# Perspectives on Nonabelian Hodge Theory (Talbot 2011)

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**Abstract**

The nonabelian Hodge correspondence provides a rich interplay of structures from topology, analysis and algebraic geometry which has spurred the curiosities of specialists and non-specialists alike. In the first part of this talk I will outline the celebrated nonabelian Hodge correspondence, due to Carlos Simpson, identifying certain representations of the fundamental group of a smooth projective complex variety with semistable "Higgs" bundles. I will discuss the consequences of this identification at the level of moduli spaces parametrizing these objects. Time permitting, I will survey more recent extensions to the characteristic $p$ or $p$--adic settings.

I will begin the second half of the lecture with a discussion of the Hodge filtration on nonabelian cohomology. Understanding the framework of filtrations set forth allows for us to view the nonabelian Hodge correspondence in a general light. Indeed, it becomes a manifestation of the following general question: when is a graded object canonically the associated graded of a filtered object? I will conclude this talk with a discussion of some work of mine bringing this perspective into the setting of higher algebra and higher categories, along with joint work with Robalo and Toën applying this perspective towards an understanding of the HKR filtration on Hochschild homology.

# 1 Part 1

## 1.1 Classical Hodge theory for real manifolds

**Remark 1.1.1.** If $X$ is a smooth manifold, we can consider the de Rham complex

$$C^\infty(X) \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots$$

where $\Omega^n_X$ denotes the real vector space of smooth differential $n$-forms on $X$ and the differential $d : \Omega^\bullet_X \to \Omega^{\bullet+1}_X$ is the exterior derivative. One of the landmark theorems in smooth manifold theory is the de Rham theorem, which tells us that the cohomology of $X$ with coefficients in $\mathbb{R}$, which purely takes into consideration the topological nature of the manifold $X$, is isomorphic to the cohomology groups of the de Rham complex:

$$H^n(X, \mathbb{R}) \cong H^n_{\mathrm{dR}}(X)\,.$$

**Definition 1.1.2** (Laplacian and Harmonic Forms)**.** If $(X, g)$ is an Riemannian manifold, then the presence of the metric $g$ allows one to also define an adjoint operator

$$\delta : \Omega^\bullet_X \to \Omega^{\bullet-1}_X\,,$$

which lowers the cohomological degree by one. We can then define the **Laplacian** $\Delta : \Omega^\bullet_X \to \Omega^\bullet_X$ as

$$\Delta = \delta d + d\delta\,.$$

We say that a differential $n$-form is **harmonic** if $\Delta\omega = 0$. We will denote the real vector space of harmonic $n$-forms of $X$ by $\mathcal{H}^n(X)$.

**Theorem 1.1.3** (The Hodge Theorem)**.** There is an isomorphism

$$H^n_{\mathrm{dR}}(X) \cong \mathcal{H}^n(X)\,.$$

Together with the de Rham theorem this tells us that each real cohomology class on $X$ has a unique harmonic representative.

## 1.2 Classical Hodge theory for complex projective varieties

**Remark 1.2.1.** Let $X$ be a smooth projective complex variety, or more generally a Kähler variety. The complex structure on the tangent bundle $TX$ of $X$ gives rise to a decomposition of the complexified differential forms

$$\Omega^n_X \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=n} \Omega^{p,q}_X \qquad \text{where } \Omega^{p,q}_X := (\Omega^{1,0}_X)^{\wedge p} \wedge (\Omega^{0,1}_X)^{\wedge q}.$$

In local holomorphic coordinates $(z_1, \ldots, z_n)$, the space $\Omega^{1,0}_X$ consists of holomorphic 1-forms $\omega = \sum c_j dz_j$ where $c_j : \mathbb{C}^n \to \mathbb{C}$ are holomorphic functions. The space $\Omega^{0,1}_X$ consists of antiholomorphic 1-forms $\alpha = \sum c_j d\bar{z}_j$, and it follows from the Cauchy-Riemann equations that any holomorphic change of local coordinates preserves $\Omega^{0,1}_X$ and $\Omega^{1,0}_X$ respectively.

**Remark 1.2.2.** The complexified de Rham differential $d$ decomposes into a holomorphic and anti-holomorphic part

$$d = \partial + \bar{\partial}$$

where

$$\partial : \Omega^{p,q} \to \Omega^{p+1,q} \quad \text{and} \quad \bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}\,.$$

This allows for a refined notion of harmonic forms, as well as formal adjoints $\partial^\dagger$ and $\bar{\partial}^\dagger$, and one defines $\Delta_d, \Delta_\partial, \Delta_{\bar{\partial}}$ analogously to the real case. These operators are related by the following theorem:

**Theorem 1.2.3** (Kähler identities)**.**

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}\,,$$

**Remark 1.2.4.** As a consequence, the three potentially distinct definitions of being harmonic are equivalent, so we denote the vector space of harmonic $(p,q)$-forms by $\mathcal{H}^{p,q}(X)$.

**Theorem 1.2.5** (Hodge Decomposition)**.** There is a decomposition

$$H^n(X;\mathbb{C}) \simeq \bigoplus_{p+q=n} H^{p,q}(X),$$

where $H^n(-)$ denotes singular cohomology. We can further can identify

$$H^{p,q}(X) = H^p_{\mathrm{Dol}}(X;\Omega^q_X) = \mathcal{H}^{p,q}(X)\,,$$

where $H^p_{\mathrm{Dol}}$ denotes Dolbeault cohomology.

**Remark 1.2.6.** This data yields a finite decreasing filtration $F_\bullet$ on $H^{p,q}$, together with a conjugate filtration $\overline{F}_\bullet$ on $H^{p,q}$, where

$$F_p(H) \cap \overline{F}_q(H) = H^n \qquad\qquad n \coloneqq p+q.$$

This yields a notion of **pure Hodge structure of weight** $n$.

## 1.3 The nonabelian Hodge correspondence

**Remark 1.3.1.** By Serre's GAGA theorem, there is a correspondence between holomorphic and complex algebro-geometric data for nice enough sheaves. Note that in what we have done above:

- The singular cohomology groups $H^n(X,\mathbb{C})$ depend on the topology of the space $X$.
- The de Rham cohomology groups $H^n_{\mathrm{dR}}(X)$ depend on the smooth structure of the manifold $X$.
- The decomposition into harmonic forms uses the holomorphic (or complex algebro-geometric) structure.

Thus we are mixing data from three very different areas of mathematics. Nonabelian Hodge theory can be viewed as a categorification of this interplay of ideas. Let us first recall a (rudimentary) version of the Riemann–Hilbert correspondence.

**Theorem 1.3.2** (The Riemann-Hilbert correspondence)**.**

There is an equivalence of categories

$$\left\{\begin{matrix}\text{Local systems of complex}\\ \text{vector bundles on } X\end{matrix}\right\} \rightleftharpoons \left\{\begin{matrix}\text{Complex vector bundles on } X\\ \text{with a flat connection}\end{matrix}\right\}.$$

**Remark 1.3.3.** Objects on the left-hand side can be identified with monodromy representations $\rho : \pi_1(X) \to \mathrm{GL}_n(\mathbb{C})$, while objects on the right-hand side can be thought of as pairs $(E,\nabla)$ where $E$ is a complex vector bundle and $\nabla : E \to E \otimes \Omega^1_X$ is a connection such that

$$\nabla^2 = 0 \qquad\qquad \text{and} \qquad\qquad \nabla(rs) = sd(r) + \nabla(s)r\,.$$

We want $H^1(X; \mathrm{GL}_n(\mathbb{C}))$ to be the space of such representations.

How do we make sense of the Hodge decomposition here? The answer: we will use so-called **Higgs bundles**. This is a pair $(E, \theta)$ where $E$ is a holomorphic bundle and $\theta : E \to E \otimes \Omega^1_X$ is an $\mathcal{O}_X$-linear map with $\theta \wedge \theta = 0$.

**Remark 1.3.4.** This structure first arose in the work of Hitchin in studying self-duality equations on Riemann surfaces, motivated by ideas from particle physics – analogous versions of *Higgs fields* describe the Higgs boson particle.

The following notion interpolates between Higgs bundles and flat bundles:

**Definition 1.3.5** (Harmonic bundles)**.** A **harmonic bundle** on $X$ is a smooth complex vector bundle $E$ with differential operators $\partial$ and $\bar{\partial}$ along with algebraic (or holomorphic) operators

$$\theta, \bar{\theta} \in H^0(X, \mathrm{End}(E) \otimes \Omega^1_X)\,.$$

**Remark 1.3.6.** One can fix a Hermitian metric so that $\partial + \bar{\partial}$ is a unitary connection and $\theta + \bar{\theta}$ is self-adjoint. Next, one sets

$$D = \partial + \bar{\partial} + \theta + \bar{\theta} \quad \text{and} \quad D'' = \bar{\partial} + \theta\,.$$

With the above conditions $(E, D)$ is a vector bundle with flat connection and $(E, \bar{\partial}, \theta) = (E, D'')$ is a Higgs bundle with $\theta \wedge \theta = 0$.

**Remark 1.3.7.** The operator $\bar{\partial}$ defines a holomorphic structure on $E$ by the Koszul-Malgrange theorem, yielding a holomorphic bundle. Given a bundle with flat connection and a Hermitian metric, one can define the data $D''_K$ needed for a Higgs structure, but then one needs to solve the system of PDEs $(D''_K)^2 = 0$. Conversely, given a Higgs bundle $E$, namely $\theta'' = \theta + \bar{\partial}$, one can define a connection $D_K$ and if $(D_K)^2 = 0$ then this is a flat connection

**Definition 1.3.8** (Stability and Semistability)**.** A bundle $E$ is **semistable** if for every coherent subsheaf $F \subset E$,

$$\frac{\deg(F)}{\mathrm{rank}(F)} \leq \frac{\deg(E)}{\mathrm{rank}(E)}\,.$$

$E$ is **stable** if this inequality is strict.

**Definition 1.3.9.** A flat bundle $E$ is **irreducible** if it does not admit a nonzero proper flat subbundle, and $E$ is **semisimple** if it decomposes as a direct sum of irreducible flat bundles.

**Theorem 1.3.10** (Nonabelian Hodge theorem (Simpson))**.** Let $X$ be a smooth complex projective variety.

- There are equivalences of categories:

$$\{\text{Harmonic bundles on } X\} \rightleftharpoons \{\text{Semisimple flat bundles on } X\} \rightleftharpoons \{\text{Semisimple } \pi_1(X)\text{-representations}\}\,.$$

- There is an equivalence of categories:

$$\left\{\text{Harmonic bundles on } X\right\} \rightleftharpoons \left\{\begin{matrix}\text{Direct sums } \bigoplus_i E_i \text{ of stable Higgs bundles} \\ E_i \text{ with } c_1(E_i)=c_2(E_i)=0\end{matrix}\right\}.$$

- The resulting equivalence

$$\left\{\text{Semisimple } \pi_1(X)\text{-representations}\right\} \rightleftharpoons \left\{\begin{matrix}\text{Direct sums } \bigoplus_i E_i \text{ of stable Higgs bundles} \\ E_i \text{ with } c_1(E_i)=c_2(E_i)=0\end{matrix}\right\}.$$

  extends to an equivalence

$$\left\{\text{All } \pi_1(X)\text{-representations}\right\} \rightleftharpoons \left\{\begin{matrix}\text{Semistable bundles with} \\ \text{vanishing Chern classes}\end{matrix}\right\}.$$

**Remark 1.3.11.** Semistability, along with the vanishing of the Chern classes $\mathrm{ch}_1$ and $\mathrm{ch}_2$, is precisely the condition needed for

$$D^2 = (\partial + \bar{\partial} + \theta + \bar{\theta})^2 = 0\,.$$

In the other direction, the associated tensor vanishes only if the bundle with flat connection is semisimple.

**Remark 1.3.12.** For some historical context, the first part of the theorem is due to Corlette and Donaldson, using the work of Eells and Sampson. The second part is a generalization of the theorem of Narasimhan and Seshadri. In the case when $\theta = 0$, this follows work of Uhlenbeck-Yau, Hitchin, Beilinson-Deligne, and others.

**Remark 1.3.13.** How can one view all of the above in light of the classical Hodge correspondence? Recall that in the classical case we have a decomposition

$$H^1(X, \mathbb{C}) \cong H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X)\,.$$

In other words, a cohomology class on the left hand side can be thought of as a pair $(e, \xi)$ where $e \in H^1(X, \mathcal{O}_X)$ and $\xi$ is a holomorphic 1-form. Similarly, in the noncommutative version one studies the nonabelian cohomology set[1] $H^1(X, \mathrm{GL}_n(\mathbb{C}))$ and its decomposition

$$H^1(X, \mathrm{GL}_n(\mathbb{C})) \cong H^1(X, \mathrm{GL}_n(\mathcal{O}_X)) \oplus H^0(X, \mathrm{End}(E) \otimes \Omega^1_X)\,.$$

The decomposition tells us that a class on the left hand side can be thought of as a pair $(E, \theta)$ where $E$ is a holomorphic bundle and $\theta : E \to E \otimes \Omega^1_X$ is a holomorphic map such that $\theta \wedge \theta = 0$.

[1] We shall see that it is really better to think of this as a moduli space/stack than a set.

## 1.4 Moduli spaces

**Remark 1.4.1.** A key aspect of the nonabelian Hodge correspondence lies in how the associated moduli spaces interact. We should really think of $H^1(X, \mathrm{GL}_n(\mathbb{C}))$ as a *moduli space* or *stack*, so let's define exactly what we mean by this.

**Definition 1.4.2** (Representability)**.** Consider a functor

$$F : \mathsf{C} \to \mathsf{Groupoids}$$

from the category $\mathsf{C}$ of discrete commutative $k$-algebras to the category of groupoids. We say that such a functor is **representable by a scheme/space/stack** if there is a scheme/space/stack $X$ for which

$$F(A) \cong \mathrm{Hom}_{\mathsf{C}}(\mathrm{Spec}(A), X)$$

These are also typically required to satisfy descent with respect to a topology on $\mathsf{C}$.

**Example 1.4.3** (Quotient stack)**.** Let $X$ be a scheme with an action of some group scheme $G$. Then we can form the quotient stack $X/G$ as the realisation of the simplicial groupoid formed by the bar resolution

$$X/G \coloneqq |X \leftleftarrows X \times G \substack{\leftarrow \\ \leftarrow \\ \leftarrow} \cdots| \,.$$

**Remark 1.4.4.** There are also other versions of quotient stacks that are relevant in mathematics – one can also take GIT quotients, and Spec $R^G$ of the ring of invariants.

We can use the functor of points point on view to define the Betti moduli space, the de Rham moduli spaces, and the Higgs moduli space.

**Construction 1.4.5.** Let $\Gamma$ be a finitely generated group. One defines a representing scheme parameterizing representations $R(\Gamma, n) = \mathrm{Map}(\Gamma, \mathrm{GL}_n)$ by sending

$$A \mapsto \mathrm{Hom}(\Gamma, \mathrm{GL}_n(A)) \,.$$

The reductive group $\mathrm{GL}_n$ acts on $R(\Gamma, \mathrm{GL}_n)$ and we let **the Betti moduli scheme** be the quotient

$$\mathcal{M}(\Gamma, \mathrm{GL}_n) = R(\Gamma, n)/\,\mathrm{GL}_n \,.$$

In particular, we will write

$$\mathcal{M}(X, n) = \mathcal{M}(\pi_1(X^{\mathrm{an}}), \mathrm{GL}_n) \,,$$

where the $X^{\mathrm{an}}$ denotes the associated analytic space.

**Construction 1.4.6.** Fix a point $x \in X$. The **de Rham moduli scheme** $R_{\mathrm{dR}}(X, n)$ assigns to a scheme $Y$ the set of isomorphism classes $(V, D)$ of vector bundles of rank $n$ with flat connection and a "frame" $\alpha : V|_x \cong \mathbb{C}^n$. This also admits an action by the algebraic group $\mathrm{GL}_n$ and one defines the **de Rham quotient stack** as

$$\mathcal{M}_{\mathrm{dR}}(X,n) = R_{\mathrm{dR}}(X,n)/\,\mathrm{GL}_n\ .$$

**Construction 1.4.7.** Let us denote by $R_{\mathrm{Dol}}(X,x,n)$ the moduli scheme of semistable Higgs bundles with vanishing Chern classes and "frame" at $x$. Again, this has a $\mathrm{GL}_n$ action, and we obtain the Higgs moduli space by taking the quotient stack

$$\mathcal{M}_{\mathrm{Dol}}(X,n) = R_{\mathrm{Dol}}(X,x,n)/\,\mathrm{GL}_n\ .$$

**Remark 1.4.8.** Originally, the construction of these moduli spaces used GIT quotients, and not quotient stacks, as done here. These moduli spaces are the associated coarse moduli spaces for the relevant quotient stacks (they represent the functors $\pi_0$ of the stacks).

The identification at the level of moduli spaces/stacks are rather subtle. As a consequence of the Riemann–Hilbert correspondence between holomorphic systems of ODE's and their monodromy representations, we have the following result:

**Proposition 1.4.9.** There are isomorphisms between the associated complex analytic spaces:

$$R_{\mathrm{dR}}(X,n)^{\mathrm{an}} \simeq R_B(X,n)^{\mathrm{an}} \quad \text{and} \quad \mathcal{M}_{\mathrm{dR}}(X,n)^{\mathrm{an}} \simeq \mathcal{M}_B(X,n)^{\mathrm{an}}\,.$$

**Theorem 1.4.10.** The correspondence provides an isomorphism of sets between the underlying points of $\mathcal{M}_{\mathrm{dR}}(X,n)$ and $\mathcal{M}_{\mathrm{Dol}}(X,n)$, which yields a homeomorphism between the underlying topological spaces:

$$\mathcal{M}_{\mathrm{dR}}(X,n)^{\mathrm{top}} \cong \mathcal{M}_{\mathrm{Dol}}(X,n)^{\mathrm{top}}\,.$$

## 1.5 The $\mathbb{C}^*$-action

**Remark 1.5.1.** There is a natural action of $\mathbb{C}^*$-action on Higgs bundles given by

$$z\cdot(E,\theta) = (E, z\theta)\,.$$

Via the equivalence of categories, we can transport this to an action on semistable flat bundles. As a consequence, the moduli space $\mathcal{M}_{\mathrm{Dol}}(X,n)$ acquires a $\mathbb{G}_m$-action.

**Proposition 1.5.2.** Semisimple flat bundles fixed by a $\mathbb{G}_m$ action are precisely those which underlie *complex variations of Hodge structure*.

**Definition 1.5.3.** A representation $\rho$ of the fundamental group $\pi_1(X)$ is **rigid** if any nearby representation in $R_B(X,n)$ is conjugate to it.

**Conjecture 1.5.4** (Simpson's motivicity conjecture)**.** Rigid representations $\rho$ are direct factor in the monodromy representation of a *motive*, i.e a family of varieties over $X$.

**Remark 1.5.5.** Recent work of Esnault-Groechenig making progress towards this conjecture, by finding a model over integers, and then studying the $p$-curvatures of the relevant connections modulo $p$. This utilizes characteristic $p$ versions of the non-abelian Hodge theorem due to Ogus-Vologodsky, and more recent works of Lan-Sheng-Zuo in the $p$-adic setting.

# 2 Part 2

**Remark 2.0.1.** We want to consider an analog of the Hodge filtration for nonabelian cohomology. Recall that in the classical case, a Hodge decomposition correspond to decreasing filtrations on $H^*(X;\mathbb{C})$. Think of $H^1(X, \mathrm{GL}_n(\mathbb{C}))$ as a complex linear mapping stack

$$\mathcal{M}_B(X,n) := \mathrm{Maps}(\pi_1(X), \mathrm{GL}_n)/\,\mathrm{GL},$$

What is the analogue to the Hodge filtration on this?

**Proposition 2.0.2** (Rees construction)**.** Consider a pair $(V, F)$ where $V$ is a complex vector space, and $F$ is a complete decreasing exhaustive filtration of $V$. We will map $F^*(V)$ to $\xi(V, F)$, which will be a submodule of $V \otimes \mathbb{C}[t, t^{-1}]$ generated by $t^{-p}F^p(V)$. This is like a $\mathbb{C}[t]$-lattice in $V \otimes \mathbb{C}[t, t^{-1}]$ and thus a module over $\mathbb{C}[t]$. Equip it with a Galois action to get a free sheaf over $\mathbb{A}^1 = \mathrm{Spec}\, \mathbb{C}[t]$ with a $\mathbb{G}_m$-action.

**Remark 2.0.3.** The upshot is that we'll get an equivalence of categories

$$\left\{ \begin{matrix} \mathbb{G}_m\text{-equivariant vector bundles} \\ \text{over } \mathbb{A}^1 \end{matrix} \right\} \rightleftharpoons \left\{ \text{Filtered vector spaces} \right\} .$$

**Definition 2.0.4** (Graded and filtered stacks)**.** A **graded stack** is a stack over the classifying stack $\mathsf{B}\mathbb{G}_m$, and a **filtered stack** is a stack over $\mathbb{A}^1/\mathbb{G}_m$.

**Remark 2.0.5.** We realize $\mathbb{A}_1/\mathbb{G}_m$ as the realization of a simplicial object:

$$\mathbb{A}_1/\mathbb{G}_m = \left| \mathbb{A}^1 \to A^1 \times \mathbb{G}_m \rightrightarrows \cdots \right|.$$

There are pullback squares

$$\begin{array}{ccccc} X & \longrightarrow & \tilde{X} & \longleftarrow & X^{\mathrm{gr}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{1} & \mathbb{A}^1/\mathbb{G}_m & \xleftarrow[0]{} & \mathsf{B}\mathbb{G}_m \end{array}$$

This endows $R\gamma(X, \mathcal{O}_X)$ with a filtration.

## 2.1 Hodge Filtration on Nonabelian Cohomology

**Remark 2.1.1.** We realize $H^1(X, \mathrm{GL}_n(\mathbb{C}))$ as a stack and take the de Rham stack $\mathcal{M}_{\mathrm{dR}}(X, n)$ over Spec $\mathbb{C}$. A filtration on the de Rham stack is then the stack $\tilde{\mathcal{M}}$ over $\mathbb{A}^1/\mathbb{G}_m$ fitting into a pullback square

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{dR}}(X,n) & \longrightarrow & \widetilde{\mathcal{M}} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{\;1\;} & \mathbb{A}^1/\mathbb{G}_m \end{array}$$

Similarly, one obtains the moduli stack of Higgs bundles $\mathcal{M}_{\mathrm{Dol}}$ as

$$\mathcal{M}_{\mathrm{Dol}}(X,n) \simeq \mathcal{M} \underset{\mathbb{A}^1/\mathbb{G}_m}{\times} \mathsf{B}\mathbb{G}_m .$$

The idea (going back to Deligne) is to now construct a 1-parameter family.

**Definition 2.1.2** ($\lambda$-Connection)**.** Fix a smooth complex vector bundle $E$. A **$\lambda$-connection** on $E$ is an operator $\nabla_\lambda : E \to E \otimes \Omega^1_X$ such that

$$\nabla_\lambda(rf) = \lambda d(r) f + r\nabla(f) \quad \text{and} \quad \nabla_\lambda^2 = 0 \,.$$

where $r$ is a coefficient, $f$ is a section of $E$, and $d$ is the de Rham differential.

**Remark 2.1.3.** Note that if $\lambda = 0$ this reduces to $\nabla(rf) = r\nabla(f)$. Setting $\theta := \nabla_0$ then precisely yields the data of a Higgs field. For $\lambda = 1$ the above definition recovers the notion of a flat connection.

**Proposition 2.1.4** (Key!)**.** A harmonic bundle $(E, D, D'')$, where $D, D''$ are operators, gives rise to a family of flat $\lambda$-connections with

- A flat part $(E_1, D_1) = (E, 0)$
- A Higgs part $(E_0, \nabla_0) = (E, \theta)$

**Theorem 2.1.5** (Simpson)**.** Let $S$ be a scheme over $\mathbb{A}^1$, then there is a functor

$$(\lambda : S \to \mathbb{A}^1) \mapsto (E, \nabla, \alpha)$$

where

- $E$ is a bundle over $X \times S$,
- $\nabla$ is a $\lambda$-connection, and
- $\alpha : E|_x \simeq \mathbb{C}^n$ is a frame.

This functor is representable by a scheme $R_{\mathrm{Hodge}}(X, n) \to \mathbb{A}^1$ which yields a map

$$\mathcal{M}^{\mathrm{Hodge}}(X,n)/\,\mathrm{GL}_n \to \mathbb{A}^1.$$

Setting $\mathcal{M} \coloneqq \mathcal{M}^{\text{Hodge}}(X, n)$, $\mathcal{M}$ admits a $\mathbb{C}^*$ action $\mathbb{C}^* \simeq \mathbb{G}_m$ we get

$$\mathcal{M}/\mathbb{G}_m \to \mathbb{A}^1/\mathbb{G}_m.$$

When $\lambda$ is invertible we recover flat connections, and so there is a pullback:

$$\begin{array}{ccc} \mathcal{M}_{\text{dR}} \times \mathbb{G}_m & \longrightarrow & \mathcal{M} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \end{array}$$

Link to Diagram

Using that Spec $k \xrightarrow{0} \mathbb{A}^1$, pulling back recovers $\mathcal{M}^{\text{Hodge}}$.

**Remark 2.1.6.** The conditions of semistability and vanishing Chern classes together allow us to lift a category of objects with an intrinsic notion of grading (e.g. $\mathcal{M}^{\text{Dol}} \to \mathsf{B}\mathbb{G}_m$) to a category with an intrinsic notion of a filtration (e.g. over $\mathbb{A}^1/\mathbb{G}_m$).

## 2.2 Higher Algebra

**Remark 2.2.1.** Given a stable $\infty$-category $\mathsf{C}$, one can make sense of filtered objects in $\mathsf{C}$ and hence of **filtered spectra**, which can be thought of as sequential diagrams

$$\cdots \longrightarrow E(n+1) \longrightarrow E(n) \longrightarrow E(n-1) \longrightarrow \cdots$$

By spectral algebraic geometry, affines are connective $\mathbb{E}_\infty$-rings. We have

- Spec $\mathbb{S}$
- Spec $\mathbb{S}[\mathbb{N}] \simeq \mathbb{A}^1$
- Spec $\mathbb{S}[\mathbb{Z}] \simeq \mathbb{G}_m$

Note that there are two notions of affine line in spectral algebraic geometry, but Spec $\mathbb{S}[\mathbb{N}]$ is the more commonly used one.

We can thus define $\mathbb{A}^1/\mathbb{G}_m$, as well as its $\infty$-category of quasicoherent sheaves as the totalisation

$$\mathsf{QCoh}(\mathbb{A}^1/\mathbb{G}_m) = \text{Tot}\left(\mathbb{S}[\mathbb{N}]\text{-}\mathsf{Mod} \to \mathbb{S}[\mathbb{N} \times \mathbb{Z}]\text{-}\mathsf{Mod} \rightrightarrows \cdots\right).$$

**Theorem 2.2.2** (M.)**.** There is a symmetric monoidal equivalence of categories

$$\mathsf{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathsf{Sp}^{\text{fil}},$$

where the right-hand side is the category of filtered spectra equipped with the Day convolution product.

**Remark 2.2.3.** Pulling back a quasicoherent sheaf from $\mathbb{A}^1/\mathbb{G}_m$ is taking a colimit of underlying objects, whereas pulling back from $\mathsf{B}\mathbb{G}_m$ amounts to taking the associated graded. As a consequence, we can import nonabelian Hodge theory into the setting of higher categories.

**Question 2.2.4.** Suppose $\mathsf{C}^0 \subseteq \mathsf{C}$ is a "graded $\infty$-category". When does $\mathsf{C}^0$ lift to a category trivial over $\mathsf{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$?

## 2.3 Hochschild Homology

**Remark 2.3.1.** The following is joint work with Marco Robalo and Bertrand Toën.

**Definition 2.3.2** (Hochschild homology)**.** Recall that **Hochschild homology** of a commutative $k$-algebra $R$ can be defined as

$$\mathrm{HH}(R/k) \coloneqq S^1 \otimes_k R\,.$$

In terms of derived algebraic geometry, we have

$$\mathrm{Spec}\ \mathrm{HH}(R/k) = \mathrm{Maps}(S^1, \mathrm{Spec}\ R)$$

**Remark 2.3.3.** The Hochschild–Kostant–Rosenberg (HKR) theorem can be used to construct a complete decreasing filtration on Hochschild homology. The existence of the HKR-filtration can also be explained as coming from a filtered circle.

**Theorem 2.3.4** (Moulinos-Robalo-Toën (MRT))**.** One can construct a filtered group stack

$$S^1_{\mathrm{fil}} \to \mathbb{A}^1/\mathbb{G}_m$$

acting on a filtered loop stack

$$\mathcal{L}_{\mathrm{fil}}(k) = \mathrm{Maps}_{/(\mathbb{A}^1/\mathbb{G}_m)}(S^1_{\mathrm{fil}}, X)$$

which fit into a diagram of the following form:

$$\begin{array}{ccccc}
\mathcal{L}X & \longrightarrow & \mathcal{L}_{\mathrm{fil}}(X) & \longleftarrow & \mathrm{Spec}\ \check{\mathrm{Sym}}_k(\mathbb{L}_k[-1]) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}\ k & \longrightarrow & \mathbb{A}^1/\mathbb{G}_m & \longleftarrow & \mathsf{B}\mathbb{G}_m
\end{array}$$

where the left recovers Hochschild homology and the right recovers the de Rham algebra.

**Remark 2.3.5.** The top-right corner is a derived scheme whose cohomology admits the structure of a **mixed complex**. Note that there is an $S^1$ action on $\mathcal{L}(X)$ and an $S^1_{\mathrm{fil}}$ action on $\mathcal{L}_{\mathrm{fil}}(X)$. We have a lift of the mixed complex to a filtered object whose underlying complex is related to Hochschild homology.

**Remark 2.3.6** (Perhaps slightly cryptic!)**.** The filtered loop space $\mathcal{L}_{\mathrm{fil}}(X)$ is related via a 2-fold bar construction to the degeneration of moduli spaces $\mathcal{M}_{\mathrm{dR}} \rightsquigarrow \mathcal{M}_{\mathrm{Dol}}$.