Methods and Mysteries in the Construction of Ring Spectra (Talbot 2017)

Speaker: Jeremy Hahn
Notes by Ariana Goh and D. Zack Garza

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Abstract

Key to the solution of the Kervaire invariant 1 problem was the construction of a certain 'designer' EO cohomology theory. I will survey the techniques, based on work of innumerable homotopy theorists, used to construct BP, tmf, $K(n)$, EO and other designer, structured ring spectra. I will end with recent work building an E3 form of BP $\langle n \rangle$, joint with Dylan Wilson, and many open questions.

1 Part 1

1.1 Setup: structured ring spectra

Remark 1.1.1. The 2017 Talbot was on structured ring spectra, i.e. "brave new algebra", where we study $\mathbb{E}_n$-ring spectra. The foundations of this field are taken care of by May, EKMM, Lurie, and many more. One may check out the survey Commutative ring spectra by Birgit Richter.

Remark 1.1.2. The key objects we'll be considering:

- $\mathbb{S}$, the initial $\mathbb{E}_\infty$-ring spectrum, whose homotopy groups record the stable homotopy groups of spheres.
- Thom $\mathbb{E}_\infty$-ring spectra: $\mathrm{MO}, \mathrm{MSO}, \mathrm{MSpin}, \mathrm{MString}, \mathrm{MU}$, etc. The sphere fits into this pattern as framed bordism.

We construct *other* $\mathbb{E}_\infty$-rings primarily to study these motivating examples.

Example 1.1.3. The following is a basic example of this paradigm of studying these bordism ring spectra using other ring spectra. One that shows up naturally is the $\mathbb{E}_\infty$-ring MU, for which

$$\pi_*\mathrm{MU} \cong \mathbb{Z}[x_1, x_2, \cdots] \qquad \text{where } |x_i| = 2i,$$

a polynomial ring with infinitely many generators in even degrees. After localizing at a prime p, the localized spectrum $\mathrm{MU}_{(p)}$ splits into sum of suspensions of BP where

$$\pi_*\mathrm{BP} = \mathbb{Z}_{(p)}[v_1, v_2, \cdots] \qquad \text{where } |v_i| = 2p^i - 2,$$

where $\mathbb{Z}_{(p)}$ denotes the p-local integers. A modern way [1] of constructing BP is as the quotient

$$\mathrm{BP} = \frac{\mathrm{MU}_{(p)}}{\left\langle x_j \;\middle|\; j \neq p^i - 1 \right\rangle}.$$

BP is a bit of a "designer" spectrum that is not as geometric in origin as MU, but after p-localizing we see that BP cohomology is essentially equivalent to MU

[1] BP was first constructed by hand via obstruction theory by Brown and Peterson.

cohomology. Moreover BP^* is easier to work with, as it has simpler homotopy groups with fewer generators in fewer degrees. Since MU^* splits into a sum of copies of BP^*, an advantage of working with BP is that formulas become more manageable and concrete in many cases.

1.2 Understanding BP

One aspect of MU that distinguishes it from an arbitrary cohomology theory is that MU^* admits power operations, i.e. MU is an $\mathbb{E}_\infty$-ring spectrum.

Question 1.2.1. Can the power operations on MU^* be accessed using BP^*? I.e., is BP and $\mathbb{E}_\infty$ ring spectrum, and does the splitting of MU into BP summands preserve this structure?

Answer 1.2.2. No! See Lawson for $p = 2$ and Senger for odd primes, who prove that BP not an $\mathbb{E}_{2p^2+4}$-ring spectrum. As a result, we can't understand all power operations in MU^* just using BP^*. However,it turns out that understanding the power operations that come from the $\mathbb{E}_4$ structure is sufficient for many applications:

Theorem 1.2.3 (Basterra, Mandell)**.** BP is an $\mathbb{E}_4$-algebra retract of MU.

Remark 1.2.4. There is a general paradigm here: when studying $\mathbb{E}_\infty$-ring spectra, one is often lead to study other (less canonical) ring spectra built by obstruction theory, and there are subtle relationships between geometric spectra like MU and designer spectra like BP that respect only part of the $\mathbb{E}_\infty$ structure.

Question 1.2.5 (Some open questions)**.**

1. Is BP an $\mathbb{E}_5$-MU-algebra? More generally, what is the exact structure here?[2]
2. While BP is an $\mathbb{E}_2$-algebra, is it a Disk_2-algebra? [3] In particular, can one take factorization homology (i.e. integrate) against unframed manifolds of dimension two or higher?[4]

1.3 Chromatic homotopy

Remark 1.3.1. Studying other $\mathbb{E}_\infty$ rings naturally leads to problems in obstruction theory. Perhaps the most important $\mathbb{E}_\infty$ ring is $\mathbb{S}$, which we study via chromatic homotopy theory. The basic strategy is the following:

1. Resolve $\mathbb{S}$ by other $\mathbb{E}_\infty$ rings, namely the $K(n)$-local spheres $L_{K(n)}\mathbb{S}$.

   That the Morava K-theories $K(n)$ exist, are $\mathbb{E}_1$-rings, and that one can take take the Bousfield localization at $\mathbb{S}$ and obtain an $\mathbb{E}_\infty$ring. This requires a great deal of foundational theory to set up, which has thankfully

[2] Many have thought about this, so it's perhaps not the best entry point for a new researcher!

[3] Any knowledge of how to put such an algebra structure on a ring would be very interesting! This is a less well-studied aspect.

[4] This would correspond to a trivial S^1-action.

been worked out. It is useful precisely because of the chromatic convergence theorem, and one can build a tower whose associated graded is these $K(n)$-local spheres.

2. Resolve $L_{K(n)}\mathbb{S}$ for a fixed n by the Hopkins-Miller EO-theories, which are $\mathbb{E}_\infty$ rings of the form $\mathbb{E}_n^{hG}$ for G a finite group acting on the height n Morava E-theory $\mathbb{E}_n$. These EO theories coming from finite group actions are supposed to be the basic building blocks of the $K(n)$-local spheres and in turn $\mathbb{S}$.

Question 1.3.2 (Big question, subject of active work)**.** In general, can one construct the resolutions in (2) above? In other words, can one always resolve the $K(n)$-local sphere by these Hopkins-Miller EO theories?

Remark 1.3.3. By work of Goerss-Henn-Mahowald-Rezk, this can be done at the prime 3 in height 2 (and some other explicit examples), and it's expected that this is generally possible.

A recent triumph of obstruction theory is that the building block EO-theories have been built. These "designer homotopy types" are not easy to build, and a large part of the 2017 Talbot was explaining how one constructs these building blocks of the $K(n)$-local spheres.

1.4 Building EO-theories

Remark 1.4.1. How one builds EO theories:

- The EO-thories are supposed to come from group actions on height n Morava E-theory, so build E_n as a homotopy commutative ring. This can be done using the Landweber exact functor theorem, which is not too difficult, but this is still a far cry from an $\mathbb{E}_\infty$-ring.
- Promote E_n to an $\mathbb{E}_\infty$ring using obstruction theory.
  - See Robinson, Goerss-Hopkins, Lurie, Pstragowski and VanKoughnett. In particular, Goerss-Hopkins construct G-actions (by finite groups G) by $\mathbb{E}_\infty$ring maps, and taking homotopy fixed points for these actions that yield the EO-theories.
  - The process is roughly the following: build an $\mathbb{E}_\infty$ ring in the homotopy category of spectra Sp, which is by definition a homotopy commutative ring. Then do this in the homotopy 2-category of Sp, then the homotopy 3-category of Sp, and so on.

    This uses that an ∞-category is a sequence of n-categories, and one checks if there is any obstruction to lifting this one layer of homotopy coherence at a time.

Question 1.4.2 (Another big question)**.** Can one compute $\pi_*\mathrm{EO}$ for $\mathrm{EO} \coloneqq \mathbb{E}_n^{hG}$ for various n and G?

Example 1.4.3. The key to the Kervaire invariant one question is computing $\pi_*\mathbb{E}_4^{hC_8}$, and captures information about diffeomorphism classes of exotic spheres.

Such computations are related to unsolved problems in obstruction theory, despite the fact that it seems as though the obstruction theory is "done" in the sense that we've already built these EO-theories.

Observation 1.4.4. In practice, these EO theories (which are all $K(n)$-local by definition) seem to be $K(n)$-localizations of nice connective ring spectra.

Example 1.4.5. At the prime $p = 2$, it turns out that

$$E_1^{hC_2} = \mathrm{KO}_2 = L_{K(1)}(\mathrm{ko}).$$

As a result, one doesn't necessarily need to 2-complete KO, since it arises naturally as the $K(1)$-localization of something else.

Similarly,

$$E_2^{hG_{24}} = L_{K(2)}(\mathrm{tmf}),$$

where tmf exists before localizing and is even connective. In light of these facts, we can regard ko and tmf as "connective E_∞ lifts" of $E_1^{hC_2}$ and $E_2^{hG_{24}}$.

Remark 1.4.6. These lifts are closer to geometry than EO theories – there is an $\mathbb{E}_\infty$ ring map due to Ando-Hopkins-Rezk

$$\mathrm{MString} \to \mathrm{tmf}.$$

This lands in tmf , which is connective, instead of (say) its $K(2)$ localization. Moreover, since MString is itself a bordism $\mathbb{E}_\infty$-ring, it is also connective.

One major goal in this area would be to understand MString and (for example) the direct sum decomposition of its 2-completion. We expect that it should involve summands that look like tmf. Anderson, Brown and Peterson[5] proved that 2-localized MSpin splits into copies of ko and Eilenberg-MacLane spectra HG, so one might expect that things like tmf are direct summands of geometrically defined bordism $\mathbb{E}_\infty$-rings.

Observation 1.4.7 (due to Hu-Kriz and Hill-Hopkins-Ravenel)**.** Using sparsity, the easiest way to compute $\pi_*\mathrm{EO}$ is to compute $\pi_*\mathrm{eo}$ where eo is a good connective lift of EO.

Note that $\pi_*\mathrm{tmf}$ is finitely-generated in each degree, and it's useful to work with something "small" in computations – for example if you're trying to rule out differentials in spectral sequences, it's advantageous to have fewer spurious elements around that could support differentials. A nice way to organize the computations of $\pi_*\mathrm{EO}$ is to understand them as the localizations of more canonical lifts with better finiteness properties.

[5] See A-B-P 66, A-B-P 67

1.5 Connective lifts

Question 1.5.1. Can one make highly structured connective eo-theories with

$$L_{K(n)}\mathrm{eo} = \mathrm{EO} = \mathbb{E}_n^{hG}$$

for various and n and G? In other words, can we produce lifts like tmf rather than just the localization $L_{K(2)}\mathrm{tmf}$?

This is a fascinating question, and there have been varying levels of success constructing these.

Question 1.5.2. Noting that $\mathrm{tmf}_{(2)}$ is a connective eo-theory, how is it built as an $\mathbb{E}_\infty$-ring?

Remark 1.5.3. For full details, see the tmf book or Lurie's Elliptic Cohomology. We'll just discuss a sketch here. We can write 2-localized tmf as a truncation

$$\mathrm{tmf}_{(2)} = \tau_{\geq 0} L_2 \mathrm{tmf}_{(2)},$$

where the latter is built out of a finite resolution involving involving the chromatic localizations

- $L_{K(2)}\mathrm{tmf}$
- $L_{K(1)}\mathrm{tmf}$
- $L_{\mathbb{Q}}\mathrm{tmf}$

Each of these terms individually is understandable, since we can use obstruction theory to make things $K(1)$ or $K(2)$ locally (or rationally), but the gluing procedure that requires more work. Here L_2 is the second stage of the chromatic tower, obtained by gluing together the above three objects. The basic strategy is to take the monochromatic pieces above, which are relatively easy to make and work with, and find a way to glue them together.

Remark 1.5.4. A general idea pursued in many examples by Lawson is that one can try to make eo as a connective cover of some L_n-local object. [6] Moreover, the most satisfactory construction of these eo-theories would be some program like this.

This worked very well for tmf, which is height 2, and there is currently partial progress at height 3. In particular, this program is not yet able to construct a connective version of $\mathbb{E}_4^{hC_8}$, which was needed in the Kervaire invariant one problem.

Part of the issue with this idea is that all of the techniques used here seem to work equally well for building $\mathbb{E}_\infty$ as $\mathbb{E}_n$ rings for any particular finite n – so for example, it's no harder to build tmf as an $\mathbb{E}_\infty$-ring than an $\mathbb{E}_1$-ring.

At heights beyond 3, the work of Lawson and Senger suggests that it is hard to build $\mathbb{E}_\infty$ rings and may be more reasonable to build $\mathbb{E}_4$ rings (e.g. BP). If

[6] See Lawson, Berhrens-Lawson and TAF (topological automorphic forms).

these techniques were to work at all, they're designed to build $\mathbb{E}_\infty$ rings, but necessarily not the seemingly easier to build $\mathbb{E}_4$-rings.

Remark 1.5.5. An alternate idea that let Hill-Hopkins-Ravenel solve Kervaire Invariant One, and recently developed by Beaudry-Hill-Shi-Zeng, constructs a connective version of $\mathbb{E}_4^{hC_8}$, which was a holy grail making it possible to do computations. However, with this construction, it's less clear how much structure there is on the object, which is a reasonable open question.

They construct it using the following procedure:

- Put a C_8 action on $\mathrm{MU}^{\otimes 4}$ by viewing this as a norm $N_{C_2}^{C_8}\mathrm{MU}_{\mathbb{R}}$

  The norm here gives a way of lifting a C_2 action on one tensor factor to a C_8 action on 4 tensor factors.

- Quotient by some elements, possibly losing structure, to obtain a quotient Q, where the connective lift of $E_4^{hC_8}$ is Q^{C_8}.

This uses the C_8 action from above, and one just needs to check that taking the quotient preserves this action. Note that these quotients can be destructive when it comes to maintaining $\mathbb{E}_n$ ring structures. [7] The fact that Q actually is a connective cover of $E_4^{hC_8}$ involves choosing elements judiciously.

1.6 Open questions and research directions

Question 1.6.1. Some natural questions that arise here:

- What group actions (with various amounts of structure, e.g. $\mathbb{E}_\infty$) act on tensor powers $\mathrm{MU}^{\otimes m}$?

- What structure exists on quotients of such tensor powers?

Remark 1.6.2. If one answers these, one would have built structured models for connective versions of Hopkins-Miller theories. As per a note by Ravenel, if one can construct a certain C_3 action on $\mathrm{MU}^{\otimes 2}$, then one could prove the (currently unsolved) 3-primary version of Kervaire invariant one.

More generally, if one really understood these two questions, one could compute fixed points[8] of Morava E-theories. This hasn't been formulated precisely yet due to the following:

Question 1.6.3. Does $\mathrm{MU}^{\otimes m}$ admit any G-actions beyond those which come from norms?

[7] It'd be interesting to know how much structure is lost here!

[8] Here the homotopy fixed points, although the strict fixed points should roughly be the connective cover of the homotopy fixed points.

2 Part 2

2.1 Structure on quotients of BP

Remark 2.1.1. If we take $\mathrm{MU}^{\otimes m}$ with a group action, possibly by $\mathbb{E}_\infty$-ring maps, what's leftover after taking the quotient? Recall that this can be a destructive procedure with respect to maintaining coherent $\mathbb{E}_n$-ring structures.

Question 2.1.2. Recall that BP is an $\mathbb{E}_4$-ring spectrum with

$$\pi_*\mathrm{BP} \cong \mathbb{Z}_{(p)}[x_1, x_2, \cdots].$$

What structure exists on quotients of BP?

Remark 2.1.3. Note that this is only a basic form of the previous questions – this doesn't deal with complications arising from tensor powers of MU or any group actions. Any progress here would lead to many natural next questions, like generalizing tools to take into account group actions. There has been recent progress, some of which is ripe for generalization – for example, by making things equivariant.

Example 2.1.4. Take

$$\mathrm{BP}\langle n\rangle \coloneqq \frac{\mathrm{BP}}{\langle v_{n+1}, v_{n+2}, \cdots\rangle} \implies \pi_*\mathrm{BP}\langle n\rangle \cong \mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_n].$$

Note that this isn't completely well-defined since there are many choices for generators – for example, one could replace v_2 with $v_2 + v_1$ to yield an equally valid presentation.

2.2 Main theorems

Theorem 2.2.1 (Baker-Jeanneret)**.** For any choice of indecomposable generators $v_{n+1}, v_{n+2}, \cdots$, the spectrum $\mathrm{BP}\langle n\rangle$ is an $\mathbb{E}_1$-BP-algebra.

Theorem 2.2.2 (H-Wilson)**.** There exists a specific choice of generators $v_{n+1}, v_{n+2}, \ldots$ such that $\mathrm{BP}\langle n\rangle = \frac{\mathrm{BP}}{\langle v_{n+1}, v_{n+2}, \cdots\rangle}$ has a $\mathbb{E}_3$-BP-algebra structure.

Remark 2.2.3. We'll discuss a bit how this theorem is proved. This is a relatively new result, and it'd be exciting to try to take it and use it in an equivariant setting.

Proposition 2.2.4. If $x \in \pi_{2\ell}\mathrm{BP}$ is any class in $\pi_*\mathrm{BP}$, then $\mathrm{BP}/\langle x\rangle$ is an $\mathbb{E}_1$-BP algebra.

This says you can freely mod out by any generator and still obtain an $\mathbb{E}_1$ structure.

2.3 Proof of proposition

Let $S^0[a_{2\ell}] = S^0 \oplus S^{2\ell} \oplus S^{4\ell} \oplus \cdots$ denote the free $\mathbb{E}_1$-ring on $S^{2\ell}$. There is an $\mathbb{E}_1$ ring map

$$\psi : S^0[a_{2\ell}] \to \mathrm{BP},$$

which hits x, following from the fact that this is a free $\mathbb{E}_1$-ring. We have

$$\mathrm{BP}/\langle x\rangle = \mathrm{BP} \otimes_{S^0[a_{2\ell}]} S^0$$

where we use the augmentation map $\varepsilon : S^0[a_{2\ell}] \to S^0$. In other words, we consider BP as a module over the free $\mathbb{E}_1$-ring $S^0[a_{2\ell}]$ and this tensor product is one definition of what the quotient is, which clearly has the same homotopy groups as $\mathrm{BP}/\langle x\rangle$.

It suffices to prove the following lemma:

Lemma 2.3.1. $S^0[a_{2\ell}]$ and the map ψ can be made $\mathbb{E}_2$.

In other words: the free $\mathbb{E}_1$-ring on an even degree sphere, which is an $\mathbb{E}_1$-algebra but not necessarily $\mathbb{E}_2$, does in fact admit an $\mathbb{E}_2$ structure. Moreover, the map coming from its universal property as a free $\mathbb{E}_1$ ring lifts to an $\mathbb{E}_2$ map.

In this case, we know that BP is not just a module over $S^0[a_{2\ell}]$, but in fact an $\mathbb{E}_1$-algebra over it, which is enough to give the tensor product an $\mathbb{E}_1$ structure.

Proof. (In the case $\ell = 1$):
Let $\mathsf{Free}_{\mathbb{E}_1}(-)$ denote taking the free $\mathbb{E}_1$-ring. We have

$$\begin{aligned} S^0[a_2] &\cong S^0 \oplus S^2 \oplus S^4 \oplus \cdots \\ &\cong \mathsf{Free}_{\mathbb{E}_1}(S^2) \\ &\cong \Sigma^\infty_+ \Omega S^3 \\ &\cong \Sigma^\infty_+ \Omega^2 \mathrm{HP}^\infty && \text{using } S^3 \cong \Omega\mathbb{HP}^\infty, \end{aligned}$$

which produces an $\mathbb{E}_2$ structure on the original free $\mathbb{E}_1$ ring.

For the map, note that there is a particularly simple filtration

$$S^7 \to S^4 \cong \mathrm{HP}^1 \to \mathrm{HP}^2 \to \mathrm{HP}^3 \to \cdot \to \mathrm{HP}^\infty,$$

where the initial S^7 comes from attaching an 8-cell along a 7-sphere. This yields a filtration of $\mathbb{E}_2$-rings

$$\Sigma^\infty_+ \Omega^2 S^4 \cong \Sigma^\infty_+ \Omega^2 \mathrm{HP}^1 \to \Sigma^\infty_+ \Omega^2 \mathrm{HP}^2 \to \cdots \to \Sigma^\infty_+ \Omega^2 \mathrm{HP}^\infty \cong \Sigma^\infty_+ \Omega S^3.$$

To produce a $\mathbb{E}_2$-ring map $\Sigma^\infty_+ \Omega S^3 \to \mathrm{BP}$, one produces maps out of each filtered pieced:

$$
\begin{tikzcd}
\Sigma^\infty_+ \Omega^2 S^4 \arrow[r] \arrow[d] & \mathrm{BP} \\
\Sigma^\infty_+ \Omega^2 \mathrm{HP}^2 \arrow[d] \arrow[ur, dashed, bend right] & \\
\Sigma^\infty_+ \Omega^2 \mathrm{HP}^3 \arrow[d] \arrow[uur, dashed, bend right] & \\
\vdots \arrow[d] & \\
\Sigma^\infty_+ \Omega S^3 \arrow[uuuur, dashed, bend right] &
\end{tikzcd}
$$

Link to Diagram

The top map is easy to produce, using that the domain is a free $\mathbb{E}_2$-algebra.

At each stage, the obstruction to lifting is a map out of a free $\mathbb{E}_2$ algebra on an odd degree class in $\pi_*\mathrm{BP}$. This is because going from HP^2 to HP^3 involves adding a cell in even dimensions, whose boundary is a sphere in odd dimension. Since HP^∞ has an even cell decomposition as a space, $\Sigma^\infty_+\Omega^2\mathrm{HP}^\infty$ has an even cell decomposition as an $\mathbb{E}_2$-algebra. Then at each stage, the obstruction to taking the extension is controlled by an element of $\pi_*\mathrm{BP}$, which is concentrated in even degrees, so these maps can always be lifted. ■

Remark 2.3.2. So these free $\mathbb{E}_1$-rings are secretly $\mathbb{E}_2$ rings (although not free as $\mathbb{E}_2$-rings) which have a simple presentation as an $\mathbb{E}_2$-ring that makes them easy to map into objects with even-degree homotopy. This gives an intuitive idea why we can quotient BP out and still retain an $\mathbb{E}_1$-ring structure.

2.4 Existence of $K(n)$ as an $\mathbb{E}_1$-$\mathbb{S}$-algebra

Remark 2.4.1. To wrap things up, we'll demonstrate a new technique that yields a relatively easy prove that $\mathrm{BP}\langle n\rangle$ can be made $\mathbb{E}_3$.

Theorem 2.4.2. Connective $k(n)$ exists as an $\mathbb{E}_1$-$\mathbb{S}$ algebra.

Remark 2.4.3. We have

$$\pi_* k(n) = \mathbb{F}_p[v_n] \quad \text{where } |v_n| = 2p^n - 2,$$

where $k(n)$ can be thought of as the following quotient:

$$k(n) \cong \frac{\mathrm{BP}}{\langle p, v_1, v_2, \ldots, v_{n-1}, v_{n+1}, v_{n+2}, \ldots \rangle}$$

The relatively simple homotopy groups imply there is a relatively simple Postnikov tower:

$$\begin{tikzcd}
 & & k(n) \arrow[d] & & \\
 & & \vdots \arrow[d] & & \\
v_n^2 & \Sigma^{4p^n-4}\mathbb{F}_p \arrow[r] & \tau_{\leq 4p^n-4}k(n) \arrow[d] & & \\
v_n & \Sigma^{2p^n-2}\mathbb{F}_p \arrow[r] & \tau_{\leq 2p^n-2}k(n) \arrow[d] & & \\
 & & \mathbb{F}_p \arrow[r, "Q_n"] & \Sigma^{2p^n-2}\mathbb{F}_p \arrow[r, "Q_n"] & \Sigma^{4p^n}
\end{tikzcd}$$

Link to Diagram

Here the first two k-invariants are listed at the bottom of the tower, and it turns out that all primary k-invariants in this tower are the Milnor operation Q_n.

How would you know that a Postnikov tower of this form existed if you didn't already know about how it arose as a quotient of BP? How could you build a spectrum with this specific tower? One can use the following procedure:

- To build $\tau_{2p^n-2}k(n)$, one just needs to identify

$$Q_n \in \pi_0 \operatorname{Hom}(\mathbb{F}_p, \Sigma^{2p^n-1}\mathbb{F}_p).$$

  So one needs to identify $Q_n \in \pi_* \operatorname{Hom}(\mathbb{F}_p, \mathbb{F}_p) \cong \pi_*\mathcal{A}_p$, the mod p Steenrod algebra.

- To build $\tau_{\leq 4p^n-4}k(n)$, one needs to check that $Q_n^2 = 0$ in $\mathcal{A}_p$, which is an Adem relation.

Note that understanding $\pi_* \operatorname{Hom}(\mathbb{F}_p, \mathbb{F}_p)$ as a group lets one build $\tau_{\leq 2p^n-2}k(n)$, since it points to a specific element (here Q_n) needed to build a 2-stage Postnikov tower. However, building $\tau_{\leq 4p^n-4}k(n)$ requires knowing $\pi_*\mathcal{A}_p$ as a *ring*, along with enough information about its multiplication and the Adem relation $Q_n^2 = 0$. Note that $\mathcal{A}_p$ parameterizes 2-stage Postnikov towers this is equivalent to a map $\mathbb{F}_p \to \Sigma^\ell \mathbb{F}_p$ from $\mathbb{F}_p$ into some suspension of $\mathbb{F}_p$.

The new idea here is that $\mathcal{A}_p$ doesn't just parameterize k-invariants (and thus 2-stage Postnikov towers, encoded in its homotopy groups), but rather if one access other higher structures well enough, this can be used to build spectra. Since $\mathcal{A}_p$ is an $\mathbb{E}_1$ ring, understanding its ring structure would allow building $k(n)$ completely as a spectrum.

Remark 2.4.4. There is a general procedure to build $k(n)$ as an $\mathbb{E}_1$ ring instead of a spectrum: one should write down the object parameterizing 2-stage Postnikov towers in the category of $\mathbb{E}_1$ rings, i.e. 2-stage towers that also happen to be $\mathbb{E}_1$-rings. This is known as the $\mathbb{E}_1$**-center** $\mathcal{Z}_{\mathbb{E}_1}(\mathbb{F}_p)$, also known as $\mathrm{THC}(\mathbb{F}_p)$, the topological Hochschild cohomology of $\mathbb{F}_p$.

The $\mathbb{E}_1$-center is also known to be an $\mathbb{E}_2$ ring and if one understands its $\mathbb{E}_2$ structure well, one can build $\mathbb{E}_1$ rings that are more complicated than 2-stage Postnikov towers.

Remark 2.4.5. Bokstedt proved that $\pi_* \mathrm{THC}(\mathbb{F}_p)$ is concentrated in even degrees. Thus given any class $x_{2\ell} \in \pi_{2\ell} \mathrm{THC}(\mathbb{F}_p)$ parameterizing some 2-stage $\mathbb{E}_1$ ring, by the previous theorem there is an $\mathbb{E}_2$ ring map

$$S^0[a_{2\ell}] \to \mathrm{THC}(\mathbb{F}_p).$$

This is some statement about an Adem relation on $\mathrm{THC}(\mathbb{F}_p)$, i.e. information about its $\mathbb{E}_2$-ring structure. Unwinding this yields another proof that Morava K-theory exists as an $\mathbb{A}_\infty$ or $\mathbb{E}_1$ ring.