Trisections and Flat Surface Bundles

GOATS 2
June 6, 2020

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Main Results

Theorem (W. 2018) There is a $((2g+1)(2h+1)+1;2g+2h)$ trisection of $\Sigma_g \times \Sigma_h$

Theorem (W. 2018) The trisection genus of $\Sigma_g \times \Sigma_h$ is

$$(2g+1)(2h+1)+1$$

- There is an algorithm for drawing a trisection diagram of $\Sigma_g \times \Sigma_h$ that realizes this bound

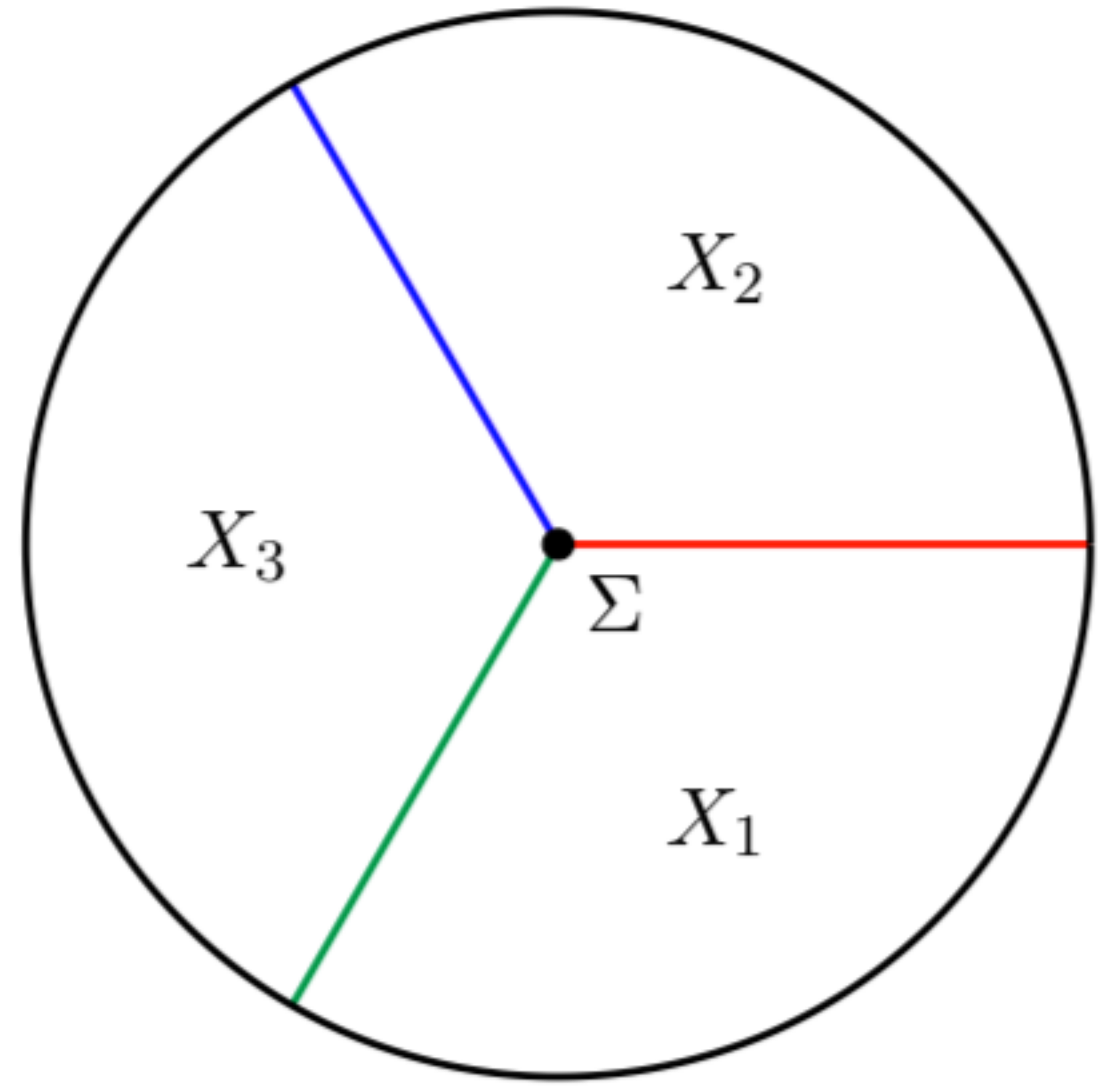
Theorem (W. 2020) If X is a flat Σ_h-bundle over Σ_g, then there is a $((2g+1)(2h+1)+1;2g+2h)$ trisection of X

Trisections

- Trisection: decomposes a 4-manifold into three simple pieces (4-d 1-handlebodies)

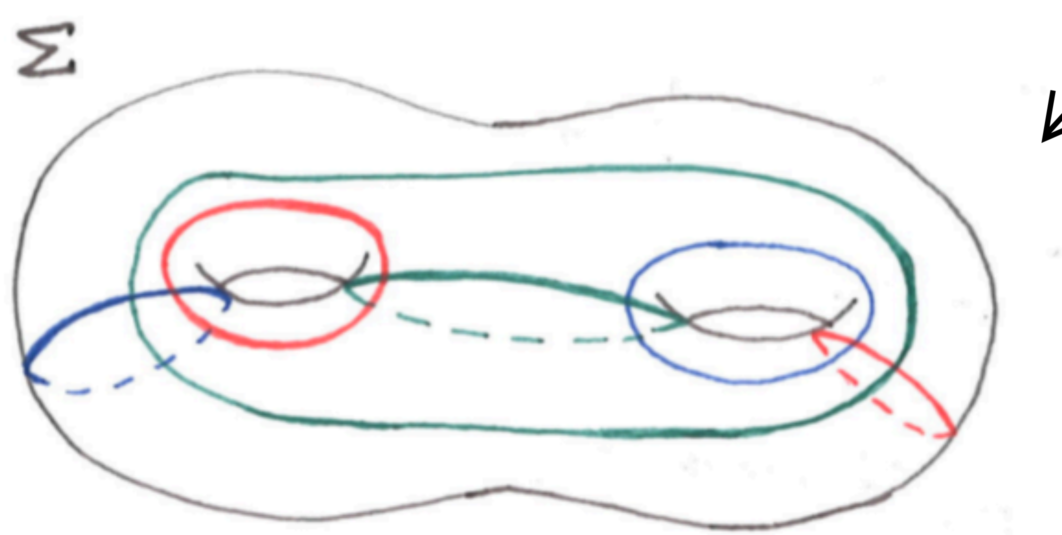
- Existence: Gay/Kirby 2012

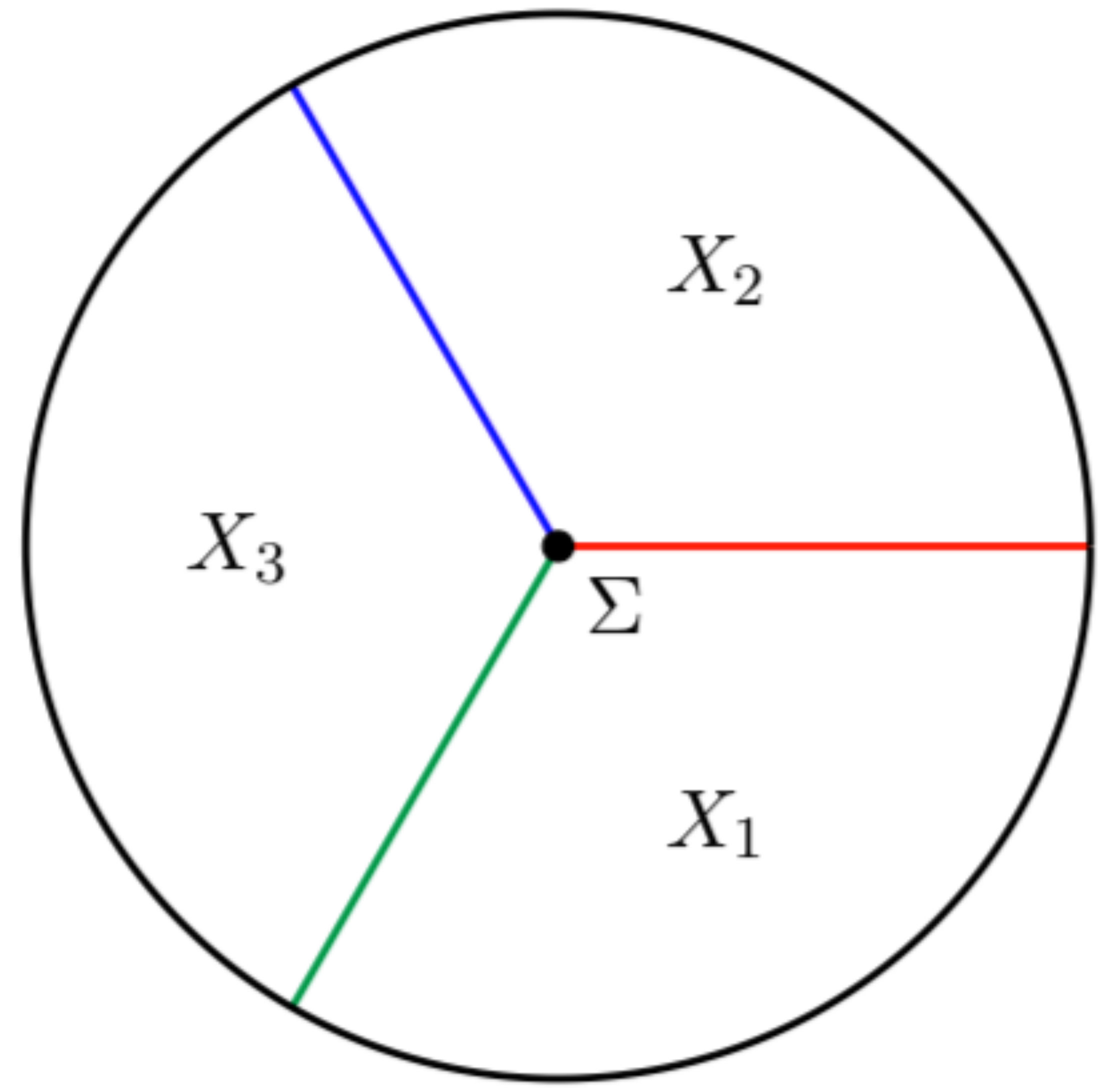
  - smooth
  - closed
  - orientable
  - connected



Trisections

- Trisection: decomposes a 4-manifold into three simple pieces (handlebodies)
- Existence: Gay/Kirby 2012
- Diagrams:





Past Work on Trisections

- Existence: smooth, orientable, closed (Gay/Kirby 2012); smooth, orientable, compact (Castro/Gay/Pinzón 2016, 2018)
- Genus 2 trisections standard (Meier/Zupan 2014)
- Lefschetz fibrations (Gay 2015, Castro/Ozbagci 2017, Baykur/Saeki 2017)
- Diagrams: 3-manifold bundles over S^1 (Koenig 2017)
- Diagrams: spun 4-manifolds (Meier 2017)

Theorem (W. 2018) There is a $((2g+1)(2h+1)+1; 2g+2h)$ trisection of $\Sigma_g \times \Sigma_h$

- X_i has genus $2g+2h$ as a 4-d handlebody; $X_1 \cap X_2 \cap X_3$ is a genus $(2g+1)(2h+1)+1$ surface

- Decompose Σ_g into three $(4g+2)$-gons B_1, B_2, B_3
  - pairwise intersect at $2g+1$ edges
  - triply intersect at $4g+2$ vertices

- Take three disjoint disks N_1, N_2, N_3 in Σ_h
  - define $\nu_i = B_i \times N_i$, for $i = 1, 2, 3$

- With $p : \Sigma_g \times \Sigma_h \to \Sigma_g$ the natural projection map, define

$$X_i = \overline{p^{-1}(B_i) \setminus \nu_i} \cup \nu_{i+1} = (B_i \times \overline{\Sigma_h \setminus N_i}) \cup (B_{i+1} \times N_{i+1})$$

- $\Sigma_g = T^2$
- $g = 1$
- three hexagons
- trisection surface is roughly a neighborhood of this 1-skeleton
- replace vertices by $\Sigma_h \setminus (\cup_{i=1}^3 N_i)$ and edges by $B_\delta \times \partial N_i$

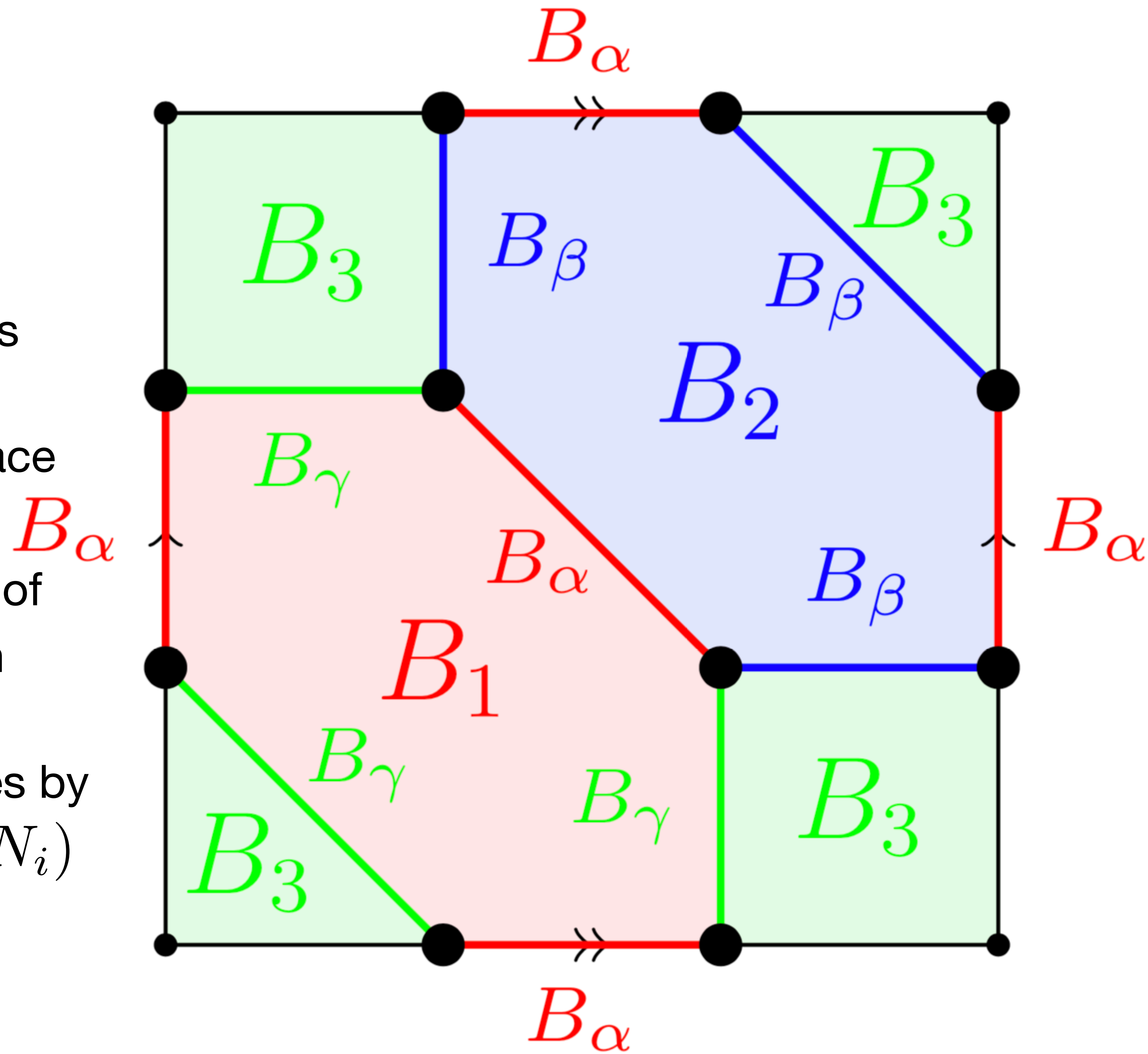


Diagram Construction

Example: $S^2 \times T^2 = \Sigma_g \times \Sigma_h$

- $(2g+1)(2h+1)+1 = (1)(3)+1 = 4$

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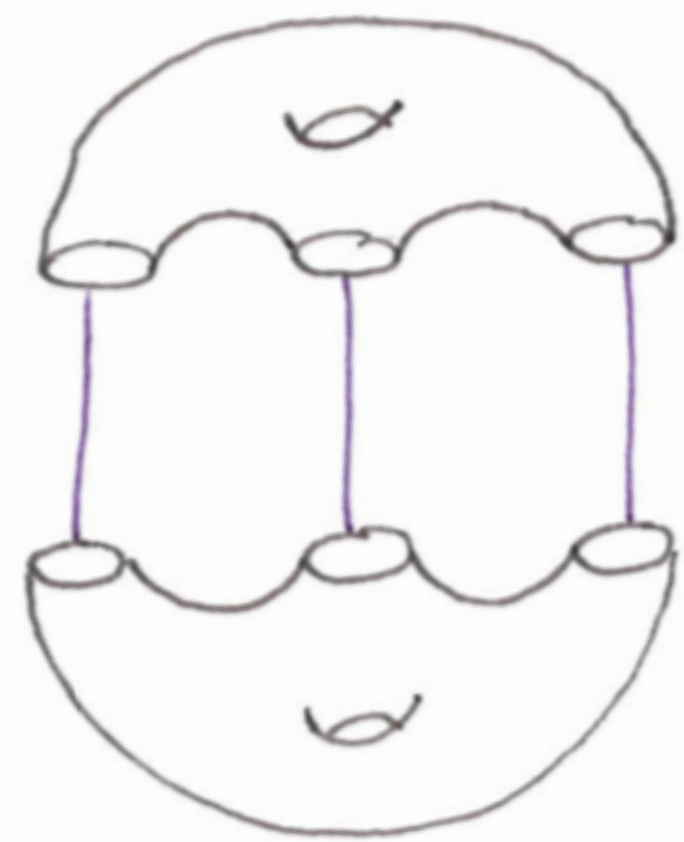


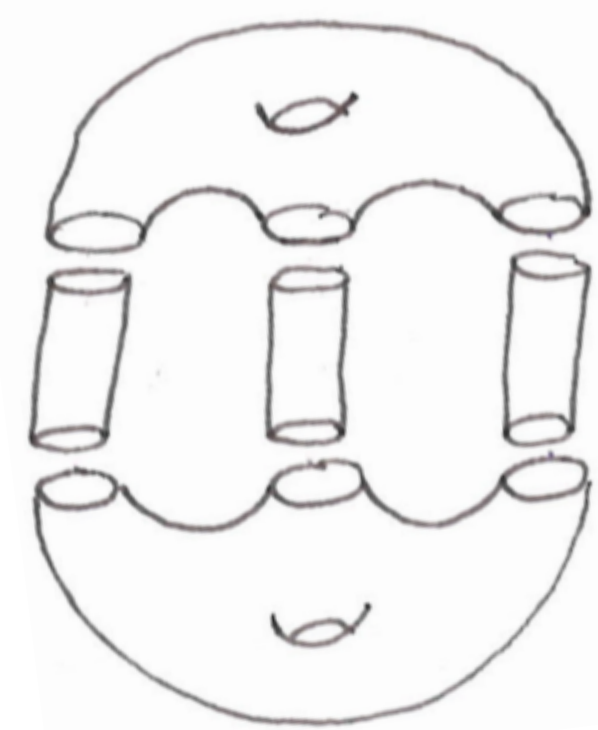


Diagram Construction

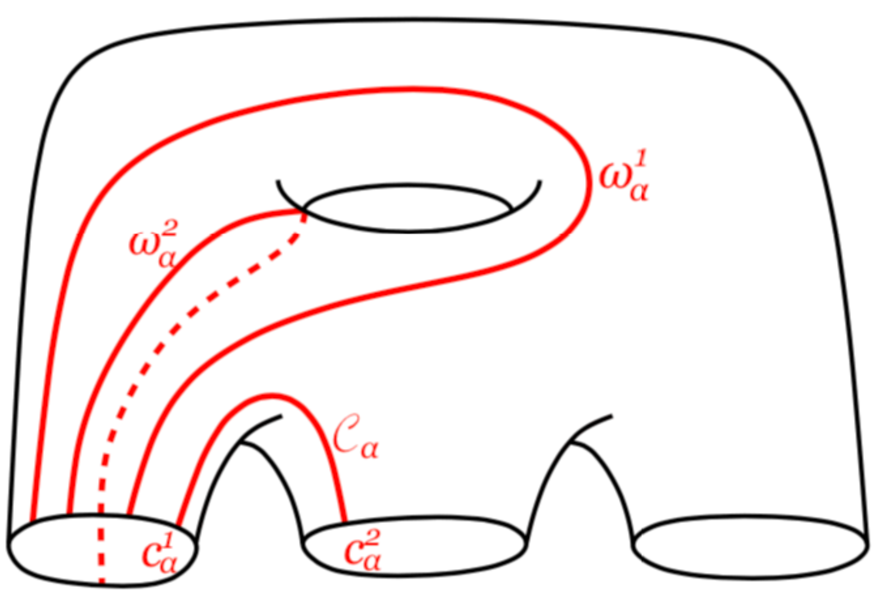
Example: $S^2 \times T^2 = \Sigma_g \times \Sigma_h$

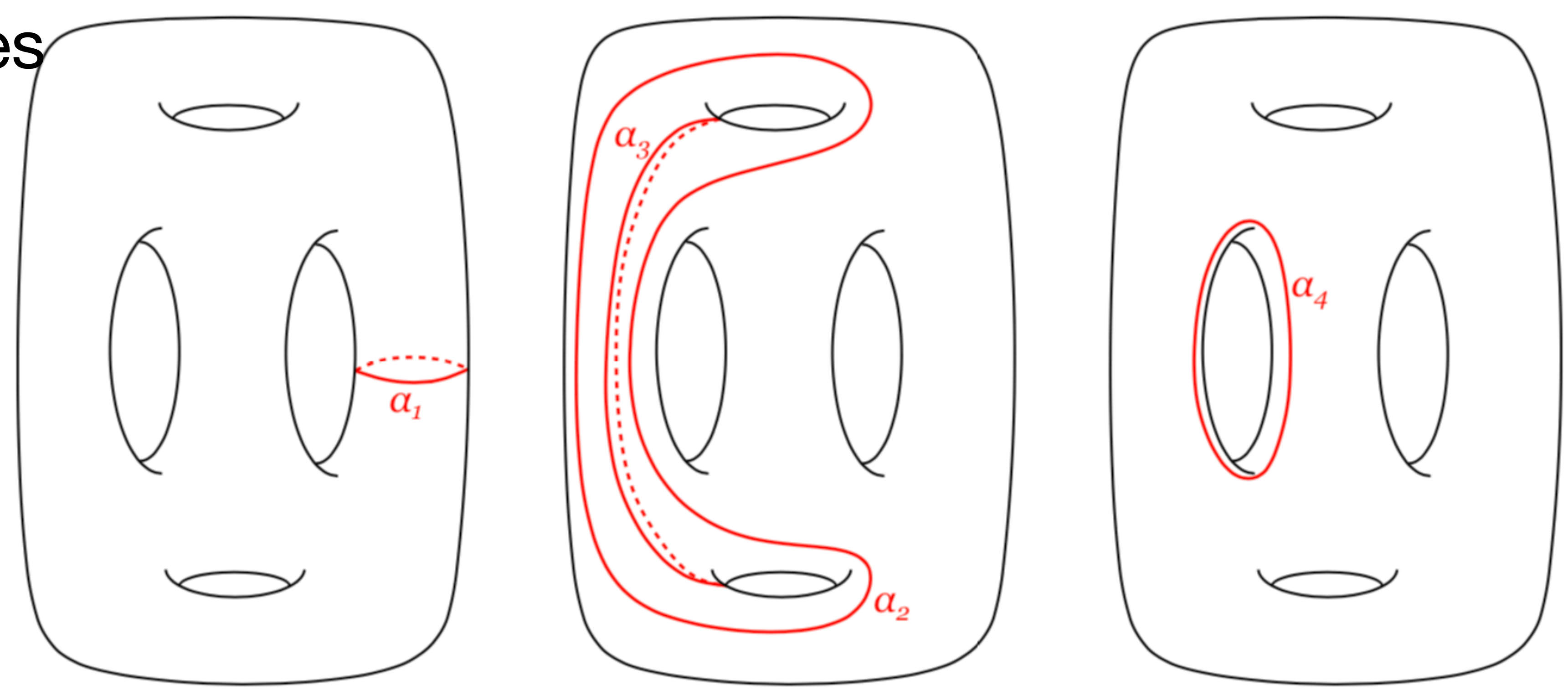
- $(2g+1)(2h+1)+1 = (1)(3)+1 = 4$
- We need to start placing curves
  - build curves from arcs in Σ_h
- We will use the structure of Σ as the union of:
  - {vertices} $\times \overline{\Sigma_h \setminus (\cup_{i=1}^3 N_i)}$
  - $B_\alpha \times \partial N_1$
  - $B_\beta \times \partial N_2$
  - $B_\gamma \times \partial N_3$

Curve Algorithm

Example: $S^2 \times T^2$

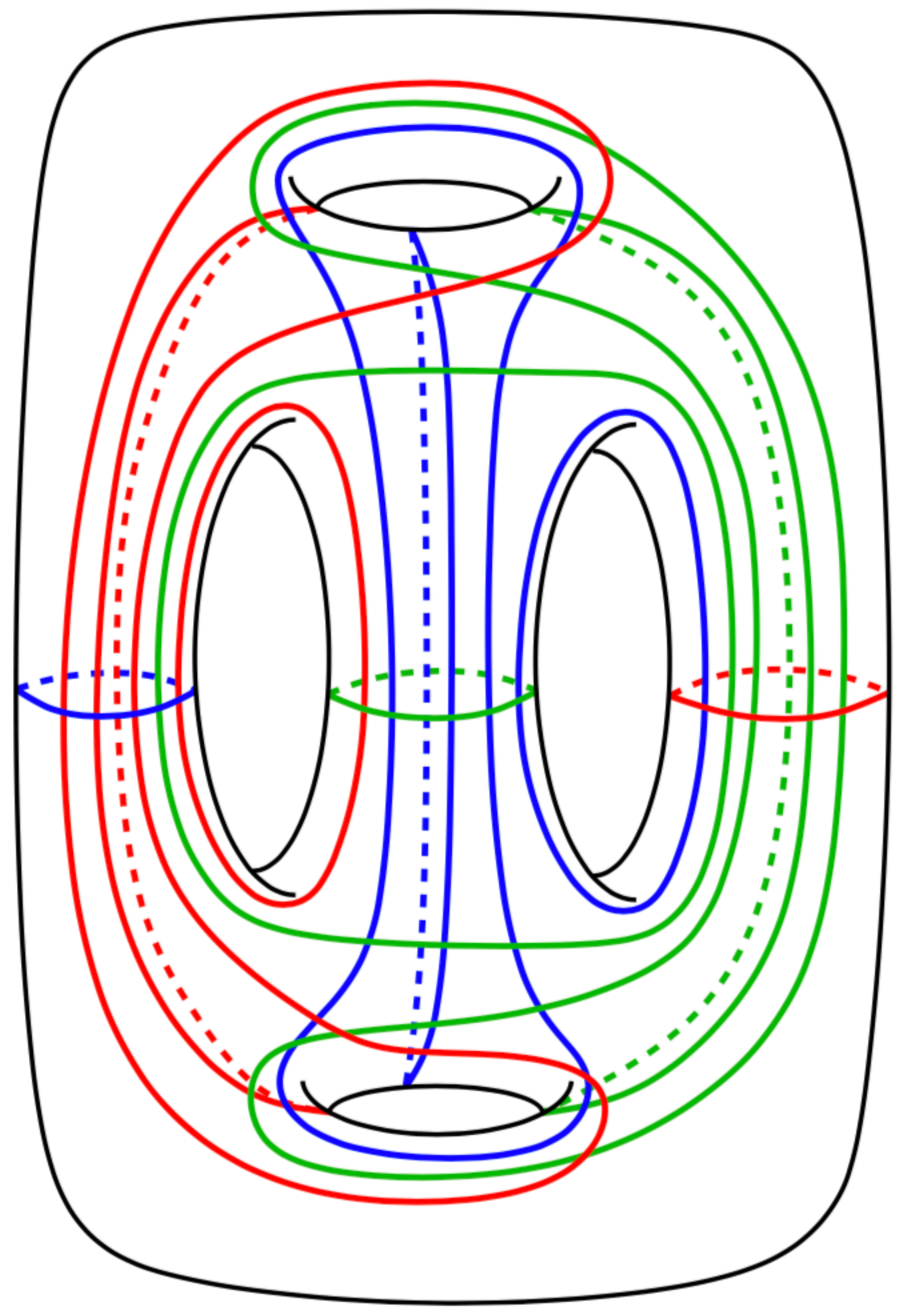
- Build curves from arcs:
- $\cup\omega_\alpha^i$ cut $\Sigma_h \setminus \cup N_i$ into a pair of pants; endpoints are in ∂N_1
- $\mathcal{C}_\alpha$ has endpoints in ∂N_1 and ∂N_2 . Connect things up.
- add meridional curves





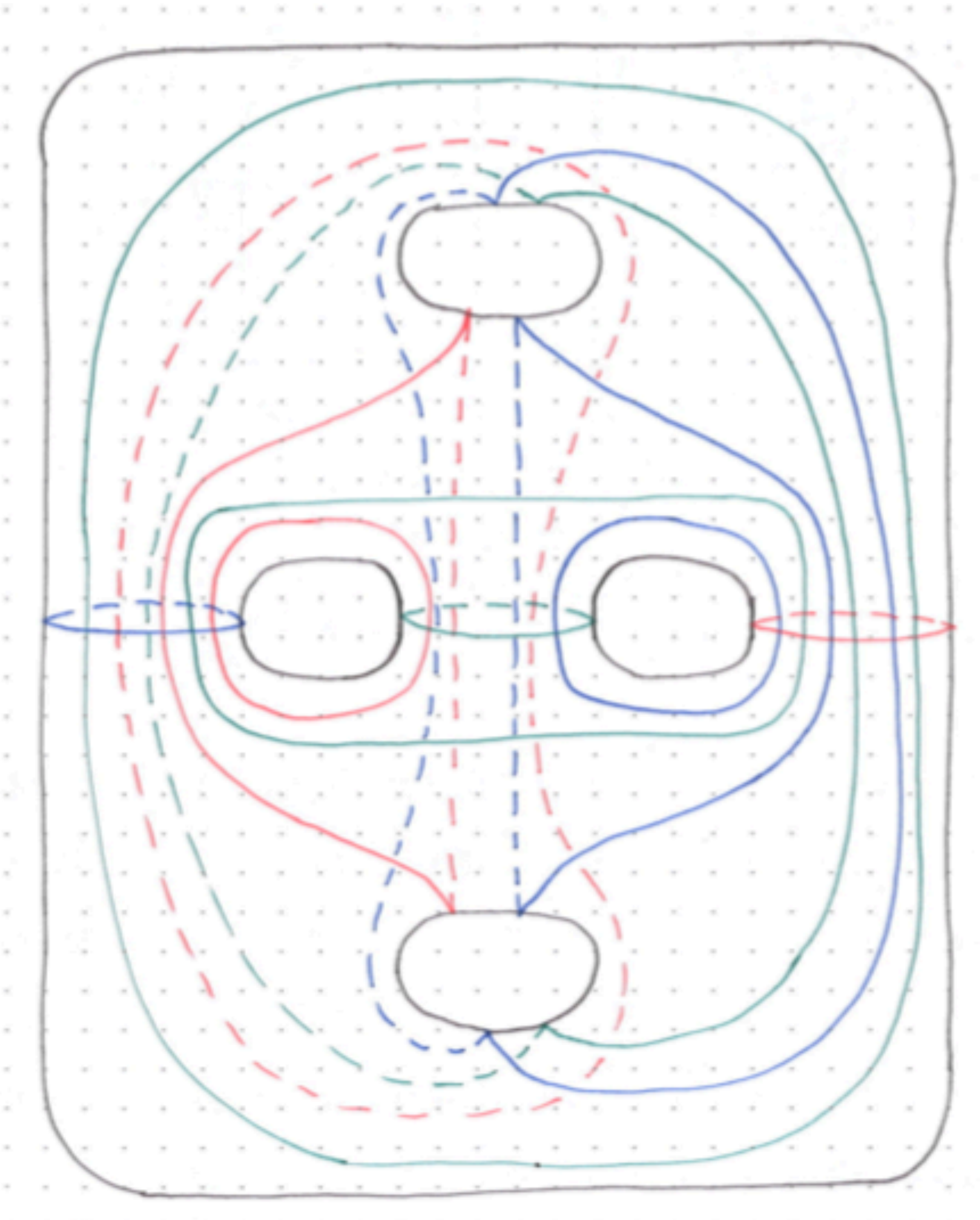
Curve Algorithm

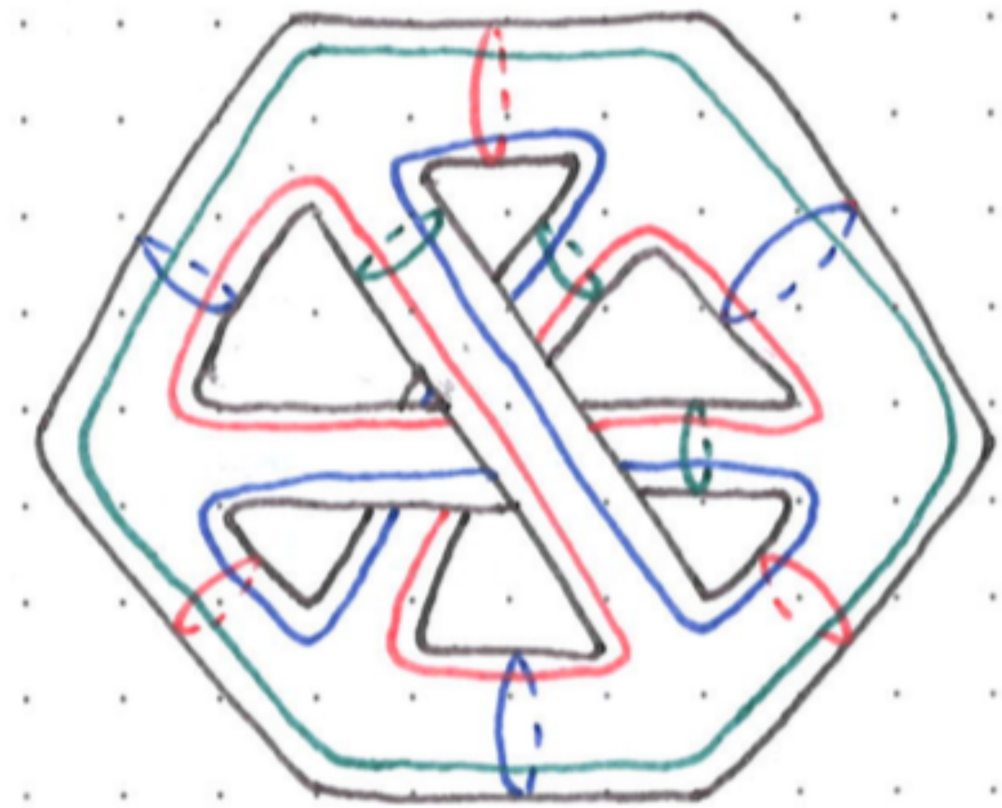
Example: $S^2 \times T^2$

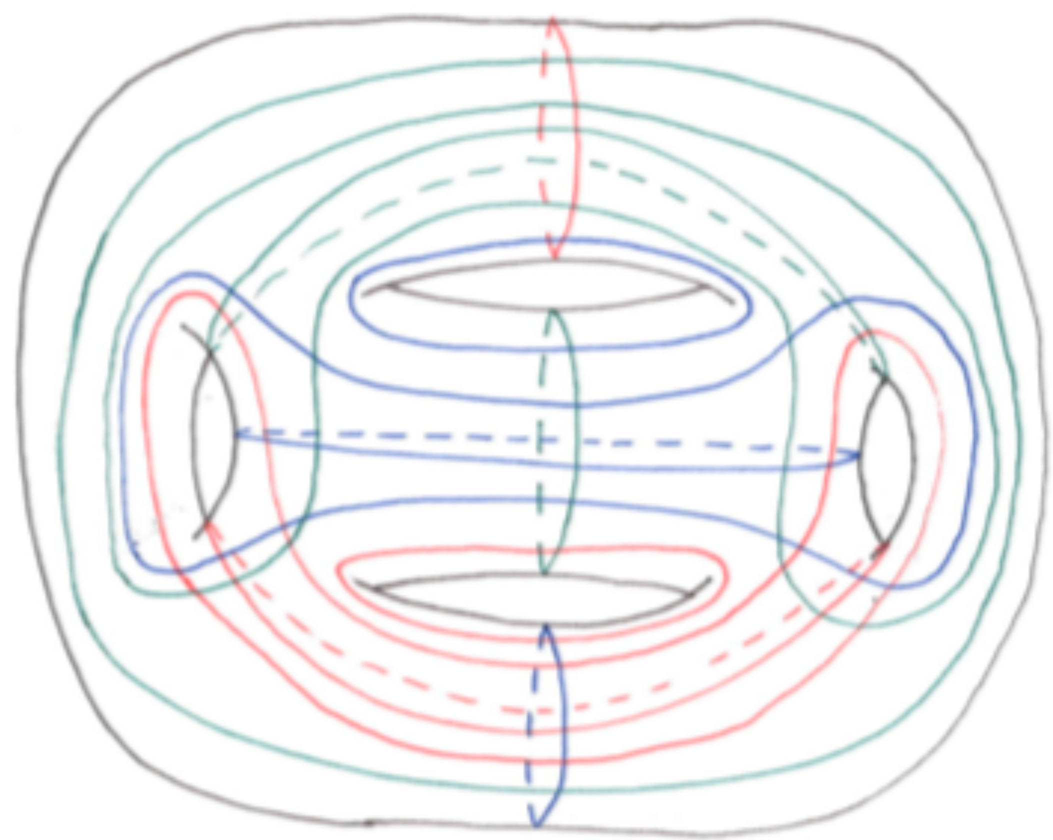


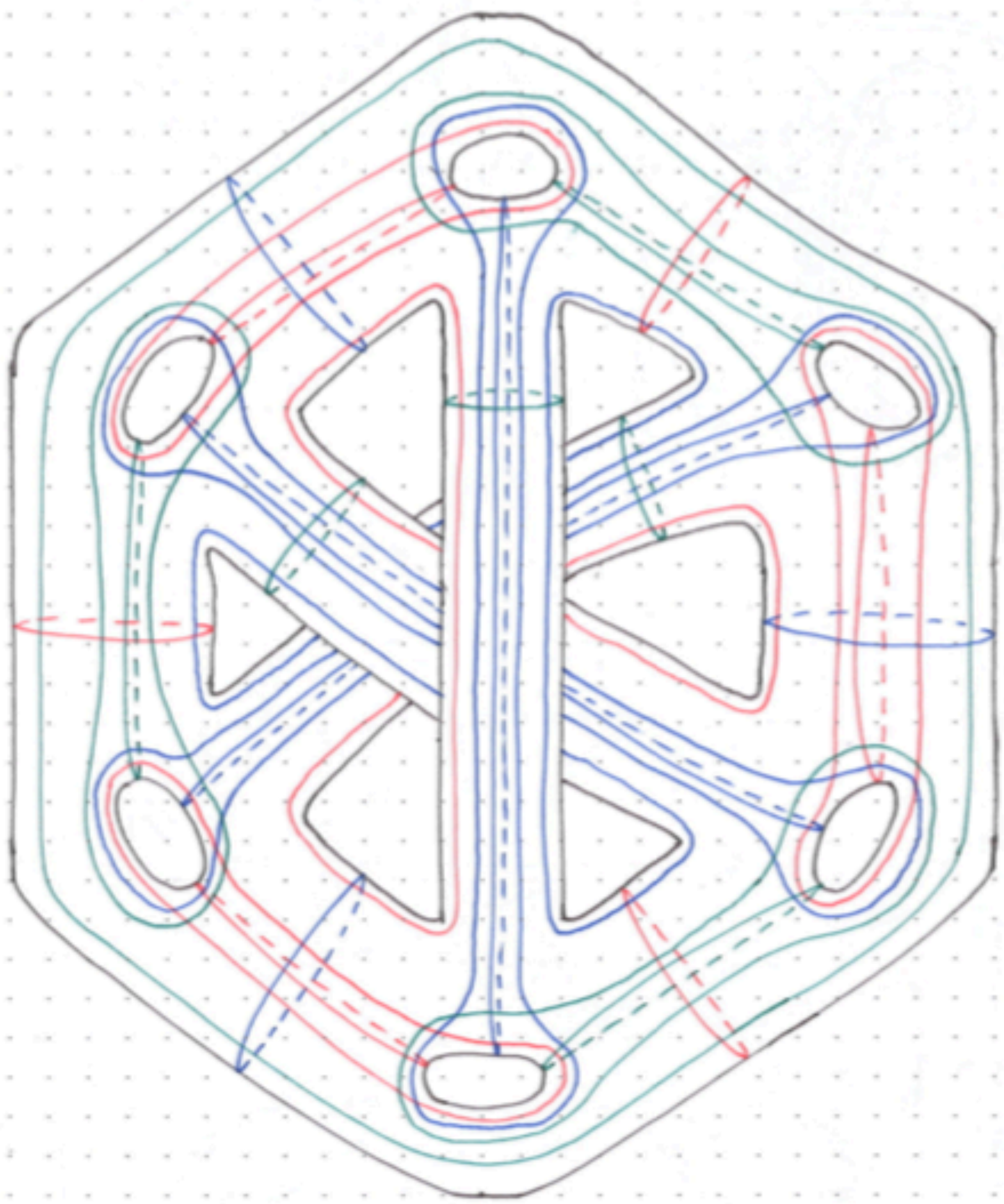
Nontrivial Bundles: Flatness

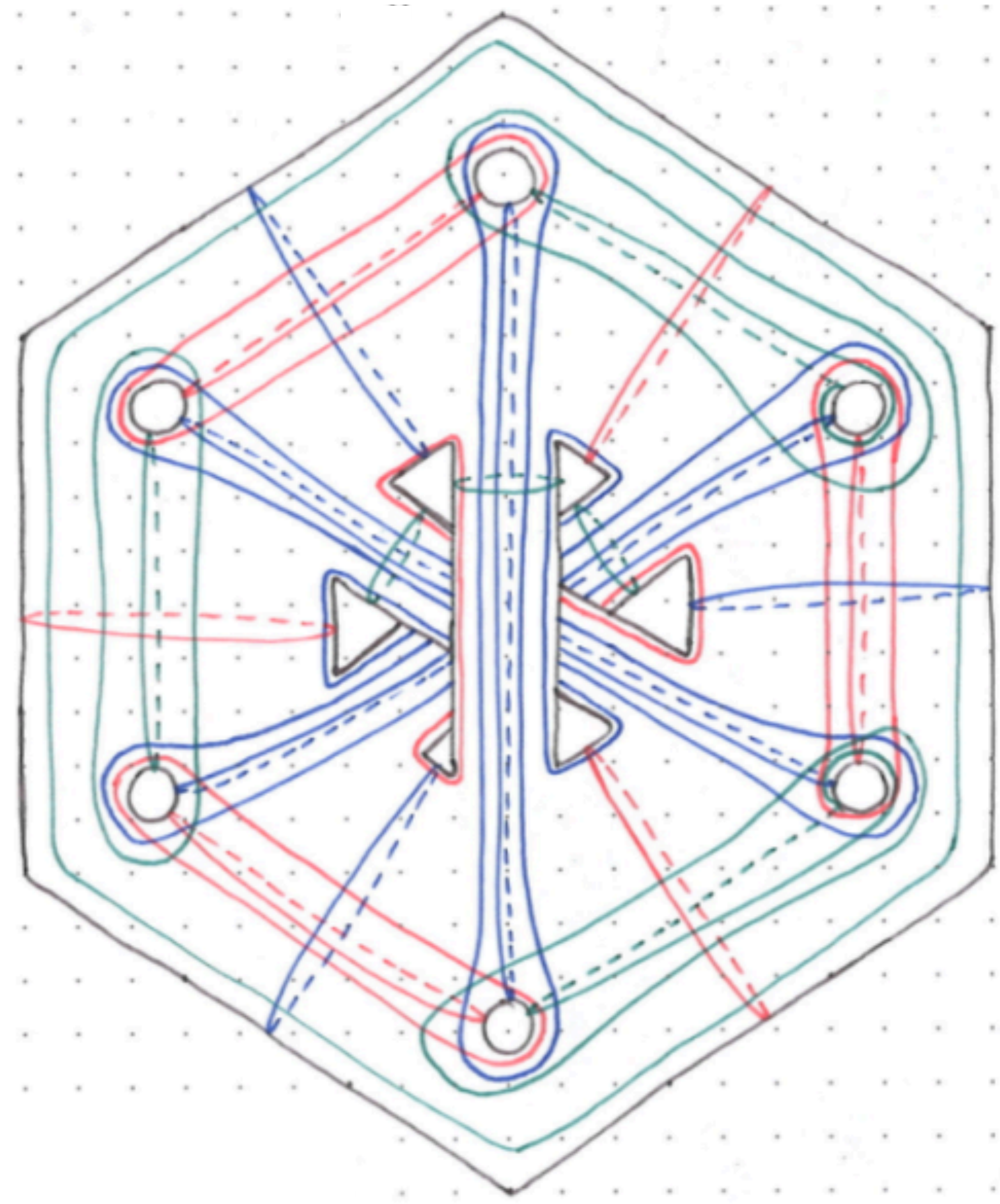
- When the bundle incorporates twisting, some of the arcs we used will be replaced by their image under a diffeomorphism of Σ_h
- For the algorithm to work, we need the bundle to be flat:
  - constructed from $D \times \Sigma_h$ by edge identifications in D
  - identify (edge $\times \Sigma_h$) with (edge $\times \varphi(\Sigma_h)$)

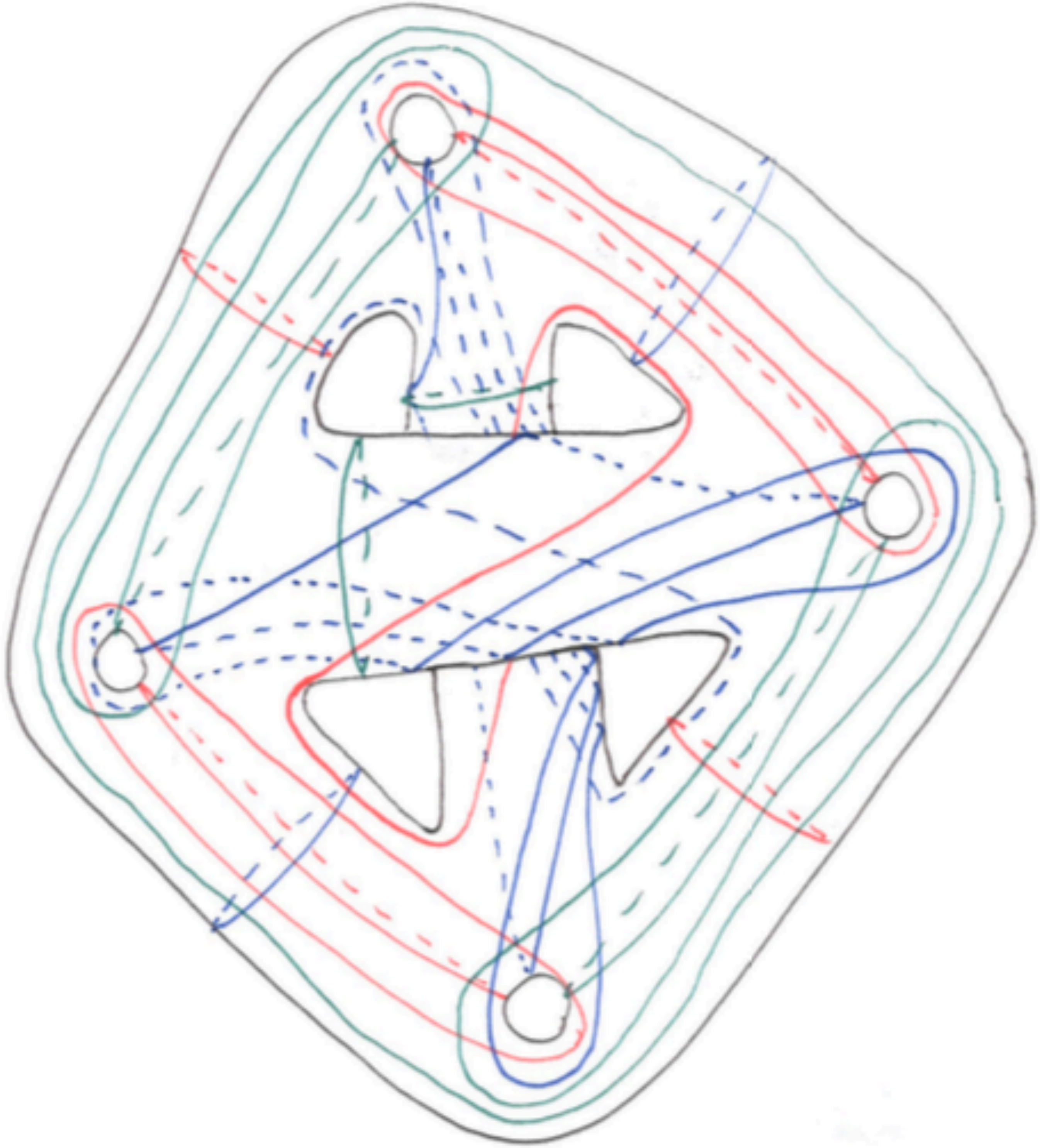


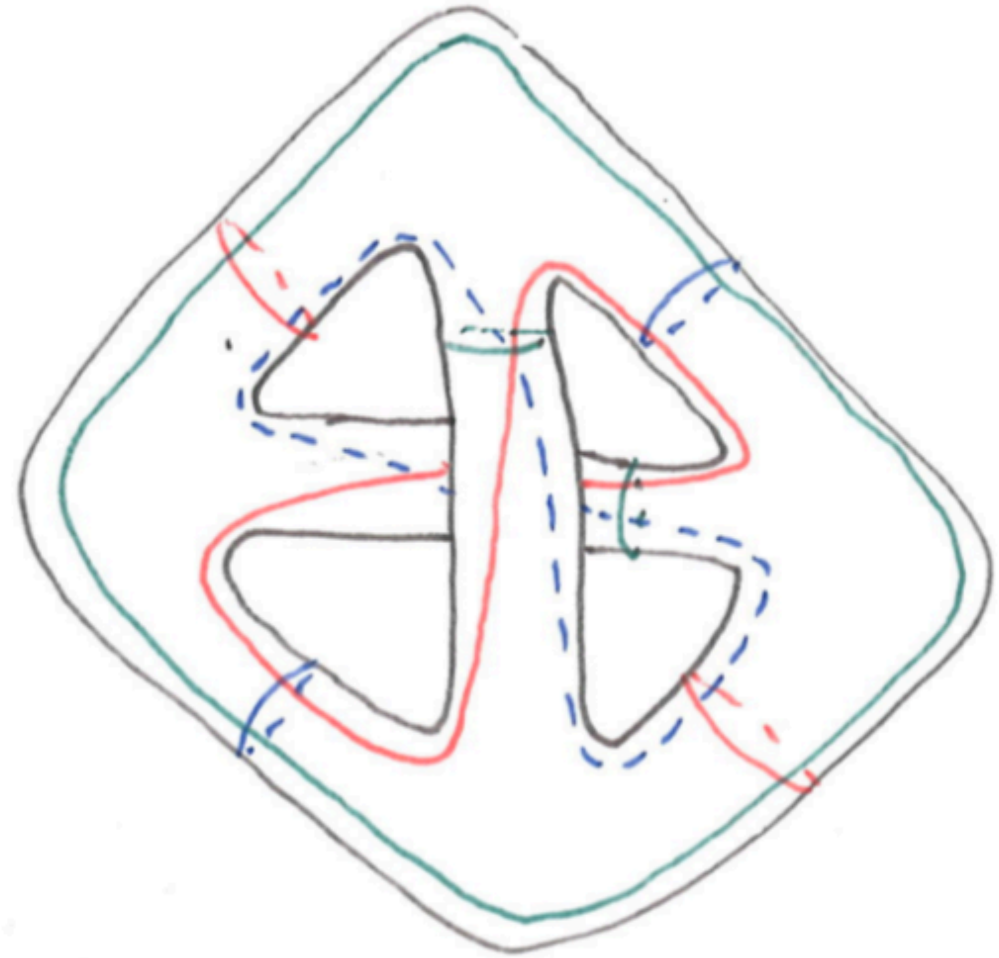


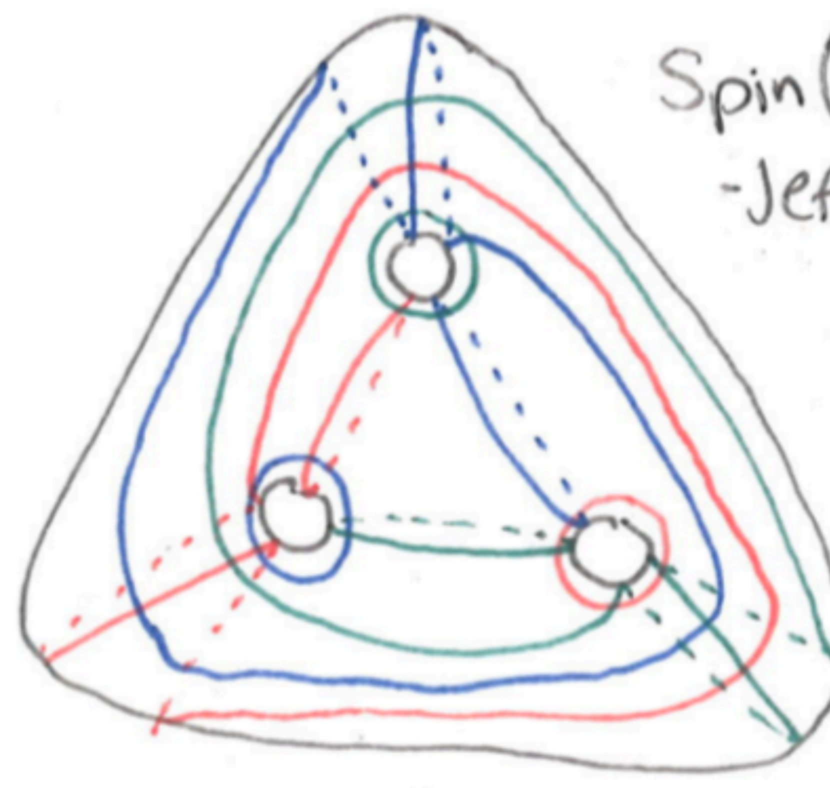












Spin($\mathbb{RP}^3$)
-Jeff Meier