UMD Geometry Festival

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D. Zack Garza

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Contents

1 UMD Geometry Festival

These are some sketches of notes I took as an attendee at the 2019 Geometry Festival at the University of Maryland. I don't expect this to be particularly enlightening for anyone other than myself, but for those reading, perhaps this can at least give you a flavor of the topics covered. Be warned that there are certainly many mistakes, both in notation as well as my own understanding and interpretation of the topics. I hope to not misrepresent anything; please reach out if you find any glaring errors!

Also note that these only cover the first two days of talks, there were two or three on the final day that are not included here.

2 Fluid Mechanics and Geometry (Yann Brenier)

Outline:

- Geometric interpretation of Euler equations for incompressible fluids (Arnold 1966)
- Discrete fluids, combinatorial optimization, generalized incompressible flows
- Generalized minimizing geodesics with probability and convexity tools (1989-2012)
- The initial value problem, relates to the completeness of certain manifolds (2018)

2.1 Euler Equations

First PDEs written in modern style: Euler, 1757.

The Euler flow is an incompressible fluid confined to a compact Riemannian manifolds, evolving according to the Euler equations. (It wasn't clear how to extend definition of Riemannian manifolds to infinite dimensions.) Evolve along geodesics, need to be volume-preserving. Look at action on gridded-up version of torus T^2.

Discrete version of incompressible motion - acts by permutation, i.e. a finite series of permutations on gridded cells. Compute "cost" of flow by adding squares of displacements, which increases if any step moves any given grid by a large amount.

There is some solution that gives the final permutation at the lowest cost – the possibilities are finite, but large. This will be our "geodesic", i.e. optimal transport.

Example: find the minimal flow between $[1, 2, \cdots n]$ and $[n, n-1, \cdots, 2, 1]$. Transpositions get it done in 12 steps. This is nice because it passes to the limit nicely, which ends up looking like a solution to Euler's equation. Observe pattern of each number "bouncing" off wall, while its neighboring number does so in a symmetric fashion the yields a rectangle.

Well-known: in 1D, there is no non-trivial solution.

In passing to the continuous limit, this solution can be improved by a factor of $\frac{\pi}{12}$. Actual solution in continuous case is known; all trigonometric functions. This shape is reflected in the finite numeric simulations.

This generalized solution has been discovered in an entirely different framework involving random walks on the symmetric group (Virag et al). The least action principle ruling the original problem is the limit of a large deviation principle ruling the random exchange of adjacent cells in the discrete model. This principle says that looking at Brownian motion to a point, conditioned on the motion being in a neighborhood of that point, converges to a geodesic in the limit as the noise goes to zero.

2.2 Generalized Flows

A generalized incompressible flow on a compact Riemannian manifold D is a probability measure μ on paths ξ_t such that μ has finite average energy

$$E_\mu \int_0^T \frac{1}{2}\left|\frac{d\xi}{dt}\right|^2 dt$$

where we are taking the expectation of this integral.

Main results on minimizing geodesic (since 1992): let $\mu_{0,T}$ be a probability measure on $D \times D$ such that the projections $\mu_0 = \mu_t = \mathcal{L}_D$, the Lebesgue measure on D. This measure is spanned by at least on generalized incompressible flow μ of minimal energy.

In fact, there is a unique pressure distribution $\nabla p(t, x)$ that relates to these solutions by an explicit equation, where we view this as an acceleration field. The uniqueness here is surprising. In Arnold's classical framework, approximate minimizing geodesics may not converge in the any classical sense but do converge to generalized solutions when the dimension is at least 3 *(Shnirelman 1985)*. In dimension 2, there is something to do with symplectic forms that prevents this. Note that there is no similar results for the dynamics of rigid bodies, i.e. geodesic curves along $SO(3)$ *(Y.B. 2012)*. Seems that the $d = 2$ case is generally open.

Smoothness of these solutions is important.The existence of a unique ∇p follows from the convexity of the minimizing geodesic problem. Main open equation: is ∇p smooth? We currently know it is L^2 locally (1999/2008) and we have data for which p is not better than locally semi-convex in x.

Last questions: what about the initial value problem? (Y.B. 2018)

A priori, convex minimization techniques are hopeless for the IVP. For a generalized incompressible flow μ with finite average energy, it does not make sense to prescribe any initial velocity $\left|\frac{d\xi}{dt}\right|_{t=0}$ for $\mu-$a.e. paths. But if we look at such a flow in minimal energy, we can set up a different equation that is tractable. The dual convex minimization problem is always solvable, and can uniquely recover the smooth classical solutions to the Euler equations for a short enough T. This can be seen as a kind of non-commutative optimal transport problem involving fields on non-negative symmetric matrices, which are of current interest.

We currently don't know if the Euler equations have global solutions, or even if they break down in finite time! Wide open, one of the main problems in non-linear PDEs.

Generally looking at space of volume-preserving diffeomorphisms, dense in the set of certain product measures? Gives a natural weak closure: goes by the name of bi-stochastic measures?

3 Moment Maps in Symplectic and Kahler Geometry (Dietmar Salamon)

Let (X, ω) be a symplectic manifold acted on by a lie group G. There is a moment map $\mu : X \to \mathcal{O}\{^*$ (?), and we can consider a quotient space $X//G = \mu^{-1}(0)/G$ where we look at where the moment map vanishes and quotient out by a group action. This is an orbifold, and in fact a manifold and symplectic.

We can look at the Kahler case (X, μ, J) and for $G \in U(n)$ we can complexify to obtain $G^C \in G(n, \mathbb{C})$. and $X/G^C \cong X^{ss}/\sim \cong X//G$. We'd like to know the behavior of maps passing through the zero set of the moment map. To this end, we use Mumford weights:

$$W_\mu(x, \xi) = \lim_{t\to\infty} \langle \mu(\exp(it\xi)x), \xi\rangle$$

where we follow a Hamiltonian flow to infinity and obtain a number.

If any of the weights are negative, the points can not be semistable, which falls out of a specific bound on the Mumford weights. Thus we have

Theorem:

$$\sup_{\xi\neq 0} \frac{-W_\mu(x,\xi)}{|\xi|} \leq \inf_{g\in G} |\mu(gx)|$$

Corollary: x is semistable $(\mu(gx) = 0) \iff \forall \xi, W_\mu(x, \xi) \geq 0$.

This is at the heart of **geometric invariant theory** (Hilbert, Mumford, Kempf, Ness, Kirwan).

The key tool is studying the gradient flow of the moment map, or more precisely $\frac{1}{2}\mu^2$, about which one can say many interesting things.

Example: let

$$J_n = \left\{ J^{2n\times 2n} \mid J^2 = -I \right\} = \left\{ g \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} g \mid g \in SL(2n, \mathbb{R}) \right\}$$

which are compatible with orientation. We can take the tangent space to obtain

$$T = \left\{ J \mid \hat{J}J + J\hat{J} = 0 \right\} = \left\{ [\xi, \hat{J}] \mid \xi \in \mathfrak{sl}(2, \mathbb{R}) \right\},$$

and get a symplectic form $\Omega_J(A,B) = \frac{1}{2}\text{Trace}(AJB)$. Then the moment map is given by

$$\Omega_J([\xi, J], \hat{J}) = \left\langle d\mu(), (J)\hat{J}, \xi \right\rangle = -\text{Trace}(\xi\hat{J}).$$

Can look at frame bundle, get structure group $SL(2n,\mathbb{R})$. Let M^{2n} be an oriented manifold with a volume form, then $J(M)$ is the space of almost-complex structures on M. Can repeat the above construction to get a symplectic form that is volume-preserving by integrating a differential form over the endomorphism bundle. This ends up being compatible with the group action (the group of volume-preserving diffeomorphism), so the question becomes whether or not it's a Hamiltonian action.

We want to get a map that yields a exact 2-form, we'll just show one that yields a closed form – the Ricci form (?).

Theorem: The action of G^{Ox} on $J(M)$ is Hamiltonian with moment map $J(M) \to \Omega^2(M)$ given by $J \mapsto 2\text{Ricc}_{G,J}$.

Choose ∇ a torsion-free connection on TM where $\nabla_G = 0$, then $\text{Ricc}_{G,J} = \frac{1}{2}(\tau_J^\nabla + d\lambda_J^\nabla)$ where d is the covariant derivative. Although we make this choice, the result Ricci form does not depend on it.

Think about the case where M is Calibi-Yau (and/or Teichmüller space?). Can do similar construction, but the Ricci form takes values in some affine space and needs a correction, and you lose equivariance.

Note that this is not a Kahler form. Look at Yau's theorem. Look at the Chern class of $S^1 \times S^3$. There is a construction that yields both a complex form and a canonical symplectic form on the Teichmüller space $\mathcal{T}_0(M)$, and we can find an explicit formula for it:

$$\Omega_J(\hat{J}_1, \hat{J}_2) = \int_M \left(\frac{1}{2}\text{tr}(\hat{J}_1 J \hat{J}_2) - f_1 g_2 + f_2 g_1\right) \rho_J$$

where ρ_J is such that $\text{Ricc}_{\rho_J, J} = \omega$.

Recovers something about Kahler-Einstein metric, the moment map can also be roughly seen as a Kahler-Ricci potential. Yields some logarithmic variant of the Kahler-Ricci flow. See the Dehn functional, Donaldson framework.

4 Zero Sets of Laplace Eigenfunctions (Aleksandr Logunov)

Let M be a closed Riemannian manifold, we will look at eigenfunctions of the Laplacian on on M.

Example: Eigenfunctions on S^2 are restrictions of homogeneous harmonic polynomials functions $f : \mathbb{R}^3 \to S^2$, which has a basis of relatively simple polynomials where the eigenvalues are related to their degrees.

Two elementary questions:

- Does the number of critical points of eigenfunctions φ_λ,

  $$C_{\phi,\lambda} = \{x \ni \nabla\varphi_\lambda(x) = 0\},$$

  tend to infinity as $\lambda \to \infty$?

- Flat eigenfunctions: is there a sequence of eigenfunctions on S^2 such that

  $$\max_M |\varphi_\lambda(x)| \leq C\|\varphi_\lambda\|_2?$$

- E.g. on S^1, all eigenfunctions $\sin(ax+b)$ are flat.
- Such eigenfunctions exist on S^{2d-1}.
- Sarnak's conjecture: there are no such sequences on S^2.

Nodal domains: a theorem of Courant (1923) says that the k-th eigenfunction has at most k nodal domains (where they are ordered by size of eigenvalue). A result of Stern/Lewey shows that there are spherical harmonics of any odd degree with only two nodal domains.

Yau's conjecture:

$$c\sqrt{\lambda} \le H^{n-1}(Z_{\varphi_\lambda}) \le C\sqrt{\lambda}.$$

Shown for algebraic manifolds and real-analytic manifolds, still open in the Riemannian case. Recent result: proves lower bound in dimensions above 2, some improvements closer to conjectured in dimension 2.

Nadirashvili's conjecture: Let u be a non-constant harmonic function in $\mathbb{R}^3$. Does $\mu(\{u=0\}) = 0$? Believed we need to understand this for harmonic functions before understanding the zero sets of Laplacians. By 2016 work, yes, and there is a uniform lower bound (?).

An old trick: can translate questions about zero sets of eigenfunctions $\varphi + \lambda\varphi = 0$ into zero sets of harmonic functions $\Delta u = 0$, given by defining $u(x,t) = \varphi(x)e^{\sqrt{\lambda}t}$.

Work of Donnelly-Fefferman bounds the growth of Laplace eigenfunctions. There is a harmonic analog of Yau's conjecture, $H^{n-1}(Z_\varphi \bigcap B_1) \le CN(B_1)$ where H is the Hausdorff measure and N is the doubling index; Yau's would follow from this by setting $C = \sqrt{\lambda}$.

On the scale $c/\sqrt{\lambda}$, you can "shake" the eigenfunctions so they look like harmonic functions; i.e. there is a quasi-conformal change of coordinates that sends the zero set the eigenfunctions to the zero set of a harmonic function.

Landis conjecture: let $\Delta u + Vu = 0$ be an elliptic equation, where V is a bounded potential $|V| < 1$. If $|u(x)| \le \exp(-|x|^{1+\varepsilon})$, then u is identically zero. Can also be formulated in terms of the maximum number of nodal curves intersecting at a point. WIP: this is true for real potentials. Solutions behave very differently between the real and complex cases. Any counterexample would necessarily have many zeros.

5 The Geometry and Arithmetic of the World's Smallest Calibi-Yau Threefolds (Jim Bryan)

Definition: A compact complex Kahler manifold X of dimension d is Calibi-Yau if $c_1(TX) = 0$, where c_1 is the first Chern class (i.e. $\det T^\vee X = K_X \cong X \times \mathbb{C}$, i.e. the canonical is trivial) $\iff$ there exists a holomorphic global d-form $\iff$ there exists a Ricci flat Kahler metric and $h^{k,0}(X) = 0$ unless $k = 0, d$.

- If $d = 1$, X is an elliptic curve, topologically only one type (a torus)
- If $d = 2$, X is a $K3$ surface, topologically only one type
- If $d = 3$, X is a CY-threefold, there are >5 million

There are two interesting Hodge numbers: - $h^{1,1}(X)$: the dimension of the Kahler cone, or dually the number of independent holomorphic curve classes - Never zero if Kahler - $h^{2,1}(X)$: the space of independent infinitesimal complex deformations - Mirror symmetry swaps these two

Definition: A rigid CY3 X has $h^{2,1}(X) = 0$.

Interested in a few aspects of CY3s:

- What combinations of values $(h^{1,1}, h^{2,1})$ are possible?

  - Mirror symmetry falls out of tabulating these
  - Physicists interested in small ones

- How small can these combinations be? (By summing these two numbers)
- Can we compute the Donaldson-Thomas/Gromov-Witten/"string partition functions" (generating functions for counts of curves)

  - What are the modular properties of these functions?

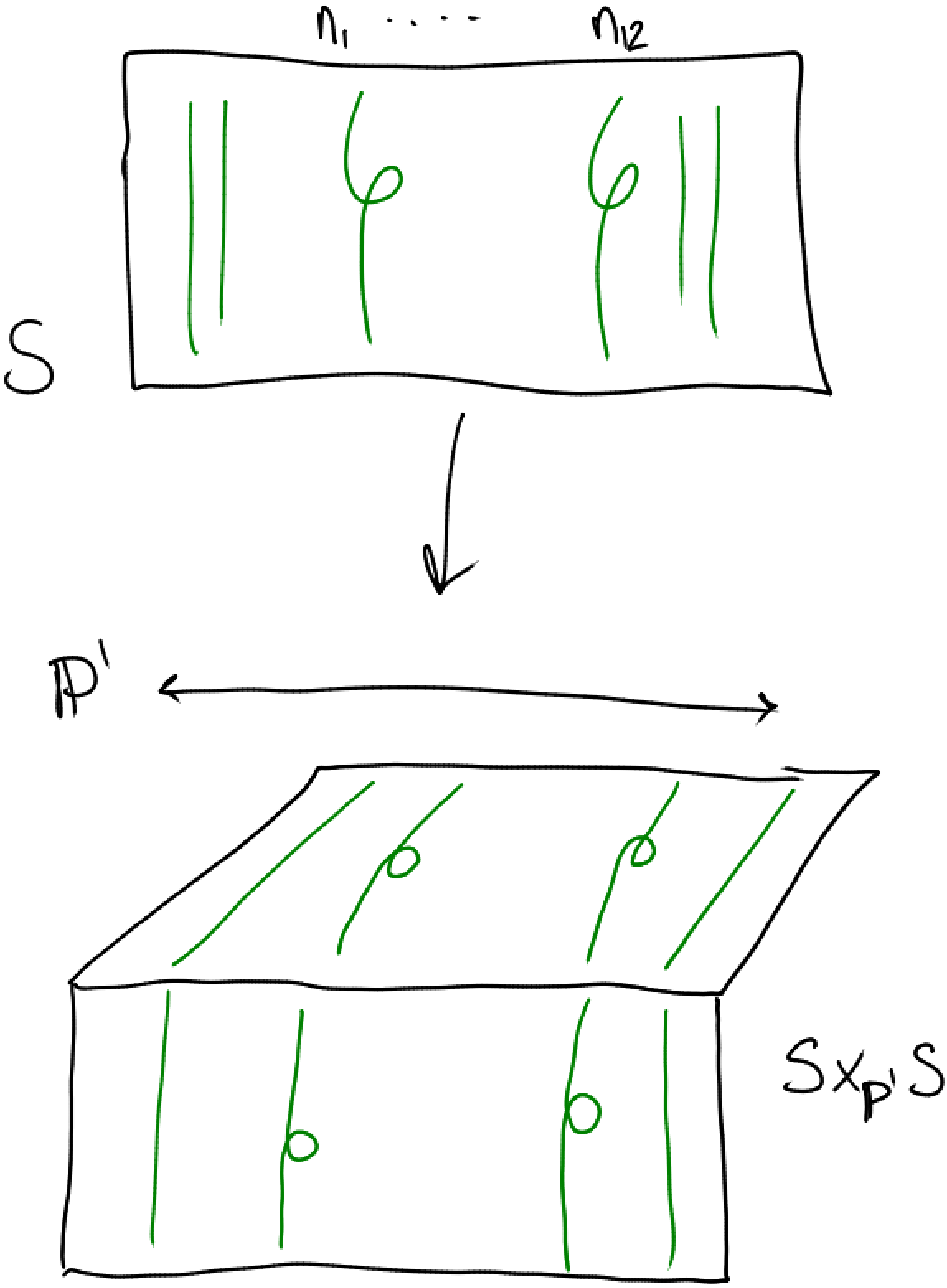
- Arithmetic: rigid CY3s over $\mathbb{Q}$ are modular forms (much like Wile's theorem showing elliptic curves are modular)

Today:

- Construct rigid CY3s (banana manifolds) of (0,4) and (0,2)
- Compute Donaldson-Thomas partition function and associated Gromov-Witten potential, a genus 2 Siegal modular form of weight $2g-2$.
- Compute the modular forms (L-series)

5.1 The Generic Banana Manifold

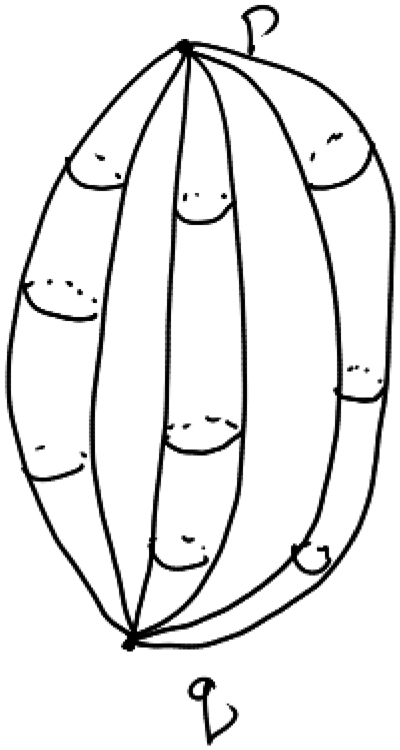
Start with a generic hypersurface $S \subset \mathbb{CP}^2 \times \mathbb{CP}^1$ of degree $(3,1)$. Projecting onto $\mathbb{P}^2$ yields $Bl_{9\text{pts}}\mathbb{CP}^2$, onto $\mathbb{P}^1$ yields an elliptic fibration (projects onto an elliptic curve) which has 12 nodal fibers.



Now to make a threefold: take the fiber product $\hat{X} = S \times_{\mathbb{P}^1} S$ which has 12 conifold singularities, which we can resolve by taking $X = Bl_\Delta(\hat{X})$, where Δ is the diagonal. This yields a CY3 with $h^{2,1}(X) = 8, h^{1,1}(X) = 20$. The fibers are generically products of elliptic curves $E \times E$, which has 12 singular fibers.

(Note that we're blowing up along a Cartier divisor and not a Weil divisor, which gives the singularities local resolutions.)

We also have $E_{\text{sing}} = \mathbb{C}^\times \times \mathbb{C}^\times \times B$ where $B = C^1 \bigcup C^2 \bigcup C^3$ where $C_i = \mathbb{P}^1$ and $C_i \bigcap C_j = \{p, q\}$ and $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. This thing is a toric variety?

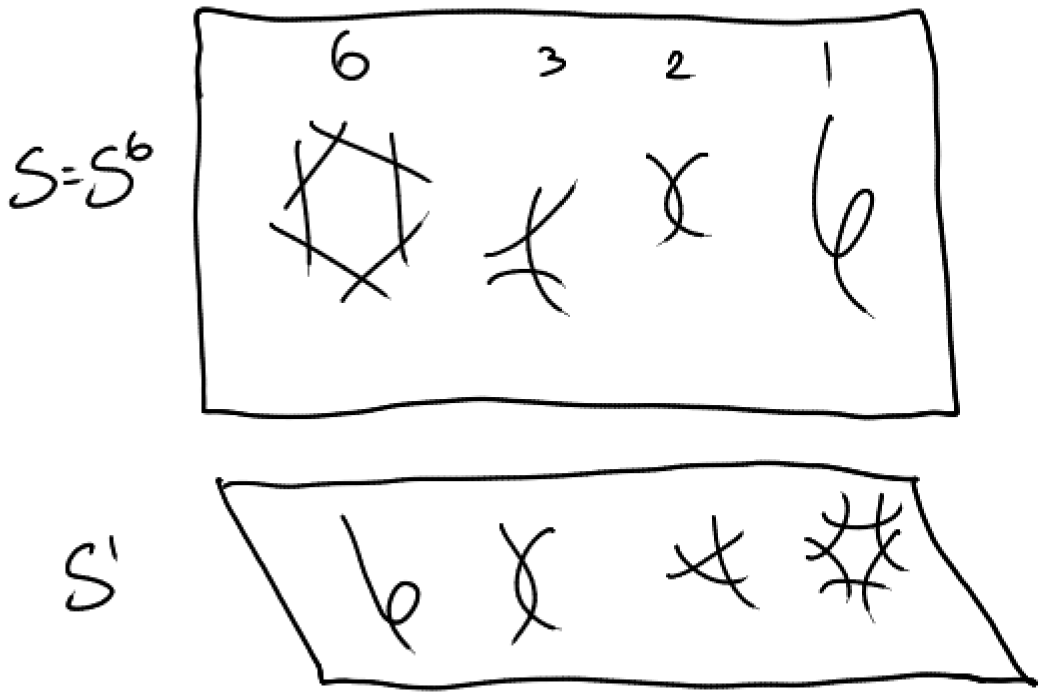


Banana mfd.

We can do enumerative geometry here by looking at the fibers of $X \xrightarrow{\pi} \mathbb{P}^1$, where $\beta \in \ker \pi \cong \mathbb{Z}^3 = H_2(X; \mathbb{Z})$ is given by $\beta = d_1[c_1] + d_2[c_2] + d_3[c_3]$. Can write out a generating function $Z^{DT}(Q_1, Q_2, Q_3, p)$ where the coefficients are the DT invariants, i.e. the virtual count of curves given by the holomorphic euler characteristic. This expands to a product where the coefficients and exponents are themselves coefficients of modular forms. The exponents appearing are the fourier coefficients of a specific modular form?

The associated GW potentials are genus 2 Siegel modular forms F of weight $2g - 2$. Can think of this as a function on the moduli space of genus 2 modular curves, and the weight is like the Chern class?

A rigid example: start with an extremal elliptic surface S^6:



This is the universal family for moduli of elliptic curves with a 6-torsion point, and has a $\mathbb{Z}/6\mathbb{Z}$ action. Let S' be a base change that reverses the order of the fibers, and take $S \times_{\mathbb{P}^1} S'$. The $\mathbb{Z}/6\mathbb{Z}$ action is now free and this is a 6×4 conifold. If we blow up the quotient along these points to get an X_6, this is a rigid CY3 where $X \xrightarrow{\pi} \mathbb{P}^1$ has 3 banana fibers and 1 section, $h^{2,1} = 0$, and $h^{1,1} = 4$. The theorem (in progress) is that Z of these fibers is similar to the previous one, and the F_qs are now Siegel for certain congruent subgroups of $\mathrm{Sp}_2(\mathbb{Z})$.

As it turns out, there was an additional $\mathbb{Z}/2\mathbb{Z}$ action, yielding a $\mathbb{Z}/12\mathbb{Z}$, and quotienting by this yields 2 banana fibers and 2 doubled fibers which are each $\mathbb{P}^1$ bundles over a special elliptic curve. Moreover, this is tied for the smallest one, at $h^{2,1} = 0, h^{1,1} = 2$.

Where does the modularity come from? Topological vertex, Schur functions, but not satisfying – we don't have good explanations of where this should be coming from.

There aren't any threefolds where we can completely compute the partition functions, we have a hard time computing with the sections (although the fibers are okay).

6 Boundary Operator Associated to Sigma_k Curvature (Yi Wang)

Define the k-Hessian energy
$$\int_\Omega u\sigma_k D^2 u \, dx.$$

When $k = 1$ this recovers the Dirichlet energy $\int_\Omega -u\Delta u$. On a Riemannian manifold, we are instead interested in the Shouten tensor which is a combination of the Ricci curvature and the scalar curvature. Can decompose the Riemann curvature tensor into a conformally invariant Weyl curvature W and and certain product.

We want to look at conformally invariant operators, and the most natural one is the conformal Laplacian $L_g = -\Delta + c_n R_g$. Under a conformal change of metric, we can look at the Yamabe problem which asks for a decomposition with $\hat{R}$ constant. This was solved in the 80s. It is a variational problem.

On 4-manifolds, there is a Chern-Gauss-Bonnet formula, which involves the L_2 norm of the Weyl tensor (local conformal invariant) and the other is a constant times σ_2, and so

$$\int_{X^4} \sigma_2 \, dv_g$$

is a conformal invariant.

This leads to a generalized Yamabe problem $\sigma_k =$ constant. This is 2nd order elliptic PDE. This problem is variational (solutions are critical points of an energy functional) if $k = 1, 2$ or g is locally conformally flat. Can write this functional, $\int \sigma_k \operatorname{vol}_g$, and can write an Euler-Lagrange equation for this to determine the gradient of this functional.

Open questions: - Is there a Dirichlet principle for the Shouten tensor or the operator $\sigma_k(D^2 u)$?

Why should this be possible? The k-Hessian energy is pointwise non-negative when $u = 0$ on the boundary, so it is a generalized Dirichlet energy by integration by parts. Can obtain a Sobolev inequality for it when Σ is $(k-1)$-convex (i.e. this holds for the second fundamental form) and $2k < n$. This gives an embedding into some L^p space. For $2k = n$ obtain an Orlicz-type inequality; for $2k > n$ obtain embeddings into Holder spaces.

The well-known Dirichlet principle: for any u where $u|_{\partial M} = f$ and $\int_\Omega |\nabla u|^2 dx \geq \int_\Omega |u_f|^2 dx$ where u_f is a harmonic extension of f that solves

$$\begin{cases} \Delta u = 0 & \text{on } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

The main result is that for smooth domain with boundary, there is a multilinear differential operator along with a multilinear functional that is symmetric in its inputs that recovers the k-Hessian. This functional looks like

$$Q = -\int_\Omega u L_k + \oint_{\partial\Omega} B_k$$

where L_k is a k-linear map and B_k is a boundary operator.

Examples involve the Laplace-Beltrami operator.

The symmetric property in this theorem yields two nice properties: the functional's Euler-Lagrange equation is the one wanted earlier, and it is convex. With some assumptions, this will yield the desired Dirichlet principle for the σ_k curvature.

The boundary operator involves some generalization of the mean curvature, called H_k which is a linear combination of the Shouten tensor and the second fundamental form. The L is an extension of the conformal Laplacian operator. When $k = 1$, this recover Escoba Yamabe's functional.

7 Spectral Asymptotics on Stationary Spacetimes (Steven Zelditch)

Want to look at a step function that jumps at eigenvalues by the multiplicity of each eigenvalues. Recall the wave equation $\Box u = 0$, and introduce the propagator $e^{-it\sqrt{-\Delta}}$ which is a pseudo-differential operator that propagates time zero solutions.

Can look at the tempered distribution

$$\mathrm{Tr} V(t) = \sum_{j=0}^{\infty} e^{it\lambda_j} = e_0(t) + \sum_L e_L(t)$$

were L are taken in the "length spectrum".

The singular support of e_0 is $\{0\}$ and of e_L is $\{L\}$. Essentially amounts to take the fourier transform of the counting function and looking at its singularities. Need to assume that the sets of closed geodesics are nondegenerate (critical point for the length function on the loop space). Note that when you look at the family of closed geodesics of a given length, the number will depend on the geometry. A sphere has a manifold's worth, a torus a single parameter family, while on a hyperbolic surface each length is isolated.

Any closed geodesic is in the unit cotangent bundle $S\Sigma$, so pick a transversal. Can look at first return map, take the derivative to obtain the linear Poincare map. Shouldn't have 1 as an eigenvalue, otherwise you could deform the geodesic (condition is equivalent to being nondegenerate).

Major milestone: able to reduce complex determinants of Hessians of phase functions to (essentially) linear algebra. Some kind of "symbol calculus" of fourier operators.

General question: instead of solving Einstein's field equations, how do we instead let a wave equation evolve on a curved spacetime?

Uncurved spacetime is given by $\mathbb{R} \times \Sigma$ with a metric $-dt^2 + h_\Sigma$, the Euclidean metric? The generalization here is to globally hypberolic stationary spacetimes, want a compact Cauchy hypersurface. Globally hyperbolic is a condition used frequently in general relativity. We have $M^{3,1} \cong \mathbb{R} \times \Sigma$ topologically (not metrically), and so every causal curves (tangent vector has lorentz norm strictly positive?) intersects a given hypersurface Σ exactly once.

Moreover $\Box$ is essentially the Laplace-Beltrami operator on this space. Stationary means there exists a timelike Killing vector field, i.e. timelike flows are isometries. The killing field is our stand-in for $\frac{\partial}{\partial t}$, since there is no preferred time coordinate. This is a strong assumption, since the cosmological constant was found to be positive (i.e. expanding spacetime). What is the propagator, the eigenvalues, the lengths of closed geodesics?

Can look at $z = \frac{\partial}{\partial t}$ and $Dz = \frac{1}{i}\frac{\partial}{\partial t}$ and we can recover the Laplacian by looking at the eigenvalues of special solutions, $Du_J = \lambda u_j$, and the u_j span $\ker \Box_g$ the Klein-Gordon operator. Want to keep everything internal with Lorentzian geometry, so don't want to choose a particular Cauchy hypersurface extrinsically.

Can look at space of null-geodesic $N = \{\gamma \ni g(\dot\gamma, \dot\gamma) = 0\}$ (equivalent to light rays) which is a symplectic manifold. Define $\mathrm{char}\Box = \{\xi \in T^\vee M \ni \sigma_\Box(x, \xi) = 0\}$, then let $\mathbb{R}$ act on this by Hamiltonian flows to obtain $N = \mathrm{char}\Box/\mathbb{R}$. Leads to studying e^{tZ} the killing flow on N. Can get a bundle of the original space over the orbits of the killing flow, which has a natural connection θ where the metric on the base space is induced by the metric in the total space. The metric does not need to be integrable; this is equivalent to requiring the space to be static instead of stationary.

The theorem: look at the spectrum of $D_z|_{\ker \Box}$. Want to define a trace, so need an inner product on $\ker \Box$. This has a natural symplectic vector space structure, since it's a solution to variational problem (?), so we can just add an almost-complex structure (see books on QFT in curved spacetime, positive and negative frequency solutions, Robert Wahl?). Can use the energy-stress tensor to define such an inner product, this will be independent of the choice of a cauchy hypersurface.

Theorem: Tr $e^{tZ}\Big|_{\ker \Box}$ is essentially the same summation formula as earlier, where the sum is over the periodic orbits of the killing flow, and this is something that makes sense for any dynamical system.

Picture to keep in mind: can evolve a hypersurface using the wave equation $\Sigma_0 \to e^{tZ}\Sigma_0$ and then look at the killing flow $e^{tZ}\Sigma_0 \to \Sigma_0$. This gives an operator that you can take the trace of.