```yaml
date: 2022-06-12 18:21
modification date: Sunday 12th June 2022 18:21:43
title: "2022 Talbot Talk Outline V3"
aliases: [2022 Talbot Talk Outline V3]
tags: projects/talbot-talk
status: in-progress
```


Last modified date: 2022-06-14 16:10

- Tags:
  - #todo/untagged
- Refs:
  - #todo/add-references
- Links:
  - #todo/create-links

2022 Talbot Talk Outline V3

- Todos and questions to ask
  - Are any particular theorems necessary for later talks?
  - Slogans for theorems
  - Clarifications on proofs: need to mark questions to ask Inna.
- Major goals to hit in talk:
  - Discuss Q1, Larsen-Lunts/Gromov question on piecewise isomorphism
  - Discuss Q2, $\mathrm{Ann}(\mathbb{L}) =_? 0$ and why we care.
  - Discuss Borisov's result relating it to ψ_n
  - State and sketch Thm A: description of $\mathsf{K}(\mathcal{V})$ and the sseq
  - State and **prove** Thm B: what the sseq measures
  - State and sketch Thm C: how Q1 and Q2 are linked
  - State and sketch Thm D: partially characterize $\mathrm{Ann}(\mathbb{L})$
  - State Thm E: strong link to birational geometry.
  - Discuss unknowns, open questions, conjectures.
- Things to prove
  - Thm A, if time. Just show the calculation if short on time.
  - Thm B, to get a handle of d_r and ∂.

- Possibly skip proof of Lem 3.2 if short on time?
  - Thm C, sketch proof (lots of auxiliary objects)
  - Thm D, maybe okay to skip diagram chase? Emphasize how to get elements in $\ker \psi_n$.

Preliminaries

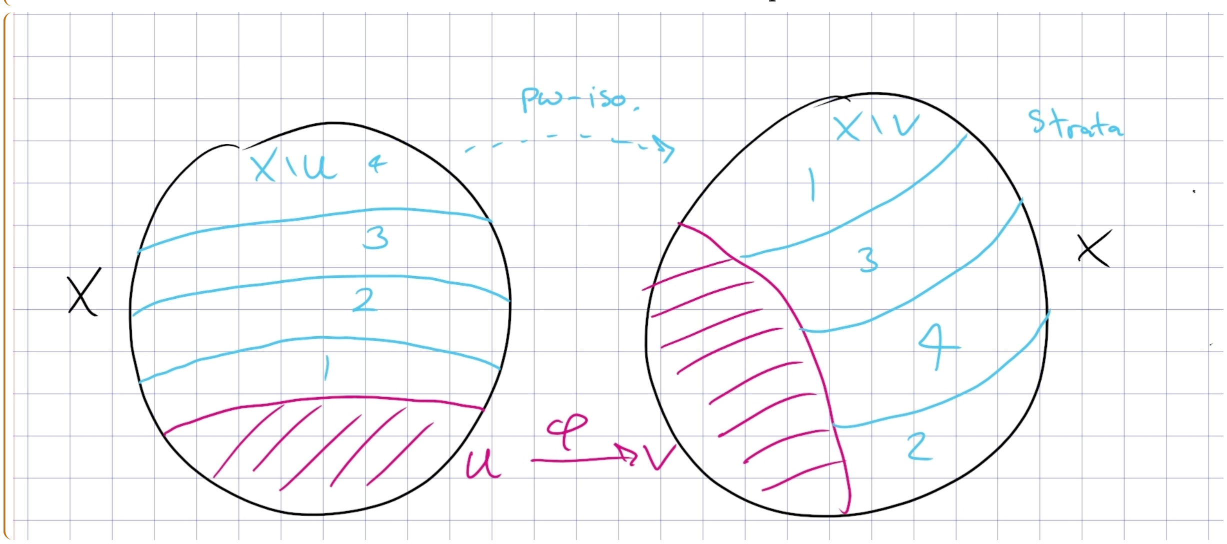
- Where we are:
  - Yesterday: classical scissors congruence.
  - Today: $\mathrm{SC} \to \mathsf{K}$, i.e. how can we encode/detect scissors congruence in the language of K theory using assemblers.
  - Tomorrow: $\mathsf{K} \to \mathrm{SC}$: enriching motivic measures, generalizing assemblers to other cut-and-paste problems, towards a topological approach on a generalized Hilbert's 3rd problem.
- Conventions:
  - k is a field.
  - A **variety** $X_{/k}$ means a reduced separated scheme of finite type over $\operatorname{Spec} k$.
  - A **stratification** of a space X is given by a partition $X = \biguplus_{i \in I} X_i$ into locally closed subsets over a poset I such that for each $j \in I$ we have

$$\overline{X_j} \subset \biguplus_{i \leq j} X_i$$

  - The parts X_i are called the *strata* of the stratification.
  - X, Y are isomorphic iff they are isomorphic in $\mathsf{Sch}_{/k}$.
  Write this as $X \cong Y$.
  - Induced by ring morphisms on an open affine cover. Not quite a morphism of ringed spaces!
  - The model for Sp we use is symmetric spectra of simplicial sets, take stable model structure with levelwise cofibrations.
  - $\mathcal{V} = \mathcal{V}_k$ is the aseembler of varieties over k and closed inclusions (locally closed embeddings).
  - $\mathsf{K}_0(\mathcal{V})$ is the **Grothendieck group of varieties** as in Michael's talk (Talk 7).
  - $\mathbb{L} = [\mathbb{A}^1_{/k}]$ is the **Lefschetz motive**, the class of the affine line.
  -

$$\operatorname{Ann}(\mathbb{L}) := \ker(\mathsf{K}_0(\mathcal{V}) \xrightarrow{\cdot \mathbb{L}} \mathsf{K}_0(\mathcal{V}))$$

  where $\cdot \mathbb{L}$ is the map induced by $X \mapsto X \underset{k}{\times} \mathbb{A}^1_{/k}$.
  - CA fact: $\mathbb{L}$ is a zero divisor $\iff \operatorname{Ann}(\mathbb{L}) = 0$.
- Examples of working with $\mathbb{L}$.
  - If $\mathcal{E} \to X$ is a rank n vector bundle (Zariski-locally trivial fibration with fibers $\mathbb{A}^n$) then $[\mathcal{E}] = [X] \cdot [\mathbb{A}^n] = [X] \cdot \mathbb{L}^n$.

- X, Y are **birational** iff there is an isomorphism $\phi : U \xrightarrow{\sim} V$ of dense open subschemes. **Write this as $X \overset{\sim}{\dashrightarrow} Y$.**
    - So in equations ϕ is given by rational functions.
    - Birational maps: "almost isomorphisms" which allow not just polynomial but rational functions, and are isomorphisms away from an exceptional set of e.g. poles or a branch locus
    - Motivations: MMP!
- X, Y are **stably birational** iff $X \times \mathbb{P}^N \overset{\sim}{\dashrightarrow} Y \times \mathbb{P}^M$ for some N, M. **Write this as $X \overset{\sim \text{Stab}}{\dashrightarrow} Y$.**
    - Lots of interesting aspects of birational geometry: $h^0(X; \Omega_X), \pi_1(X^{\text{an}}), \text{CH}_0(X)$ are stable birational invariants (see recent 2010s work of Claire Voisin)
- X, Y are **piecewise isomorphic** if there are stratifications $X = \biguplus_{i \in I} X_i$ and $Y = \biguplus_{i \in I} Y_i$ with each $X_i \cong Y_i$. **Write this as $X \underset{\text{pw}}{\cong} Y$.**
    - Think of this as cut-and-paste equivalence for varieties.
    - Note $X \underset{\text{pw}}{\cong} Y \implies [X] = [Y] \in \mathsf{K}_0(\mathcal{V})$.
    - If $X \overset{\sim}{\dashrightarrow} Y$ and additionally $X \setminus U \cong Y \setminus V$, then $X \underset{\text{pw}}{\cong} Y$ and $[X] = [Y]$.
    - 

Motivation

Reference: Zak17b, Annihilator of the Lefschetz Motive

- Summary of big questions:
    - When is $\mathsf{K}_0(\mathcal{V}) \to \mathsf{K}_0(\mathcal{V})[\frac{1}{\mathbb{L}}]$ injective? So are equations in the localization still valid in the original ring?
    - What does equality in $\mathsf{K}_0(\mathcal{V})$ mean geometrically? What does an equation in this ring mean?

- Summary of big structural questions about $\mathsf{K}_0(\mathcal{V})$ we're looking at in this paper:

Q1: Larsen-Lunts/Gromov, PW Isos

- There is a filtration on $\mathsf{K}_0(\mathcal{V}_k)$ where gr_n is induced by the image of

$$\operatorname{gr}_n \mathsf{K}_0(\mathcal{V}) = \operatorname{im}\left(\frac{\mathbb{Z}\left[X \middle| \dim X \leq n\right]}{\langle [X] = [Y] + [X \setminus Y] \rangle} \xrightarrow{\psi_n} \mathsf{K}_0(\mathcal{V}_k) \right)$$

- Q, Gromov: if $U, V \hookrightarrow X$ with $X \setminus U \cong X \setminus V$, how far are U and V from being birational?
- Q, Larsen-Lunts: $[X] = [Y] \overset{???}{\implies} X \underset{\text{pw}}{\cong} Y$?
- **Answer**: No! Borisov and Karzhemanov construct counterexamples for $k \hookrightarrow \mathbb{C}$, Inna shows that this fails for *convenient* fields.
- **Conjecture**: this is almost true, and the only obstructions come from $\operatorname{Ann}(\mathbb{L})$.
- **Conjecture**: for certain varieties, $[X] = [Y] \implies X, Y$ are **stably birational**.
- Encode these as injectivity of ψ_n, so $\ker \psi_n = 0$ -- when does $X \overset{\sim}{\dashrightarrow} Y$ extend to $X \underset{\text{pw}}{\cong} Y$?

Q2: $\operatorname{Ann}(\mathbb{L}) \overset{?}{=} 0$

- When is $\operatorname{Ann}(\mathbb{L})$ nonzero?
  - Important for motivic measures, rationality questions.
- Answer (Borisov): $\mathbb{L}$ generally **is** a zero divisor, Borisov and Karzhemanov elements in $\operatorname{Ann}(\mathbb{L})$ and seemingly coincidentally constructs elements in $\ker \psi_n$.
  - In case not covered in previous talk
  - Shows an equality in K_0:

  Theorem 2.13. *The cut-and-paste conjecture of Larsen and Lunts fails.*

  Proof. The equality

  $$[X_W](L^2-1)(L-1)L^7 = [Y_W](L^2-1)(L-1)L^7$$

  implies that trivial $GL(2,\mathbb{C}) \times \mathbb{C}^6$ bundles over X_W and Y_W have the same class in the Grothendieck ring. If it were possible to cut them into unions of isomorphic varieties, then $X_W \times GL(2,\mathbb{C}) \times \mathbb{C}^6$ would be birational to $Y_W \times GL(2,\mathbb{C}) \times \mathbb{C}^6$. This implies that X_W and Y_W are stably birational, and thus birational, in contradiction with Proposition 2.2. □

  - Shows that certain bundles over X, Y are birational, so X, Y are stably birational
  - Picks a special mirror pair where stably birational implies birational
  - Show the bundles are pw-iso, so stably birational.
  - Use that X, Y are known *not* to be birational.
- Q: How and why are $\operatorname{Ann}(\mathbb{L})$ and $\ker \psi_n$ related?

Outline of Results

- Slogans for what's shown in this paper:
  - **Thm A:** Constructs a stable (filtered) homodtopy type $\mathsf{K}(\mathcal{V})$ where gr $\mathsf{K}(\mathcal{V})$ is simpler than gr $\mathsf{K}_0(\mathcal{V})$.
  - **Thm B**: The associated spectral sequence is an obstruction theory for birational auts extending to pw auts (so detects $\ker \psi_n$ for various n)
  - **Thm C:** Q1 and Q2 are linked: elements in $\operatorname{Ann}(\mathbb{L})$ yield elements in $\ker(\psi_n)$.
  - **Thm D**: Partial characterizations of $\operatorname{Ann}(\mathbb{L})$.
  - **Thm E**: Identification of $\mathsf{K}_0(\mathcal{V})/\left\langle \mathbb{L} \right\rangle$ in terms of stable birational geometry.
- Conclusions:
  - Elements in $\operatorname{Ann}(\mathbb{L})$ always produce elements in $\ker \psi_n$

Theorems

Thm A: There is a homotopical enrichment of $\mathsf{K}_0(\mathcal{V})$ with a simple associated graded

Theorem

Let

- $\mathcal{V}_k^{(n)}$ be the nth filtered assembler of $\mathcal{V}$ generated by varieties of dimension $d \leq n$.
- $\operatorname{Aut}_k\ k(X)$ be the group of birational automorphisms of the variety X.
- B_n be the set of birational isomorphism classes of varieties of dimension $d = n$.

There is a spectrum $\mathsf{K}(\mathcal{V})$ such that $\mathsf{K}_0(\mathcal{V}) := \pi_0 \mathsf{K}(\mathcal{V})$ coincides with the Grothendieck group of varieties discussed previously, and $\mathcal{V}^{(n)}$ induces a filtration on the $\mathsf{K}(\mathcal{V})$ such that

$$\operatorname{gr}_n \mathsf{K}(\mathcal{V}) = \bigvee_{[X] \in B_n} \Sigma_+^\infty \mathbf{B} \operatorname{Aut}_k\ k(X),$$

with an associated spectral sequence

$$E_{p,q}^1 = \bigvee_{[X] \in B_n} \left(\pi_p \Sigma_+^\infty \mathbf{B} \operatorname{Aut}_k\ k(X) \oplus \pi_p \mathbb{S} \right) \Rightarrow \mathsf{K}_p(\mathcal{V})$$

Note that the $p = 0$ column converges to $\mathsf{K}_0(\mathcal{V})$.

Proof

- Define $\mathcal{V}^{(n.n-1)} = \mathsf{Var}^{\dim=n}_{/k} \cup \{\emptyset\}$, the varieties of dimension *exactly* n.
- Zak17b Thm. 1.8: extract cofibers in the filtration to see the associated graded:

$$
\begin{tikzcd}
\vdots & \\
K(\mathcal{V}^{(n)}) \arrow[u, hook] \arrow[r, two heads] & K(\mathcal{V}^{(n,n-1)}) \\
\vdots \arrow[u, hook] \arrow[r, two heads] & \vdots \\
K(\mathcal{V}^{(2)}) \arrow[u, hook] \arrow[r, two heads] & K(\mathcal{V}^{(2,1)}) \\
K(\mathcal{V}^{(1)}) \arrow[u, hook] & \\
\mathrm{Fil} & \mathrm{gr}
\end{tikzcd}
$$

- Finish by a computation:

$$
\begin{aligned}
\mathsf{K}(\mathcal{V}^{(n,n-1)}) &\simeq \tilde{\mathsf{K}}(\mathcal{V}^{(n,n-1)}) \\
&\simeq \mathsf{K}(\mathsf{C}) \\
&\simeq \mathsf{K}\left(\bigvee_{\alpha\in B_n} \mathsf{C}_{X_\alpha}\right) \\
&\simeq \bigvee_{\alpha\in B_n} \mathsf{K}(\mathsf{C}_{X_\alpha}) \\
&\cong \bigvee_{\alpha\in B_n} \Sigma^\infty_+ \mathbf{B}\operatorname{Aut}_k k(X_\alpha) \qquad \text{Zak17a} \\
&:= \bigvee_{\alpha\in B_n} \Sigma^\infty_+ \mathbf{B}\operatorname{Aut}(\alpha).
\end{aligned}
$$

where

- $\tilde{\mathsf{K}}(\mathcal{V}^{(n,n-1)})$: the full subassembler of irreducible varieties.
  - **Why the reduction works:** general theorem (Zak17b Thm. 1.9) on subassemblers with enough disjoint open covers
- $\mathsf{C} \leq \mathcal{V}^{(n,n-1)}$: subvarieties of some X_α representing some α, as α ranges over B_n.
  - **Why the reduction works:** apply (Zak17b Thm. 1.9) again
- C_{X_α} is the subassembler of only those varieties admitting a (unique) morphism to X_α for a fixed α.
  - **Why the reduction works:** each nonempty variety admits a morphism to exactly one X_α representing some α -- otherwise, if $X \mapsto X_\alpha, X_\beta$ then X_α and X_β are forced to be birational (the morphisms are inclusions of dense opens) implying $\alpha = \beta$
  -
- $\operatorname{Aut}(\alpha) := \operatorname{Aut}_k k(X)$ for any X representing $\alpha \in B_n$.

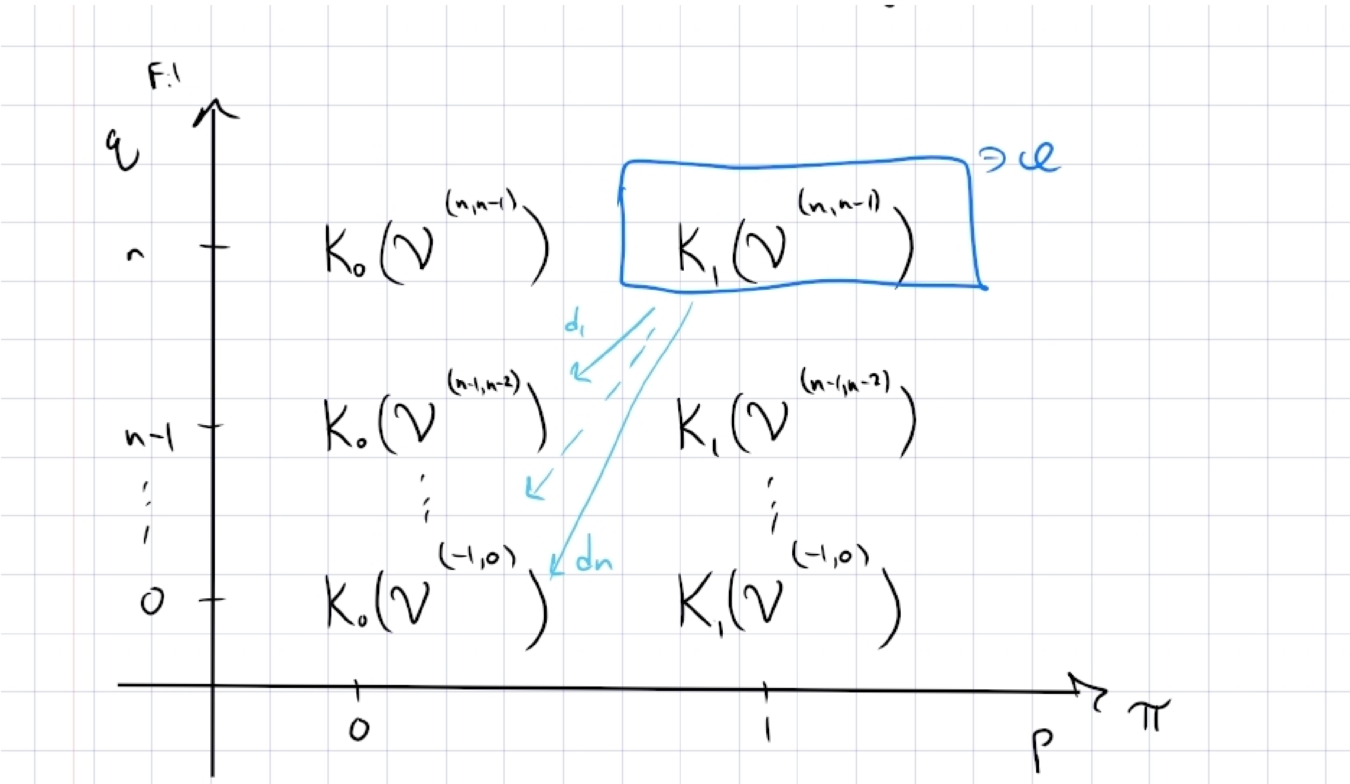
Thm B: the spectral sequence measures $\ker\psi_n$ and how birational morphisms can fail to extend to piecewise isomorphisms

Theorem

There exists nontrivial differentials d_r from column 1 to column 0 in some page of $E^* \iff \bigcup_n \ker\psi_n \neq 0$ (ψ_n has a nonzero kernel for some n),

> More precisely, $\phi \in \operatorname{Aut}_k k(X)$ extends to a piecewise automorphism $\iff d_r[\phi] = 0 \quad \forall r \geq 1.$

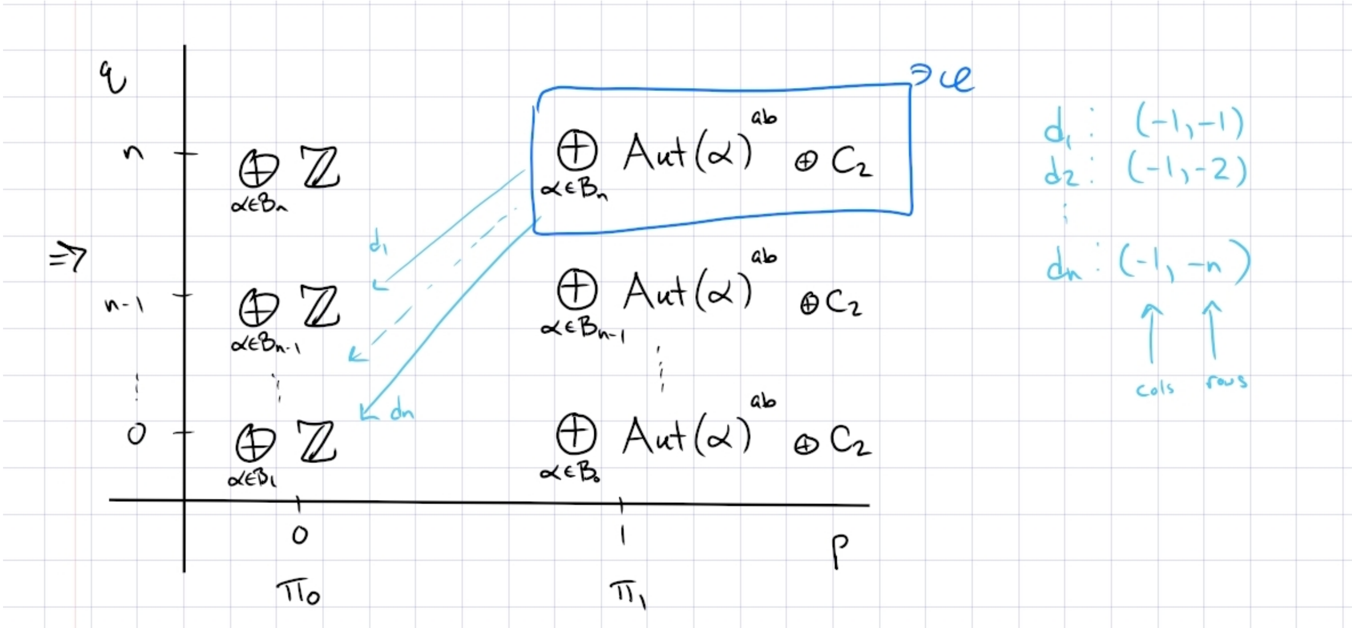
Before proving, a look at this spectral sequence:



Compute

$$\begin{aligned} \mathsf{K}_p(\mathcal{V}^{(n,n-1)}) &:= \pi_p \mathsf{K}(\mathcal{V}^{(n,n-1)}) \\ &\simeq \pi_p \bigvee_{\alpha \in B_n} \Sigma^\infty_+ \mathbf{B}\operatorname{Aut}(\alpha) \\ &\cong \bigoplus_{\alpha \in B_n} \pi_p \Sigma^\infty_+ \mathbf{B}\operatorname{Aut}(\alpha) \end{aligned},$$

and use $\pi_p \Sigma^\infty_+ \mathbf{B}G$ is $\mathbb{Z}$ for $p = 0$ and $G^{\mathrm{ab}} \oplus C_2$ for $p = 2$ to identifty



There is a boundary map ∂ coming from the connecting map in the LES in homotopy of a pair for the filtration.

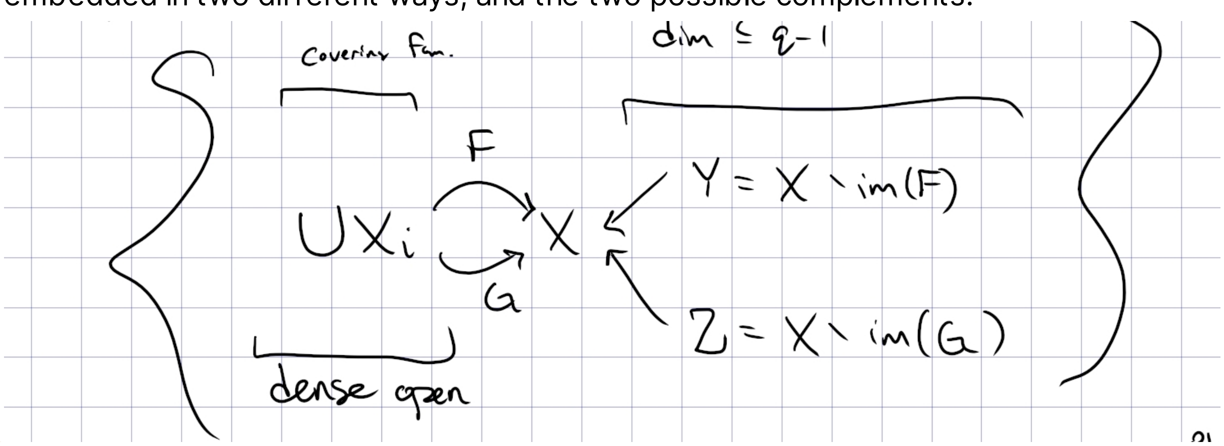
Lemma 3.2 (Let's understand K_1!)

If $\phi \in \mathrm{Aut}(\alpha)$ for $\alpha \in B_q$ is represented by $\phi : U \to V$ then

$$\partial[\phi] = [X \setminus V] - [X \setminus U] \quad \in \mathsf{K}_0(\mathcal{V}^{(q-1)})$$

Proof of Lemma

- In general, $x \in \mathsf{K}_1(\mathcal{V}^{(q,q-1)})$ corresponds to data: X a variety, a dense open subset embedded in two different ways, and the two possible complements:



- (ZakB Prop 3.13) shows that for this data,

$$\partial[x] = [Z] - [Y] \in \mathsf{K}_0(\mathcal{V}^{(q-1)})$$

- For ϕ, we can represent it with the data:

$$
\begin{tikzcd}
& & Y := X \setminus U \arrow[dl] \\
U \arrow[rr, bend left, "i_U"] \arrow[dr, dashed, "\phi"', "\sim"] & & X \\
& V \arrow[ur, "i_V"'] & & Z := X \setminus V \arrow[ul]
\end{tikzcd}
$$

- Then $\partial[\phi] = [Z] - [Y] = [X \setminus V] - [X \setminus U]$ as desired.

Proof of Theorem

$\implies$: suppose ϕ extends to a piecewise automorphism.

- Then $[X \setminus U] = [X \setminus V] \in \mathsf{K}_0(\mathcal{V}^{q-1})$ since $X \setminus U \xrightarrow{\sim} X \setminus V$ by assumption
- By Lem 3.2 above,

$$\partial[\phi] = [X \setminus V] - [X \setminus U] = 0$$

- (Zak17B Lemma 2.1): d_1 and higher d_r are built using ∂, so $\partial(x) = 0 \implies d_r(x) = 0$ for all $r \geq 1$ (permanent boundary).

$\impliedby$: suppose $d_r[\phi] = 0$ for all $r \geq 1$.

- Since $d_1[\phi] = 0$ in particular,

$$[X \setminus U] = [X \setminus V] \in \mathsf{K}_0(\mathcal{V}^{(q,q-1)})$$

since $d_1 = \partial \circ p$ for some map p.
- An inductive argument allows one to write $X = U_r \uplus X'_r = V_r \uplus Y'_r$ where

$$U_r \underset{\text{pw}}{\cong} V_r, \quad \dim X'_r, \dim Y'_r < n - r, \quad \partial[\phi] = [Y'_r] - [X'_r]$$

- Take $r = n$ to get

$$\dim X'_n, \dim Y'_n < 0 \implies X'_n = Y'_n = \emptyset \quad \text{and} \quad X = U_n = V_n$$

- Then

$$\partial[\phi] = [\emptyset] - [\emptyset] = 0 \implies \phi \text{extends.}$$

- A general remark on why $\partial[\phi] = 0$ implies it extends:
  - $\partial[\phi]$ measures the failure of ϕ to extend to a piecewise isomorphism:
$$\partial[\phi] = 0 \implies [X \setminus V] = [X \setminus U] \implies \exists \psi : X \setminus V \underset{\text{pw}}{\cong} X \setminus U$$
  - If additionally $U \cong V$ then $\phi \uplus \psi$ assemble to a piecewise automorphism of X.

Thm C: There is a direct link between $\bigcup_{n \geq 0} \ker \psi_n$ and $\operatorname{Ann}(\mathbb{L})$

Theorem C

Let k be a **convenient field**, e.g. $\operatorname{ch} k = 0$.
Then $\mathbb{L}$ is a zero divisor in $\mathsf{K}_0(\mathcal{V}) \implies \psi_n$ is not injective for some n.

Short: For k convenient

$$\operatorname{Ann}(\mathbb{L}) \neq 0 \implies \bigcup_n \ker \psi_n \neq \emptyset.$$

Proof

- Strategy: contrapositive. Suppose $\ker \psi_n = 0$ for all n. Write $\mathcal{V} := \mathcal{V}_k$.
- There is a cofiber sequence
$$\mathsf{K}(\mathcal{V}) \overset{\cdot\mathbb{L}}{\hookrightarrow} \mathsf{K}(\mathcal{V}) \overset{\ell}{\twoheadrightarrow} \mathsf{K}(\mathcal{V}/\mathbb{L})$$
where $\mathcal{V}/\mathbb{L}$ is a "cofiber assembler" (Zak17b Def 1.11)
- Take the LES to identify $\ker(\cdot\mathbb{L})$ with $\operatorname{coker}(\ell)$:
$$\Rightarrow \operatorname{Ker}\left(K_0(\mathcal{V}) \xrightarrow{\cdot\mathbb{L}} K_0(\mathcal{V})\right) = \operatorname{Coker}\left(K_1(\mathcal{V}) \xrightarrow{\ell_1} K_1(\mathcal{V}/\mathbb{L})\right)$$
$$\begin{tikzcd} \cdots \arrow[r, dashed] & K_1(\mathcal{V}) \arrow[r, "\cdot\mathbb{L}_1"] & K_1(\mathcal{V}) \arrow[r, "\ell_1"] & \overset{\text{coker}}{K_1(\mathcal{V}/\mathbb{L})} \arrow[dlll, bend left] \\ \overset{\text{ker}}{K_0(\mathcal{V})} \arrow[r, "\cdot\mathbb{L}_0"] & K_0(\mathcal{V}) \arrow[r, "\ell_0"] & K_0(\mathcal{V}/\mathbb{L}) & \end{tikzcd}$$
- Reduce to analyzing

$$\mathrm{coker}(E^\infty_{1,q} \to \tilde{E}^\infty_{1,q})$$

where $\tilde{E}$ is an auxiliary sseq.

- Suppose all α extend, then all differentials from column 1 to column 0 are zero.
- The map $E^r \to \tilde{E}^r$ is surjective for all r on all components that survive to E^∞.
- All differentials out of these componenets are zero, so $E^\infty \twoheadrightarrow \tilde{E}^\infty$.
- Then $\mathsf{K}_1(\mathcal{V}) \xrightarrow{\ell} \mathsf{K}_1(\mathcal{V}/\mathbb{L})$ (a surjection), making $0 = \mathrm{coker}(\ell) = \ker(\cdot\mathbb{L})$ so $\mathbb{L}$ is not a zero divisor.

Thm D: Equality in K_0 doesn't imply PW iso and elements in $\mathrm{Ann}(\mathbb{L})$ give rise to elements in $\bigcup \ker \psi_n$.

Theorem

Suppose that k is a *convenient* field. If $\chi \in \mathrm{Ann}(\mathbb{L})$ then $\chi = [X] - [Y]$ where

$$[X \times \mathbb{A}^1] = [Y \times \mathbb{A}^1] \quad \text{but } X \times \mathbb{A}^1 \underset{\text{pw}}{\not\cong} Y \times \mathbb{A}^1.$$

Thus elements in $\mathrm{Ann}(\mathbb{L})$ give rise to elements in $\bigcup \ker \psi_n$.

Proof (can omit)

- Let $\chi \in \ker(\cdot\mathbb{L})$ and pullback in the LES to $x \in \mathsf{K}(\mathcal{V}^{(n)}/\mathbb{L})$ where n is minimal among filtration degrees:

  · Let $\chi \in \mathrm{Ker}(\cdot\mathbb{L})$ Choose x of min filtration deg n

  $$\begin{tikzcd}
  K_1(\mathcal{V}^{(n-1)}) \arrow[r, "\cdot\mathbb{L}_1"] & K_1(\mathcal{V}^{(n)}) \arrow[r, "\ell_1"] & K_1(\mathcal{V}^{(n)}/\mathbb{L}) \arrow[dll, "\partial"] \\
  K_0(\mathcal{V}^{(n-1)}) \arrow[r, "\cdot\mathbb{L}_0"] & K_0(\mathcal{V}^{(n)}) \arrow[r, "\ell_0"] & K_0(\mathcal{V}^{(n)}/\mathbb{L})
  \end{tikzcd}$$

  coker, ker

  $\chi \mapsto 0$
- Write $\partial[x] = [X] - [Y]$ with X, Y of minimal dimension.
- By (LS10 Cor 5),

$$\begin{aligned}[X \times \mathbb{A}^1] = [Y \times \mathbb{A}^1] &\implies \dim X + 1 = \dim Y + 1 \\ &\implies \dim X = \dim Y = d\end{aligned}$$

- Claim: d is small: $d < n - 1$.
- Done if this claim is true: proceed by showing X and Y are not piecewise isomorphic by showing $\ker \psi_n$ is nontrivial by a diagram chase.

Proving the claim:

- **Claim**: If $\mathbb{L}([X] - [Y]) \in \ker ?$ then we can produce an element in $\ker \psi_n$.
  - Diagram chase:

    ① ②

    $$X \dashrightarrow [X]-[Y] \in \operatorname{im} \partial^{(n-1)} \longmapsto \mathbb{L}([X]-[Y]) \neq 0$$

    $$\begin{tikzcd} K_1(\mathcal{V}^{(n-1)}/\mathbb{L}) \arrow[r, "\partial^{(n-1)}"] \arrow[d] & K_0(\mathcal{V}^{(n-2)}) \arrow[r, "\cdot\mathbb{L}_{n-2}"] \arrow[d, "i_*^{n-2}"] & K_0(\mathcal{V}^{(n-1)}) \arrow[d, "i_*^{n-1}"] \\ K_1(\mathcal{V}^{(n)}/\mathbb{L}) \arrow[r, "\partial^{(n)}"] & K_0(\mathcal{V}^{(n-1)}) \arrow[r, "\cdot\mathbb{L}_{n-1}"] & K_0(\mathcal{V}^{(n)}) \end{tikzcd}$$

    ④ $\in \ker i_*^{n-1}$

    $$i_*^{n-2}([X]-[Y]) \in \operatorname{im} \partial^{(n)} \longmapsto 0$$

    ③

    1. $[X] - [Y] \notin \operatorname{im}(\partial)$ by the minimality of n for x, noting $\partial[x] = [X] - [Y]$.
    2. By exactness $\operatorname{im} \partial = \ker(\cdot\mathbb{L})$, so $\mathbb{L}([X] - [Y]) \neq 0$.
    3. By choice of n, $i_*(\mathbb{L}([X] - [Y])) \in \operatorname{im} \partial = \ker(\cdot\mathbb{L})$ in bottom row, so $\mathbb{L}([X] - [Y]) = 0$ in bottom-right.
    4. Commutativity forces $\mathbb{L}([X] - [Y]) \in \ker i_*^{n-1}$.
  - Thus $\mathbb{L}([X] - [Y])$ corresponds to an element in $\ker \psi_n$. (???)

Thm E: K-theory $\operatorname{mod} \mathbb{L}$ models stable birational geometry

Theorem

There is an isomorphism

$$\mathsf{K}_0(\mathcal{V}_{\mathbb{C}}) / \langle \mathbb{L} \rangle \xrightarrow{\sim} \mathbb{Z}[\mathsf{SB}_{\mathbb{C}}] \qquad \in \mathbb{Z}\text{-Mod}.$$

Proof: omitted.

Closing Remarks

- What did we accomplish:
  - Established a precise relationship between Q1 and Q2.
- Unknowns:
  - What fields are convenient?
  - What is the associated graded for the filtration induced by ψ_n?
  - Is there a characterization of $\operatorname{Ann}(\mathbb{L})$?
  - (Interesting) What is the kernel of the localization $\mathsf{K}_0(\mathcal{V}_k) \to \mathsf{K}_0(\mathcal{V}_k)[\frac{1}{\mathbb{L}}]$?
  - Does ψ_n fail to be injective over every field k?

Conjecture (A Correction to Q1 on $\ker \psi_n$)

Conjecture. Suppose that X and Y are varieties over a convenient field k such that $[X]=[Y]$ in $K_0\left(\mathcal{V}_k\right)$. Then there exist varieties X' and Y' such that $[X'] \neq [Y']$, $\left[X' \times \mathbb{A}^1\right]=\left[Y' \times \mathbb{A}^1\right]$, and $X \mathrm{I}\left(X' \times \mathbb{A}^1\right)$ is piecewise isomorphic to $Y \mathrm{I}\left(Y' \times \mathbb{A}^1\right)$

Short: If $[X]=[Y]$, there exist X', Y' st

- $[X'] \neq [Y']$
- $[X' \times \mathbb{A}^1]=[X']\mathbb{L}=[Y']\mathbb{L}=[Y' \times \mathbb{A}^1]$
- $X \coprod X' \times \mathbb{A}^1 \underset{\text{pw}}{\cong} Y \coprod Y' \times \mathbb{A}^1$

- If the conjecture holds, when X, Y are not birational but are *stably* birational, then the error of birationality is measured by a power of $\mathbb{L}$.
- Possibly contingent upon conjecture:

$$[X] \equiv [Y] \bmod \mathbb{L} \implies X \overset{\sim\text{Stab}}{\dashrightarrow} Y .$$