On the unstable Gromov-Lawson-Rosenberg conjecture

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Definition (Scalar curvature)

Let (M, g) be a connected Riemannian manifold. The scalar curvature $\mathbf{S}$ *of (M, g) assigns to each point of M a real number defined by the local geometry. Precisely,* $\mathbf{S} = \mathrm{tr}_g(\mathrm{Ric})$.

Examples

- $\mathbb{E}^n$ has constant scalar curvature equal to 0.
- S^n of radius r has constant scalar curvature equal to $\frac{n(n-1)}{r^2}$.

Positive scalar curvature

Question

When does M admit a metric g of positive scalar curvature κ?

In dimension 2 this is completely solved.

Theorem (Gauss-Bonnet)

Let M be a compact two-dimensional Riemannian manifold, then

$$\kappa = \int_M \mathbf{S} dA = 4\pi\chi(M).$$

The Euler characteristic (a topological invariant) is an obstruction to the geometric problem.

- Let M be a closed spin manifold and X a spinor bundle.
- Let $L^2(M, X)$ denote the space of L^2 functions on the space of sections $M \to X$.

$$L^2(M, X) = \left\{ f : M \to X \colon \int_M ||f(x)||^2 dx < \infty \right\}$$

- Let $D : L^2(M, X) \to L^2(M, X)$ be the Dirac operator.
- $D^2 = \Delta + \kappa/4$ and $\Delta \geq 0$.

- Now, $\kappa > 0$ implies D^2 invertible.
- Hence, D invertible.
- Define $\mathrm{Index(D)} = \dim\ker(\mathrm{D}) - \dim\operatorname{coker}(\mathrm{D})$.
- So $\mathrm{Index(D)} = 0$

Corollary (Lichnerowicz)

$\mathrm{Index(D)} \neq 0$ *implies* M *does not admit a metric with* $\kappa > 0$.

The Atiyah-Singer index theorem

Theorem (Atiyah-Singer)

If M is a closed spin $4k$-manifold then $\mathrm{Index(D)} = \hat{\mathrm{A}}(\mathrm{M})$.

Here $\hat{A}(M)$ is the "A-hat genus of M", a topological invariant.

A more general obstruction

Theorem (Rosenberg)

Let M be a closed spin n-manifold and G a discrete group. Let $u : M \to BG$ be a continuous map. If M admits a metric of positive scalar curvature, then $\alpha[M, u] = 0 \in KO_n(C_r^* G)$.

Here $\alpha : \Omega_n^{\mathrm{Spin}} \to KO_n(C_r^* G)$ is the index of the Dirac operator.

The Baum-Connes conjecture identifies $KO_n(C_r^* G)$ with $KO_n^G(E_{\mathcal{FIN}} G)$ a topological invariant.

If G is torsion-free then the conjecture states the "assembly map" $\mu_{\mathbb{R}} : KO_n(BG) \to KO_n(C_r^* G)$ is an isomorphism.

Conjecture (Gromov-Lawson-Rosenberg)

Let M be a closed spin n-manifold, $n \geq 5$ with $\pi_1 M = G$. Suppose that $u : M \to BG$ induces the identity on G, then M admits a metric of positive scalar curvature if and only if $\alpha[M, u] = 0 \in KO_n(C_r^* G)$.

The conjecture has been verified for:

- All simply connected M [Stolz].
- When $\pi_1(M)$ is finite with periodic cohomology [Botvinnik-Gilkey-Stolz].
- $G = \pi_1(M)$ is torsion free discrete and $\dim BG \leq 9$ [Joachim-Schick].
- $\pi_1(M)$ is a Fuchsian group [Davis-Pearson].

There is a counterexample due to T. Schick with $\pi_1(M) = \mathbb{Z}^4 \oplus \mathbb{Z}_3$.

Theorem (H.)

Let G be either

- $\mathrm{PSL}_2(\mathbb{Z}[1/p])$ *for* $p \equiv 11 \pmod{12}$,
- *or a lattice in* $\mathrm{PSL}_2(\mathbb{C})$ *with all finite subgroups cyclic,*

then G satisfies the GLR conjecture.

Why is $p \equiv 11 \pmod{12}$?

A group G has property:
(M) if every finite subgroup is contained in a unique maximal finite subgroup.
(NM) if every maximal finite subgroup is self normalising.

$\mathrm{PSL}_2(\mathbb{Z}[1/p])$ only satisfies (M) and (NM) when $p \equiv 11 \pmod{12}$.

When G satisfies (M) and (NM), the p-chain spectral sequence of Davis and Lück is very well behaved.

The p-chain spectral sequence, some homological algebra and the Baum-Connes assembly map give us a commutative diagram.

$$\begin{array}{ccccccc}
\widetilde{ko}_{n+1}(X) & \longrightarrow & \bigoplus_{(H)\in\Lambda} \widetilde{ko}_n(BH) & \longrightarrow & \widetilde{ko}_n(B\Gamma) & \longrightarrow & \widetilde{ko}_n(X) \\
\downarrow p & & \downarrow \mu_{\mathbb{R}}\circ p & & \downarrow \mu_{\mathbb{R}}\circ p & & \downarrow p \\
\widetilde{KO}_{n+1}(X) & \longrightarrow & \bigoplus_{(H)\in\Lambda} \widetilde{KO}_n(C^*_r(H;\mathbb{R})) & \longrightarrow & \widetilde{KO}_n(C^*_r(\Gamma;\mathbb{R})) & \longrightarrow & \widetilde{KO}_n(X).
\end{array}$$

where $X = E_{\mathcal{FIN}}G/G$. In the $\mathrm{PSL}_2(\mathbf{Z}[1/p])$ case this is a wedge of spheres. The result follows with a bit of work from here.

A more general theorem

Theorem (H.)

Let G be discrete satisfying (M), (NM), the Baum-Connes conjecture, and such that all finite subgroups are cyclic (or generalised quaternion). Let $X = E_{\mathcal{FIN}}G/G$. If $p : \widetilde{ko}_n(X) \to \widetilde{KO}_n(X)$ is an isomorphism for all $n \geq 5$, then G satisfies the GLR conjecture.

An open question

Question

Does $\mathrm{PSL}_2(\mathbf{Z}[1/p])$ *satisfy the GLR conjecture for* $p \not\equiv 11 \pmod{12}$*?*

Thank you for listening!