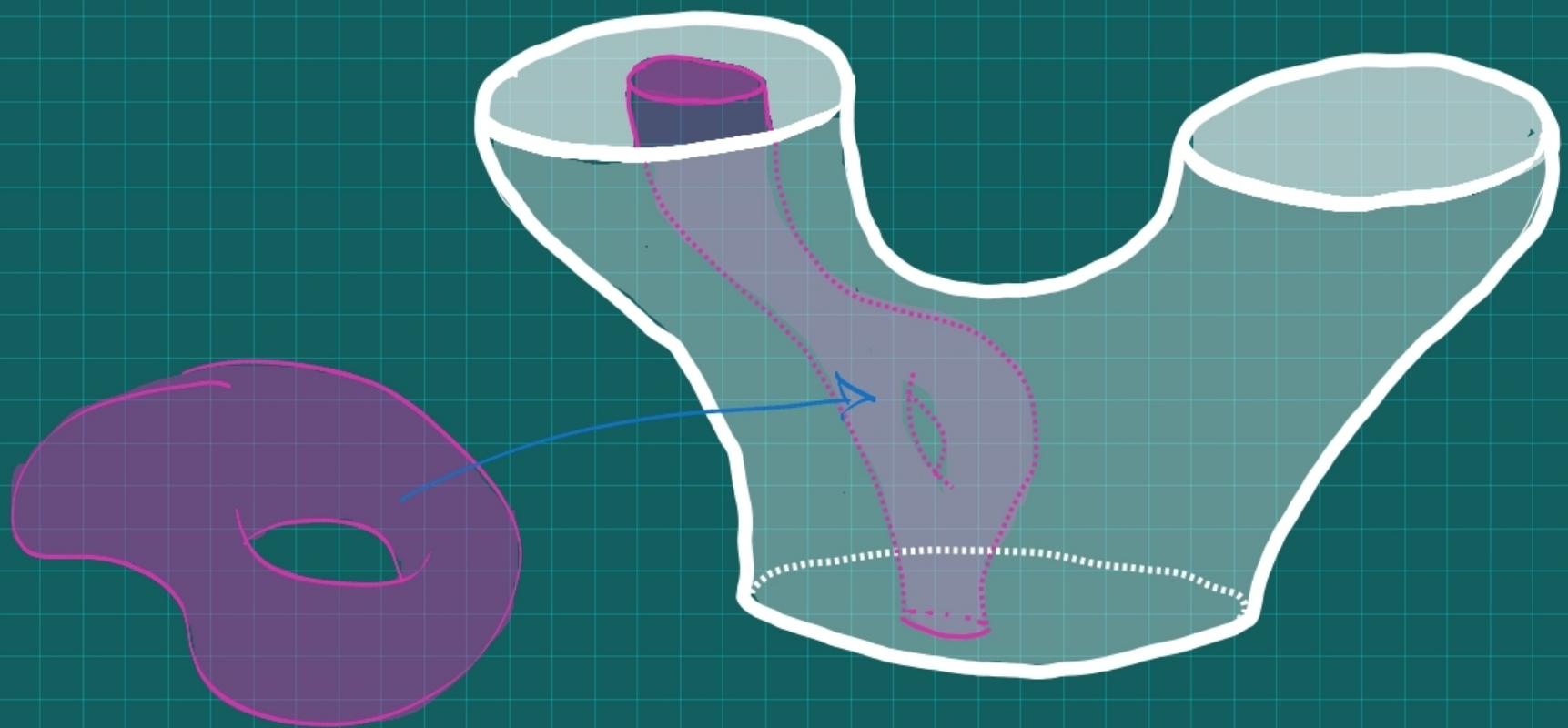


Reading Seminar

J-Holomorphic Curves



Overview & Goals

Eg: $\mathrm{Gr}_k(\mathbb{C}^\infty)$

- Long-term goal: define the Gromov-Witten invariants $\mathrm{GW}(M, A, g, n)$

For (M, ω) compact symplectic, $[A] \in H_2^{\text{sing}}(M; \mathbb{Z})$, define a moduli space

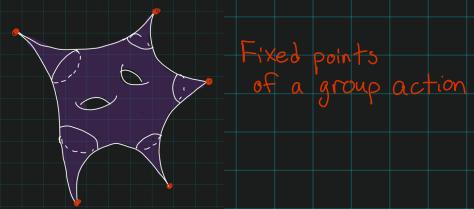
$$\overline{\mathcal{M}}_{g,n}(A, J) := \left\{ \begin{array}{l} J\text{-holomorphic curves } \Sigma \hookrightarrow M \text{ with } [\Sigma] = [A] \in H_2 \\ \text{where } g(\Sigma) = g \text{ and } [x_1, \dots, x_n] \in \Sigma^{x^n} \text{ are } n \text{ marked pts} \end{array} \right\} / \sim$$

\downarrow
Forget A, J &
 J -Holo condition

$f(x_i) \in M$

$$\overline{\mathcal{M}}_{g,n} = \left\{ \text{Stable curves } \Sigma \hookrightarrow M \text{ with } g(\Sigma) = g \text{ & } n \text{ marked pts} \right\}$$

A smooth Deligne-Mumford stack \leadsto a \mathbb{C} -orbifold. For $g=0$: a compact smooth \mathbb{C} -mfld!



- Rough def.

$$\mathrm{GW}(M, A, g, n) : H^*(M; \mathbb{Q})^{\otimes n} \otimes_{\mathbb{Z}} H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q}) \longrightarrow \mathbb{Q}$$

$$[\alpha_1, \alpha_2, \dots, \alpha_n] \otimes \beta \mapsto \int_{\overline{\mathcal{M}}_{g,n}(A, J)} \prod_{j=1}^n \mathrm{ev}_j^*(\alpha_j) \cup \pi^*(\beta^{\mathrm{PD}})$$

Idea: Push cohom classes into $\overline{\mathcal{M}}_{g,n}$
and "cap with fundamental class"
(integrate over moduli space)

$= \# \left\{ \begin{array}{l} J\text{-holomorphic curves } \Sigma_g \\ \text{with } n \text{ marked pts } x_i \\ \text{where } X_i \text{ intersects a cycle} \\ X_i \subseteq H_2(M) \text{ dual to } \alpha_i \\ \text{and } [\pi \Sigma_g] = [\beta] \in H_2(\overline{\mathcal{M}}_{g,n}) \end{array} \right\}$

$$\cdot \text{Example: } \# \left\{ \begin{array}{l} \text{Lines in } \mathbb{P}^3 \text{ intersecting} \\ \text{4 (generic) lines} \end{array} \right\} = \mathrm{GW}_{L, 4}^{\mathbb{P}^3(\mathbb{C})} (c^2, c^2, c^2, c^2) = 2$$

- These will only depend on the deformation type of (M, ω)

Can take 1-param. families
 $\gamma : I \rightarrow \mathcal{D}^2(M)$
 $t \mapsto \omega_t$

(M, ω_t)

of "semipositive" sympl. mfds

- In $\dim M=4$, depend only on the diffeomorphism type and are iso to SW invariants.
(Counterexamples in $\dim M=6$)

- Remarks

- $g=0$: Quantum Cohomology & Frobenius manifolds

- Can produce a TQFT (somehow)

- Mirror Symmetry: conjecturally gives a 2nd way to compute GW.

Ch. 3: Moduli Spaces & Transversality

- Ch. 3 Goal: Show that for generic J , $\mathcal{M}^*(A, \Sigma, J) \in \text{sm Mfd}_{\mathbb{C}}^{<\infty}$ (for simple curves, easier transversality)

- 3.1: Moduli space of simple curves

- 3.2: Thom-Smale transversality

$\hookrightarrow \mathcal{M}^*(A, \Sigma, J)$ is a smooth mfd when the linearized operator is surjective $\forall J$ -hol. curves

- 3.3: Examples in dim 4, surjectivity $\forall J$ (use RR instead of Sard-Smale)

- 3.4: Moduli spaces with pointwise constraints

- Recall from Han's talks:

- $(M, \omega, J) \in \text{Mfd}^{2n}(S_p)$, J an ω -tame almost \mathbb{C} -struct on M

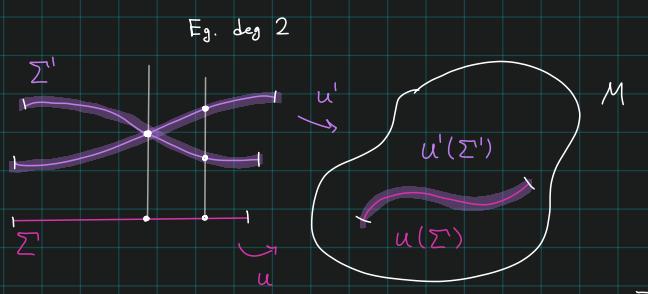
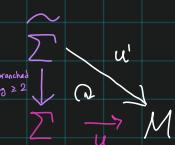
- (Σ, j_Σ, dv) a Riemann sfc, compact

- The CR operator $\bar{\partial}_J = \frac{1}{2} (Jd - dj_\Sigma)$, $\text{Ker } \bar{\partial}_J = J$ -Holomorphic curves.

- $u \in C^\infty(\Sigma, M)$ a curve/solution

- $\mathcal{M}^*(A, \Sigma, J) := \left\{ u \in C^\infty(\Sigma, M) \cap \text{Ker } \bar{\partial}_J \mid \begin{array}{l} [u(\Sigma)] = A \in H_2(M; \mathbb{Z}) \\ \text{& } u \text{ simple} \end{array} \right\}$

Where u is not simple iff \exists

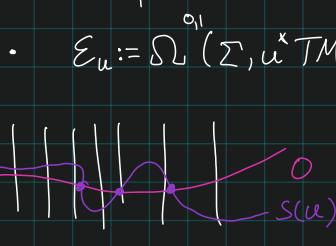


(factors through a ramified curve)

$$\{ T\Sigma \rightarrow u^* TM \}$$

$$E_u := \Omega^0(\Sigma, u^* TM) \rightarrow (\mathcal{E})$$

$$\begin{matrix} u^* TM & \xrightarrow{\quad} & TM \\ \downarrow & \downarrow & \downarrow \\ \Sigma & \xrightarrow{u} & M \end{matrix}$$



$$\mathcal{M}(A, \Sigma, J) = \mathcal{Z}(s)$$

obtain \mathcal{M}^* by taking $B \cap \{u \mid \text{somewhere inj.}\}$

$$\begin{aligned} \mathcal{E} &\xrightarrow{\quad} T_u B \\ \downarrow & \quad ds \\ B &\xrightarrow{\quad} T_{(u,0)} E = T_u B \oplus E_u \\ \downarrow & \quad \pi_B \\ T_u B &\xrightarrow{\quad} E_u \\ \text{:= } Du: T_u B &\rightarrow E_u \\ &= Du: \Omega^0(\Sigma; u^* TM) \rightarrow \Omega^0(\Sigma; u^* TM) \\ &\quad (\text{Complete to Sobolev spaces } W^{k,p}) \end{aligned}$$

The vertical differential

Transversality $\Leftrightarrow Du$ is surjective

- Du is Fredholm: $\dim_{\mathbb{R}} \ker Du$ & $\dim_{\mathbb{R}} \text{coker } Du < \infty$, im Du closed,

$\text{Ind}(Du) := \dim \ker Du - \dim \text{coker } Du < \infty$ is well-defined.

$$\text{RR} = n(2 - 2g(\Sigma)) + 2c_1(u^* TM). \quad (= \chi(H_{\bar{\partial}}^*(u^* TM)), \Gamma(\Omega^{p,q}) \xrightarrow{\bar{\partial}} \Gamma(\Omega^{p,q+1}))$$

- Upshot: For J a regular value of

$$\mathcal{M}^*(A, \Sigma, J) = \{ (u, J) \in \mathcal{M}^* \times \mathcal{T}^e \}, \quad \mathcal{T}^e = \{ J \in C^e(TM, TM) \}$$

$\downarrow \pi$
universal moduli space

with the C^∞ topology

$\pi^{-1}(J) = \mathcal{M}^*(A, \Sigma, J)$ is a fin dim manifold (implicit fn thm for Banach mfds)

and $T_{(u,0)} \mathcal{M}^*(A, \Sigma, J) = \ker Du$

Sard-Smale: Regular values are (Baire) 2nd category in \mathcal{T}^e

3.3: Regularity

$$\begin{aligned} D_u &: T_u B \rightarrow \mathcal{E}_u \\ &= D_u: \Omega^0(\Sigma; u^* TM) \rightarrow \Omega^{0,1}(\Sigma; u^* TM) \end{aligned}$$

- Def (3.1.4): J is regular for A $\Leftrightarrow D_u$ is surjective $\forall u \in \mathcal{M}^*(A, \Sigma, J)$

- A point $p := (u, J) \in \mathcal{M}^*(A, \Sigma, J)$ is regular $\Leftrightarrow T_p \mathcal{M}^*(A, \Sigma, J)$

$d\pi_p \downarrow$ surjects

$T_J J$

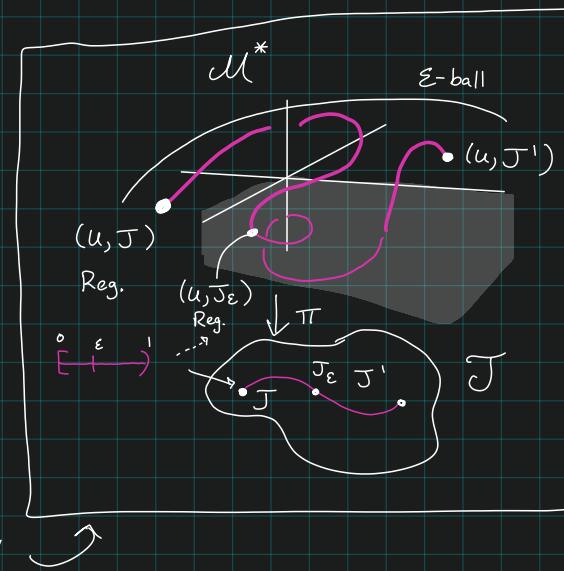
$\Rightarrow \mathcal{M}^*(A, \Sigma, J)$

$$\begin{array}{c} t \mapsto (u_t, J_t) \\ [0, \varepsilon) \xrightarrow{\gamma} \Sigma \\ t \mapsto J_t \\ 0 \mapsto J \end{array}$$

Smooth paths can be lifted

So if (u, J) is regular

then $\exists (u, J')$ regular nearby



⚠ (u, J) not regular may imply no nearby regular pts!

- Main Thm (3.3.4): Let $(\overset{\text{sphere}}{\Sigma}, j_\Sigma) \hookrightarrow (M^+, J)$ be a holo. embedded sphere with $\Sigma^2 := \Sigma \cdot \Sigma := p$

Then J is regular for $A := [\Sigma]$ $\Leftrightarrow p \geq -1$.

$$\tilde{M} \xrightarrow{M \in \text{Mfd}(S^p)}$$

- Main Thm (3.3.5): Let $\tilde{A} := [S^2 \times \text{pt}] \in H_2(\overset{\sim}{S^2 \times M}; \mathbb{Z})$.

Then $\forall J \in \mathcal{J}(M, \omega)$, $\widetilde{J} := i \times J$ is regular for \tilde{A}

self-intersection

- Summary of other lemmas: Criteria for surjectivity

- 3.3.1: $c_1(L_k) \geq -1$ when...

$$u^* TM = \bigoplus L_k$$

for $P_c^1 \xrightarrow{u} M$ & J integrable

- 3.3.2: $c_1(L_k) \geq -1$ when...

$\text{Bun}_{\mathbb{C}}$ $\left[\begin{array}{l} \mathcal{E} \quad \text{with } \mathcal{E} = \bigoplus L_k, \quad D: \Omega^0(P; \mathcal{E}) \rightarrow \Omega^{0,1}(P; \mathcal{E}) \text{ an } \mathbb{R}\text{-linear CR operator,} \\ \downarrow \\ P_c^1 \end{array} \right.$

- 3.3.3: $c_1(u^* TM) \geq 1$ when...

$$(M^+, J) \text{ almost complex, } u \in \text{Imm}^\infty(P_c^1, M)^{J\text{-Hol}}$$

- Rest of the chapter: Examples of non-regular curves.

$$\left\{ \begin{array}{c} u^* TM \xrightarrow{\quad} TM \\ \downarrow \\ \Sigma \rightarrow M \end{array} \right\}$$

3.3: Regularity

• Fix $\Sigma := \mathbb{P}_{\mathbb{C}}^1 \cong S^2$, write $c_1(L) := \langle c_1(L), [\Sigma] \rangle$, assume J is integrable

• Thm (Grothendieck): in $Bun_{\mathbb{C}}^r(\text{Holo})$,

$$\begin{array}{ccc} \mathbb{C}^r \rightarrow \mathcal{E} & \xrightarrow{\sim} & \mathbb{C}^r \rightarrow \bigoplus_{k=1}^r \mathcal{O}_{\mathbb{P}^1}(n_k) \\ \downarrow & \text{splitting principle} & \downarrow \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{array} \quad \text{unique} \quad \Rightarrow \quad u^* TM = \bigoplus_{k=1}^r L_k.$$

• Lemma (3.3.1): If $c_1(L_k) \geq -1 \ \forall k$ then D_u is surjective.

• Proof (1: AG)

• Thm (RR, C.1.10 part 3): If $\mathcal{E} \in Bun_{\mathbb{C}}(\text{Holo})$ and $F \subseteq \mathcal{E}$ is a sub-bundle,

$$D_u \text{ is surjective} \Leftrightarrow \overbrace{\mu(\mathcal{E}, F)}^{\text{Maslov index}} + 2 \chi(\Sigma) > 0$$

$$\text{Moreover } \mu(\mathcal{E}, \emptyset) = 2 \langle c_1(\mathcal{E}), [\Sigma] \rangle \text{ when } \partial\Sigma = \emptyset.$$

• Now expand:

$$\begin{aligned} \mu(u^* TM, \emptyset) + 2 \chi(\Sigma) &\rightarrow \Sigma \text{ a sphere} \Rightarrow H^*(S^2; \mathbb{Z}) = \mathbb{Z}[x]/\langle x^2 \rangle, |x| = 2 \\ &= \mu\left(\bigoplus_{k=1}^2 L_k, \emptyset\right) + 4 \end{aligned} \quad \Rightarrow \chi(\Sigma) = 1 - 0 + 1 = 2$$

$$= \sum_{k=1}^2 \mu(L_k, \emptyset) + 4 \quad \xrightarrow{\text{Fact from appendix}}$$

$$= \sum_{k=1}^2 2 \cdot c_1(L_k) + 4$$

$$> 0 \iff 2c_1(L_1) + 2c_1(L_2) > -4$$

$$\iff c_1(L_1) + c_1(L_2) > -2$$

$$\iff c_1(L_1) > -1 \ \& \ c_1(L_2) > -1. \quad \blacksquare$$

• Proof (2: Symplectic)

• J integrable $\Rightarrow D_u = \bar{\partial}_J$ the Dolbeault derivative (determined by \mathbb{C} -struct on M)

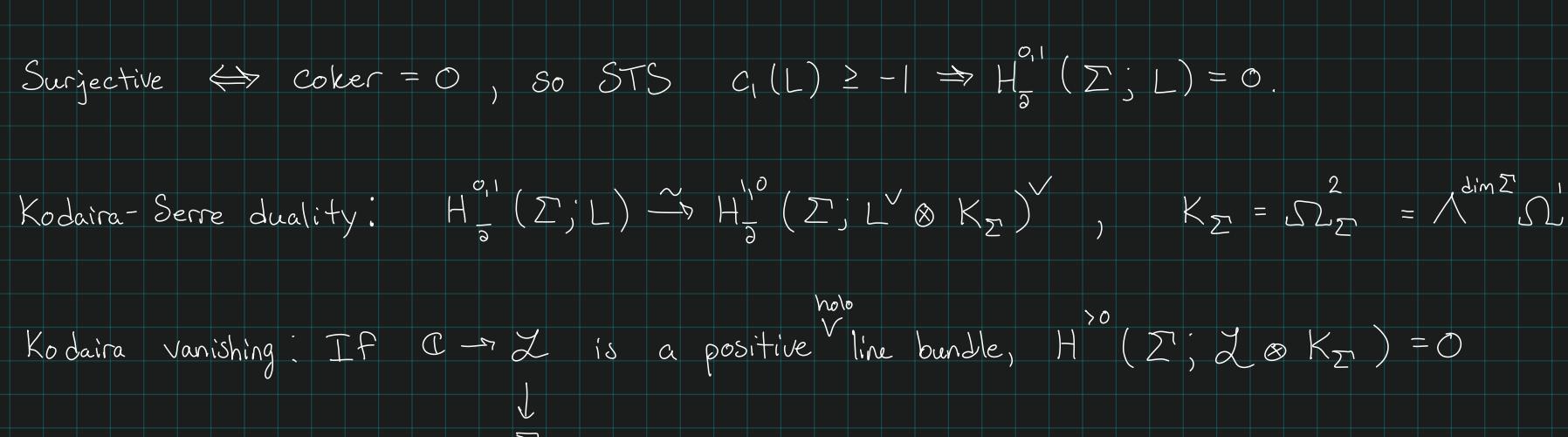
$\hookrightarrow D_u$ respects the splitting $u^* TM \cong \bigoplus L_k$

$$\text{Coker}(A \xrightarrow{f} B) \cong B/\text{im } A$$

def: $A \xrightarrow{f} I$ \downarrow terminal

$$B \xrightarrow{g} B/A$$

$$\text{coker}\left(\Omega^0(\Sigma; L) \xrightarrow{D_u = \bar{\partial}_J} \Omega^{0,1}(\Sigma; L)\right) = H^{0,1}_{\bar{\partial}_J}(\Sigma; L)$$



• Surjective $\Leftrightarrow \text{coker} = 0$, so STS $c_1(L) \geq -1 \Rightarrow H^{0,1}_{\bar{\partial}_J}(\Sigma; L) = 0$.

• Kodaira-Serre duality: $H^{0,1}_{\bar{\partial}_J}(\Sigma; L) \xrightarrow{\sim} H^{1,0}_{\bar{\partial}_J}(\Sigma; L^\vee \otimes K_\Sigma)^\vee$, $K_\Sigma = \bigwedge^2 \Omega_\Sigma^\wedge = \bigwedge^{\dim \Sigma} \Omega_\Sigma^\wedge$

• Kodaira Vanishing: If $\mathcal{L} \rightarrow \Sigma$ is a positive $\overset{\text{holo}}{\wedge}$ line bundle, $H^{>0}(\Sigma; \mathcal{L} \otimes K_\Sigma) = 0$

• Compute $c_1(L^\vee \otimes K_\Sigma) = c_1(L^\vee) + c_1(K_\Sigma)$

$$= c_1(L^\vee) - c_1(T\Sigma)$$

$$= c_1(L^\vee) - e(TS^2) \quad \text{since } c_1 \text{ is a top class}$$

$$= c_1(L^\vee) - (1 + (-1)^2)$$

$$= c_1(L^\vee) - 2$$

$$= -c_1(L) - 2$$

$$\text{So } c_1(L^\vee \otimes K_\Sigma) < 0 \iff -c_1(L) - 2 < 0$$

$$\iff -c_1(L) < 2$$

$$\iff c_1(L) > -2$$

$$\iff c_1(L) \geq -1. \quad \square$$