Reading Seminar: J-Holomorphic Curves, Section 3.3

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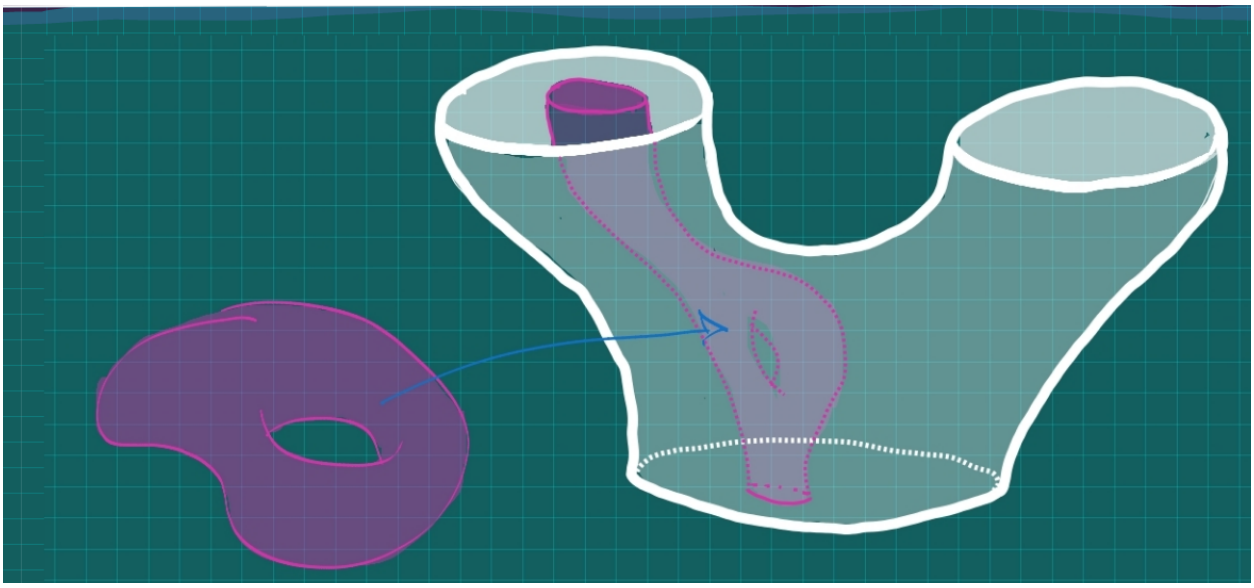


1 | Goal: The Gromov-Witten Invariants

⚠ **Warning 1.0.1**
These are extremely rough and imprecise notes taken for a reading seminar. They're very likely full of mistakes and misunderstandings, so please use at your own risk.

Remark 1.0.2: Our long-term goal: define the **Gromov-Witten** invariants of (say) a symplectic 4-manifold, i.e. a complex symplectic surface. These will be notated something like $\mathrm{GW}(M, A, g, n)$, where

- (M, ω, J) is a symplectic manifold with an almost-complex structure J,
- $A \in H_2(M; \mathbb{Z})$ is a fixed homology class,
- g, n denote the genus and number of marked points of a complex curve (Riemann surface) $\Sigma_{g,n}$ we're mapping into M.

The point of these invariants is to take some kind of enumerative count of rigid complex curves (J*-holomorphic curves*) embedded into M, up to some notion of isomorphism so that we can make this a finite count. The J-holomorphic condition is essentially that these curves are in the kernel of some differential operator that generalizes the usual Cauchy-Riemann equations from 1-dimensional complex analysis. The actual count will be the number of curves that are homologically equivalent to the fixed class A.

The rough program is to form some moduli space of curves, "rigidify" it enough to get it to be a finite-dimensional smooth manifold, and try to relate the curve count to the dimension of the moduli space. We'll then relate that dimension to the analytic *index* of a Cauchy-Riemann operator D, and use a mix of analysis, topology, and algebraic geometry to compute this index (either by computing kernels/cokernels directly or relating them to other invariants like integrals, Chern classes, etc).

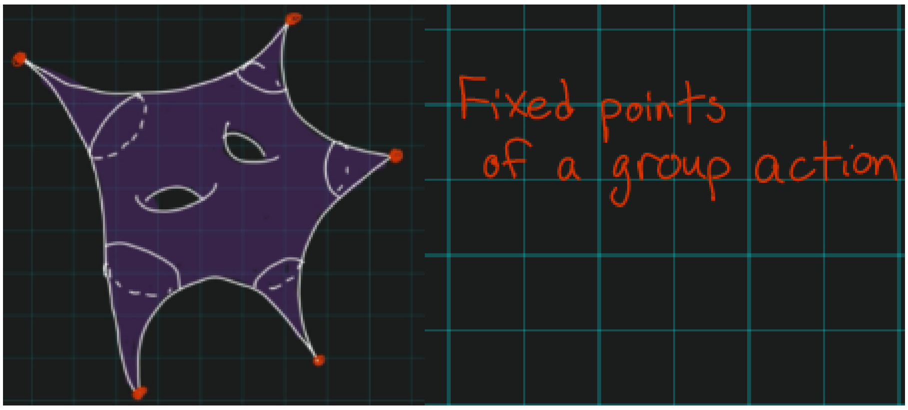
Note: I'll be writing $\mathrm{Sym}^n(X)$ *everywhere for convenience, but I don't know if this is true on the nose – in the older literature these maps are all defined to land in* $M^{\times n}$ *instead.*

Remark 1.0.3: Fix M, A, J, g, n. We'll define $\overline{\mathcal{M}}_{g,n}(A, J)$ to be the moduli space of embedded J-holomorphic curves $\Sigma \overset{f}{\hookrightarrow} M$ with $[\Sigma] = [A]$ in $H_2(M; \mathbb{Z})$ where $g(\Sigma) = g$ and $[x_1, x_2, \cdots, x_n] \in \mathrm{Sym}^n \Sigma$ is a collection of n (distinct?) marked points.

Remark 1.0.4: There is a well-studied moduli space $\overline{\mathcal{M}}_{g,n}$, *the* moduli space of curves, which shows up in a thousand different guises with a thousand different names. It is a very good space that a lot of people like. We'll define it here to be the moduli space of **stable** embedded curves $\Sigma \hookrightarrow M$ with $g(\Sigma) = g$ where again Σ carries n marked points. The stability condition is something we'll cover later.

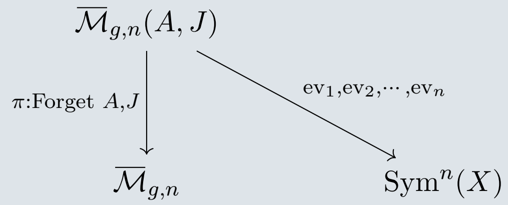
In full generality, $\overline{\mathcal{M}_{g,n}}$ is a smooth Deligne-Mumford stack – i.e. a somewhat complicated algebro-geometric object. One slogan that may be helpful: the category of schemes isn't closed under quotients, but schemes embed into stacks and stacks are closed under taking quotients. So you might think of a stack as the quotient of a scheme by a non-free group action, where you might even take an algebraic group instead of just a Lie group or something. A Deligne-Mumford stack will just be a stack that is stratified by quotient stacks.

Remark 1.0.5: It may be helpful to think of $\overline{\mathcal{M}}_{g,n}$ as a complex *orbifold.* As a first approximation, an *orbifold* is just a manifold with some exceptional collection of singular "orbifold points", which are the fixed points of some group action. Here's the cartoon I usually have in mind:



For genus $g = 0$, $\overline{\mathcal{M}}_{0,n}$ will be a smooth compact complex manifold of finite dimension, so we can run arguments from smooth/differential topology if we just map in J-holomorphic spheres.

Definition 1.0.6 (Gromov-Witten invariants, preliminary definition)
There are two natural maps floating around:



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The evaluation maps ev_j are coming from the fact that every point in the moduli space upstairs carries the data of an embedding $f : \Sigma_{g,n} \hookrightarrow M$, and if x_i is a marked point we can just push it forward and look at $f(x_i) \subseteq M$. So morally speaking, the GW invariants will be defined as

$$\mathrm{GW}(M, A, g, n) : H^*(M; \mathbb{Q})^{\otimes_{\mathbb{Q}} n} \otimes_{\mathbb{Q}} H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q}) \to \mathbb{Q}$$
$$(\alpha_1, \alpha_2, \cdots, \alpha_n) \otimes \beta \mapsto \int_{\overline{\mathcal{M}}_{g,n}(A,J)} \prod_{j=1}^{n} \mathrm{ev}_j^*(\alpha_j) \smile \pi^*(\beta^{\mathrm{PD}}),$$

where here the product denotes the n-fold cup product in (say, singular) cohomology.

Remark 1.0.7: The output will be the number of embedded J-holomorphic curves Σ

- Where $g(\Sigma) = g$
- With n marked points x_i
- Where x_i intersects a cycle $X_i \subseteq H_2(M; \mathbb{Q})$
- Where X_i is dual to α_i
- Where $[\pi(\Sigma)] = [\beta] \in H_2(\overline{\mathcal{M}}_{g,n})$.

So roughly the number of curves representing the homology class β, and we get it by "integrating over the moduli space" in the sense of capping against a fundamental class.

⚠ **Warning 1.0.8**

This description is partially a cartoon! It will work in certain special cases, but $\overline{\mathcal{M}}_{g,n}$ doesn't have an honest fundamental class in general to integrate against. Some hard work of e.g. Fantechi constructs a "virtual fundamental class" that (I think) more faithfully captures this idea.

Example 1.0.9 ***(Uses):*** GW invariants can be used to get at classical enumerative problems. For example, we can compute the number of lines in $\mathbb{P}^3(\mathbb{C})$ intersecting 4 generic lines as

$$\mathrm{GW}^{\mathbb{P}^3}_{L,4}(c^2, c^2, c^2, c^2) = 2.$$

Remark 1.0.10: The GW invariants will only depend on the **deformation type** of (M, ω). In particular, we'll be able to take 1-parameter families of symplectic manifolds constructed by cooking up paths

$$\gamma : I \to \Omega^2(M)$$
$$t \mapsto \omega_t,$$

all of which will have the same GW invariants, provided we start with *semipositive* symplectic manifolds and choose these paths carefully. Morally, this is moving the manifolds (M, ω) around in the moduli space, just in a controlled way (along semipositive families) as opposed to just wiggling in an ε ball in $\overline{\mathcal{M}}_{g,n}(A, J)$.

Remark 1.0.11: A remarkable (and hard?) theorem is that in real dimension 4, the GW invariants only depend on the *diffeomorphism type* of the manifold, and can detect non-diffeomorphic smooth manifolds. They are also equal to the Seiberg-Witten invariants in this dimension. This is not a general phenomenon though – there are counterexamples in dimension 6 where neither of these statements hold.

Remark 1.0.12: The later chapters of the book discuss some applications to other topics. I'll just relay the words here, in case anything is meaningful to you all, since I don't know much about them yet myself:

- For $g = 0$, GW is related to quantum cohomology and Frobenius manifolds. There is some theorem about proving the associativity in quantum cohomology.
- There is *some* way to produce a TQFT in this setting as well, and lots of people like these.
- Mirror symmetry is supposed to give a 2nd way to compute these invariants. I think the symplectic side covered here corresponds to the "A side", and conjecturally there is a "B side" mirror with the same GW invariants. The book is a little old now, so I don't know how conjectural this still is.

2 | Ch. 3: Moduli Spaces and Transversality

Remark 2.0.1: Our goal for this chapter: show that for a general J, the moduli space $\mathcal{M}^*(A, \Sigma, J)$ is a smooth complex manifold of finite dimension. The asterisk here corresponds to taking only *simple curves* – this doesn't seem to be a necessary condition, but is meant to make transversality arguments simpler. Here's a rough outline of the sections:

- 3.1: Defines the moduli space of simple curves
- 3.2: Discusses Thom-Smale transversality, and shows that $\mathcal{M}^*$ is a smooth manifold when the (linearized) Cauchy-Riemann operator D is surjective for all J-holomorphic curves.
- 3.3 (today): Discusses examples of **regularity** in dimension 4, along with some sufficient conditions to determine if your favorite almost complex structure J is sufficiently regular. The usual proofs lean on things like the Sard-Smale theorem, but here we'll use some AG techniques like the Riemann-Roch theorem to check these conditions.
- 3.4: Discusses moduli spaces with pointwise constraints.

Remark 2.0.2: Some structures to recall from Han's talks:

- (M, ω, J) will be a $2n$-dimensional symplectic manifold, with $\omega \in \Omega^2(M)$ a symplectic form, J an ω-tame almost-complex structure on M, so $J \in \operatorname{End}(TM)$ with $J^2 = -\operatorname{id}$.

- (Σ, j_Σ, dV) will be a Riemann surface with an almost-complex structure and dV its volume form. Note that in dimension 2, all almost-complex structures are *integrable* in the sense that they come from an honest complex structure, so we'll always think of j_Σ as an actual complex structure.

  - The Cauchy-Riemann operator $\bar{\partial}_J \approx \frac{1}{2}(Jd - dj_\sigma)$, where I'm being **very** loose with this definition! Just recall that (exercise) the usual Cauchy-Riemann equations can be generalized by $dJ - Jd = 0$ where J is the standard complex structure on $\mathbb{C}^n$, and here we just allow the two complex structures to vary in the domain/codomain.
  - Also $\ker \bar{\partial}_J$ are precisely the J-holomorphic curves, so solutions to this generalized Cauchy-Riemann equation.

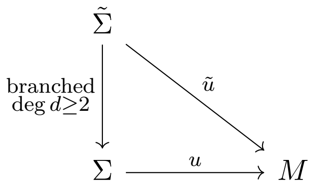
- $u \in C^\infty(\Sigma, M)$ will be a smooth map representing a solution. Note that we wanted Sobolev completions to some $W^{k,p}$ in order to apply PDE theory to u. In particular, we'll want the linearized $\bar{\partial}_J$ to be a Fredholm operator so that it has a well-defined index

$$\operatorname{ind}(D) := \dim_{\mathbb{R}} \ker(D) - \dim_{\mathbb{R}} \operatorname{coker}(D).$$

  These are supposed to be like "operators that are invertible up to finite-dimensional noise", and such operators (and their indices) are stable under small perturbations.

- $\mathcal{M}^*(A, \Sigma, J)$ will be $u \in C^\infty(\Sigma, M) \cap \ker \bar{\partial}_J$ with $[u(\Sigma)] = A \in H_2(M; \mathbb{Z})$ and u a **simple** curve.

  - Simple curves are defined by the following condition: a curve $u : \Sigma \to M$ is *not simple* iff there exists a branched cover $\tilde{\Sigma} \to \Sigma$ of degree $d \geq 2$ and an embedding $\tilde{u} : \tilde{\Sigma} \to M$ making the following diagram commute:



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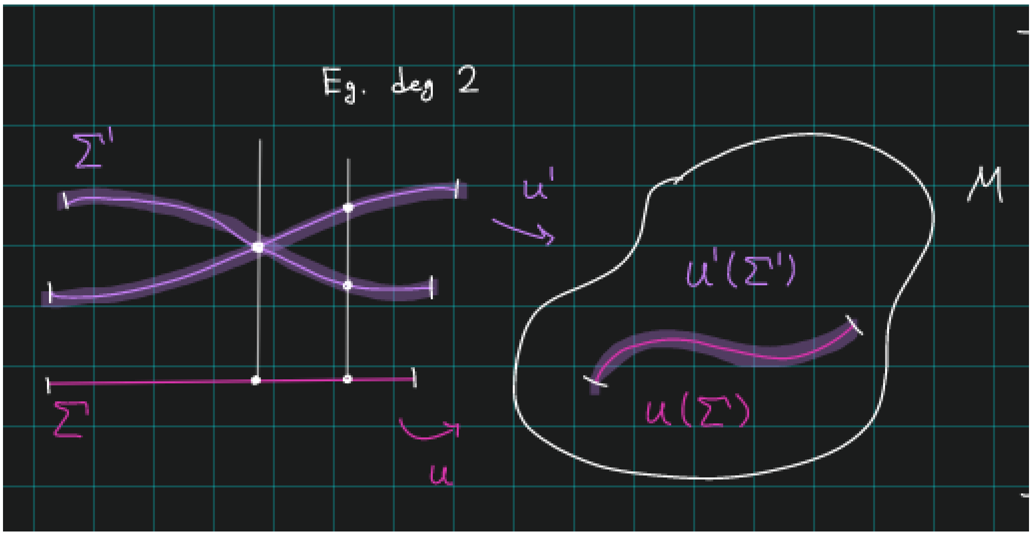
This is the condition that Σ doesn't factor through a ramified curve, here's a cartoon for a non-simple curve where $d = 2$:



Here they both have the same image, so represent the same embedded curve, but $\tilde{\Sigma}$ has a branch point over Σ near the center. Non-simple curves will correspond to orbifold points in $\mathcal{M}(A, \Sigma, J)$, and the theorem is that simple curves are generic in this moduli space.

- We've pulled back the tangent bundle of M in the following way:

$$\begin{array}{ccc} u^*TM & \dashrightarrow & TM \\ \downarrow & \lrcorner & \downarrow \\ \Sigma & \longrightarrow & M \end{array}$$

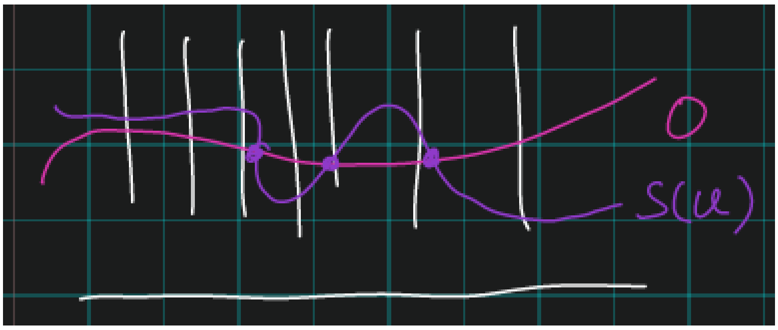
Link to Diagram

- We've constructed a bundle with a global section?

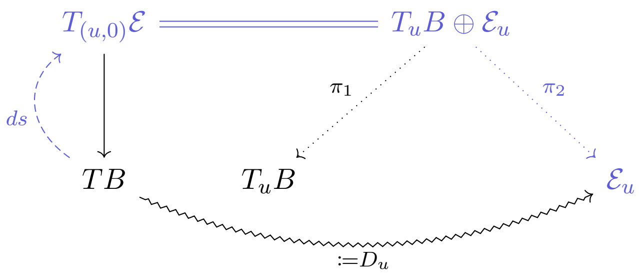
$$\begin{array}{ccc} \mathcal{E}_u := \Omega^{0,1}(\Sigma, u^*TM) & \longrightarrow & \mathcal{E} \\ & & \downarrow \; \uparrow \, s(u) := (u, \bar{\partial}_J(u)) \\ & & B := \left\{ u \in C^\infty(\Sigma, M) \,\middle|\, [u(\Sigma)] = A \right\} \end{array}$$

Link to Diagram

Our moduli space $\mathcal{M}(A, \Sigma, J)$ will be the zero section of this bundle, and we'll obtain $\mathcal{M}^*$ by intersecting the base space with the solutions u that are **somewhere injective**. It seems like we'll somehow need to perturb sections to get them to be transverse to the zero section:



- Given $\mathcal{E} \to B$, we've taken tangent spaces of everything and cooked up a map $D_u : TB \to \mathcal{E}_u$:



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This could be identified as a map

$$D_u : \Omega^0(\Sigma; u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM).$$

- By Riemann-Roch, we identified

  $$\operatorname{Ind}(D_u) = n(2 - 2g(\Sigma)) + 2c_1(u^*TM),$$

  which roughly comes from taking the Euler characteristic of Dolbeault cohomology $H^*_{\bar\partial}(u^*TM)$, which ultimately came from the differential on global sections

  $$\bar\partial : \Gamma(\Omega^{p,q}) \xrightarrow{\bar\partial} \Gamma(\Omega^{p,q+1}).$$

- We wanted to allow varying the almost-complex structure J, so we defined $\mathcal{J}^\ell$ be all $J \in C^\ell(TM, TM)$ which were ℓ times continuously differentiable, where we equip this space with the smooth topology.

- We enlarged the moduli space to a *universal moduli space* $\mathcal{M}^*(A, \Sigma, \mathcal{J}^\ell)$ which fibers over $\mathcal{J}^\ell$:

$$\begin{array}{c} \mathcal{M}^*(A, \Sigma, \mathcal{J}^\ell) \\ \Big\downarrow \pi \\ \mathcal{J}^\ell \end{array}$$

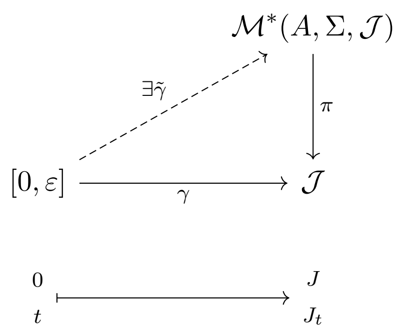
Link to Diagram

- The upshot was that $\pi^{-1}(J) = \mathcal{M}^*(A, \Sigma, J)$ is our original moduli space, and we can apply the implicit function theorem for (infinite-dimensional) Banach manifolds to conclude this is a finite-dimensional submanifold.
- Somehow we also show that $T_{(u,0)}\mathcal{M}^*(A, \Sigma, J) = \ker D_u$.
- We also use the Sard-Smale theorem to show that in $\mathcal{J}^\ell$, the regular values of π^{-1} are Baire 2nd category (a countable intersection of open dense sets), so "generic" in an appropriate sense.
  - Useful example: $\mathbb{Q} \subset \mathbb{R}$ is 1st category but the irrationals are 2nd category.

2.1 3.3: Regularity

Definition 2.1.1 (Regular)
In light of the previous discussion, we'll say that J is **regular** for A iff D_u is surjective for all $u \in \mathcal{M}^*(A, \Sigma, J)$ where the J is fixed. A point $p := (u, J) \in \mathcal{M}^*$ will be a **regular point** iff $T_p\mathcal{M}^* \xrightarrow{d\pi_p} T_{\pi(p)}\mathcal{J}$ is surjective, where now we let J vary in $\mathcal{J}$.

Remark 2.1.2: Recall that $\mathcal{J}$ is the space of all almost-complex structures on M. A consequence of regularity is that any smooth one-parameter family $[0, 1] \to \mathcal{J}$ can be ε-lifted in the sense that there is a commutative diagram



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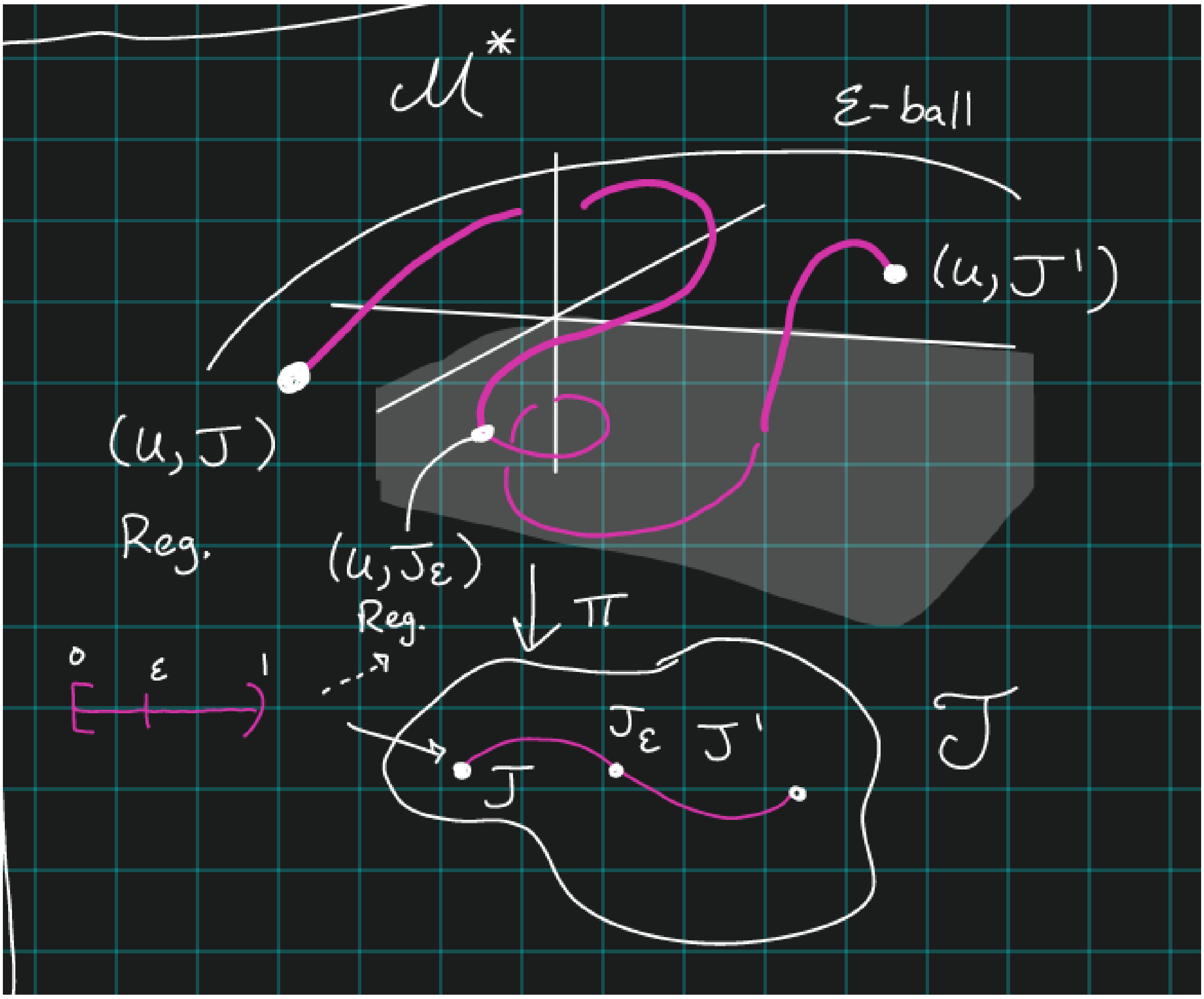
Note: here I just mean $\gamma(0) = J$ and $\gamma(t) := J_t$.

Moreover if (u, J) is regular, then the lifts (u, J_t) along the path upstairs will still be regular nearby:



So although we can't freely perturb regular values in the moduli space, we can take one-parameter families J_t in $\mathcal{J}$ as "controlled deformations" of almost-complex structures and lift them to controlled deformations upstairs.

Warning 2.1.3
If the point (u, J) is *not* regular, then there may not be any nearby regular points in the universal moduli space $\mathcal{M}^*(A, \Sigma, \mathcal{J})$.

2.2 Overview of Main Theorems

Remark 2.2.1: The two main theorems of this section describe sufficient conditions for regularity and how to produce a regular almost-complex structure.

Theorem 2.2.2 ***(3.3.4).***
Let $g(\Sigma) = 0$ and $\dim_{\mathbb{R}} M = 4$, and consider the J-holomorphically embedded sphere $(\Sigma, j_\Sigma) \hookrightarrow (M^4, J)$. Letting $p := \Sigma^2 := \Sigma \cdot \Sigma$ be the self-intersection number of Σ, then J is regular for $A := [\Sigma]$ if and only if $p \geq -1$.

Theorem 2.2.3 ***(3.3.5).***
Let $\tilde{A} := [S^2 \times \mathrm{pt}]$ as a class in $H_2(\tilde{M}; \mathbb{Z})$ where $\tilde{M} := S^2 \times M$. Then for all $J \in \mathcal{J}(M, \omega)$, the almost-complex structure $\tilde{J} := i \times J$ is regular for $\tilde{A}$, where i is the standard complex structure on S^2.

Remark 2.2.4: The following is a summary of the other lemmas in this chapter, which are useful on their own but also used to prove the above two theorems.

- 3.3.1: If J is integrable and $\mathbb{CP}^1 \xrightarrow{u} M$, then $u^*TM = \oplus L_k$ decomposes as a sum of line bundles and J is regular iff $c_1(L_k) \geq -1$ for all k, where c_1 denotes the Chern number.

- 3.3.2: If $\mathcal{E} \to \mathbb{CP}^1$ is *any* bundle, not just u^*TM, and there exists a decomposition $\mathcal{E} = \oplus L_k$ into line bundles, and if $D : \Omega^0(\mathbb{CP}^1; \mathcal{E}) \to \Omega^{0,1}(\mathbb{CP}^1; \mathcal{E})$ is any $\mathbb{R}$-linear Cauchy-Riemann operator that preserves the decomposition in the sense that $D(L_k) \subseteq L_k$ for all k, then D_u is surjective iff $c_1(L_k) \geq -1$ for all k.

- 3.3.3: If (M, ω, J) is any 4-dimensional symplectic manifold and J is any almost-complex structure (not necessarily integrable) and $u : \mathbb{CP}^1 \to M$ is an *immersed* J-holomorphic sphere, then D_u is surjective if $c_1(u^*TM) \geq 1$.

The rest of the section involves examples and constructions.

3 | 3.3: Regularity Calculations

Remark 3.0.1: Fix $\Sigma := \mathbb{CP}^1$, which is homeomorphic to S^2. For notation, we'll write $c_1(L) := \langle c_1(L),\ [\Sigma] \rangle$ for L a line bundle. where we're using the intersection pairing

$$\langle -,\ - \rangle : H^2(M; \mathbb{Q}) \otimes_{\mathbb{Q}} H_2(M; \mathbb{Q}) \to \mathbb{Q}.$$

Theorem 3.0.2 ***(Splitting Principle (Grothendieck)).***
Every complex holomorphic line bundle of rank r over $\mathbb{CP}_1$ decomposes uniquely into a direct sum of line bundles:

$$
\begin{array}{ccc}
\mathbb{C}^r & \longrightarrow & \mathcal{E} \\
 & & \downarrow \\
 & & \mathbb{CP}^1
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{C}^r & \longrightarrow & \bigoplus_{k=1}^{\ell} \mathcal{O}_{\mathbb{CP}^1}(n_k) \\
 & & \downarrow \\
 & & \mathbb{CP}^1
\end{array}
$$

Link to Diagram

These bundles are holomorphically isomorphic.

Remark 3.0.3: AG break: $\mathcal{O}_X(a_k)$ needs some explanation! If $\mathcal{O}_X$ is the structure sheaf (so regular functions), then $\mathcal{O}(n) := \mathcal{O}(1)^{\otimes n}$, and $\mathcal{O}(1)$ will be the **Serre twisting sheaf**, sometimes referred to as the **hyperplane bundle**. To describe this, note that we first have a tautological bundle over the Grassmannian over $\mathbb{C}^n$ where the fiber over a point (corresponding to a subspace V) is V itself regarded as a subset of $\mathbb{C}^k \subseteq \mathbb{C}^n$.

$$
\begin{array}{ccc}
F_{[W]} := W \subset \mathbb{C}^k & \longrightarrow & \gamma := \{([W], W)\} \subset \mathrm{Gr}_k(\mathbb{C}^n) \times \mathbb{C}^k \\
 & & \downarrow \\
 & & [W] \in \mathrm{Gr}_k(\mathbb{C}^n)
\end{array}
$$

Link to Diagram

Taking $k = 1$, we can identify $\mathbb{CP}^n := \mathrm{Gr}_1(\mathbb{C}^{n+1})$ as the space of lines in $\mathbb{C}^{n+1}$ to get the **tautological line bundle** which defines $\mathcal{O}(-1)$:

$$
\begin{array}{ccc}
\mathbb{C} & \longrightarrow & \gamma := \mathcal{O}_{\mathbb{P}^n}(-1) \\
 & & \downarrow \\
 & & \mathbb{CP}^n
\end{array}
$$

Link to Diagram

Note that the fiber above a line is just the line itself. This lets us get $\mathcal{O}(-k)$ for any k; to get positive numbers just define $\mathcal{O}(1) := \mathcal{O}(-1)^\vee$ as the dual bundle, where you replace each fiber F with its dual space $F^\vee := \mathrm{Hom}(F, \mathbb{C})$ as a vector space.

Remark 3.0.4: Upshot: these are relatively simple building blocks, just tensor powers and duals of an object where nothing too mysterious is going on. Moreover, for us, $u^*TM = L_1 \oplus L_2$ breaks up as *some* sum of line bundles – it doesn't actually matter which twists they are for our purposes.

3.1 Lemma 3.3.1

Lemma 3.1.1 ***(3.3.1).***
If $u^*TM \cong \oplus_{k=1}^{\ell} L_k$ and $c_1(L_k) \geq 1$ for every k, then D_u is surjective.

Remark 3.1.2: To prove this, we'll need an analytic version of Riemann-Roch:

Theorem 3.1.3 ***(Riemann-Roch, Append C.1.10, Part 3).***
If $\mathcal{E} \to \Sigma$ is a holomorphic bundle and $F \leq \mathcal{E}$ is a sub-bundle, then D_u is surjective iff

$$\mu(\mathcal{E}, F) + 2\chi(\Sigma) > 0,$$

where $\mu(-,-)$ is a **relative Maslov index**. Moreover, taking $F = \emptyset$, if $\partial\Sigma = \emptyset$ then there is a formula

$$\mu(\mathcal{E}) := \mu(\mathcal{E}, \emptyset) = 2\langle c_1(\mathcal{E}),\ [\Sigma]\rangle.$$

3.2 Proof using Riemann-Roch

Proof (of Lemma, using Riemann-Roch).
We'll first need that since Σ is a sphere, we know its cohomology ring:

$$H^*(S^2; \mathbb{Z}) \cong \mathbb{Z}[x]/\left\langle x^2 \right\rangle \qquad \text{where } |x| = 2,$$

which is only supported in degrees $d = 0, 2$. So

$$\chi(\Sigma) = 1 - 0 + 1 = 2.$$

Note that you could also just check this cellularly: S^2 has a CW complex structure with one 0-cell and one 2-cell, and you can compute χ using just the ranks of cellular chain groups instead of homology.

Strategy: take the LHS appearing in the RR formula above, we'll try to show it's positive.

$$\begin{aligned} \mu(u^*TM) + 2\chi(\Sigma) &= \mu(u^*TM) + 4 \\ &= \mu\left(\bigoplus_{k=1}^{\ell} L_k\right) + 4 \\ &= \sum_{k=1}^{\ell} \mu\left(L_k\right) + 4 \\ &= \sum_{k=1}^{\ell} 2c_1(L_k) + 4, \end{aligned}$$

and thus

$$\begin{aligned} 2\sum_{k=1}^{\ell} c_1(L_k) + 4 > 0 \\ \iff \sum_{k=1}^{\ell} c_1(L_k) > -2. \end{aligned}$$

Since the rank of u^*TM is at least 2, there are at least 2 summands. So if every $c_1(L_k) > -1$, this inequality holds, and that is sufficient for D_u to be surjective.

■

3.3 Proof using AG/Chern Classes

Proof (of lemma, using complex analytic arguments).
Since J is assumed integrable, $D_u = \bar{\partial}_J$ coincides with the Dolbeault derivative determined by the complex structure on M, and D_u respects the splitting $u^*TM \cong \bigoplus L_k$. We want to show D_u is surjective, so it thus suffices to show $\operatorname{coker} \bar{\partial}_J = 0$, where it's worth recalling a nice identification:

$$\operatorname{coker}\left(A \xrightarrow{f} B\right) \cong B/\operatorname{im} A.$$

The actual definition is taking a pushout against the terminal object in your category:

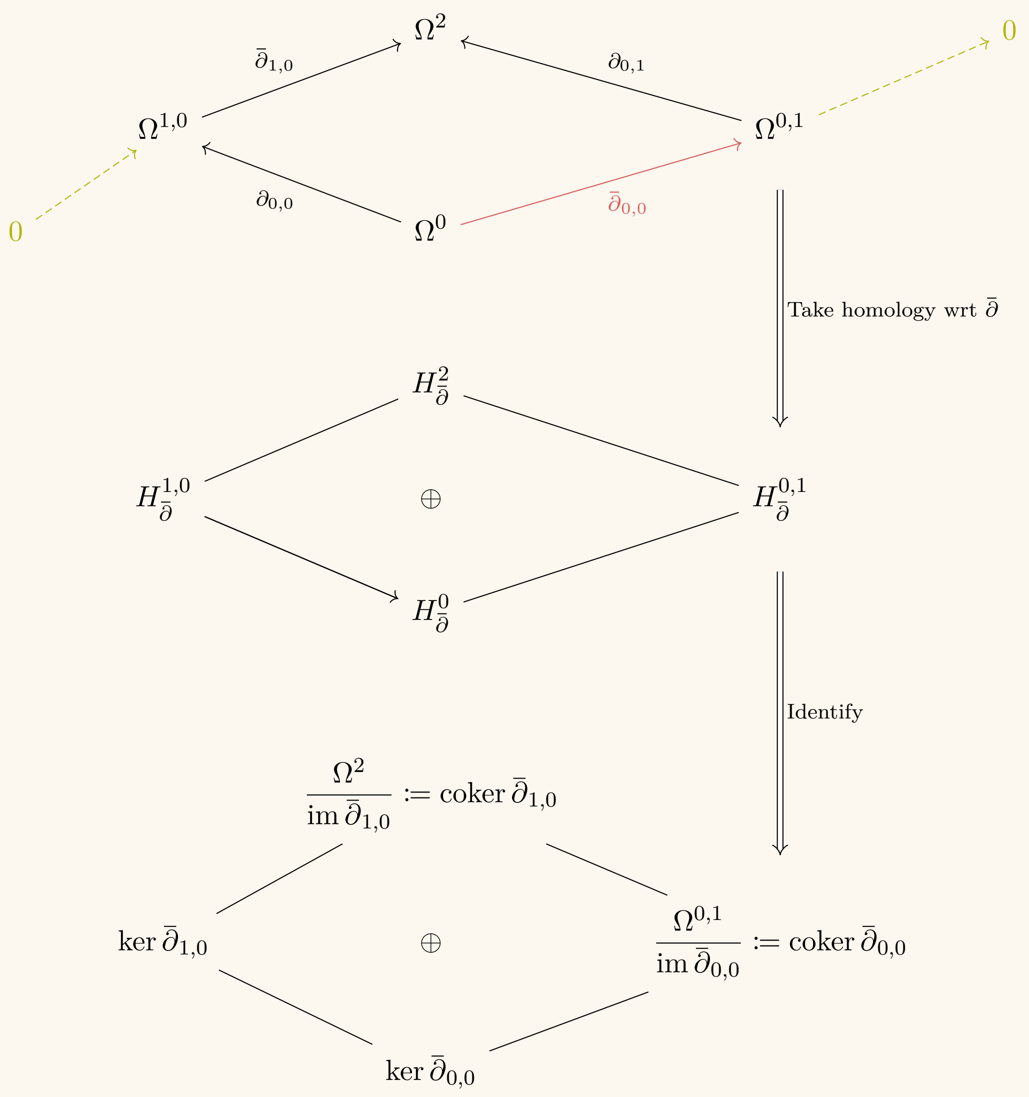
$$\begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \longrightarrow & \operatorname{coker} f := B\coprod_A 1 \end{array}$$

Link to Diagram

We can identify this as

$$\operatorname{coker}(\Omega^0(\Sigma; L) \xrightarrow{D_u = \bar{\partial}_J} \Omega^{0,1}(\Sigma; L)) := H^{0,1}_{\bar{\partial}_J}(\Sigma; L).$$

The last equality is not so obvious, but follows if you think about how this splits out in the Hodge diamond:



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The main thing to notice is that one is taking the homology with respect to $\bar{\partial}$, so the bottom-right corner of the diamond just forms a 2-term chain complex and we get a kernel/cokernel pair.

So now it suffices to show that $H^{0,1}_{\bar{\partial}}(\Sigma; L) = 0$ (for $L := L_k$ any of the bundle summands) whenever $c_1(L_k) \geq 1$ for all L_k. We'll need a definition:

Definition ***(Canonical Bundle)***
Let Ω^1_Σ be the bundle of holomorphic 1-forms on Σ. Then the **canonical bundle** is defined as

$$K_\Sigma := \bigwedge^{\dim \Sigma} \Omega^1_\Sigma = \Omega^2_\Sigma,$$

which here coincides with the bundle of holomorphic 2-forms. It is sometimes written as ω_Σ

We can now apply **Kodaira-Serre duality**:

$$H^{0,1}_{\bar\partial}(\Sigma; L) \xrightarrow{\sim} H^{1,0}_{\bar\partial}(\Sigma; L^\vee \otimes K_\Sigma)^\vee,$$

where notably we've switched from antiholomorphic forms to holomorphic forms.
We'll also need **Kodaira vanishing**: If $\mathcal{L} \to \Sigma$ is a *positive* holomorphic line bundle, then

$$H^i(\Sigma; \mathcal{L} \otimes K_\Sigma) = 0 \qquad \forall\, i > 0.$$

The book justifies the uses of this theorem here by saying $c_1(\mathcal{L})$ can be interpreted as the self-intersection number of the zero section, and mumbles something about "positivity of intersections". I'm not really sure why this works!
A related fact (maybe a consequence?) is that $\mathcal{L}$ has nonzero holomorphic sections $\iff$ $c_1(\mathcal{L}) \geq 0$, so maybe positivity is related to positivity of Chern numbers.
Now setting $\mathcal{L} := L^\vee \otimes K_\Sigma$, playing around with the logic we find that if $c_1(\mathcal{L}) < 0$ then $\mathcal{L}$ has *no* holomorphic sections, and for reasons unknown, this should imply that $H^{1,0}_{\bar\partial}(\Sigma; \mathcal{L})^\vee = 0$ and conclude the proof. In any case, let's just compute the Chern number:

$$\begin{aligned} c_1(\mathcal{L}) &= c_1(L^\vee) + c_1(K_\Sigma) \\ &= c_1(L^\vee) - c_1(T\Sigma) \\ &= c_1(L^\vee) - e_1(TS^1) && \text{since } c_1 \text{ is a top class} \\ &= c_1(L^\vee) + \left(1 + (-1)^2\right) && \text{by a well-known formula for spheres} \\ &= c_1(L^\vee) - 2 \\ &= -c_1(L) - 2. \end{aligned}$$

So now unwinding things, we have

$$\begin{aligned} c_1(\mathcal{L}) < 0 &\iff -c_1(L) - 2 < 0 \\ &\iff -c_1(L) < 2 \\ &\iff c_1(L) > -2 \\ &\iff c_1(L) \geq -1. \end{aligned}$$

which is exactly the condition appearing in the lemma. Running this same argument for every L_k concludes the proof!

■

Remark 3.3.2: Note that we've used some special facts in that last calculation:

- Using that $L^\vee \cong L^{-1}$ for line bundles, $c_1(L^\vee) = c_1(L^{-1}) = -c_1(L)$.
- I don't think $c_1(A \otimes B) = c_1(A) + c_1(B)$ in general, this must be special for $B = K$ the canonical.
- $c_1(K_X) = -c_1(TX)$ is a general fact, for complex manifolds at least. Apparently this is

obvious from Chern-Weil theory, but you can also use

$$c_1(TX) = c_1(\det TX) \coloneqq c_1\left(\bigwedge^{\text{top}} TX\right) \coloneqq c_1({K_X}^\vee) = c_1(K_X^{-1}) = -c_1(K_X).$$

- The top Chern class is always the Euler class (almost by definition) when it makes sense.