# Partial marked length spectrum rigidity for negatively curved surfaces

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GOATS

What can the periodic orbits of a dynamical system $(X, T)$ tell us about $X$?

In particular: periodic orbits of geodesic flow on a compact negatively curved surface.

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- If you just know the lengths of the periodic orbits, you don't know much.
- If you know the lengths of the periodic orbits, the homotopy class each length belongs to, and that the metric is sufficiently nice you can identify the metric $g$ up to isometry.

# What's special about the homotopy classes?

When $S$ is a compact surface with a negatively curved metric $g$, **each nontrivial free homotopy class has exactly one closed geodesic**.

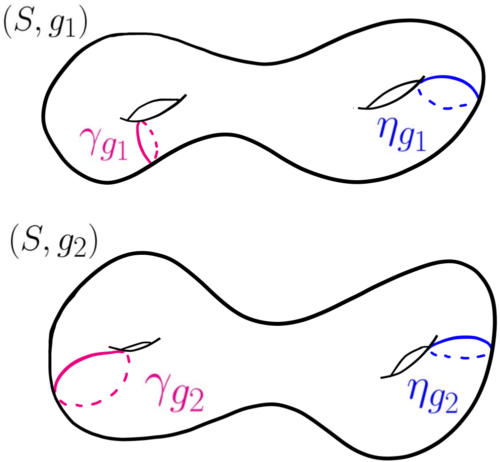
# Nonexamples in nonpositive curvature

# What's special about the homotopy classes?

When $S$ is a compact surface with a negatively curved metric $g$, **each nontrivial free homotopy class has exactly one closed geodesic**.

$\gamma_g :=$ the unique closed geodesic representing $\langle \gamma \rangle$ in $(S, g)$.

$\gamma_g$ is also the shortest curve in $\langle \gamma \rangle$.



# Dynamical information $\rightarrow$ geometric information

When $S$ is a compact surface with negatively curved metric $g$, the following collections of information are the same:

- The length of each periodic orbit of geodesic flow and the associated homotopy class for each orbit.
- The length of the closed geodesic in each homotopy class.

# Marked Length Spectrum

We encode the lengths and homotopy classes of each closed geodesic with the following function.

The **marked length spectrum** for a negatively curved Riemannian metric $g$ on a compact surface $S$ is a function

$$\begin{aligned} MLS_g : \mathcal{C} &\to \mathbb{R}^+ \\ \langle\gamma\rangle &\mapsto \text{length}_g(\gamma_g) \end{aligned}$$

where $\gamma_g$ is the unique geodesic in $\langle\gamma\rangle \in \mathcal{C}$.

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where $\gamma_g$ is the unique geodesic in $\langle \gamma \rangle \in \mathcal{C}$.

Another way to think about it:

$$MLS(S, g) = \big(\text{length}_g(\gamma_g)\big)_{\langle \gamma \rangle \in \mathcal{C}}$$

# Marked Length Spectrum Area Info

## Theorem (Croke and Dairbekov, 2004)

*Let $g_1$ and $g_2$ be two negatively curved Riemannan metrics on a compact surface $S$. If*

$$length(\gamma_{g_1}) \leq length(\gamma_{g_2})$$

*for every class $\langle \gamma \rangle \in \mathcal{C}$, then*

$$Area(S, g_1) \leq Area(S, g_2).$$

# **Partial** Marked Length Spectrum Area Info

### Theorem (S.)

*Let $g_1$ and $g_2$ be two negatively curved Riemannan metrics on a compact surface $S$* **and suppose $\mathcal{B} \subset \mathcal{C}$ has a subexponential growth rate by length.** *If*

$$length(\gamma_{g_1}) \leq length(\gamma_{g_2})$$

*for every class $\langle\gamma\rangle \in \mathcal{C} \setminus \mathcal{B}$, then*

$$Area(S, g_1) \leq Area(S, g_2).$$

# Subsets of $\mathcal{C}$ with subexponential growth rate

$\mathcal{B} \subset \mathcal{C}$ has a subexponential growth rate if

$$\lim_{T \to \infty} \frac{1}{T} \log |\mathcal{B}_T| = 0,$$

where $\mathcal{B}_T = \{\langle \gamma \rangle \in \mathcal{B} : \text{length}(\gamma_g) < T\}$.

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Examples of such sets are

- Any finite set
- $\{\langle\gamma\rangle : \gamma_g \text{ is simple}\}$
- $\{\langle\gamma\rangle : \gamma_g \text{ has fewer than } n \text{ self-intersections}\}$

# A note on exponential growth

$\mathcal{B} \subset \mathcal{C}$ grows exponentially if

$$\lim_{T \to \infty} \frac{1}{T} \log |\mathcal{B}_T| > 0,$$

where $\mathcal{B}_T = \{\langle \gamma \rangle \in \mathcal{B} : \text{length}(\gamma_g) < T\}$.

Examples of such sets are

- $\mathcal{C}$
- The set of $\varepsilon$-dense geodesics

# Marked length spectrum rigidity

## Theorem (Croke, Otal, 1990)

*Let $g_1$ and $g_2$ be two negatively curved Riemannian metrics on a compact surface $S$. Then*

$$MLS(S, g_1) = MLS(S, g_2)$$
$$\Updownarrow$$
$$g_1 = g_2.$$

i.e. the marked length spectrum uniquely determines the metric up to isometry.

# **Partial** marked length spectrum rigidity

## Theorem (S.)

*Let $g_1$ and $g_2$ be two negatively curved Riemannian metrics on a compact surface $S$.* **If $\mathcal{B} \subset \mathcal{C}$ has a subexponential growth rate by length**, *then*

$$MLS(S, g_1)\big|_{\mathcal{C}\setminus\mathcal{B}} = MLS(S, g_2)\big|_{\mathcal{C}\setminus\mathcal{B}}$$

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i.e. the **partial*** marked length spectrum uniquely determines the metric up to isometry.

# Partial marked length spectrum rigidity

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### Corollary

$$MLS(S, g_1)\big|_{\mathcal{C}\setminus\mathcal{B}} = MLS(S, g_2)\big|_{\mathcal{C}\setminus\mathcal{B}} \implies MLS(S, g_1) = MLS(S, g_2)$$

# Why the assumptions in the theorem?

## Theorem (S.)

*Let* $g_1$ *and* $g_2$ *be two* **negatively curved** *Riemannian metrics on a compact* **surface** $S$*. If* $\mathcal{B} \subset \mathcal{C}$ *has a* **subexponential growth rate** *by length, then*

$$MLS(S, g_1)\big|_{\mathcal{C}\setminus\mathcal{B}} = MLS(S, g_2)\big|_{\mathcal{C}\setminus\mathcal{B}}$$
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# Why the assumptions in the theorem?

Negative curvature guarantees that

- There is exactly one periodic orbit per homotopy class. We might lose that with nonpositive curvature.
- Closed geodesics that are almost dense grow exponentially quickly.

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# Why dimension 2?

There is something called a geodesic current, and Otal proved the following about the Liouville current, $\lambda_g$.

## Theorem (Otal, 1990)

*Let $g_1$ and $g_2$ be two negatively curved Riemannian metrics on a compact surface $S$. If $MLS(S, g_1) = MLS(S, g_2)$, then $\lambda_{g_1} = \lambda_{g_2} \iff g_1 = g_2$.*

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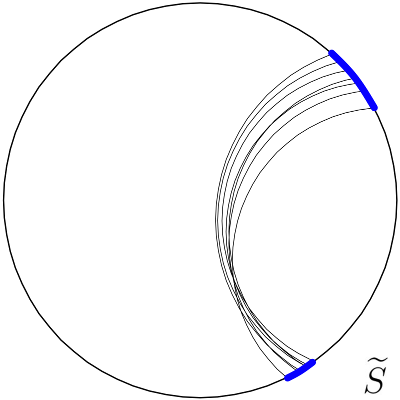
**That means that if we can reconstruct $\lambda_g$, that's enough to uniquely identify the metric $g$.**

# The boundary at infinity

The boundary at infinity of $\widetilde{S}$ is the equivalence classes of geodesics at $\infty$.

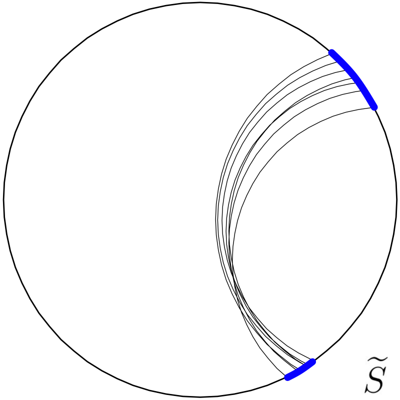
If $g$ is a negatively curved metric, then two points on the boundary of $\widetilde{S}$ uniquely identify a geodesic.



# Geodesic Currents

A **geodesic current** is a locally finite, $\pi_1(S)$-invariant Borel measure on the space of geodesics on $\widetilde{S}$, the universal cover of $(S, g)$.

Of particular interest is a current that depends on the metric: the **Liouville current** $\lambda_g$.

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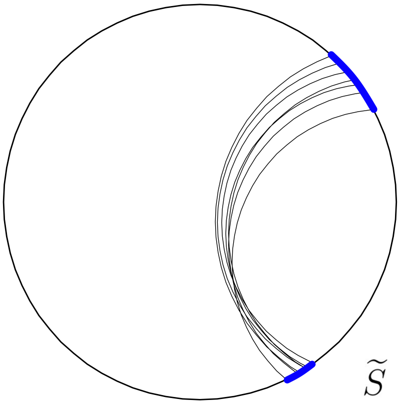
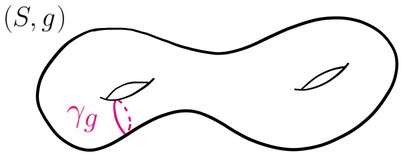
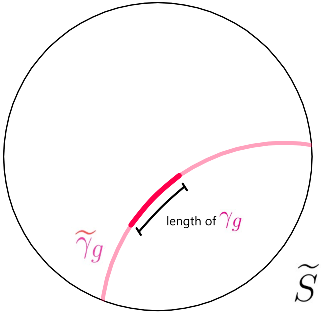


The idea: Show that you can reconstruct the Liouville current using the marked length spectrum info.

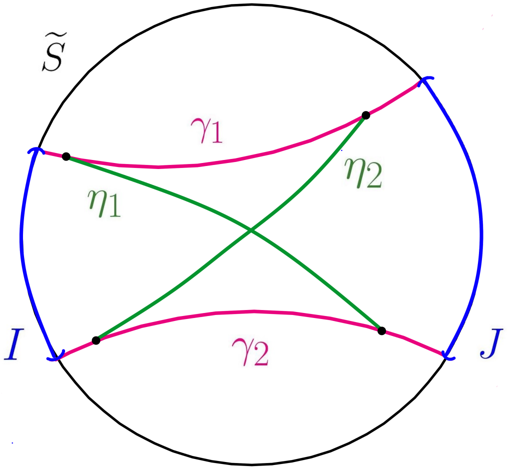



## Proposition (Bonahon, 1988)

*For any homotopy class* $\langle\gamma\rangle$*,*
*length* $\gamma_g$
$= \lambda_g($*geodesics intersecting one copy of* $\gamma_g)$

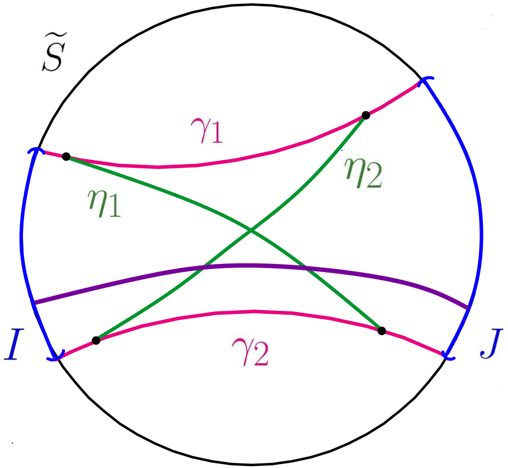


Say we want to find the measure of of the geodesics passing between $I$ and $J$.



Say we want to find the measure of of the geodesics passing between $I$ and $J$.

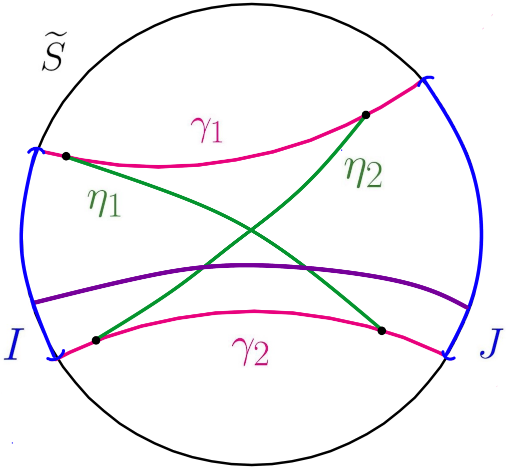
Each such geodesic intersects $\eta_1$ and $\eta_2$, but does not intersect $\gamma_1$ or $\gamma_2$.



Say we want to find the measure of of the geodesics passing between $I$ and $J$.

Each such geodesic intersects $\eta_1$ and $\eta_2$, but does not intersect $\gamma_1$ or $\gamma_2$.

$$2 \cdot \lambda_g(I \times J) = \text{length}(\eta_1) + \text{length}(\eta_2) - \text{length}(\gamma_1) - \text{length}(\gamma_2)$$

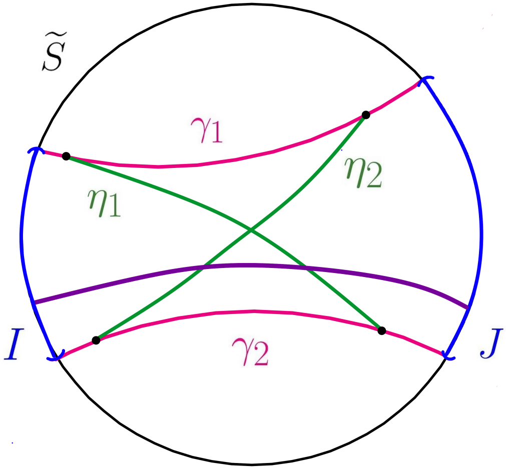


Say we want to find the measure of of the geodesics passing between $I$ and $J$.

Each such geodesic intersects $\eta_1$ and $\eta_2$, but does not intersect $\gamma_1$ or $\gamma_2$.

$$2 \cdot \lambda_g(I \times J) = \text{length}(\eta_1) + \text{length}(\eta_2) - \text{length}(\gamma_1) - \text{length}(\gamma_2)$$

This is only if $\eta_1, \eta_2, \gamma_1$, and $\gamma_2$ are each one copy of a closed geodesic **and they are not in $\mathcal{B}$**

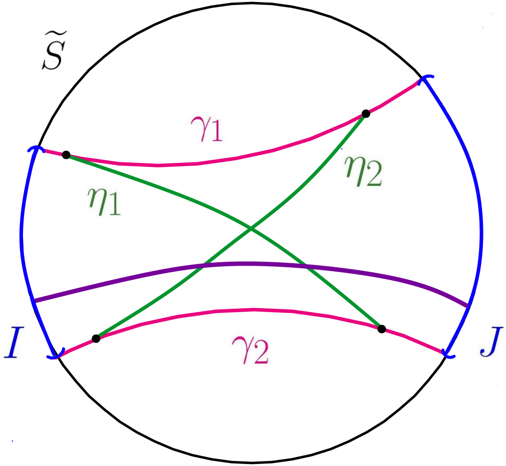


# Approximate!

Take a sequence of 4−tuples that approach the boundary.

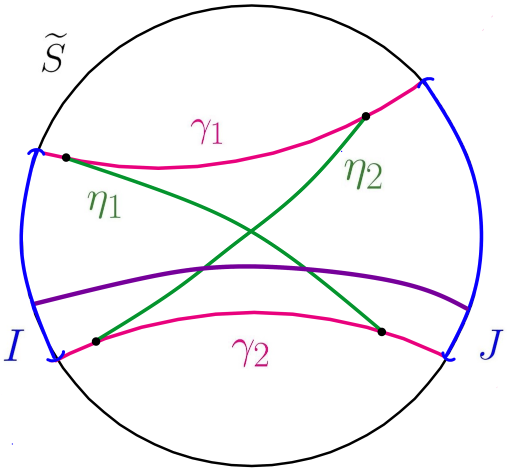
As long as we don't use any lifts of geodesics in $\mathcal{B}$, we can find $\lambda_g(I \times J)$ using $MLS(S, g)\big|_{\mathcal{C}\setminus\mathcal{B}}$.

Being able to find the measure of an arbitrary set means that we can reconstruct $\lambda_g$. That's enough to uniquely identify $g$ by an earlier proposition!



# Why dimension 2?

Notice: this "trap a geodesic" approach won't work in higher dimensions.



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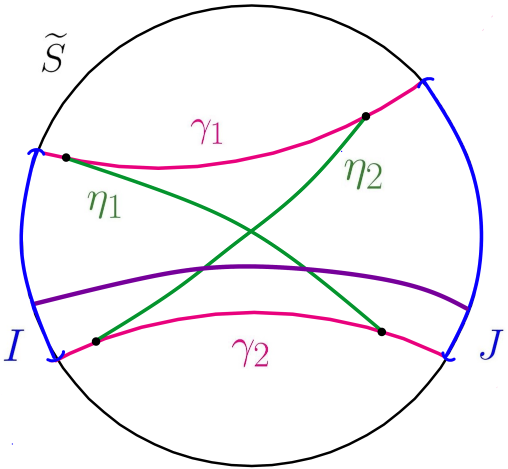
**Trouble:** What if every choice we make in this sequence has a geodesic in $\mathcal{B}$ show up?



# Why the assumptions in the theorem?

## Theorem (S.)

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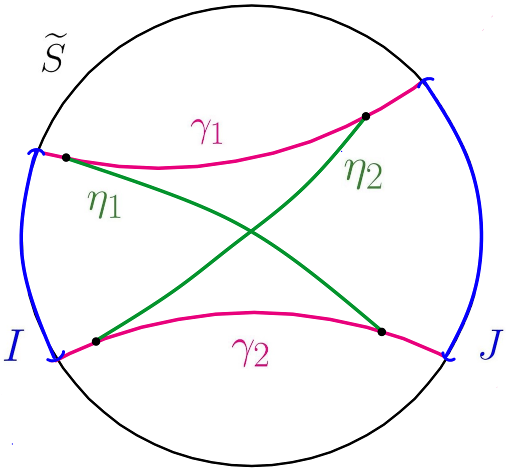
$$\Updownarrow$$
$$g_1 = g_2.$$

# Where does slow growth come in?

- Introduce an error in the boundary set.
- Show that there exists a sequence in which the number of **unique** homotopy classes that show up grows exponentially.
- Tossing out a subset that grows slowly still leaves most of this sequence.

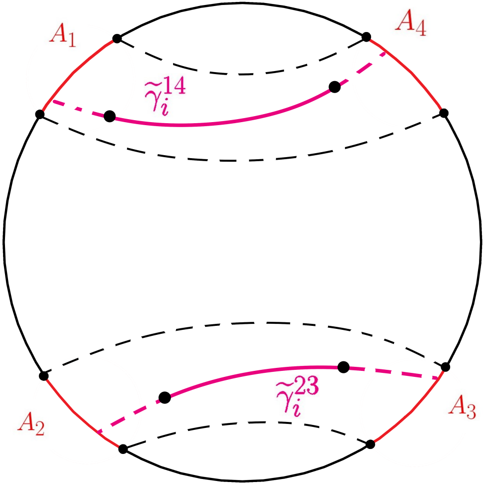


# Where does slow growth come in?

- **Introduce an error in the boundary set.**

That means we can choose $\varepsilon-$dense geodesics for our pink geodesics.

Since the set of $\varepsilon-$dense geodesics grows faster than $\mathcal{B}$ , we can always find a sequence of $\gamma_i^{14}$ and $\gamma_i^{23}$ in $\mathcal{C} \setminus \mathcal{B}$.



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