Characterizations of 2 - Homeomorphic Spaces

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Notation

Blanket assumption - all spaces under discussion are Tychonoff (i.e completely regular and Hausdorff).

Note: A space X is **completely regular** if for all closed $A \subset X$ and all $x \in X \setminus A$, there is a continuous $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $f(A) \equiv 0$.

All ordinals have the order topology, and ω is the smallest infinite ordinal

Also, if X and Y are spaces that are homeomorphic, then we will write $X \approx Y$.

Definition of 2 - Homeomorphism

Definition

(Arhangel'skii and Maksyuta, 2018)
Let X and Y be spaces. X and Y are **2 - homeomorphic** (denoted $X \overset{2}{\approx} Y$) if there exist H_X closed in X and H_Y closed in Y such that $H_X \approx H_Y$, and if we let $U_X = X \setminus H_X$ and $U_Y = Y \setminus H_Y$, then $U_X \approx U_Y$.

Note: Unlike homeomorphism, 2 - homeomorphism is not an equivalence relation on a collection of spaces - it has **major** transitivity problems.

One Avenue of Research

Problem

What are the spaces X that are 2 - homeomorphic to some space Y that has property P?

We first consider compactness. Note that ω and $\omega + 1$ are 2 - homeomorphic, so that compactness is not preserved by 2 - homeomorphism (spoiler alert: not much is!).

A Compactness Result

Theorem

A space X is 2 - homeomorphic to some compact space Y if and only if $S_X = \{p \in X : X$ is not locally compact at $p\}$ is compact.

Note: Tychonoff spaces are exactly the spaces which can be embedded into compact Hausdorff spaces - in particular, if a space X can be embedded densely into a compact Hausdorff space Y, we call Y a **compactification** of X. (e.g. $\omega + 1$ is a compactification of ω).

For a given space X, in the collection of all compactifications of X there is a maximal element (in the sense of embeddings, at least), which is called the **Stone - Čech compactification** of X, and is denoted βX.

A Neat Equivalence

If X is a non-compact space, then βX will contain points not in X. In other words, the set $X^* \equiv \beta X \setminus X \neq \emptyset$, where X^* is called the **Stone - Čech remainder** of X.

Theorem

A space X is 2 - homeomorphic to some compact space Y if and only if X^* is locally compact.

Scattered Spaces

Definition

A space X is **scattered** if every non-empty subset of X contains an isolated point of X.

Definition

Let X be a scattered space. The **Cantor - Bendixson Height** of X (denoted $CB(X)$) is defined as follows:

1. Let $X_0 = \varnothing$
2. For an ordinal $\lambda > 0$, let $X_\lambda =$ the set of isolated points of $X \setminus \bigcup_{\mu<\lambda} X_\mu$

Then $CB(X) = \sup\{\lambda : X_\lambda \neq \varnothing\}$.

Discrete Spaces, and a Rare Preserved Property

Lemma

If $X \overset{2}{\approx} Y$, and X is scattered, then Y is scattered.

Note: This is true because being scattered is a rare property that is inherited by both subspaces **and** finite unions.

Theorem

Let X be a discrete space. $Y \overset{2}{\approx} X$ if and only if Y is scattered with $CB(Y) \leq 2$, and $|Y| = |X|$.

Three Scattered Spaces

Note: Our previous example of ω and $\omega + 1$ shows that being a discrete space is not preserved by 2 - homeomorphism. If we look at ω and $\omega \cup \{p\}$, where p is a point in ω^*, we see that neither weight nor character are preserved by 2 - homeomorphism. We also see that 2 - homeomorphic spaces that are not homeomorphic can have homeomorphic Stone - Čech compactifications.

A Question About Scattered Spaces

A discrete space can only "jump" one level under 2 - homeomorphism - does this hold for other scattered spaces?

Problem

If X is scattered with $CB(X) = n$ for some $n \in \omega$, and Y is a (necessarily scattered) space such that $Y \overset{2}{\approx} X$, is there a finite upper bound on $CB(Y)$?

Theorem

Let $n \in \omega$. If $CB(X) = n$, and $Y \overset{2}{\approx} X$, then $CB(Y) \leq 2n$.

Note: If X is a scattered space with $CB(X) = n$, then there is no guarantee that X will be 2 - homeomorphic to a scattered space Y with $CB(Y) = 2n$, but we do know that, for any $n \in \omega$, a space exists that does achieve this maximum result.