Condensed-matter physics ← bordism ← the Adams spectral sequence

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Overview

- From SPTs to bordism
- Computing bordism with the Adams spectral sequence
- Crystalline phases

What is bordism?

- Diffeomorphism classes of closed n-manifolds form a commutative monoid under disjoint union
- Mod out by the submonoid of manifolds M which are the boundary of a compact $(n+1)$-manifold W
- This is an abelian group, called the *bordism group*, and denoted Ω_n

What is bordism?

- Variant: choose $G \to \mathrm{GL}_n(\mathbb{R})$, equip M and W with compatible *G-structures*. Denoted Ω_n^G
  - A G-structure on M is data of a lift of the transition functions on M, by default valued in $\mathrm{GL}_n(\mathbb{R})$, to G
- Also, "bordism of G-manifolds with a map to X" – M bounds and the map extends to a map $W \to X$
  - This defines a generalized homology theory

What is bordism?

- Computing bordism groups is a classical problem in algebraic topology
  - Thom, Pontrjagin, Wall, Anderson-Brown-Peterson, Quillen, ...
- Unoriented bordism groups begin $\mathbb{Z}/2$, 0, $\mathbb{Z}/2$ (generated by $\mathbb{RP}^2$), 0, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ ($\mathbb{RP}^4$ and $\mathbb{RP}^2 \times \mathbb{RP}^2$), $\mathbb{Z}/2$ (SU_3/SO_3)
- Oriented bordism groups begin $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$ (generated by $\mathbb{CP}^2$), $\mathbb{Z}/2$ (SU_3/SO_3)

Ok, but why?

Theorem (Freed-Hopkins)

There is an isomorphism between the abelian group of deformation classes of reflection-positive, invertible, n-dimensional topological field theories (IFTs) on manifolds with G-structure and $\mathrm{Tors}(\Omega_n^G) \oplus \mathrm{Free}(\Omega_{n+1}^G)$.

- Unlike general quantum field theory, *topological field theory* is rigorously, mathematically formalized by Atiyah-Segal
- “Reflection-positive” (equivariance data under orientation-reversal) occurs in all physics-motivated examples
- “Invertible” is special: these are the simplest examples

SPTs

- Condensed-matter physicists study *topological phases of matter*, including classification questions
  - e.g., take some alloy, cool it to a certain temperature, maybe apply a magnetic field…
  - the material then behaves strangely (e.g. "particles" that aren't bosons or fermions)
- Full classification is very difficult, so restrict to *symmetry-protected topological* (SPT) phases
  - These are the simplest examples: can be combined with another phase to obtain a trivial phase
  - Concretely modeled by Hamiltonians built from combinatorial data on a manifold
- These form an abelian group (once dimension and symmetry type are fixed), and computing these groups has been a focus of recent research in condensed-matter physics

From SPTs to invertible field theories

- Ansatz: the low-energy effective theory of an SPT phase is an invertible topological field theory
- And should be some sort of equivalence between SPT phases and IFTs
- Point: the ansatz would mean Freed-Hopkins' theorem also computes groups of SPTs!
- However, making this ansatz into rigorous math is a difficult open problem

Testing this ansatz

- Compute what the ansatz predicts in a range of examples using bordism
- Compare with physicists' computations by other methods
- Taken up by Freed-Hopkins and Jonathan Campbell. Conclusion: the answers agree!

The running example

- In the next part of the talk, we'll discuss how to use the Adams spectral sequence to compute bordism groups
- Each step will be implemented on the running example of G-bordism where $G = \mathrm{Pin}^+ \ltimes C_n$.
  - Can think of this as sort of like mixing the data of a spin structure and a cover $M' \to M$ with structure group D_{2n}
  - This bordism corresponds to an interesting kind of SPT, but I care for a slightly different reason, which I'll talk about later
- Good expository reference: Beaudry, Campbell, "A guide for computing stable homotopy groups"

The Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(X;\mathbb{F}_2),\mathbb{F}_2) \Longrightarrow \pi_{t-s}(X)_2^\wedge$$

- In the next few slides, we'll explain this notation
- Briefly: LHS approximates the RHS well, and is easier to calculate.
- Imperfect approximation: have to calculate *differentials* and *hidden extensions*

The Steenrod algebra $\mathscr{A}$

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(X;\mathbb{F}_2),\mathbb{F}_2) \Longrightarrow \pi_{t-s}(X)_2^\wedge$$

- This is the (graded, noncommutative) algebra of *stable mod* 2 *cohomology operations*, i.e. natural transformations $H^*(-;\mathbb{F}_2) \to H^{*+k}(-;\mathbb{F}_2)$ that commute with suspension
- Generated by *Steenrod squares* $\mathrm{Sq}^k : H^* \to H^{*+k}$, $k \geq 0$
- Thus $H^*(X;\mathbb{F}_2)$ is a graded $\mathscr{A}$-module, and pullback maps are $\mathscr{A}$-module maps

Ext groups

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(X;\mathbb{F}_2),\mathbb{F}_2) \Longrightarrow \pi_{t-s}(X)_2^\wedge$$

- Given an algebra R and left R-modules M, N, can define an abelian group $\mathrm{Ext}_R^s(M,N)$ of sequences
  $N \to P^s \to \cdots \to P^1 \to M$, up to a notion of equivalence.
  - Alternatively: $\mathrm{Ext}_R^k(-,N)$ is the right derived functor of $\mathrm{Hom}_R(-,N)$
- If R is graded, define $\mathrm{Ext}_R^{s,t}(M,N) := \mathrm{Ext}_R^s(M,\Sigma^t N)$
  - i.e. shift the grading of N up by t

Completion

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(X;\mathbb{F}_2),\mathbb{F}_2) \Longrightarrow \pi_{t-s}(X)_2^\wedge$$

- This means that you only recover the 2-*completion* of the homotopy groups
- Under reasonable circumstances, determines the free part and the 2-torsion part
- No information on p-torsion when p is odd
  - When $\pi_*(X)$ computes bordism groups, there are other, generally easier, ways to compute odd-primary torsion

Simplifying the problem

- Assume your bordism question is $\Omega_*^{\mathrm{Spin}}(X)$ for some X
- Anderson-Brown-Peterson: spin bordism determined by *ko*-theory, some other stuff
  - Below degree 8, spin bordism is isomorphic to *ko*-theory!
- Conveniently, $\widetilde{H}^*(ko;\mathbb{F}_2) \cong \mathscr{A} /\!/ \mathscr{A}(1)$, so can apply a change-of-rings theorem

Simplifying the problem

- Upshot: in degrees ≤ 7, the E_2-page for $\Omega_*^{\mathrm{Spin}}(X)$ is isomorphic to
$$\mathrm{Ext}^{s,t}_{\mathscr{A}(1)}(H^*(X;\mathbb{F}_2),\mathbb{F}_2)$$
- $\mathscr{A}(1) = \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle$ is *much smaller* than $\mathscr{A}$

Justifying the assumption

- Want to write bordism questions as spin bordism of something!
- Large class of examples: *twisted spin structures*
  - Pick a (virtual) vector bundle $V \to X$
  - A (V, X)-twisted spin structure on $E \to M$ is a map $f : M \to X$ and a spin structure on $E \oplus f^*V$
- “Spinlike” G-structures, e.g. spin^c, pin$^\pm$, can often be described as twisted spin structures

Bordism of (X, V)-twisted spin structures

Proposition

The bordism groups of manifolds with an (X, V)-twisted spin structure are isomorphic to $\widetilde{\Omega}_*^{\mathrm{Spin}}(X^{V-\dim V})$

- Here $X^{V-\dim V}$ is (a shift of) the *Thom spectrum* of V
- Running example: $\mathrm{Pin}^+ \ltimes C_n$ bordism is equivalent to $(BD_{2n}, -V)$-twisted spin bordism, where V is induced from the standard 2d representation

The Thom isomorphism: a pretty cool theorem

Theorem (Thom)

There is an isomorphism of graded abelian groups
$H^*(X;\mathbb{F}_2) \xrightarrow{\cong} \widetilde{H}^{*+\dim V}(X^V;\mathbb{F}_2)$.

- Typically written $x \mapsto Ux$ (U is the *Thom class*)
- This theorem and its generalizations are used over and over in algebraic topology of manifolds

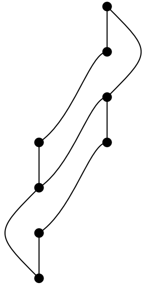
$\mathscr{A}$-action on cohomology of Thom spectra

- Determined by Stiefel-Whitney classes of V
- One way to describe: $\mathrm{Sq}^i(U) = Uw_i(V)$, together with Cartan formula and $\mathscr{A}$-action on $H^*(X;\mathbb{F}_2)$

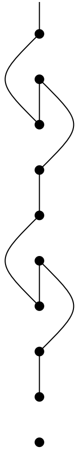
Drawing $\mathscr{A}(1)$-modules

- Draw a dot for every $\mathbb{F}_2$ summand (as an abelian group)
- Height = grading
- For $\mathrm{Sq}^1(a) = b$, draw a straight line from a to b
- For $\mathrm{Sq}^2(a) = c$, draw a curvy line from a to b

Drawing $\mathscr{A}(1)$-modules

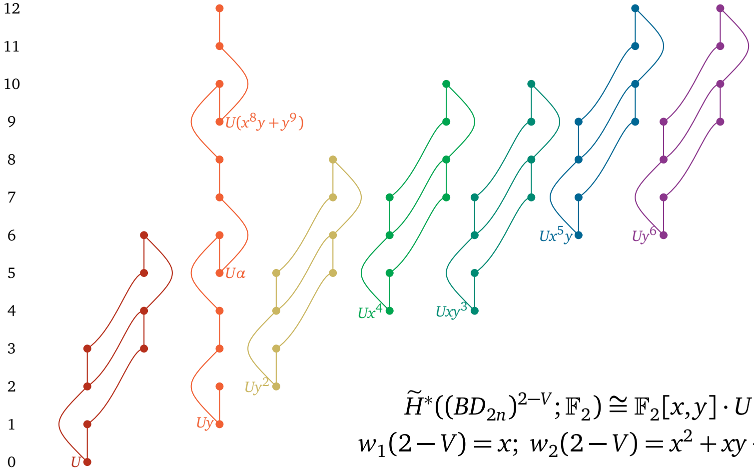


$\mathscr{A}(1)$



$H^*(\mathbb{RP}^\infty; \mathbb{F}_2)$

Drawing $\mathscr{A}(1)$-modules: the running example

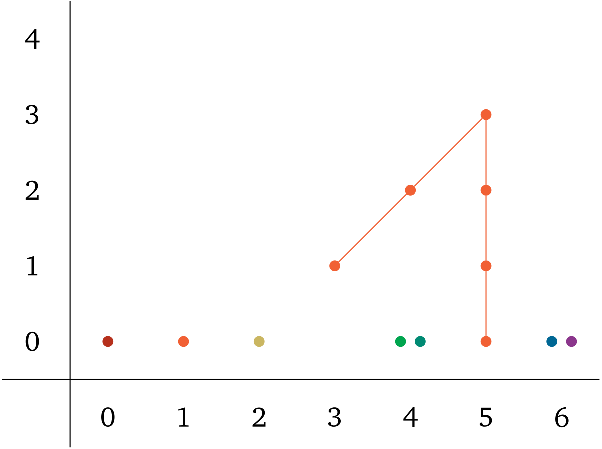


$$\widetilde{H}^*((BD_{2n})^{2-V};\mathbb{F}_2) \cong \mathbb{F}_2[x,y]\cdot U$$
$$w_1(2-V)=x;\ w_2(2-V)=x^2+xy+y^2$$

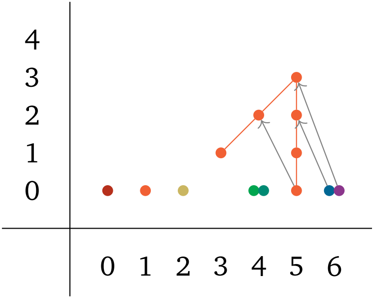
Computing Ext

- Ext commutes with direct sums: work one summand at a time
- $\mathrm{Ext}^{s,t}_{\mathscr{A}(1)}(\Sigma^k \mathscr{A}(1), \mathbb{F}_2)$: $\mathbb{F}_2$ in degree $s = 0, t = k$, 0 elsewhere
- For commonly occurring summands, can look it up (Beaudry-Campbell)
- Short exact sequence of $\mathscr{A}(1)$-modules $\Longrightarrow$ long exact sequence in Ext
- Last resort: compute directly (write down a nice projective resolution...)

The E_2-page of the running example

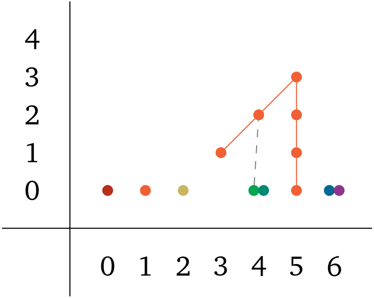


Tricks for computing differentials



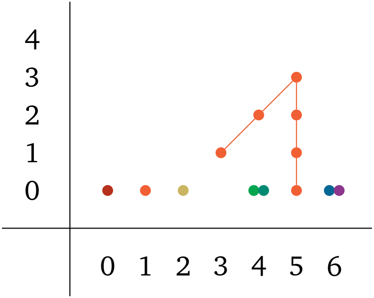
- Surprisingly effective: differentials must be h_0-, h_1-linear
- Adams SS is functorial, so map to or from something where the differential must vanish
- Use topology to infer facts about the bordism groups in question
  - In our example: *Smith homomorphism* from degree 5 to $\Omega_4^{\mathrm{Pin}^+} \cong \mathbb{Z}/16$; check this hits a generator of that $\mathbb{Z}/16$

Hidden extensions are difficult



- Still useful: functoriality, geometric topology
- Use that h_1-action lifts to action by $\eta \in \pi_* \mathbb{S}$, which is 2-torsion, to rule out some extensions by 2
  - This solves the extension problem in degree 4 of our example

The answer to our running example



$\Omega_0 = \mathbb{Z}/2$

$\Omega_1 = \mathbb{Z}/2$

$\Omega_2 = \mathbb{Z}/2$

$\Omega_3 = \mathbb{Z}/2$

$\Omega_4 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$

$\Omega_5 = \mathbb{Z}/16$

$\Omega_6 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$

What is a crystalline phase?

- Many symmetries of interest in condensed-matter physics act on space, such as the translation symmetries in a crystal structure
- Given a way of modeling phases, can ask whether phases are invariant under this symmetry
- Physicists discovered some interesting examples, leading to the notion of *crystalline symmetry-protected phases*

Freed-Hopkins' modified proposal

- Understanding the low-energy behavior of crystalline phases is not as straightforward
- Freed-Hopkins: physics suggest "topological phases on a space X" (for fixed symmetry type) behaves like a generalized homology theory (a modification of bordism)
- Crystalline phases for a group G acting on space then given by corresponding G-equivariant generalized homology theory
- In many cases of interest, can be reduced to a nonequivariant question about bordism groups

An example

- Implementing this proposal for fermionic phases with symmetry group D_{2n} ($n \equiv 2 \bmod 4$) acting on $\mathbb{R}^d$ by rotations and reflections leads to our running example
- Zhang-Wang-Yang-Qi-Gu (2019) compute this in dimension 3 for all n. They also get $\mathbb{Z}/2$!
- WIP: several other symmetry groups: some can be compared to physics calculations, others are new. Not all predictions agree