More Brieskorn Spheres Bounding Rational Balls

Oğuz Şavk, Boğaziçi University

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Homology Spheres and Homology Balls

We fix our notation as follows.

- Y : closed oriented smooth 3-manifold,
- X : compact oriented smooth 4-manifold.

Definition

1. The manifold Y is called an *integral homology sphere* if
$$H_*(Y, \mathbb{Z}) = H_*(S^3, \mathbb{Z}).$$
2. The manifold X is called an *integral homology ball* if
$$H_*(X, \mathbb{Z}) = H_*(B^4, \mathbb{Z}).$$

Question (Kirby's list, Problem 4.2)

Which integral homology spheres bound integral homology balls?

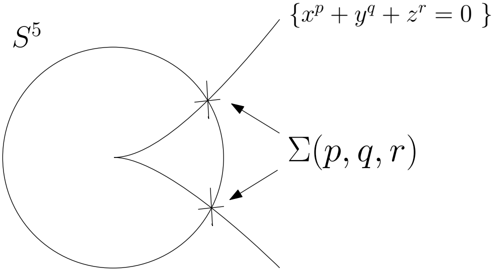
Brieskorn Homology Spheres

This question is fairly addressed to Brieskorn homology spheres.

Definition

Let p, q and r be pairwise coprime integers. Then the *Brieskorn homology sphere* $\Sigma(p,q,r)$ is the link of singularity at the origin

$$\Sigma(p,q,r) = \{x^p + y^q + z^r = 0\} \cap S^5 \subset \mathbb{C}^3.$$



Brieskorn Spheres Bounding Balls

A considerable number of Brieskorn spheres are known to bound integral homology balls.

Theorem (Akbulut-Kirby '79, Casson-Harer '81, Stern '79, Fintushel-Stern '81, Fickle '84)

- $\Sigma(2,5,7)$, $\Sigma(3,4,5)$, $\Sigma(2,3,13)$,
- $\Sigma(p, ps \pm 1, ps \pm 2)$ *for* p *odd,* $\Sigma(p, ps-1, ps+1)$ *for* p *even and* s *odd,*
- $\Sigma(p, ps \pm 1, 2p(ps \pm 1) + ps \mp 1)$ *for* p *even and* s *odd,* $\Sigma(p, ps \pm 1, 2p(ps \pm 1) + ps. \pm 2)$ *and* $\Sigma(p, ps \pm 2, 2p(ps \pm 2) + ps \pm 1)$,
- $\Sigma(2,7,19)$, $\Sigma(3,5,19)$,
- $\Sigma(2,3,25)$, $\Sigma(2,7,47)$, $\Sigma(3,5,49)$.

Brieskorn Spheres Non-trivially Bounding Balls

Definition

The manifold X is called a *rational homology ball* if

$$H_*(X, \mathbb{Q}) = H_*(B^4, \mathbb{Q}).$$

Fact

If an integral homology sphere bounds an integral homology ball, then it bounds a rational homology ball.

The reverse direction motivates the following definition.

Definition

An integral homology sphere is said to be *non-trivially* bounds a rational homology ball if it is obstructed from bounding an integral homology ball.

Brieskorn Spheres Non-trivially Bounding Balls

Now our first question turns out that

Question

Which integral homology spheres non-trivially bound rational homology balls?

Fintushel and Stern provided the first example by showing that

1. $\Sigma(2,3,7)$ bounds a rational homology ball, and
2. Rokhlin invariant μ and Neumann-Siebenmann invariant $\bar{\mu}$ of $\Sigma(2,3,7)$ is both non-zero so it does not bound an integral homology ball.

Therefore,

Theorem (Fintushel-Stern '84)

The Brieskorn sphere $\Sigma(2,3,7)$ non-trivially bounds a rational homology ball.

Homology Cobordism Groups

Definition

Let A be an abelian group. The *A-homology cobordism group* Θ^3_A is defined as follows

$$\Theta^3_A = \{A\text{-homology spheres}\}/\sim$$

where the equivalence relation is given by $Y_0 \sim Y_1$ if and only if there exits a compact oriented smooth 4-manifold X such that

$$\partial X = -(Y_0) \sqcup Y_1 \text{ and } H_*(X, A) = H_*(S^3 \times [0,1], A).$$

- The summation is given by connected sum,
- The standard sphere S^3 gives the zero element,
- The inverse element is obtained by reversing the orientation.

Homology Cobordism Groups

Fact

An A-homology sphere Y bounds an A-homology ball iff $Y \sim S^3$.

Fact

There is a canonical group homomorphism

$$\psi : \Theta^3_{\mathbb{Z}} \to \Theta^3_{\mathbb{Q}}$$

induced by inclusion.

The result of Fintushel and Stern can be interpreted as

Corollary

- *The map ψ is not injective,*
- *Further, the kernel of ψ contains a $\mathbb{Z}$ subgroup.*

Finding New Examples

The Brieskorn sphere $\Sigma(2,3,7)$ has remained the single example for more than thirty years.

The difficulty of finding another such examples is due to

1. the handle decomposition of 4-manifolds: If such an integral homology sphere exists, then the corresponding rational homology ball must contain 3-handles - $B^3 \times B^1$'s. -
2. the complexity of Kirby calculus: There is no general method for finding examples, almost all proofs present own originality.

Therefore, this challenge gives the motivation for searching more examples.

Finding New Examples

Akbulut and Larson presented the first additional examples.

Theorem (Akbulut-Larson '18)

The following Brieskorn spheres non-trivially bound rational homology balls:

- $\Sigma(2,3,19)$,
- $\Sigma(2,4n+1,12n+5)$ *for odd* n,
- $\Sigma(3,3n+1,12n+5)$ *for odd* n.

Using their technique, we found new infinite families.

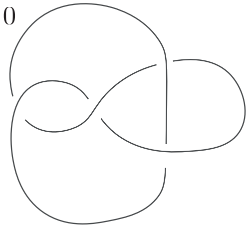
Theorem (Şavk '19)

The followings non-trivially bound rational homology balls:

- $\Sigma(2,4n+3,12n+7)$ *for even* n,
- $\Sigma(3,3n+2,12n+7)$ *for even* n.

Technique of Akbulut and Larson

Let Y be 3-manifold obtained by 0-surgery along the figure-eight knot in S^3.

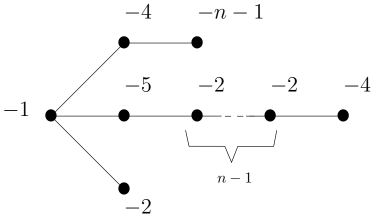
$Y =$ 

Lemma (Akbulut-Larson, 2018)

Any integral homology sphere obtained by an integral surgery on Y bounds a rational homology ball.

Proof of New Brieskorn Spheres

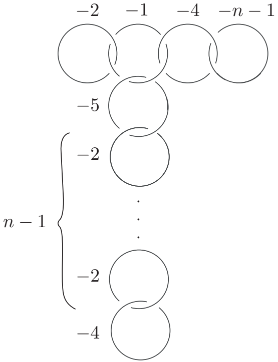
Start our proof first by noting that $\Sigma(2, 4n + 3, 12n + 7)$ and $\Sigma(3, 3n + 2, 12n + 7)$ are respectively the boundaries of plumbed 4-manifolds

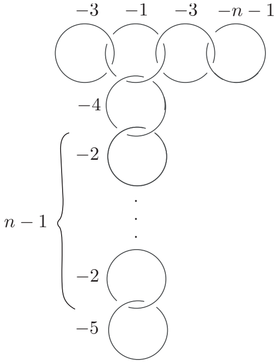


To complete the proof via previous lemma, we use the dual approach by giving integral surgeries from their plumbing graphs.

Proof of New Brieskorn Spheres

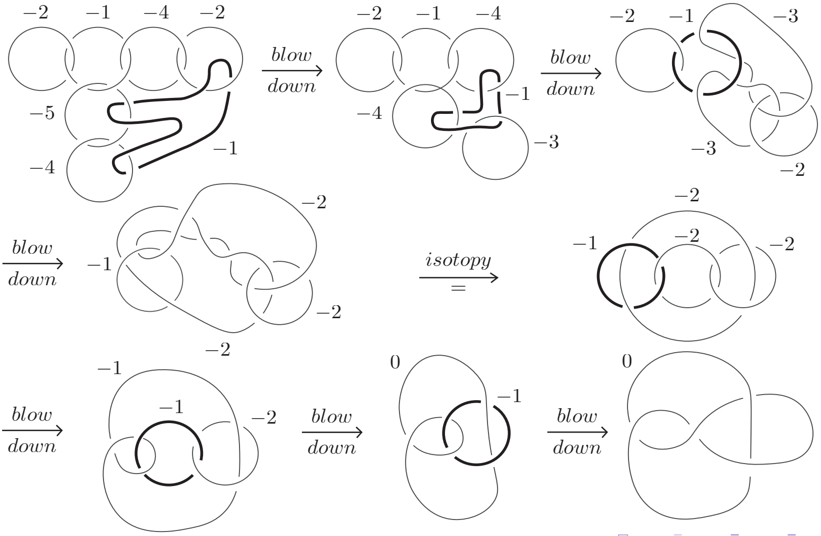
The followings are surgery diagrams of $\Sigma(2, 4n+3, 12n+7)$ and $\Sigma(3, 3n+2, 12n+7)$ respectively.



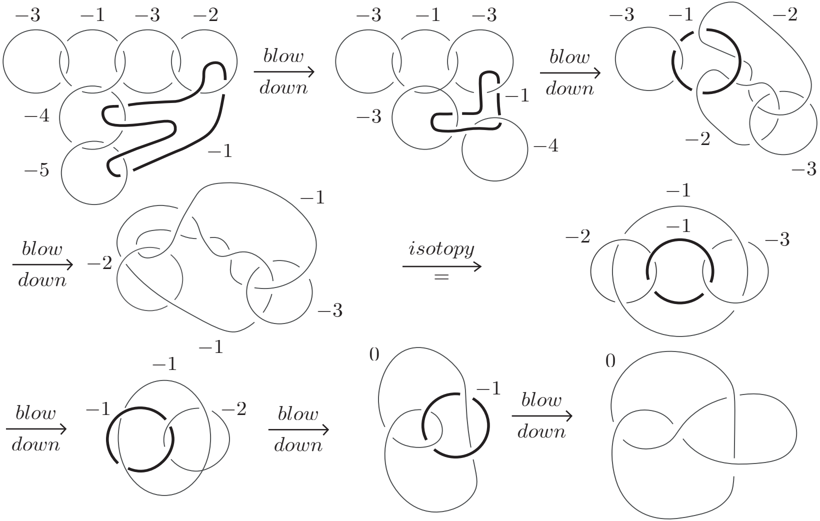


We will show that they are both obtained by (-1)-surgery on Y.

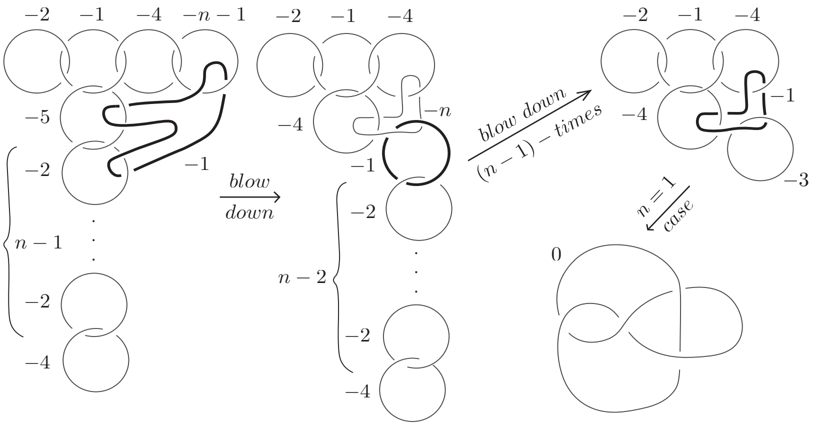
Proof of New Brieskorn Spheres: $n = 1$ cases



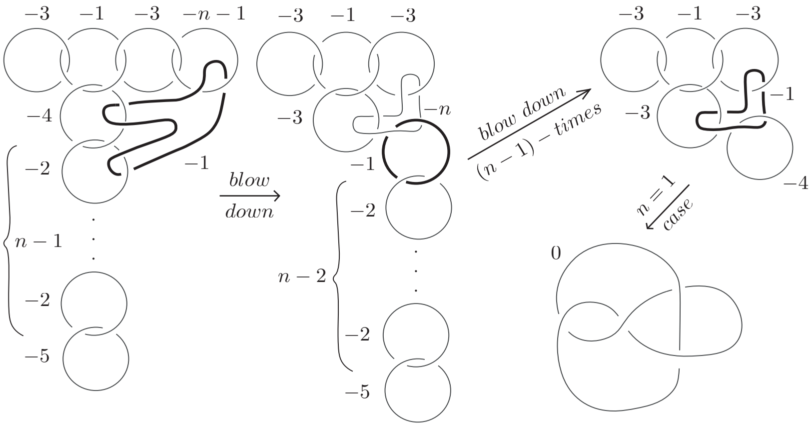
Proof of New Brieskorn Spheres: $n = 1$ cases



Proof of New Brieskorn Spheres: General cases



Proof of New Brieskorn Spheres: General cases



Proof of New Brieskorn Spheres: $\bar{\mu}$ invariant

Let $X(G)$ be the plumbed 4-manifold with $\partial X = \Sigma(p,q,r)$. There is a unique homology class $w \in H_2(X,\mathbb{Z})$, *Wu class*, satisfying

- $w.x = x.x \mod 2$ for all $x \in H_2(X,\mathbb{Z})$,
- all coordinates of w are either 0 or 1.

Then *Neumann-Siebenmann invariant* of $\Sigma(p,q,r)$ is defined to be

$$\bar{\mu}(\Sigma(p,q,r)) = -\frac{|G| + w.w}{8}$$

where $|G|$ is the number of vertices of plumbing graph G.

Plumbing graphs of $\Sigma(2,4n+3,12n+7)$ and $\Sigma(3,3n+2,12n+7)$ have both $n+5$ vertices, so $|G| = n+5$. When n is even $w.w$ are both $-n-13$ (otherwise they are $-n-5$). Thus for even n they have $\bar{\mu} = 1$. Therefore, they do not bound integral homology balls.

Further Directions

The *current* integral homology cobordism invariants cannot detect the linear independence of our spheres in $\Theta^3_{\mathbb{Z}}$.

Thus the *further* questions are

1. Does the kernel of $\psi : \Theta^3_{\mathbb{Z}} \to \Theta^3_{\mathbb{Q}}$ contain $\mathbb{Z}^\infty$ subgroup?
2. Does the kernel of $\psi : \Theta^3_{\mathbb{Z}} \to \Theta^3_{\mathbb{Q}}$ contain $\mathbb{Z}^\infty$ summand?
3. We prove that they are not integral homology cobordant to S^3. Are they homology cobordant to each other?

The *last* comment is that you can use our method to recover the classical results of Akbulut-Kirby, Casson-Harer, Stern, and Fickle.

Thanks for listening!