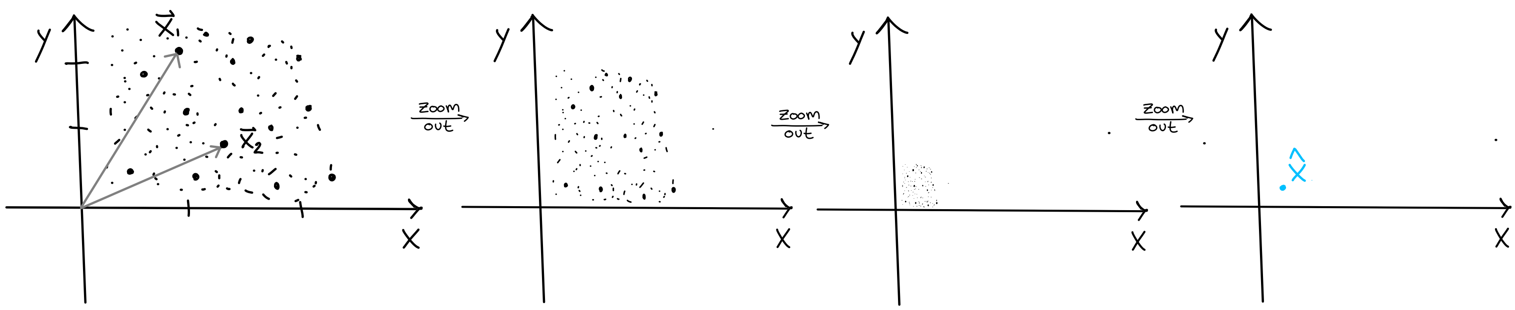
Moments

Consider a finite collection of points in $\mathbb{R}^2$,

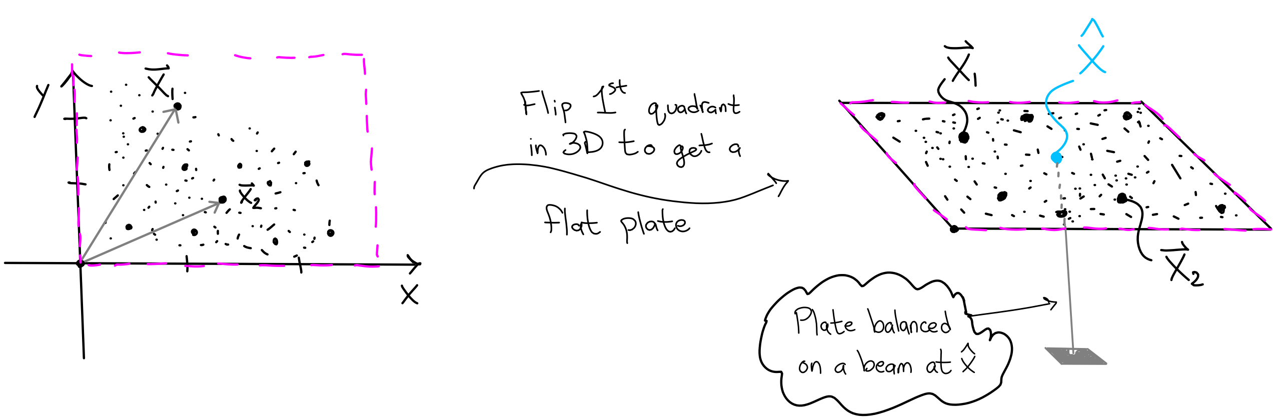
$$X = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$$

How can we replace X with a single point $\hat{x}$ that "best" represents the collection X?



If these were particles with mass, we could equivalently ask for a single point mass $\hat{x}$ that acts like their combined masses and accounts for their relative positions.

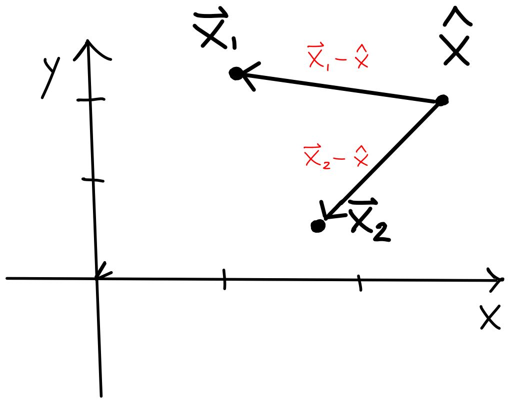
So we could think of the $\vec{x}_i$ as points with mass on a flat plate; then $\hat{x}$ should be the point where we could add a support and physically balance the plate:



What is a good choice? Idea: choose $\hat{x}$ such that all of the distances between $\hat{x}$ and $\vec{x}_i$ are minimized.

$$\hat{x} = \arg\min \underbrace{\sum_{i=1}^{n}}_{\text{Minimize all distances simultaneously}} \underbrace{\|\vec{x}_i - \hat{x}\|}_{\text{Distance between } \vec{x}_i \text{ and } \hat{x}}^2$$

Minimizing distance is equivalent to minimizing distance squared.



The solution to this minimization problem is given by

$$\hat{x} = E[X] = \frac{\sum_{i=1}^{n} \vec{x}_i}{n} \text{, the expected value of } X.$$

We can think of this as a system of n particles, each having mass $1/n$, so the total mass is n, and we can rewrite this as

$$\hat{x} = E[X] = \sum_{i=1}^{n} \underbrace{\vec{x}_i}_{\text{position}} \cdot \underbrace{\frac{1}{n}}_{\text{contribution to total mass}}$$

Towards a generalization, we can think of a "density function" which assigns each $\vec{x}_i$ its mass, i.e.

$$\rho : X \to \mathbb{R}$$
$$\vec{x}_i \mapsto m_i := \text{mass of } \vec{x}_i$$

where here $m_i = \frac{1}{n}$ for all i. We can then write

Note that if we interpret $\rho(\vec{x}_i)$ as a mass, we can define

$$M = \sum_{i=1}^{n} \rho(\vec{x}_i) \quad \text{, the \underline{mass} of } X.$$

and thus write

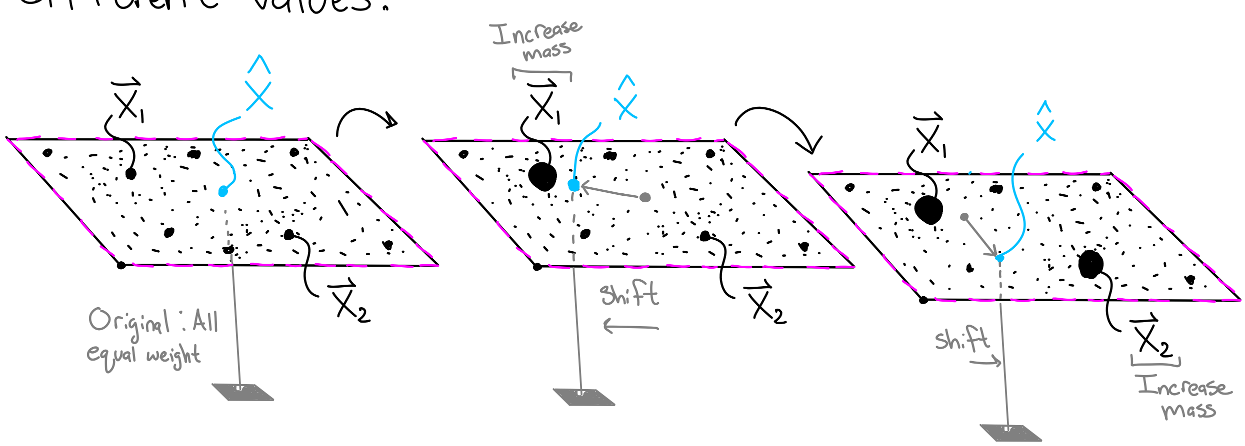
$$\hat{x} = E[X] = \sum_{i=1}^{n} \underbrace{\vec{x}_i}_{\text{position}} \cdot \underbrace{\frac{\rho(\vec{x}_i)}{M}}_{\substack{\text{contribution}\\ \text{to total mass}}}$$

$$= \frac{\sum_{i=1}^{n} \vec{x}_i \, \rho(\vec{x}_i)}{M}$$

$$\boxed{= \frac{\sum_{i=1}^{n} \vec{x}_i \, \rho(\vec{x}_i)}{\sum_{i=1}^{n} \rho(\vec{x}_i)}}$$

This is the version of the formula we will generalize.

Consider what happens when we now let $\rho(\vec{x}_i)$ take on different values:



Our definition of $\hat{x}$ now produces the <u>center of mass</u> of X.

In light of this, we define

$$\boxed{\vec{M} = \sum_{i=1}^{n} \vec{x}_i \cdot \rho(\vec{x}_i), \text{ the \underline{moment} of } X}$$

and

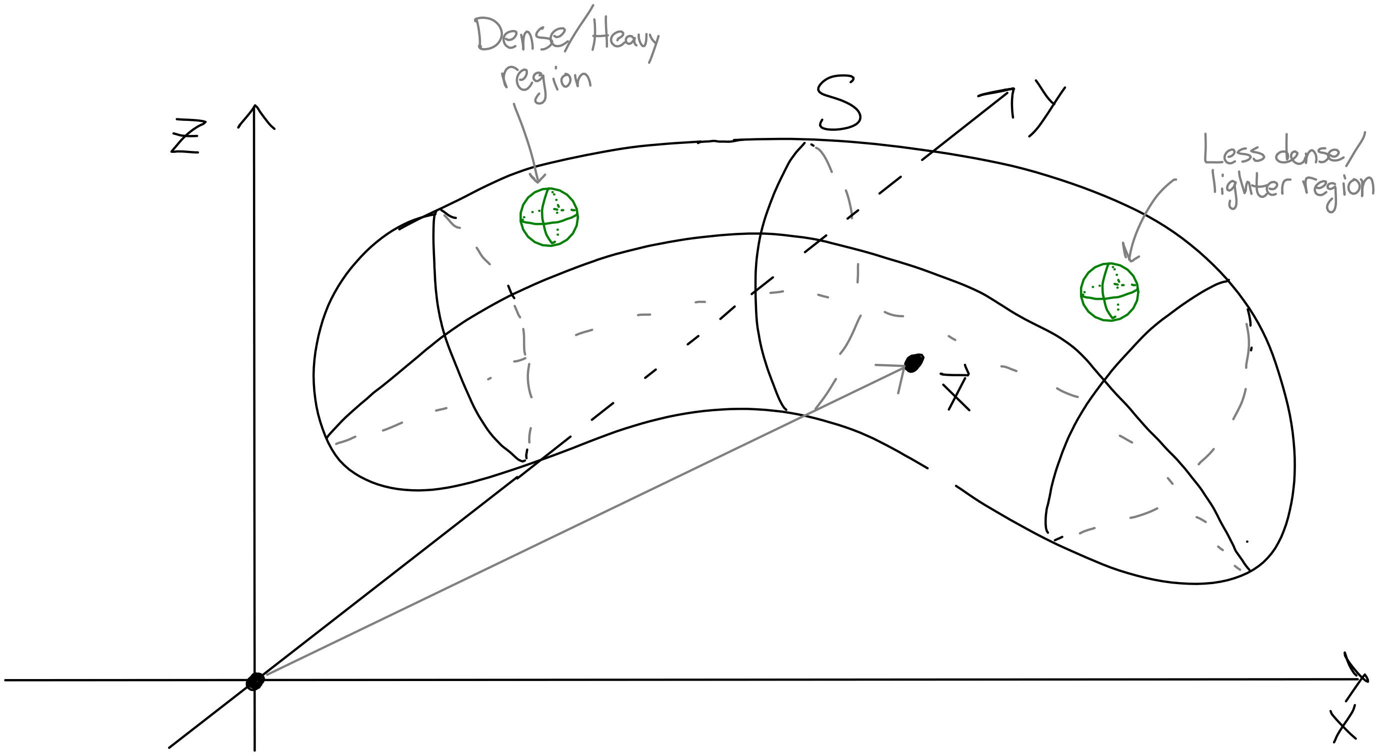
$$\boxed{\vec{C} = \frac{\vec{M}}{M} = \frac{\sum_{i=1}^{n} \vec{x}_i \cdot \rho(\vec{x}_i)}{\sum_{i=1}^{n} \rho(\vec{x}_i)} \quad \text{the \underline{center of mass} of } X}$$

The "moment" is a sum of points, each weighted by its mass, which will be an important interpretation.

We started with a finite set $X \subseteq \mathbb{R}^2$, but note that these definitions make sense for any finite $X \subseteq \mathbb{R}^n$ and any $\rho: X \to \mathbb{R}$.

Now we will generalize this from discrete sets of points to continuous infinite sets. Consider $S \subseteq \mathbb{R}^3$ a surface along with the interior region it bounds.

Imagine S formed from a metal alloy, where the density varies throughout S:



To find the center of mass of S, we can still use the "expected value" idea, where we replace sums with integrals.

We have the following analogy:

	Discrete		Continuous
$\vec{M}$ (Moment)	$\sum \vec{x}_i \, \rho(\vec{x}_i)$	$\longrightarrow$	$\int_S \vec{x} \, \rho(\vec{x}) \, dV$
M (Total mass)	$\sum \rho(\vec{x}_i)$	$\longrightarrow$	$\int_S \rho(\vec{x}) \, dV$
$\vec{C}$ (Center of mass)	$\vec{M}/M$	$\longrightarrow$	$\vec{M}/M$.

A priori, the integral for $\vec{M}$ may not make sense - it prescribes integrating a vector over a volume. However, this can be computed coordinate by coordinate. I.e, if $\vec{M} = (m_x, m_y, m_z)$, then

$m_x = \int_S x\, \rho(x,y,z)\, dV$, moment about the yz plane

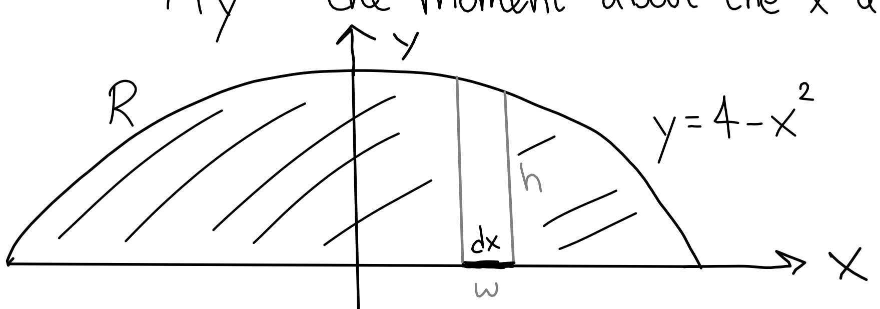
$m_y = \int_S y\, \rho(x,y,z)\, dV$, moment about the xz plane

$m_z = \int_S z\, \rho(x,y,z)\, dV$, moment about the xy plane

In other words, each coordinate of the moment is given by the expected value of that coordinate with respect to the density ρ.

<u>Example</u>: The center of mass of a planar plate R in $\mathbb{R}^2$ bounded by $y = 4 - x^2$ and $y = 0$, with density given by $\rho(x,y) = 2x^2$.

$$\text{Let } \left.\begin{aligned} M_x &= \text{the moment about the y-axis}\\ M_y &= \text{the moment about the x-axis}\end{aligned}\right] \Rightarrow \vec{C} = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$



$$\begin{aligned} dA &= h \cdot w \\ &= (4 - x^2)\, dx \end{aligned}$$

$$\text{So } M_y = \int_R y\,\rho(x,y)\,dA = \int_{-2}^{2} y\,\rho(x,y)\,(4-x^2)\,dx$$

$$= \int_{-2}^{2} y\;2x^2\,(4-x^2)\,dx$$

$$= \int_{-2}^{2} (4-x^2)\,2x^2\,(4-x^2)\,dx$$

$$= 2048/105.$$

$$M_x = \int_R x\,\rho(x,y)\,dA = \int_{-2}^{2} \underbrace{x\;2x^2}_{\text{odd}} \cdot \underbrace{(4-x^2)}_{\text{even}}\,dx \quad = \text{odd}$$

$$= 0.$$

$$M = \int_R \rho(x,y)\,dA = \int_{-2}^{2} 2x^2(4-x^2)\,dx$$

$$= 256/15$$

$$\Rightarrow \vec{c} = \left(0, \frac{2048/105}{256/15}\right). \quad \blacksquare$$

Note that in general we can take n^{th} moments about a value:

$$M^n(p) := \int_S (x-p)^n\,\rho(x)\,dV$$

If ρ is a probability distribution, then

$$M^1(0) = E[X] \xrightarrow{\text{Approx}} \text{Center of mass}$$
$$M^2(E[X]) = Var(X) \longrightarrow \text{"Moment of inertia"}$$