On the Goodwillie Derivatives of the Identity in Structured Ring Spectra

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Guiding principle

The Goodwillie derivatives of the identity functor in a suitably nice model category C (denoted $\partial_* \mathrm{Id}_{\mathsf{C}}$) should come equipped with a canonical operad structure.

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- If $\mathcal{O}$ is a reduced operad of spectra, then $\partial_* \mathrm{Id}_{\mathsf{Alg}_{\mathcal{O}}}$ is a "highly homotopy coherent" operad which is equivalent to $\mathcal{O}$ [C.]

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First, we'll recall some necessary background on functor calculus and operads.

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$$X \mapsto \mathcal{O} \circ (X) = \coprod_{n \geq 0} \mathcal{O}[n] \wedge_{\Sigma_n} X^{\wedge n}.$$

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Functor calculus in $\mathsf{Alg}_{\mathcal{O}}$ (cont.) – Taylor tower of Id

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Thm. [C]

The derivatives of the identity in $\mathrm{Alg}_{\mathcal{O}}$ posses an intrinsic "homotopy coherent" operad structure with respect to which $\partial_* \mathrm{Id} \simeq \mathcal{O}$ as operads.

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- The coaugmentation $\mathcal{O} \to C(\mathcal{O})$ which yields an equivalence of (homotopy coherent) operads $\mathcal{O} \simeq \partial_* \mathrm{Id}$. $\square$

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- Similar box product pairings can be used to provide a new description for the operad structure on $\partial_* \mathrm{Id}_{\mathsf{S}_*}$. Hope to extend this technique to other suitable model categories C to attack the "guiding principle" and better understand the relation between C to $\mathrm{Alg}_{\partial_* \mathrm{Id}_{\mathsf{C}}}$

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