Framings of Links in 3-manifolds and Torsion in Skein Modules

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Abstract

We show that the only way of changing the framing of a link by ambient isotopy in an oriented 3-manifold is when the manifold admits a properly embedded non-separating S^2. This change of framing is given by the Dirac trick, also known as the light bulb trick. The main tool we use is based on McCullough's work on the mapping class groups of 3-manifolds. We also express our results in the language of skein modules. In particular, we relate our results to the presence of torsion in the framing skein module.

Introduction

A skein module is an algebraic object associated to an oriented 3-manifold, usually constructed as a formal linear combination of embedded (or immersed) submanifolds, modulo locally defined relations. While choosing these relations one takes into account the following factors:

- whether the module we obtain is computable,
- how well the skein module distinguishes between different 3-manifolds and different links in the 3-manifolds,
- whether the skein module reflects the topology or geometry of a 3-manifold (for example, the existence of incompressible or non-separating surfaces in a manifold, or geometric decomposition of a manifold), and
- whether the skein module admits some additional structure (for example, filtration, gradation, multiplication, Hopf algebra structure, or categorification).

Torsion in Skein Modules

- In 1995, Przytycki showed that the the framing signed skein module (q-deformation of the fundamental group) has torsion if the manifold contains non-separating tori and 2-spheres.
- In 1996, Przytycki showed that the presence of non-separating closed surfaces in an oriented 3-manifold yields torsion in the q-homology skein module.
- Goal: Understand the structure of the framing skein module and check whether it detects the presence of non-separating surfaces in an oriented 3-manifold.

Outline of the Talk

- Preliminaries
- Homeomorphisms which are Dehn twists on the boundary
- Main result
- Main theorem for the framing skein module
- Future directions (time permitting)

Preliminaries

Definition (Mapping Class Group)

Let M be an orientable manifold and $Homeo(M)$ denote the space of PL orientation preserving homeomorphisms of M. Then the **mapping class group** of M, denoted by $\mathcal{H}(M)$, is defined as the space of all ambient isotopy classes of $Homeo(M)$.

Definition (Dehn Homeomorphisms)

Let $(F^{n-1} \times I, \partial F^{n-1} \times I) \subset (M^n, \partial M^n)$, where F is a connected codimension 1 submanifold, and $(F \times I) \cap \partial M = \partial F \times I$. Let $\langle \phi_t \rangle$ be an element of $\pi_1(Homeo(F), 1_F)$, that is, for $0 \leq t \leq 1$, ϕ_t is a continuous family of homeomorphisms of F such that $\phi_0 = \phi_1 = 1_F$. Define a **Dehn homeomorphism** as $h \in \mathcal{H}(M)$ by:

$$h = \begin{cases} h(x,t) = (\phi_t(x), t) & \text{if } (x,t) \in F \times I \\ h(m) = m & \text{if } m \notin F \times I \end{cases}$$

Examples of Dehn Homeomorphisms

F	$\pi_1(Homeo(F))$	Dehn homeomorphism
$S^1 \times S^1$	$\mathbb{Z} \times \mathbb{Z}$	Dehn twist about a torus
$S^1 \times I$	$\mathbb{Z}$	Dehn twist about an annulus
D^2	$\mathbb{Z}$	twist
S^2	$\mathbb{Z}/2\mathbb{Z}$	rotation about a sphere
$\mathbb{R}P^2$	$\mathbb{Z}/2\mathbb{Z}$	rotation about a projective plane
Klein bottle	$\mathbb{Z}$	Dehn twist about a Klein bottle
Möbius band	$\mathbb{Z}$	Dehn twist about a Möbius band
$\chi(F) < 0$	$\{0\}$	

Homeomorphisms which are Dehn twists on the boundary

Theorem (McCullough, 2006)

Let M be a compact orientable 3-manifold which admits a homeomorphism which is Dehn twists on the boundary about the collection $C_1, \ldots, C_n$ of simple closed curves in ∂M. Then for each i, either C_i bounds a disk in M, or for some $j \neq i$, C_i and C_j cobound an incompressible annulus in M.

Corollary

Let M' be a compact oriented 3-manifold with one of the boundary components of M' being a torus. We denote this torus by $\partial_0 M'$. Let $\widetilde{f} : M' \longrightarrow M'$ be a homeomorphism which acts nontrivially on $\partial_0 M'$ and is constant on $\partial M' \setminus \partial_0 M'$. Then $\partial_0 M'$ has a compressing disk.

Homeomorphisms which are Dehn twists on the boundary

Theorem (McCullough, 2006)

Let M be a compact orientable 3-manifold which admits a homeomorphism which is Dehn twists on the boundary about the collection $C_1, \ldots, C_n$ of simple closed curves in ∂M. Then there is a collection of disjoint imbedded disks and annuli in M, each of whose boundary circles is isotopic to one of the C_i, for which some composition of Dehn twists about these disks and annuli is isotopic to h on ∂M.

Corollary

Let M' be a compact oriented 3-manifold with some boundary components, say, $\partial_1(M'), \ldots, \partial_k(M')$ being tori. Let $\widetilde{f} : M' \longrightarrow M'$ be a homeomorphism which acts nontrivially on every $\partial_i(M')$ and which is the identity on $\partial M' \setminus \bigcup_i \partial_i(M')$. Then either $\partial_i(M')$ has a compressing disk, say D_i^2, or there is some $j \neq i$ such that there is an incompressible annulus $Ann_{i,j}$ with one boundary component on $\partial_i(M')$ and the other on $\partial_j(M')$. Furthermore, one can take these disks and annuli to be disjoint. Also, $\widetilde{f}$ restricted to $\partial M'$ is ambient isotopic to the composition (of some powers) of the Dehn homeomorphism along D_i^2 and $Ann_{i,j}$.

Main Results

Theorem (B. - Ibarra - Montoya-Vega - Przytycki - Weeks, 2020)

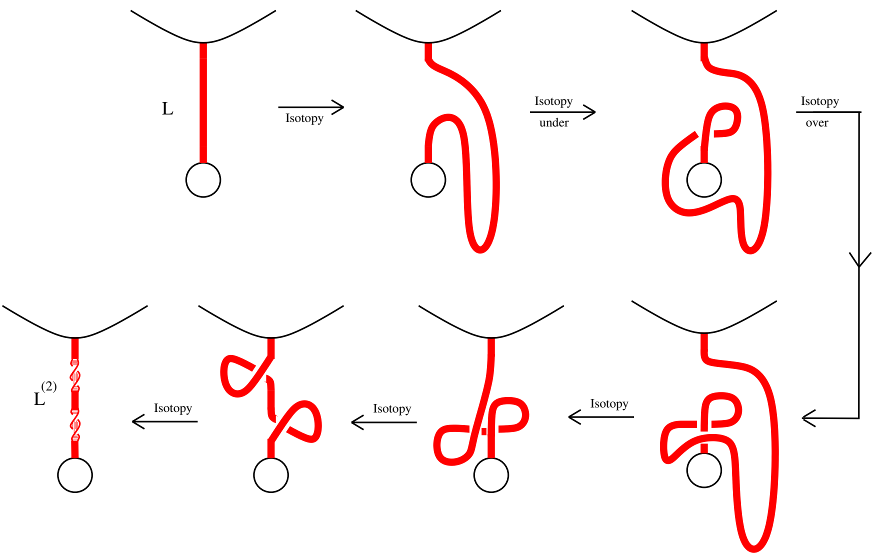
Let L be a framed link in a compact oriented 3-manifold M. The only way of changing the framing of L by ambient isotopy while preserving the components of L is when M has a properly embedded non-separating 2-sphere and either:

(i) *L intersects the non-separating 2-sphere transversely exactly once, or*

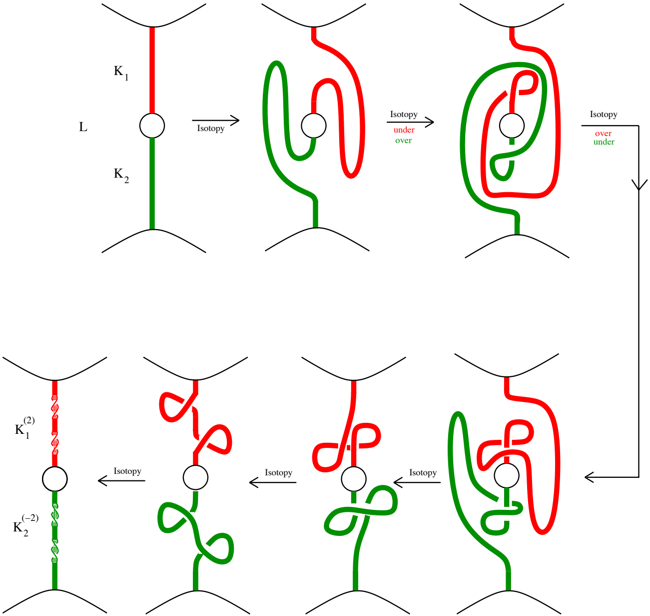
(ii) *L intersects the non-separating 2-sphere transversely in two points, each point belonging to a different component of L.*

The framing is changed by a composition of even powers of the Dehn homeomorphisms along the disjoint union of S_j^2, where S_j^2 satisfy conditions (i) or (ii).

The Light Bulb Trick



The Light Bulb Trick For Two Components of a Link



Spin Structure

Definition

A 3-manifold M is parallelizable if the tangent bundle of M is trivial, that is, there are three vector fields V_1, V_2, and V_3 which form a basis at every tangent space.

Theorem (Stiefel, 1935)

Every orientable 3-manifold is parallelizable.

- Homotopy classes of parallelizations can be identified with spin structures and spin structures form an affine space over the $\mathbb{Z}_2$-linear space $H^1(M, \mathbb{Z}_2)$.
- To every framed knot $K \subset M$, which represents an element of $H_1(M, \mathbb{Z}_2)$, we associate an element of $\pi_1(SO(3)) = \mathbb{Z}_2$ using the parallelization of M.
- By the universal coefficient theorem, $H^1(M, \mathbb{Z}_2) \cong Hom(H_1(M), \mathbb{Z}_2)$.

Non-Compact 3-Manifolds

Corollary

Our main results also hold for non-compact oriented 3-manifolds.

Proof.

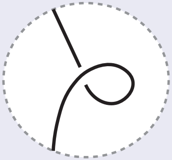
The ambient isotopy of M, which changes the framing of L can be taken to have support on a finite number of 3-balls in M.[a] Thus, the new ambient isotopy has a compact oriented 3-manifold as a support and the result follows. $\square$

[a] Hudson showed that if C is a compact subset of a manifold M and $F : M \times I \longrightarrow M$ is an ambient isotopy of M then there is another ambient isotopy $\hat{F} : M \times I \longrightarrow M$ such that $F_0 = \hat{F}_0$, $F_1 \setminus C = \hat{F}_1 \setminus C$ and there exists a number N such that the set $\{x \in M \mid \hat{F} \setminus \{x\} \times (k/N, (k+1)/N) \textit{ is not constant}\}$ sits in a ball embedded in M.

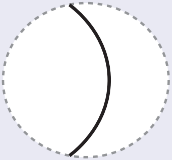
The Framing Skein Module

Definition

Let M be an oriented 3-manifold and $\mathcal{L}^{fr}$ the set of unoriented framed links in M up to ambient isotopy. Let S^{fr} be the submodule of the module $\mathbb{Z}[q^{\pm 1}]\mathcal{L}^{fr}$ generated by framing expressions of the form:



$- q$



$L^{(1)}$ $\qquad\qquad$ L

The **framing skein module** of M is defined as the quotient:

$$\mathcal{S}_0(M, q) = \mathbb{Z}[q^{\pm 1}]\mathcal{L}^{fr}/S^{fr}.$$

Main Result

Theorem (B. - Ibarra - Montoya-Vega - Przytycki - Weeks, 2020)

The framing skein module of an oriented 3-manifold M detects the presence of non-separating 2-spheres in M. In particular, $\mathcal{S}_0(M, q) = \mathbb{Z}[q^{\pm 1}]\mathcal{L}^f \oplus \bigoplus_{L \in (\mathcal{L}^{fr} \setminus \mathcal{L}^f)} \frac{\mathbb{Z}[q]}{q^2-1}$*, where* $\mathcal{L}^f$ *is composed of links which do not intersect any 2-sphere in M transversely at exactly one point.*

Proof of Theorem

(i) If L intersects a non-separating sphere S_i^2 transversely in exactly one point, say by component K_i, then τ_i^2 is ambient isotopic to the identity and twists the framing of K_i by two full twists, and thus also the framing of L.
$K_i^{(1)} - qK_i \equiv 0 \iff K_i^{(2)} \equiv q^2 K_i \iff \tau_i^2(K_i) \equiv q^2 K_i \iff$
$K_i \equiv q^2 K_i \iff (1-q^2)K_i \equiv 0.$
Therefore, in the framing skein module $(q^2-1)L = 0$. We notice that $\mathbb{Z}[q^{\pm}]\mathcal{L}^{fr}$ divided by this relation exactly gives the desired result.

(ii) If exactly two components K_i and K_j of the link L intersect a non-separating 2-sphere $S_{i,j}^2$ in one point each, then $\tau_{i,j}^2$ changes the framing of K_i by 2 and of K_j by -2. Thus, even though this Dehn homeomorphism changes the framing of L, it is invisible in the framing skein module which does not see which component is twisted. Therefore, the twists cancel algebraically in $\mathcal{S}_0(M, q)$.

☺ Thank you for listening ☺