but not compact.

28.5 (May 2017 6)

Let (X,d) be a metric space. A function $f \colon X \to \mathbb{R}$ is said to be lower semi-continuous (l.s.c) if $f^{-1}(a,\infty) = \{x \in X : f(x) > a\}$ is open in X for every $a \in \mathbb{R}$. Analogously, f is upper semi-continuous (u.s.c) if $f^{-1}(-\infty,b) = \{x \in X : f(x) < b\}$ is open in X for every $b \in \mathbb{R}$.

1. Prove that a function $f \colon X \to \mathbb{R}$ is continuous if and only if f is both l.s.c. and u.s.c.

2. Prove that f is lower semi-continuous if and only if $\liminf_{n\to\infty} f(x_n) \geq f(x)$ whenever $\{x_n\}_{n=1}^{\infty} \subseteq X$ such that $x_n \to x$ in X.

28.6 (January 2017 3)

Let (X,d) be a compact metric space. Suppose that $f_n \colon X \to [0,\infty)$ is a sequence of continuous functions with $f_n(x) \geq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in X$, and such that $f_n \to 0$ pointwise on X. Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X.

29 | Integral Calculus

29.1 1.

(June 2014 1)Define $\alpha\colon [-1,1] \to \mathbb{R}$ by

$$\alpha(x) := \begin{cases} -1 & x \in [-1,0] \\ 1 & x \in (0,1]. \end{cases}.$$

Let $f\colon [-1,1] \to \mathbb{R}$ be a function that is uniformly bounded on $[-1,1]$ and continuous at $x = 0$, but not necessarily continuous for $x \neq 0$. Prove that f is Riemann-Stieltjes integrable with respect to α over $[-1,1]$ and that

$$\int_{-1}^{1} f(x) d\alpha(x) = 2f(0)..$$

29.2 (June 2017 2)

Prove : $f \in \mathcal{R}(\alpha)$ on $[a,b]$ if and only if for any $a < c < b$, $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and on $[c,b]$. In addition, if either condition holds, then we have that

$$\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha.$$

29.3 (Spring 2017 7)

Prove that if $f \in \mathcal{R}$ on $[a,b]$ and $\alpha \in C^1[a,b]$, then the Riemann integral $\int_a^b f(x)\alpha'(x)dx$ exists and

$$\int_a^b f(x) d\alpha(x) = \int_a^b f(x)\alpha'(x)dx..$$

30 Sequences and Series (and of Functions)

30.1 (January 2006 1)

Let the power series series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ have radii of convergence R_1 and R_2, respectively.

30.2 ?

If $R_1 \neq R_2$, prove that the radius of convergence, R, of the power series $\sum_{n=0}^{\infty}(a_n + b_n)x^n$ is $\min\{R_1, R_2\}$. What can be said about R when $R_1 = R_2$?

30.3 ?

Prove that the radius of convergence, R, of $\sum_{n=0}^{\infty} a_n b_n x^n$ satisfies $R \geq R_1 R_2$. Show by means of example that this inequality can be strict.

30.4 ?

Show that the infinite series $\sum_{n=0}^{\infty} x^n 2^{-nx}$ converges uniformly on $[0, B]$ for any $B > 0$. Does this series converge uniformly on $[0, \infty)$?

30.5 (January 2006 4a)

Let

$$f_n(x) = \begin{cases} \frac{1}{n} & x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right] \\ 0 & \text{otherwise.} \end{cases}.$$

Show that $\sum_{n=1}^{\infty} f_n$ does not satisfy the Weierstrass M-test but that it nevertheless converges uniformly on $\mathbb{R}$.

30.6 ?

Let $f_n\colon [0,1) \to \mathbb{R}$ be the function defined by

$$f_n(x) := \sum_{k=1}^{n} \frac{x^k}{1+x^k}..$$

1. Prove that f_n converges to a function $f\colon [0,1) \to \mathbb{R}$.
2. Prove that for every $0 < a < 1$ the convergence is uniform on $[0,a]$.
3. Prove that f is differentiable on $(0,1)$.

January 2019 Qualifying Exam

1. Suppose that $f : [0,1] \to \mathbb{R}$ is differentiable and $f(0) = 0$. Assume that there is a $k > 0$ such that

   $$|f'(x)| \leq k|f(x)|$$

   for all $x \in [0,1]$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

Proof .
Proof. Let $0 < \delta_1 < 1$, and fix $x_1 \in (0, \delta_1]$. Since $f(x)$ is differentiable on all of $[0,1]$, $f(x)$ is differentiable on all of $(0, \delta_1)$. So by the Mean Value Theorem, there exists $x_2 \in (0, x_1)$ such that

$$f'(x_2) = \frac{f(x_1) - f(0)}{x_1 - 0} = \frac{f(x_1)}{x_1}.$$

Solving for $f(x_1)$, we get $f(x_1) = f'(x_2)x_1$. So by hypothesis, $f(x_1) = f'(x_2)x_1 \leq k|f(x_2)|x_1$. Assume for $x_1, x_2, \ldots, x_{n-1} \in (0,1)$ the following conditions are satisfied for $j \in \{1, 2, \ldots, n-1\}$.

$$\begin{aligned} x_j &\in (0, x_{j-1}) \\ f(x_{j-1}) &= f'(x_j)x_{j-1} \\ f(x_1) &\leq k^{j-1}|f(x_j)|(x_{j-1} \cdots x_2 x_1) \end{aligned}$$

I now claim that this inductive process is true for $j = n$, given that it holds for all $j \leq n$. We apply the Mean Value Theorem to find some $x_n \in (0, x_{n-1})$ such that $f'(x_n) = \frac{f(x_{n-1})}{x_{n-1}}$, then write $f(x_{n-1}) = f'(x_n)x_{n-1}$. By our inductive hypothesis, we have

$$\begin{aligned}|f(x_1)| &\leq k^{n-2}|f(x_{n-1})|(x_{n-2}\cdots x_2x_1)\\ &= k^{n-2}|f'(x_n)x_{n-1}|(x_{n-2}\cdots x_2x_1)\\ &\leq k^{n-2}(k|f(x_n)|)(x_{n-1}x_{n-2}\cdots x_2x_1)\\ &= k^{n-1}|f(x_n)|(x_{n-1}x_{n-2}\cdots x_2x_1).\end{aligned}$$

Thus our claim holds by induction. Now, since f is a continuous function on the closed interval, we can apply the Extreme Value Theorem to find some $M > 0$ for which $f(x) \leq M$ for all $x \in [0, 1]$. Thus we get

$$|f(x_1)| \leq k^n M(x_n \cdots x_1) = (kx_n)(kx_{n-1})\cdots(kx_1)M$$

for all $n \in \mathbb{N}$. If $k < \frac{1}{x_1}$, then for any $\epsilon > 0$ we can find $N \in \mathbb{N}$ sufficiently large so that $|f(x_1)| < \epsilon$. Otherwise, we set $\delta_1 < \frac{1}{k}$ so that $kx_1 < 1$.

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