joint with Bahar Acu, Agnès Gadbled, Alexandra Marinković, Emmy Murphy, Laura Starkston, & Angela Wu

[ACGMMSW] project started WISCON

Defn A 2 dim mfld X equipped with a nondegenerate closed 2 form ω will be called a symplectic mfld

EX: • $(\mathbb{R}^{2n}, dx_i \wedge dy_i)$

• $(\mathbb{C}^n, dz_i \wedge d\bar{z}_i)$

• $(\mathbb{CP}^2, \omega_{FS})$

Lets study closed symplectic manifolds (X, ω) by considering a divisor $(D, i^*\omega) \subset (X, \omega)$

(a codim 2 submfld with an induced symplectic form

$\Rightarrow$ split X into: $X \setminus D$ & nhd(D)

• Donaldson: $\exists$ such a divisor

• Giroux: Can choose a divisor so that $X \setminus D$ is a Weinstein domain

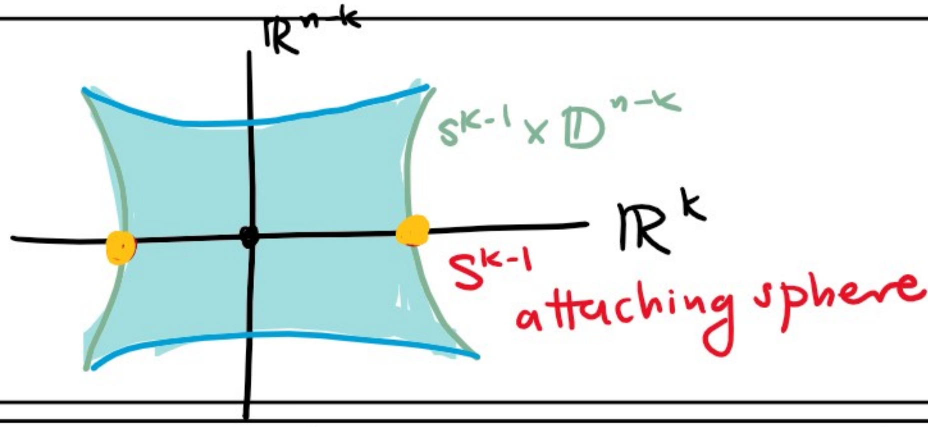
Goal: Produce explicit Weinstein handle decompositions for complement of smoothed toric divisors in toric 4-mflds

I. Handledecompositions

smooth n-mfld M:

$$M_0 \subset M_1 \subset \dots \subset M_n = M$$

M_k given by attaching k-handle $\mathbb{D}^k \times \mathbb{D}^{n-k}$ to M_{k-1} along $S^{k-1} \times \mathbb{D}^{n-k}$



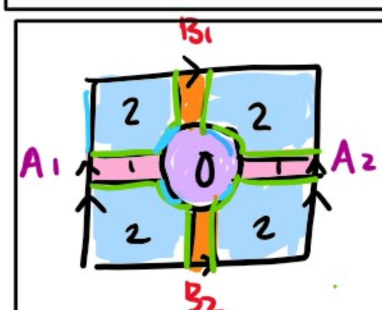
RMK: Morse function on M gives a handledecomp

	0 handle	1 handle	2 handle		
M surface	$\mathbb{D}^0 \times \mathbb{D}^2$	$\mathbb{D}^1 \times \mathbb{D}^1$  $S^0 \times \mathbb{D}^1$	$\mathbb{D}^2 \times \mathbb{D}^0$  $S^1 \times \mathbb{D}^0$		

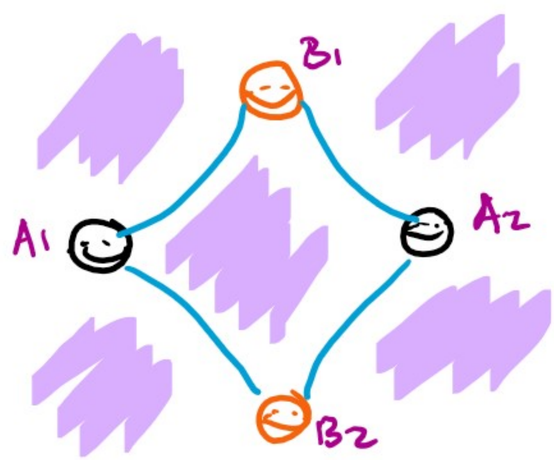
	0 handle	1 handle	2 handle	3 handle	4 handle
M 4-mfld	$D^0 \times D^4$	$D^1 \times D^3$  $S^0 \times D^3$	$D^2 \times D^2$  $S^1 \times D^2$ $n \in \mathbb{Z}$ framing	$D^3 \times D^1$	$D^4 \times D^0$

Examples

$\partial = S^1 \times S^2$



T^2



$$D^* T^2 = \left\{ (q,p) \;\middle|\; \begin{array}{l} q \in T^2,\ p \\ \text{covector} \\ |p| \leq 1 \end{array} \right\}$$

$$= T^2 \times D^2$$

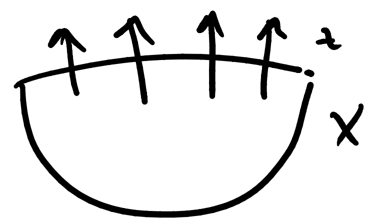
Weinstein handle decomposition

Let (X, ω) with $\partial X \neq \emptyset$, compact, exact

$\omega = d\alpha$, w/ v.f Z Liouville vector field

$$\mathcal{L}_Z \omega = \omega$$

& $Z \pitchfork \partial X$



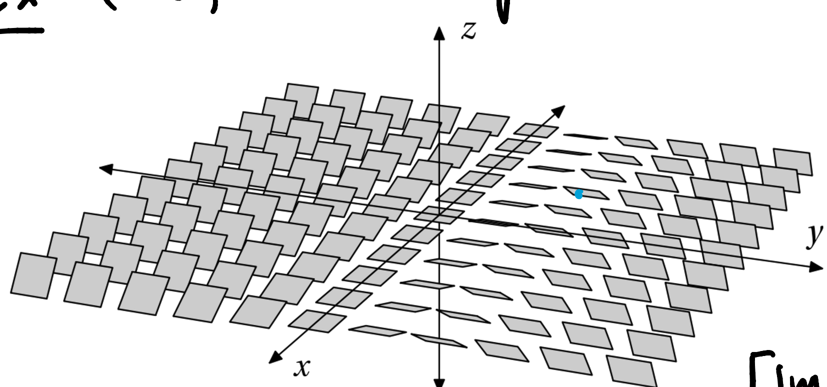
$\Rightarrow \ker(\alpha|_{\partial X})$ is a contact structure on ∂X

All of this gives us a Liouville domain

<u>Defn</u> A hyperplane distribution $\xi \subset TY^{2n+1}$

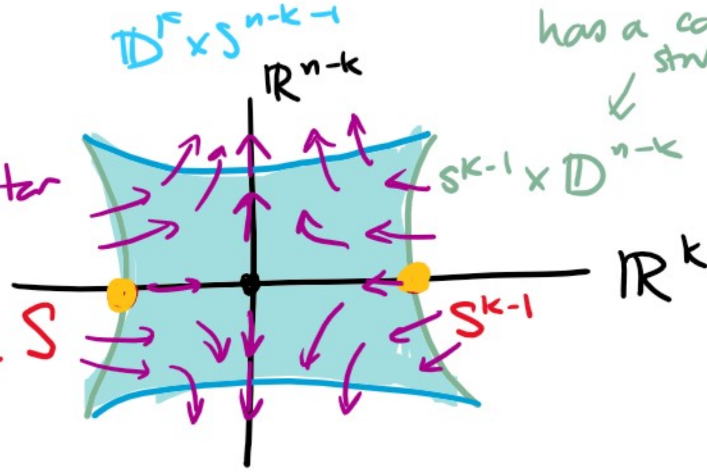
$\xi = \ker(\lambda)$ is a contact structure if $\lambda \wedge (d\lambda)^{n-1} \neq 0$

<u>Ex</u> $(\mathbb{R}^3, \ker(dz - y\,dx))$



[Image from wikipedia]

Weinstein
k-handle



Z Liouville vector field

isotropic attaching sphere S

$T_pS \subseteq \xi_p$

A Weinstein domain is a Liouville domain with a morse function ϕ st Z is gradient like wrt ϕ

[Eliashberg]
Weinstein $2n$ mfld can be built with Weinstein k-handles $k \leq n$
If $k=n$ its a critical handle. Its attached along a legendrian

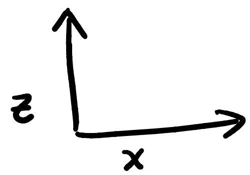
Defn: $\Lambda \subset (Y, \xi)$ is legendrian submfld if $T_p\Lambda \subset \xi_p \; \forall p \in \Lambda$ & $2\dim(\Lambda)+1 = \dim(Y)$

[Eliashberg] If $2n=4$, 2 handles are attached along a legendrian knot $\Lambda \subset \#^k(S^1 \times S^2)$ with framing $tb(\Lambda)-1$

Front projection of a legendrian knot in $(\mathbb{R}^3, \ker(dz - y\,dx))$

$F : (x, y, z) \to (x, z)$

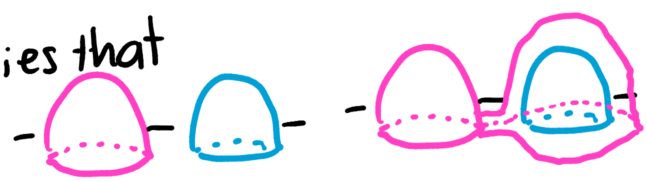
$(\mathbb{R}^3, \ker(dz - y\,dx))$



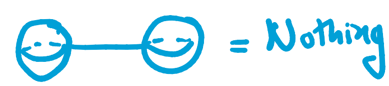
$y = \frac{dz}{dx}$ slope b/c $T_p \Lambda \subseteq \ker(dz - y\,dx)$

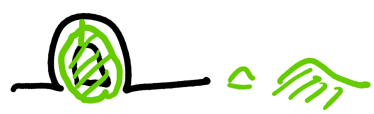
RMK

Two smooth handlebodies that differ by:



- ★ handleslides
- ★ handle cancellation
- ★ isotopy of attaching spheres





give homeomorphic manifolds

Two smooth Weinstein handlebodies that differ by

- ★ handleslides
- ★ handle cancellation
- ★ Isotropic isotopy of attaching spheres (leg Reidemeister moves, Gompf moves)

give Weinstein homotopic mflds.

[Gompf, Casals-Murphy]

II Toric mfld background.

Defn A toric 4-mfld symp 4-mfld (X, ω) w/ an effective Hamiltonian torus action

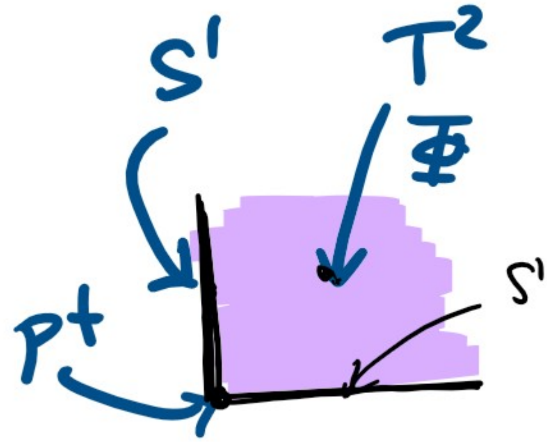
moment map

$\Phi : (X, \omega) \to \mathbb{R}^2$

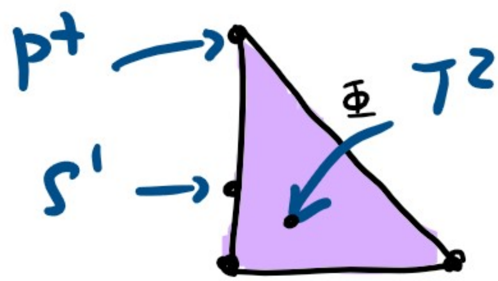
$\Phi = (\Phi_1, \Phi_2) \qquad \iota_{X_k} \omega = -d\Phi_k$

Ex $(\mathbb{C}^2, dz \wedge d\bar{z})$

$\Phi = \frac{1}{2} (|z_1|^2, |z_2|^2)$



$(\mathbb{CP}^2, \omega_{FS})$



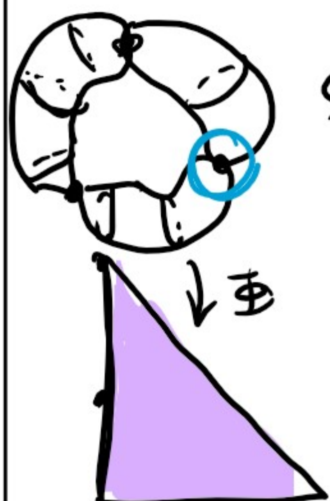
[Atiyah, Guillemin-Sternberg]

[Atiyah-Guillemin-Sternberg]

- $\Delta = \Phi(X)$ convex polytope in $\mathbb{R}^2$

  Delzant polytope

- Φ^{-1}(facets of Δ) toric divisor



$\subseteq \mathbb{CP}^2$

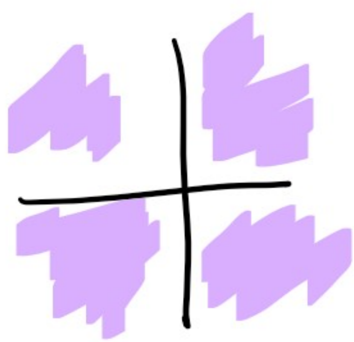
Normal crossing singularity

Note

$X \setminus D = D^* T^2$

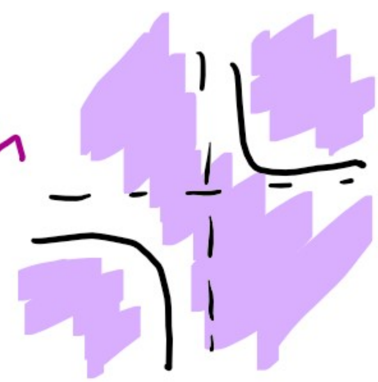
What if we smoothed D?

locally

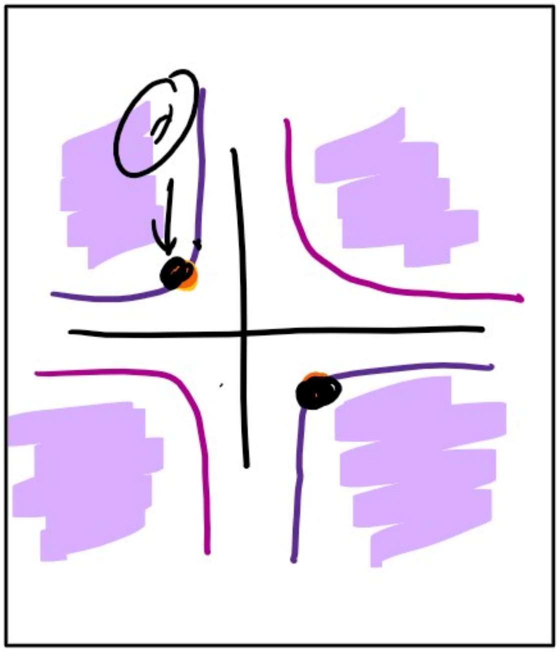


$z_1 z_2 = 0$

smooth $\longrightarrow$



$z_1 z_2 = \epsilon$



$X \setminus \tilde{D} = \mathbb{D}^* T^2 \cup$ 2handle attached along an $S^1 \times \mathbb{D}^2$ to $\partial \mathbb{D}^* T^2 = S^* T^2$

Note

$$\begin{tikzcd} S^1 \times \mathbb{D}^2 \arrow[r, hook] \arrow[d] & S^* T^2 \arrow[d] \\ S^1 \arrow[r, hook] & T^2 \end{tikzcd}$$

$\gamma_{(a,b)}$ curve

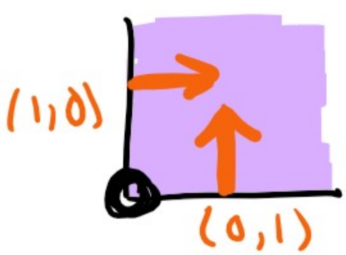
$\Lambda_{a,b}$:= conormal lift of this $\gamma_{a,b}$ curve $= \{(q,p) \in S^* T^2 \mid q \in \gamma_{a,b},\ p(v) = 0 \ \ v \in T\gamma_{a,b}\}$

THM [ACSGMMSW]

The complement of a divisor smoothed at a node has a Weinstein structure

& $\mathbb{D}^* T^2 \cup \Lambda_{a,b}$

where (a,b) = difference of the inward normals at the node

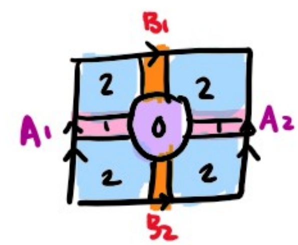


$(a,b) = (1,-1)$

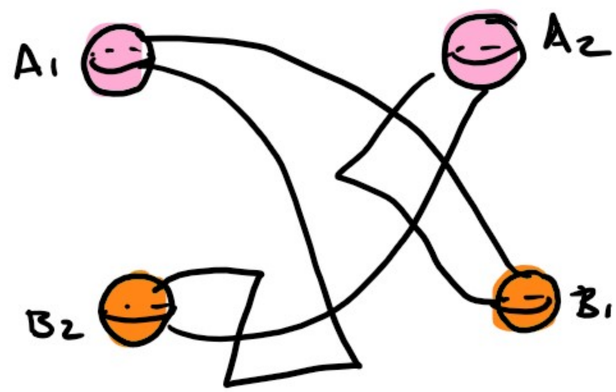
Q What is $\Lambda_{a,b}$?

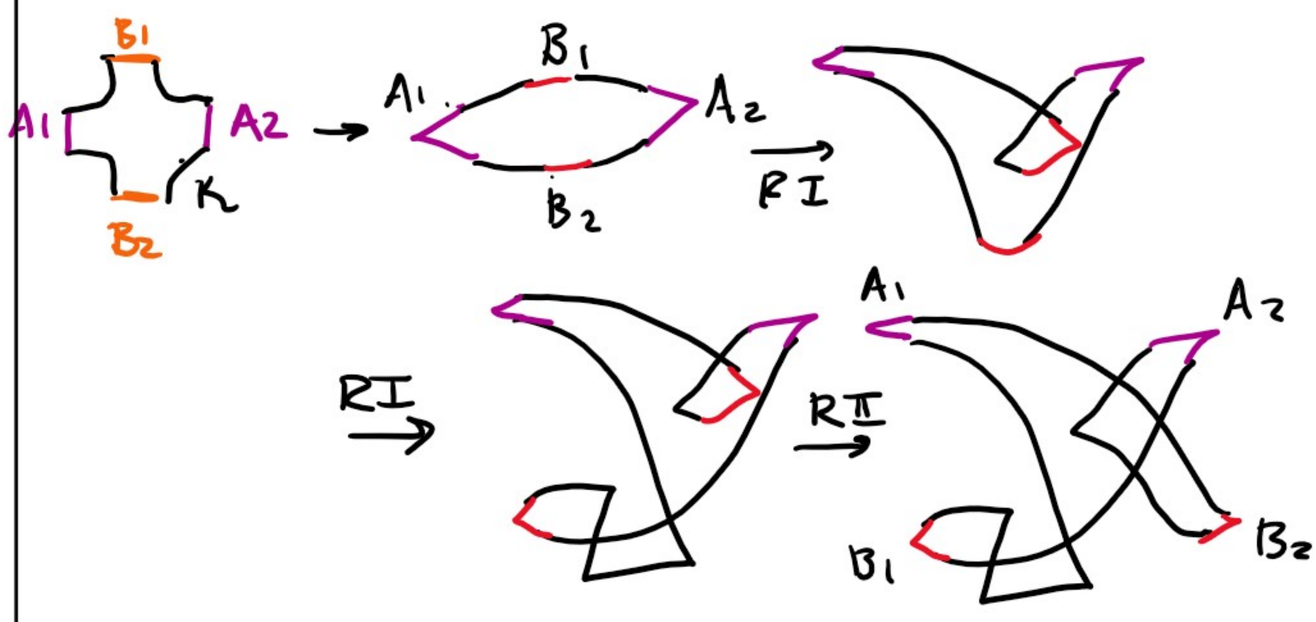
Build $\mathbb{D}^*T^2$

Gompf handlebody for T^*T^2:



T^2



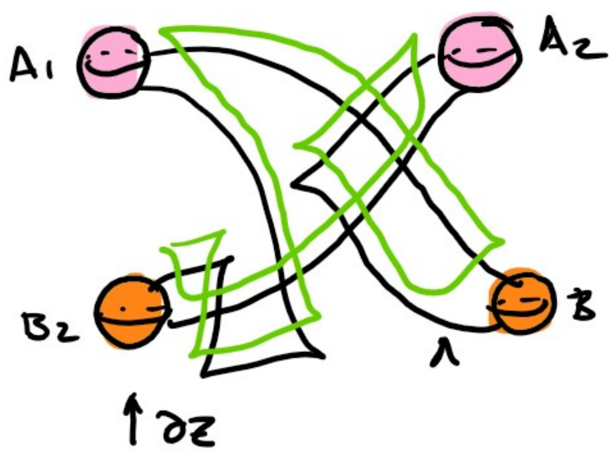


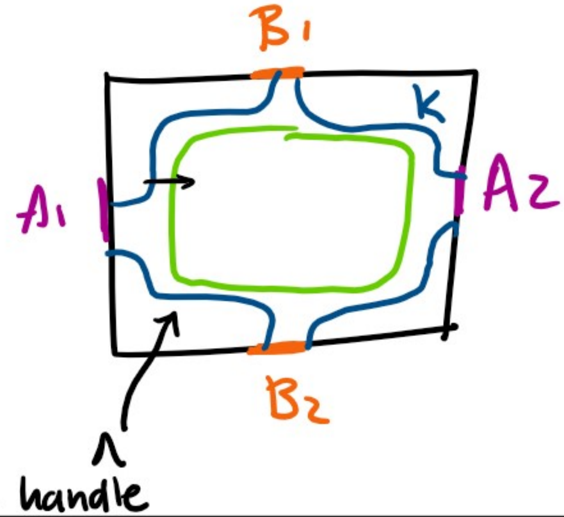
THM [AESGMMSW]
(X, ω, Z) Liouville mfld whose lagrangian skeleton is a smoothed closed lagrangian $L \Rightarrow T^*L = (X, \omega, Z)$

THM [AC-S GMMSW]
The Gompf handlebody of $T^*\Sigma_g$ induces the canonical stein structure on $T^*\Sigma_g$

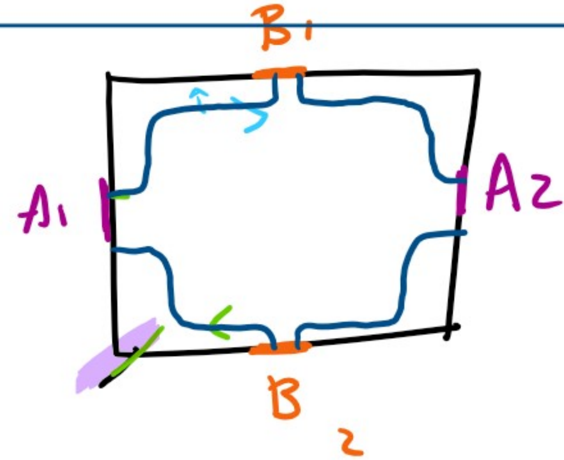
(Use $\Sigma_g = T^2$ here)

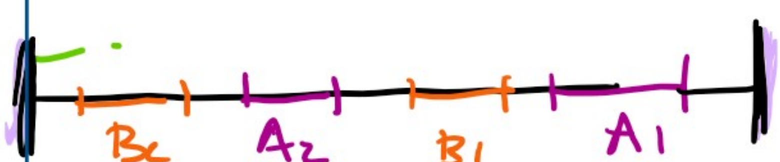
- the conormal lift of a level set $\phi = c$ positive Reeb pushoff of $\Lambda = \{p_1 = 0 = p_2, \phi(q_1, q_2) = c\}$



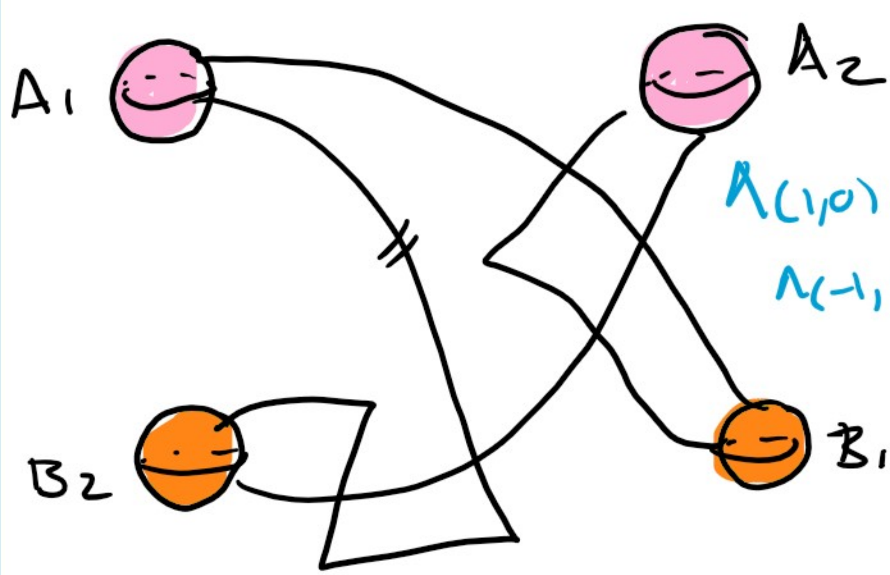


Strategy: isotope $\gamma_{a,b}$ close to Λ & use a contactomorphism $J^1(S^1) \longrightarrow U \subseteq S^*T^2$
U nbd of Λ



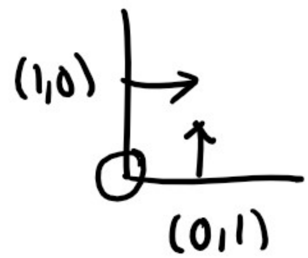


$J^1(S^1) = T^*S^1 \times \mathbb{R}$

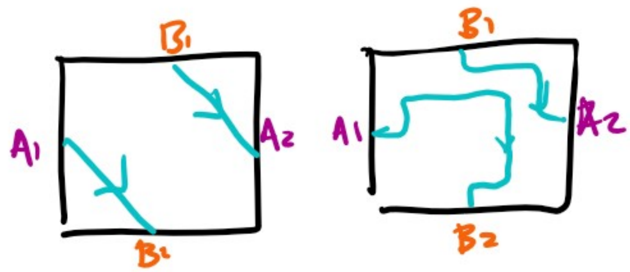


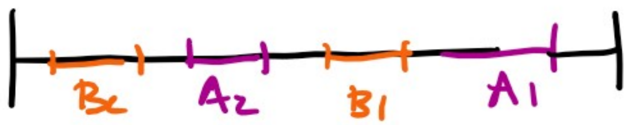
$\Lambda_{(1,0)}$
$\Lambda_{(-1,0)}$

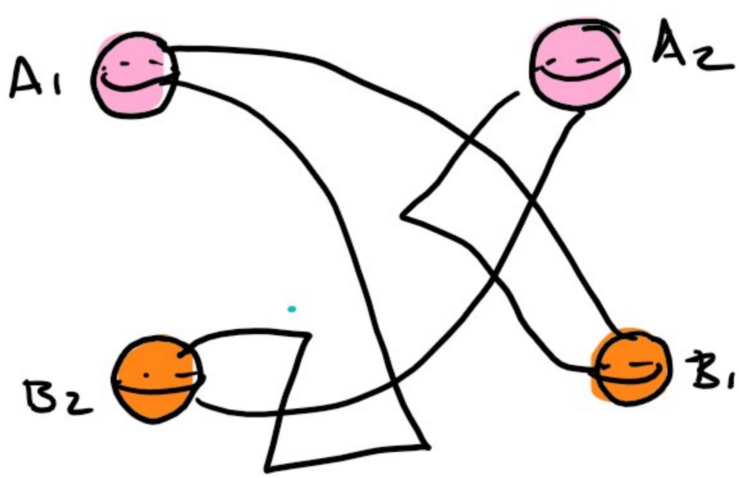
EX smooth once

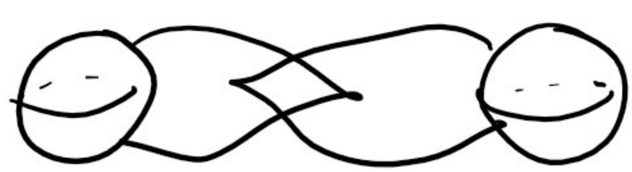


$\Lambda_{(1,-1)}$

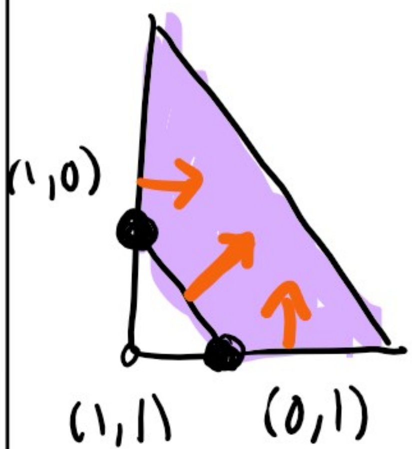






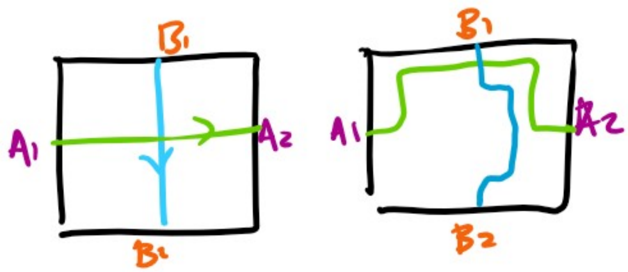
$\sim$ 

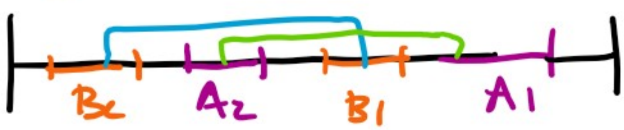
Smooth twice

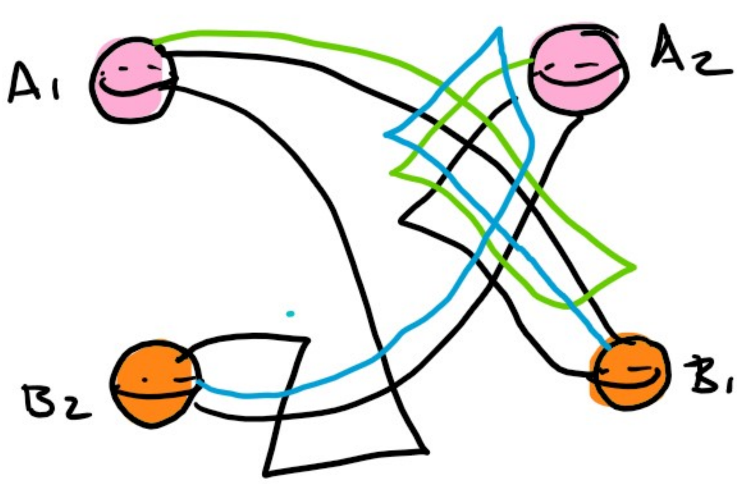


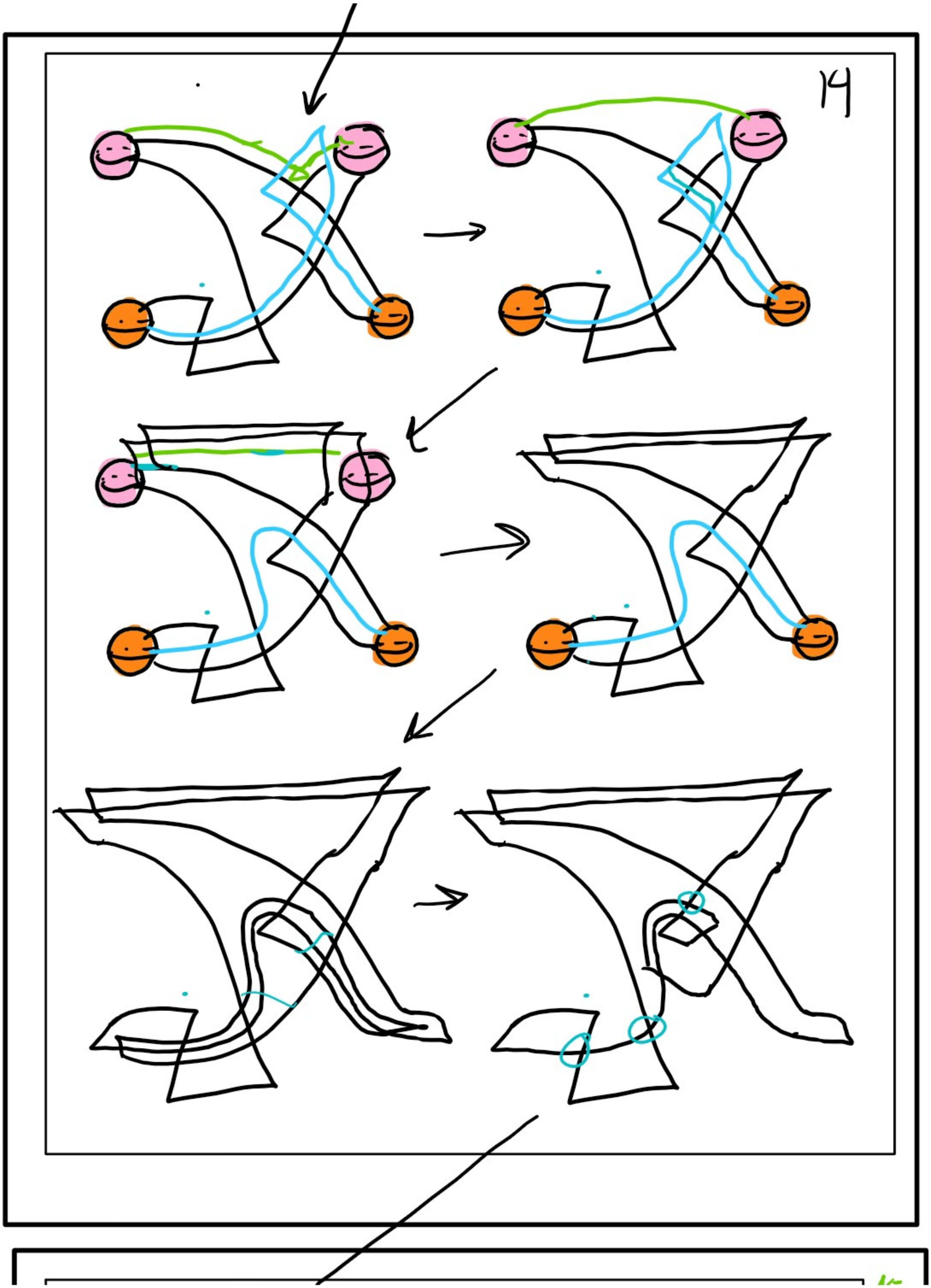
$\Lambda_{(1,0)}$ $\Lambda_{(0,1)}$

$\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$





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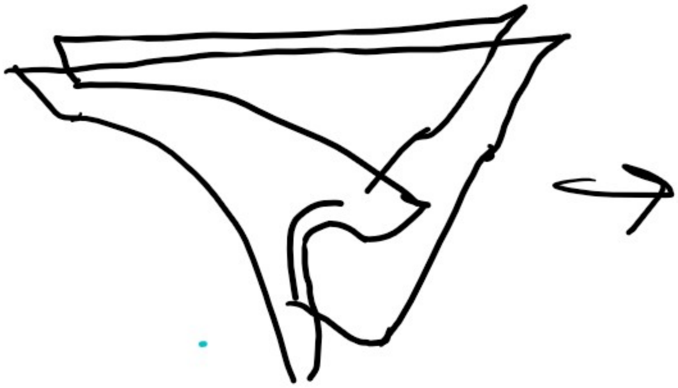
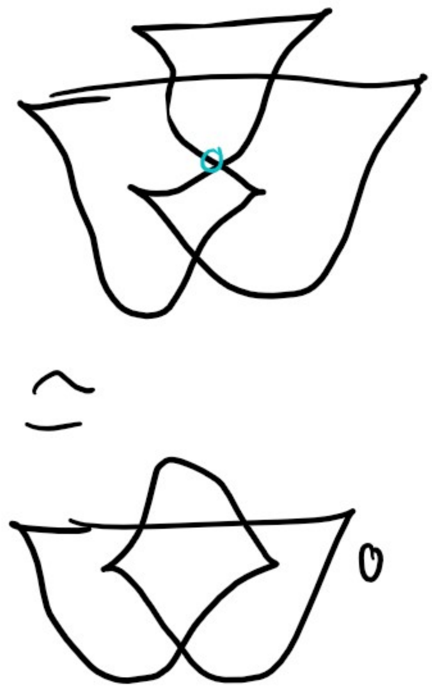




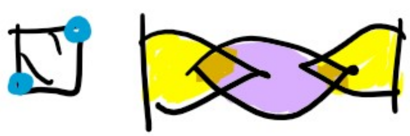
Trefoil!

Further questions:

☐ If you smooth
$\begin{pmatrix} p_1 & \cdots & p_n \\ q_1 & \cdots & q_n \end{pmatrix}$ nodes vs $\begin{pmatrix} p_1' & \cdots & p_n' \\ q_1' & \cdots & q_n' \end{pmatrix}$
when do you produce symplectomorphic complements?

$(\Delta) \xrightarrow{SL_2\mathbb{Z}} (\Delta)$

☐ What lagrangian spheres are not isotopic in $X \setminus \tilde{D}$ that <u>are</u> isotopic in X?



☐ What does the Chekanov Eliashberg DGA of tell us about MS of toric mflds?