Mathematics Subject GRE Workshop

Agenda

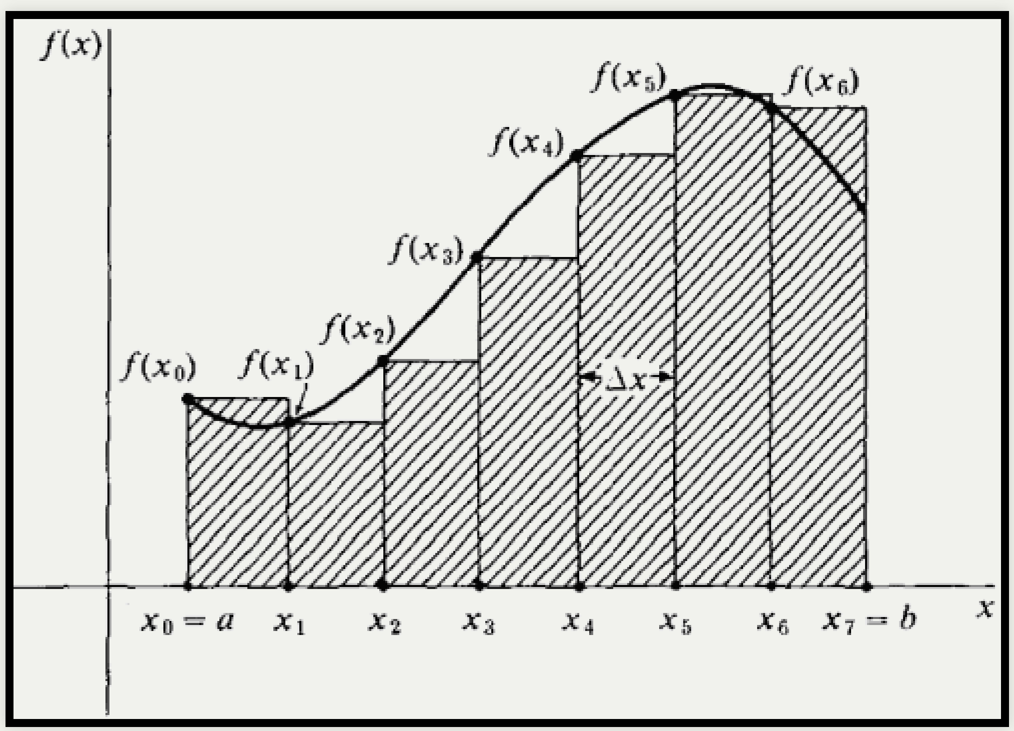
- Description of Mathematics Subject GRE
- Topics it covers
- Exam logistics
- Recommended resources
- Study techniques/tips
- Review of topics + sample problems

What is the Mathematics Subject GRE?

- Different from the Math section of the *General* GRE
- Required of graduate student applicants to many Math Ph.D. programs
- Tests a breadth of undergraduate topics

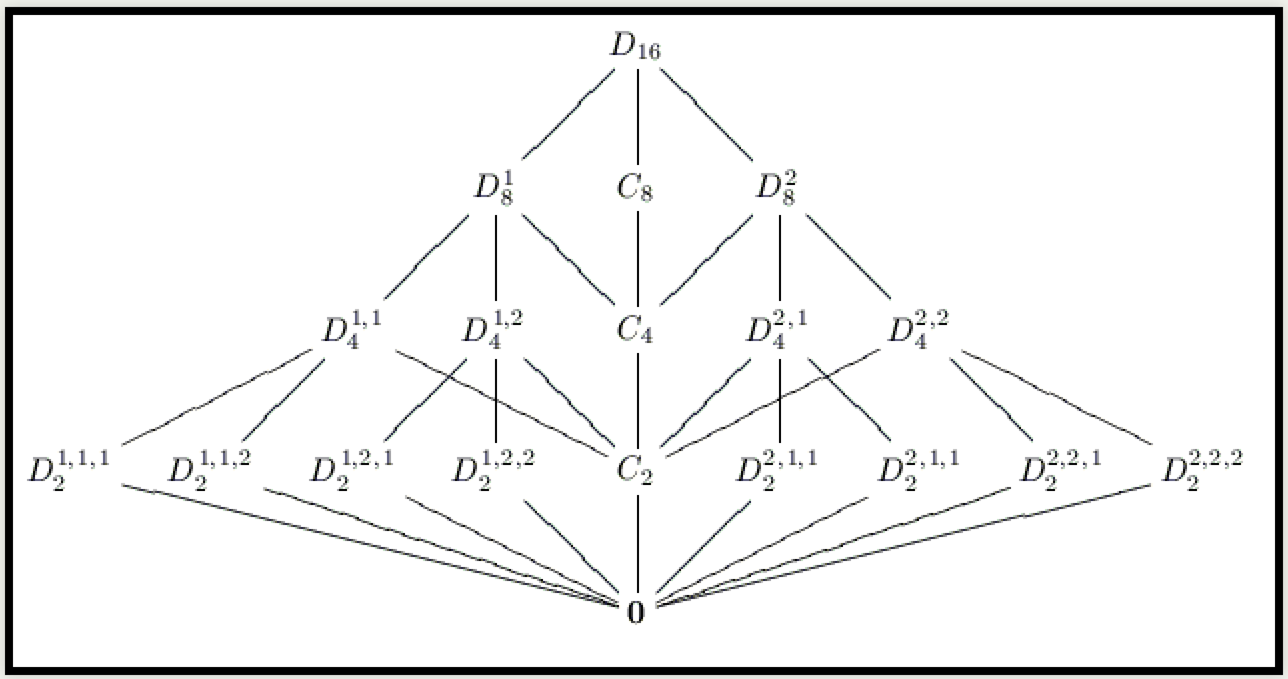
Topics

- Calculus (50%)
  - Single Variable
  - Multivariable
  - Differential Equations



“Algebra” (25%)

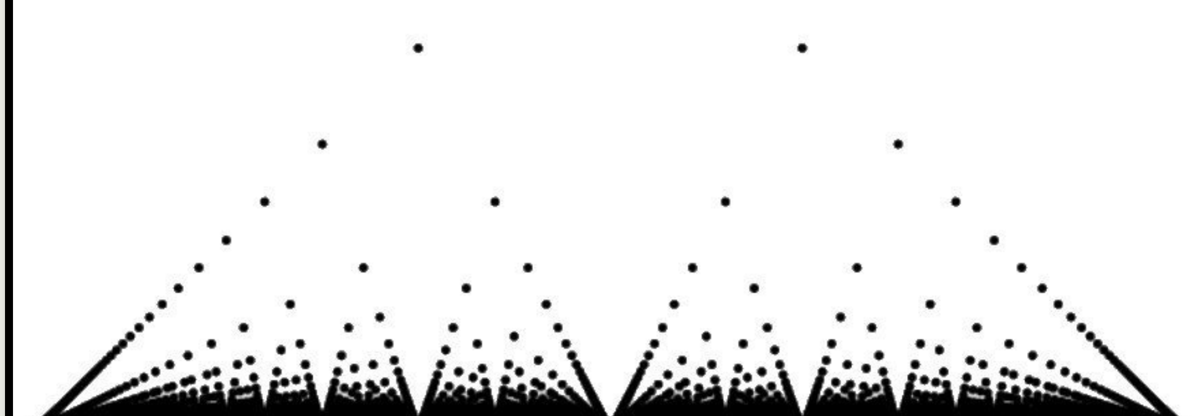
- Linear Algebra
- Abstract Algebra
- Number Theory



Mixed Topics (25%)

- Real Analysis
- Logic / Set Theory
- Discrete Mathematics
- Point-Set Topology
- Complex Analysis
- Combinatorics
- Probability

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{q} & \text{if } x \text{ is rational, } x = \frac{p}{q} \text{ in lowest terms and } q > 0 \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

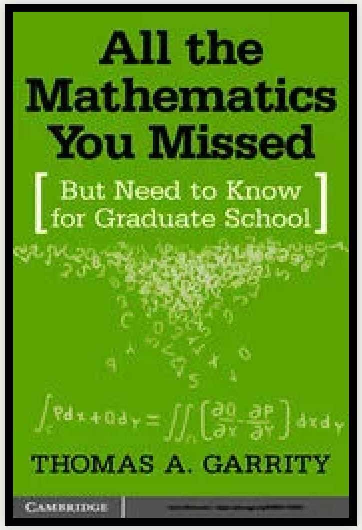


Logistics

- Multiple choice, 5 choices
- 66 questions, 170 minutes
- No downside to guessing
- Only offered 3x/year
- Need to register ~2 months in advance

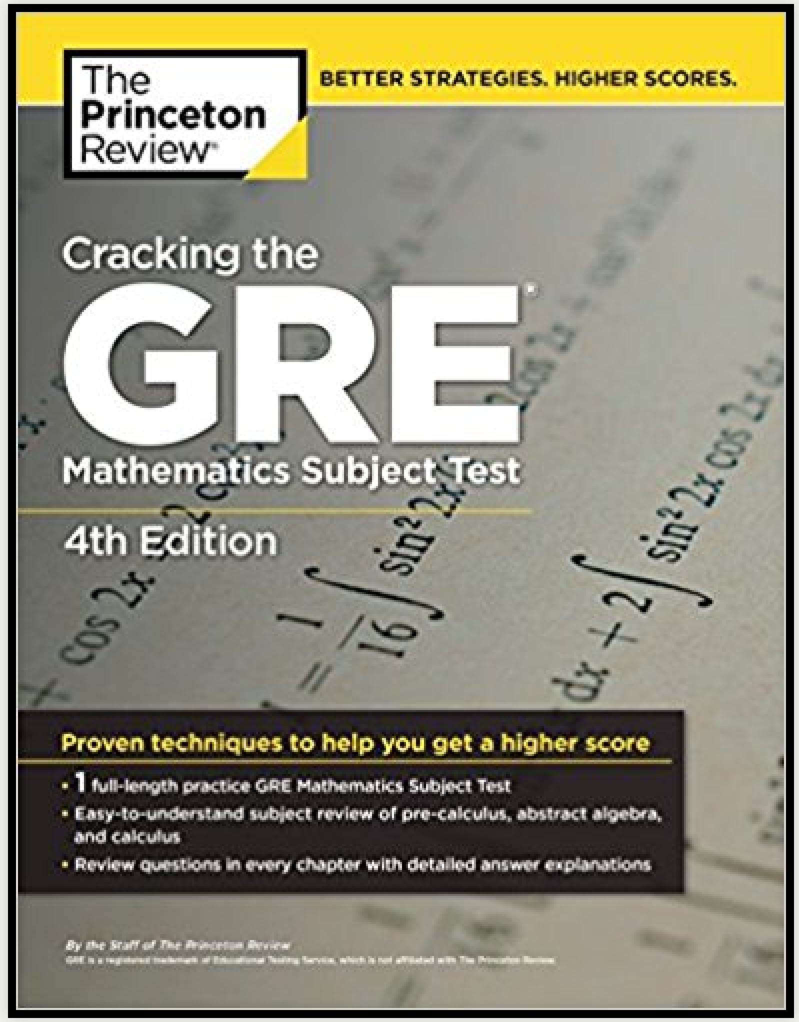
References

Garrity, All the Mathematics You Missed (But Need to Know for Graduate School)



Good high-level overview of undergrad topics.

The Princeton Review, Cracking the Math GRE Subject Test



“Calculus: The Greatest Hits”, good breadth.

Shallow treatment of Algebra, Real Analysis, Topology, Number Theory.

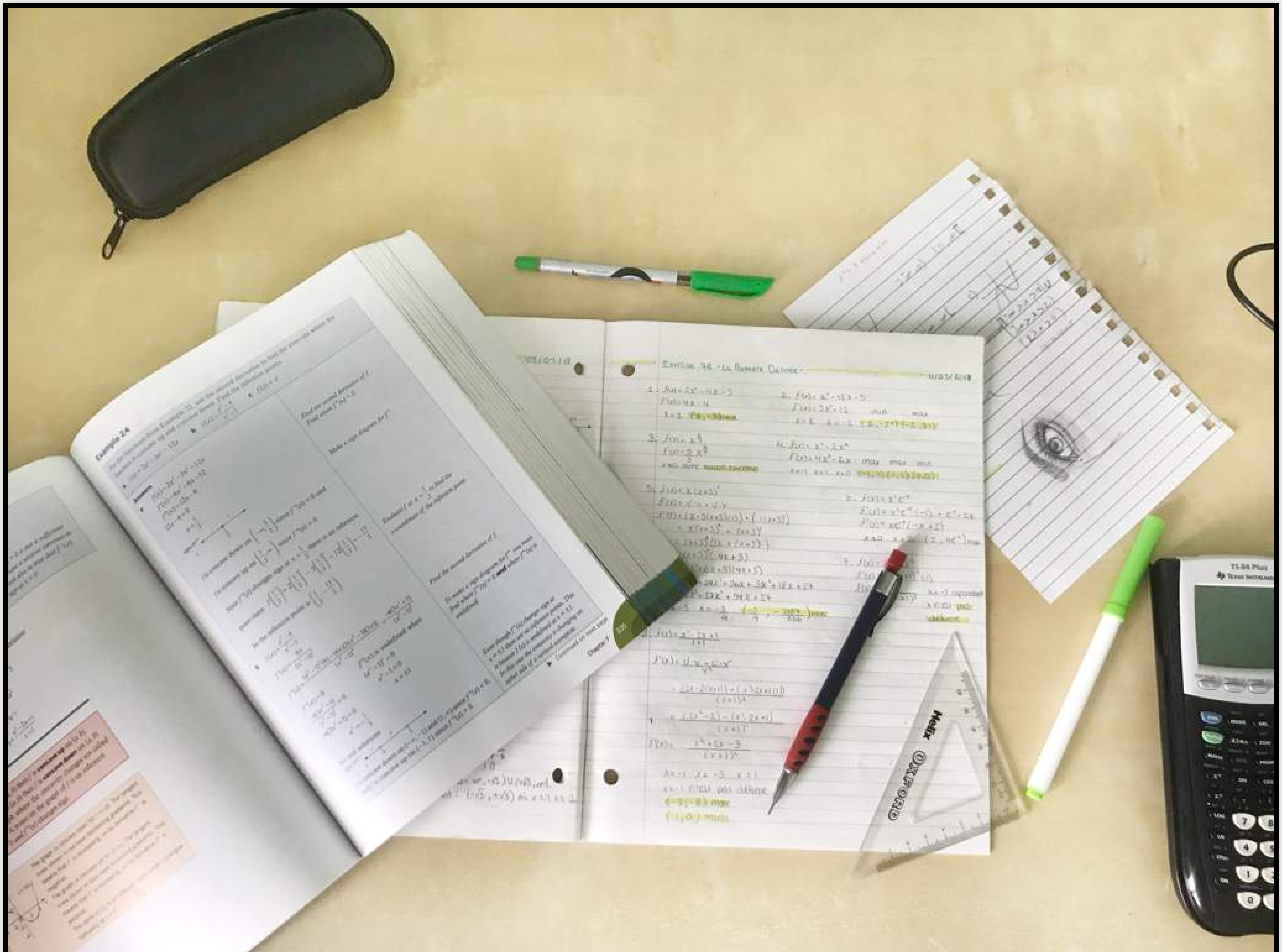
Five Official Practice Exams (with Solutions)

- GR 1268
- GR 0568
- GR 9367
- GR 8767
- GR 9768

All old and *significantly* easier than exams in recent years.

Aim for 90th percentile in < 2 hours.

General Tips



Math-Specific Tips

- Focus on lower div
- For Calculus, focus on speed: median ≤ 1 minute
- Drill *a lot* of problems
  - Seriously, a lot.
  - *Seriously.*
- Should memorize formulas and definitions
  - No time to rederive!
- Save actual exams as diagnostic tools

Study Tips

- Start early
  - Steady practice paced over 3-9 months is 100x more effective than 1 month of cramming
- Speed is important
- Spaced repetition, e.g. Anki
- Replicate exam conditions
- Build mental stamina
  - i.e. 2-3 hours of uninterrupted problem solving
- Self care!!
  - Sleep
  - Eat right

Single Variable Calculus

Differential

- Computing limits
- Showing continuity
- Computing derivatives
- Rolle's Theorem
- Mean Value Theorem
- Extreme Value Theorem
- Implicit Differentiation
- Related Rates
- Optimization
- Computing Taylor expansions
- Computing linear approximations

Integral

- Riemann sum definition of the integral
- The fundamental theorem of Calculus (both forms)
- Computing antiderivatives
  - u-substitutions
  - Partial fraction decomposition
  - Trigonometric Substitution
  - Integration by parts
  - Specific integrands
- Computing definite integrals
- Solids of revolution
- Series (see real analysis section)

Computing Limits

- Tools for finding $\lim_{x \to a} f(x)$, in order of difficulty:
  - Plug in: equal to $f(a)$ if $f \in C^0(N_\varepsilon(a))$
  - Algebraic Manipulation
  - L'Hopital's Rule (only for indeterminate forms $\frac{0}{0}, \frac{\infty}{\infty}$)
    - For $\lim f(x)^{g(x)} = 1^\infty, \infty^0, 0^0$, let
      $L = \lim f^g \implies \ln L = \lim g \ln f$
  - Squeeze theorem
  - Take Taylor expansion at a
  - Monotonic + bounded (for sequences)

Use Simple Techniques

When possible, of course.

$$\frac{a}{b+\sqrt{c}} = \frac{a}{b+\sqrt{c}}\left(\frac{b-\sqrt{c}}{b-\sqrt{c}}\right) = \frac{a(b-\sqrt{c})}{b^2-c}$$

$$\frac{1}{ax^2+bx+c} = \frac{1}{(x-r_1)(x-r_2)} = \frac{A}{x-r_1} + \frac{B}{x-r_2}$$

The Fundamental Theorems of Calculus

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\int_a^b \frac{\partial}{\partial x} f(x)\,dx = f(b) - f(a)$$

First form is usually skimmed over, but very important!

FTC Alternative Forms

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} g(t)dt = g(b(x))b'(x) - g(a(x))a'(x)$$

Commuting D and I

Commuting a derivative with an integral

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt$$

$$+ f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x)$$

(Derived from chain rule)

Set

$$a(x) = a, b(x) = b, f(x,t) = f(t) \implies \frac{\partial}{\partial x} f(t) = 0,$$

then commute to derive the FTC.

Applications of Integrals

- Solids of Revolution
  - Disks: $A = \int \pi r(t)^2 \, dt$
  - Cylinders: $A = \int 2\pi r(t)h(t) \, dt$
- Arc Lengths
  - $ds = \sqrt{dx^2 + dy^2}, \qquad L = \int \, ds$

Series

There are 6 major tests at our disposal:

- **Comparison Test**
  - $a_n < b_n$ and $\sum b_n < \infty \implies \sum a_n < \infty$
  - $b_n < a_n$ and $\sum b_n = \infty \implies \sum a_n = \infty$
  - You should know some examples of series that converge and diverge to compare to.
- **Ratio Test**

$$R = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

  - $R < 1$: absolutely convergent
  - $R > 1$: divergent
  - $R = 1$: inconclusive

More Series

- **Root Test**

$$R = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

  - $R < 1$: convergent
  - $R > 1$: divergent
  - $R = 1$: inconclusive
- **Integral Test**

$$f(n) = a_n \implies \sum a_n < \infty \iff \int_1^\infty f(x)dx < \infty$$

More Series

- **Limit Test**

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L < \infty \implies \sum a_n < \infty \iff \sum b_n < \infty$$

- **Alternating Series Test**

$$a_n \downarrow 0 \implies \sum (-1)^n a_n < \infty$$

Advanced Series

- **Cauchy Criteria**:
  - Let $s_k = \sum_{i=1}^{k} a_i$ be the k-th partial sum, then

$$\sum a_i \text{ converges } \iff \{s_k\} \text{ is a Cauchy sequence,}$$

- **Weierstrass M Test**:

$$\sum_{n=1}^{\infty} |\|f_n\|_\infty| < \infty \implies$$

$$\exists f \in C^0 \ \ni \ \sum_{n=1}^{\infty} f_n \rightrightarrows f$$

  - i.e. define $M_k = \sup\{f_k(x)\}$ and require that $\sum |M_k| < \infty$
  - "Absolute convergence in the sup norms implies uniform convergence"

Multivariable Calculus

General Concepts

- Vectors, div, grad, curl
- Equations of lines, planes, parameterized curves
    - And finding intersections
- Multivariable Taylor series
    - Computing linear approximations
- Multivariable optimization
    - Lagrange Multipliers
- Arc lengths of curves
- Line/surface/flux integrals
- Green’s Theorem
- The divergence theorem
- Stoke’s Theorem

Geometry in $\mathbb{R}^3$

Lines

$$Ax + By + C = 0,\ \mathbf{x} = \mathbf{p} + t\mathbf{v},$$

$$\mathbf{x} \in L \iff \langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle = 0$$

Planes

$$Ax + By + Cz + D = 0,\ \mathbf{x}(t, s) = \mathbf{p} + t\mathbf{v}_1 + s\mathbf{v}_2$$

$$\mathbf{x} \in P \iff \langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle = 0$$

Distances to lines/planes: project onto orthogonal complement.

Tangent Planes/Linear Approximations

Let $S \subseteq \mathbb{R}^3$ be a surface. Generally need a point $\mathbf{p} \in S$ and a normal $\mathbf{n}$.

Key Insight: The gradient of a function is normal to its level sets.

Case 1: $S = \{[x, y, z] \in \mathbb{R}^3 \mid f(x, y, z) = 0\}$

i.e. it is the zero set of some function $f : \mathbb{R}^3 \to \mathbb{R}$

- ∇f is a vector that is normal to the zero level set.
- So just write the equation for a tangent plane $\langle \mathbf{n}, \mathbf{x} - \mathbf{p}_0 \rangle$.

Tangent Planes/Linear Approximations

Case 2: S is given by $z = g(x, y)$

- Let $f(x, y, z) = g(x, y) - z$, then

$$\mathbf{p} \in S \iff \mathbf{p} \in \{[x, y, z] \in \mathbb{R}^3 \mid f(x, y, z) = 0\}.$$

- Then ∇f is normal to level sets, compute $\nabla f = [\frac{\partial}{\partial x} g, \frac{\partial}{\partial y} g, -1]$
- Proceed as in previous case.

Optimization

Single variable: solve $\frac{\partial}{\partial x} f(x) = 0$ to find critical points c_i then check min/max by computing $\frac{\partial^2}{\partial x^2} f(c_i)$.

Multivariable: solve $\nabla f(\mathbf{x}) = 0$ for critical points $\mathbf{c}_i$, then check min/max by computing the determinant of the Hessian:

$$H_f(\mathbf{a}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(\mathbf{a}) \end{bmatrix}.$$

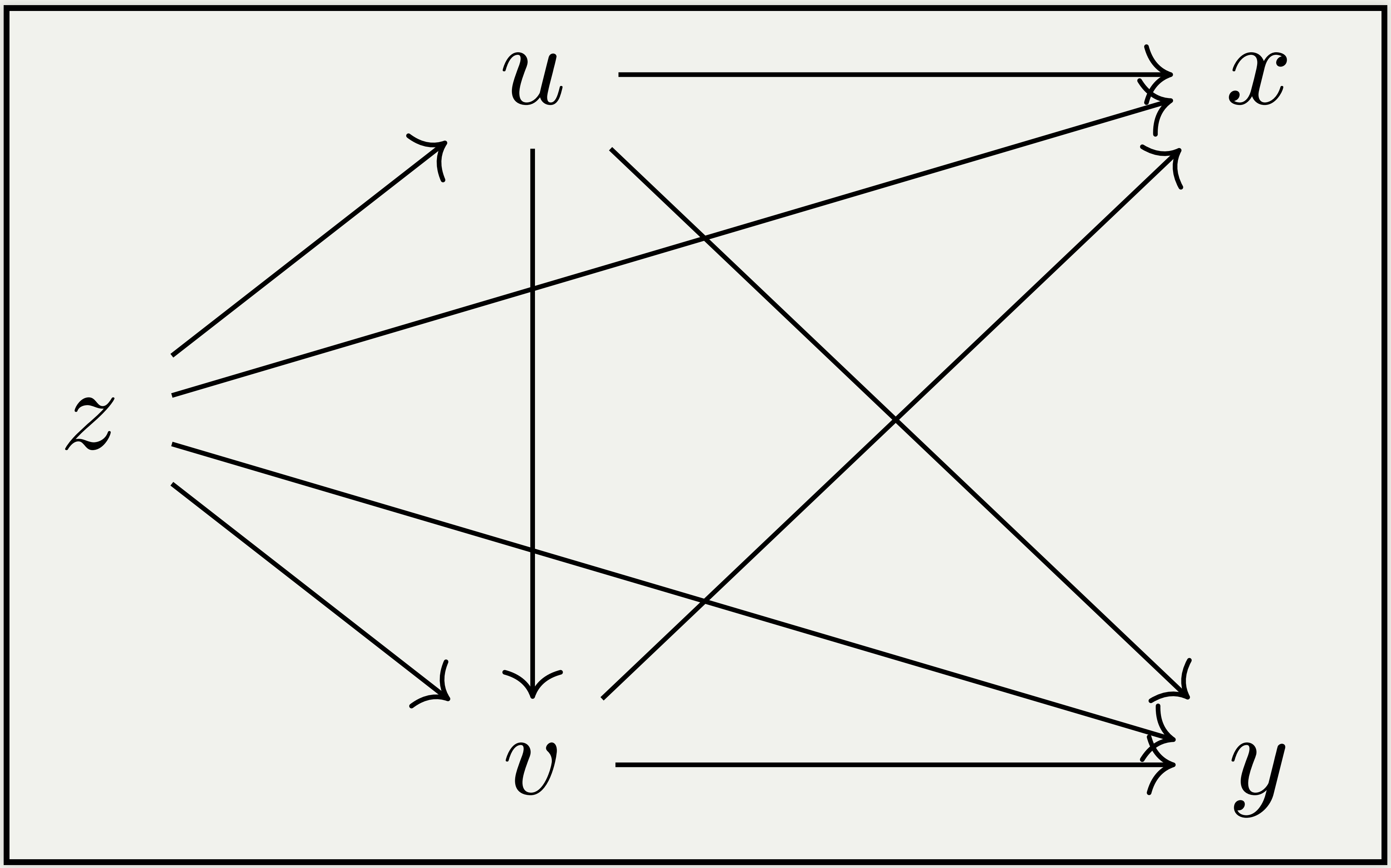
Optimization

Lagrange Multipliers:

$$\text{Optimize } f(\mathbf{x}) \text{ subject to } g(\mathbf{x}) = c$$
$$\implies \mathbf{\nabla f} = \lambda \mathbf{\nabla g}$$

- Generally a system of nonlinear equations
  - But there are a few common tricks to help solve.

Multivariable Chain Rule



Multivariable Chain Rule

To get any one derivative, sum over all possible paths to it:

$$\begin{aligned}\left(\frac{\partial z}{\partial x}\right)_{y} = &\left(\frac{\partial z}{\partial x}\right)_{u,y,v} \\ &+ \left(\frac{\partial z}{\partial v}\right)_{x,y,u}\left(\frac{\partial v}{\partial x}\right)_{y} \\ &+ \left(\frac{\partial z}{\partial u}\right)_{x,y,v}\left(\frac{\partial u}{\partial x}\right)_{v,y} \\ &+ \left(\frac{\partial z}{\partial u}\right)_{x,y,v}\left(\frac{\partial u}{\partial v}\right)_{x,y}\left(\frac{\partial v}{\partial x}\right)_{y}\end{aligned}$$

Subscripts denote variables held constant while differentiating.

Linear Approximation

Just use Taylor expansions.

Single variable case:

$$f(x) = f(p) + f'(p)(x-p)$$
$$+ f''(p)(x-a)^2 + O(x^3)$$

Multivariable case:

$$f(\mathbf{x}) = f(\mathbf{p}) + \nabla f(\mathbf{p})(\mathbf{x}-\mathbf{a})$$
$$+ (\mathbf{x}-\mathbf{p})^T H_f(p)(\mathbf{x}-\mathbf{p}) + O(\|\mathbf{x}-\mathbf{p}\|_2^3)$$

Linear Algebra

Big Theorems

- Rank Nullity:

$$|\ker(A)| + |\mathrm{im}\,(A)| = |\mathrm{domain}(A)|$$

- Fundamental Subspace Theorems

$$\mathrm{im}\,(A) \perp \ker(A^T), \qquad \ker(A) \perp \mathrm{im}\,(A^T)$$

- Compute
  - Determinant, trace, inverse, subspaces, eigenvalues, etc
  - Know properties too!
- Definitions
  - Vector space, subspace, singular, consistent system, etc

Fundamental Spaces

- Finding bases for various spaces of A:
  - $\text{rowspace} A / \text{im } A^T \subseteq \mathbb{R}^n$
    - Reduce to RREF, and take nonzero rows of $\mathrm{RREF}(A)$.
  - $\text{colspace} A / \text{im } A \subseteq \mathbb{R}^m$:
    - Reduce to RREF, and take columns with pivots from original A.

Fundamental Spaces

- $\text{nullspace}(A) / \ker A$:
  - Reduce to RREF, zero rows are free variables, convert back to equations and pull free variables out as scalar multipliers.
- Eigenspace:
  - Recall the equation:
$$\lambda \in \text{Spec}(A) \iff \exists \mathbf{v}_\lambda \ni A\mathbf{v}_\lambda = \lambda \mathbf{v}_\lambda$$
  - For each $\lambda \in \text{Spec}(A)$, compute $\ker(\lambda I - A)$

Big List of Equivalent Properties

Let A be an $n \times n$ matrix representing a linear map $L : V \to W$

TFAE:

- A is invertible and has a unique inverse A^{-1}
- A^T is invertible
- $\det(A) \neq 0$
- The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $b \in \mathbb{R}^m$
- The homogeneous system $A\mathbf{x} = 0$ has only the trivial solution $\mathbf{x} = 0$
- $\operatorname{rank}(A) = \dim(W) = n$
  - i.e. A is full rank
- $\operatorname{nullity}(A) := \dim(\operatorname{nullspace}(A)) = \dim(\ker L) = 0$

Big List of Equivalent Properties

- $A = \prod_{i=1}^{k} E_i$ for some finite k, where each E_i is an elementary matrix.
- A is row-equivalent to the identity matrix I_n
- A has exactly n pivots
- The columns of A are a basis for $W \cong \mathbb{R}^n$
  - i.e. $\operatorname{colspace}(A) = \mathbb{R}^n$
- The rows of A are a basis for $V \cong \mathbb{R}^n$
  - i.e. $\operatorname{rowspace}(A) = \mathbb{R}^n$
- $(\operatorname{colspace}(A))^{\perp} = \left(\operatorname{rowspace}(A^T)\right)^{\perp} = \{\mathbf{0}\}$
- Zero is not an eigenvalue of A.
- A has n linearly independent eigenvectors

Various Other Topics

- Quadratic forms
- Projection operators
- Least Squares
- Diagonalizability, similarity
- Canonical forms
- Decompositions (QR, VDV^{-1}, SVD, etc)

Ordinary Differential Equations

Easy IVPs

- Should be able to immediately write solutions to any initial value problem of the form

$$\sum_{i=0}^{n} \alpha_i y^{(i)}(x) = f(x)$$

  - Just write the characteristic polynomial.

Easy IVPs

- Example: A second order homogeneous equation
  $$ay'' + by' + cy = 0 \mapsto ax^2 + bx + c = 0$$
  - Two distinct roots:
    $$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$
  - One real root:
    $$y(x) = c_1 e^{rx} + c_2 x e^{rx}$$
  - Complex conjugates $\alpha \pm \beta i$:
    $$y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

More Easy IVPs

- The Logistic Equation

$$\frac{dP}{dt} = r\left(1 - \frac{P}{C}\right)P \implies P(t) = \frac{P_0}{\frac{P_0}{C} + e^{-rt}(1 - \frac{P_0}{C})}$$

- Separable

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{1}{g(y)}dy = \int f(x)dx + C$$

More Easy IVPs

- Systems of ODEs

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}(t) \implies \mathbf{x}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t}\ \mathbf{v}_i$$

for each eigenvalue/eigenvector pair $(\lambda_i, \mathbf{v}_i)$.

Less Common Topics

- Integrating factors
- Change of Variables
- Inhomogeneous ODEs (need a *particular solution*)
  - Variation of parameters
  - Annihilators
  - Undetermined coefficients
  - Reduction of Order
  - Laplace Transforms
  - Series solutions
- Special ODEs
  - Exact
  - Bernoulli
  - Cauchy-Euler

Topics: Number Theory

Definitions

- The fundamental theorem of arithmetic:

$$n \in \mathbb{Z} \implies n = \prod_{i=1}^{n} p_i^{k_i}, \quad p_i \text{ prime}$$

- Divisibility and modular congruence:

$$x \mid y \iff y = 0 \mod x \iff \exists c \ \ni \ y = xc$$

- Useful fact:

$$x = 0 \mod n \iff x = 0 \mod p_i^{k_i} \ \forall i$$

(Follows from the Chinese remainder theorem since all of the $p_i^{k_i}$ are coprime)

Definitions

- GCD, LCM

$$xy = \gcd(x, y)\,\mathrm{lcm}(x, y)$$

$$d \mid x \text{ and } d \mid y \implies d \mid \gcd(x, y)$$

$$\text{and } \gcd(x, y) = d \gcd(\frac{x}{d}, \frac{y}{d})$$

  - Also works for $\mathrm{lcm}(x, y)$
  - Computing $\gcd(x, y)$:
    - Take prime factorization of x and y,
    - Take only the distinct primes they have in common,
    - Take the minimum exponent appearing

The Euclidean Algorithm

Computes GCD, can also be used to find modular inverses:

$$
\begin{aligned}
a &= q_0 b + r_0 \\
b &= q_1 r_0 + r_1 \\
r_0 &= q_2 r_1 + r_2 \\
r_1 &= q_3 r_2 + r_3 \\
&\vdots \\
r_k &= q_{k+2} r_{k+1} + \mathbf{r_{k+2}} \\
r_{k+1} &= q_{k+3} r_{k+2} + 0
\end{aligned}
$$

Back-substitute to write $ax + by = \mathbf{r_{k+2}} = \gcd(a, b)$.

(Also works for polynomials!)

Definitions

- Coprime

$$a \text{ is coprime to } b \iff \gcd(a, b) = 1$$

- Euler's Totient Funtion

$$\phi(a) = \left|\{x \in \mathbb{N} \ni x \leq a \text{ and } \gcd(x, a) = 1\}\right|$$

  - Computing ϕ:

$$\gcd(a, b) = 1 \implies \phi(ab) = \phi(a)\phi(b)$$

$$\phi(p^k) = p^k - p^{k-1}$$

  - Just take the prime factorization and apply these.

Definitions

Know some group and ring theoretic properties of $\mathbb{Z}/n\mathbb{Z}$

- $\mathbb{Z}/n\mathbb{Z}$ is a field $\iff$ n is prime.
  - So we can solve equations with inverses: $ax = b \mod n \iff x = a^{-1}b \mod n$
- But there will always be *some* units; in general,

$$|(\mathbb{Z}/n\mathbb{Z})^\times| = \phi(n)$$

and is cyclic when $n = 1, 2, 4, p^k, 2p^k$

Chinese Remainder Theorem

The system

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ &\vdots \\ x &\equiv a_r \pmod{m_r} \end{aligned}$$

has a unique solution $x \mod \prod m_i$ iff $\gcd(m_i, m_j) = 1$ for each pair i, j.

Chinese Remainder Theorem

The solution is given by

$$x = \sum_{j=1}^{r} a_j \frac{\prod_i m_i}{m_j} \left(\left[\frac{\prod_i m_i}{m_j} \right]^{-1}_{\bmod \ m_j} \right)$$

Seems symbolically complex, but actually an easy algorithm to carry out by hand.

Chinese Remainder Theorem

Ring-theoretic interpretation: let $N = \prod n_i$, then

$$\gcd(i, j) = 1 \;\; \forall (i, j) \implies \mathbb{Z}_N \cong \bigoplus \mathbb{Z}_{n_i}$$

Theorems

- Fermat's Little Theorem and Euler's Theorem

$$a^p = a \mod p$$

$$p \nmid a \implies a^{p-1} = 1 \mod p$$

$$\text{and in general,}$$

$$a^{\phi(p)} = 1 \mod p$$

- Wilson's Theorem

$$n \text{ is prime } \iff (n-1)! = -1 \mod n$$

Advanced Topics

- Mobius Inversion
- Quadratic residues
- The Legendre/Jacobi Symbols
- Quadratic Reciprocity

Topics: Abstract Algebra

Definitions

- Group, ring, subgroup, ideal, homomorphism, etc
- Order, Center, Centralizer, orbits, stabilizers
- Common groups: $S_n, A_n, C_n, D_{2n}, \mathbb{Z}_n$, etc

Structure

- Structure of S_n
  - e.g. Every element is a product of disjoint cycles, and the order is the lcm of the order of the cycles.
  - Generated by (e.g.) transpositions
  - Cycle types
  - Inversions
  - Conjugacy classes
  - Sign of a permutation
- Structure of $\mathbb{Z}_n$

$$\mathbb{Z}_{pq} = \mathbb{Z}_p \oplus \mathbb{Z}_q \iff (p, q) = 1$$

Basics

Group Axioms

- Closure: $a, b \in G \implies ab \in G$
- Identity: $\exists e \in G \mid a \in G \implies ae = ea = a$
- Associativity: $a, b, c \in G \implies (ab)c = a(bc)$
- Inverses: $a \in G \implies \exists b \in G \mid ab = ba = e$

One step subgroup test:

$$H \leq G \iff a, b \in H \implies ab^{-1} \in H$$

Useful Theorems

Cauchy's Theorem

- If $|G| = n = \prod p_i^{k_i}$, then for each i there exists a subgroup H of order p_i.

The Sylow Theorems

- If $|G| = n = \prod p_i^{k_i}$, for each i and each $1 \leq k_j \leq k_i$ then there exists a subgroup $H_{i,j}$ for all orders $p_i^{k_j}$.
  - Note: partial converse to Cauchy's theorem.

Classification of Abelian Groups

Suppose $|G| = n = \prod_{i=1}^{m} p_i^{k_i}$

$$G \cong \bigoplus_{i=1}^{n} G_i \text{ with } |G_i| = p_i^{k_i} \text{ and}$$

$$G_i \cong \bigoplus_{j=1}^{k} \mathbb{Z}_{{p_i}^{\alpha_j}} \text{ where } \sum_{j=1}^{k} \alpha_j = k_i$$

G decomposes into a direct sum of groups corresponding to its prime factorization. For each component, you take the corresponding prime, write an integer partition of its exponent, and each unique partition yields a unique group.

Ring Theory

- Definition: $(R, +, \times)$ where $(R, +)$ is abelian and $(R, times)$ is a monoid.
- Ideals: $(I, +) \leq (R, +)$ and $r \in R, x \in I \implies rx \in I$
- Noetherian: $I_1 \subseteq I_2 \subseteq \cdots \implies \exists N \ni I_N = I_{N+1} = \cdots$
  - (Ascending chain condition)
- Differences between prime and irreducible elements
  - Prime: $p \mid ab \implies \mid a$ or $p \mid b$
  - Irreducible: x irreducible $\iff \nexists a, b \in R^\times \ni p = ab$
- Various types of rings and their relations:

$$\text{field} \implies \text{Euclidean Domain} \implies \text{PID} \implies \text{UFD} \implies \text{integral domain}$$

Topics: Real Analysis

- Properties of Metric Spaces
- The Cauchy-Schwarz Inequality
- Definitions of Sequences and Series
- Testing Convergence of sequences and series
- Cauchy sequences and completeness
- Commuting limiting operations:
    - $[\frac{\partial}{\partial x}, \int dx]$
- Uniform and point-wise continuity
- Lipschitz Continuity

Big Theorems

- **Completeness**: Every Cauchy sequence in $\mathbb{R}^n$ converges.
- **Generalized Mean Value Theorem**

$$f, g \text{ differentiable on } [a, b] \implies$$

$$\exists c \in [a, b] : [f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c)$$

  - Take $g(x) = x$ to recover the usual MVT
- **Bolzano-Weierstrass**: every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
- **Heine-Borel**: in $\mathbb{R}^n$, X is compact $\iff$ X is closed and bounded.

Topics: Point-Set Topology

General Concepts

- Open/closed sets
- Connected, disconnected, totally disconnected, etc
- Mostly topics related to metric spaces

Useful Facts

- Topologies are closed under
  - Arbitrary unions:

$$U_j \in \mathcal{T} \implies \bigcup_{j \in J} U_i \in \mathcal{T}$$

  - Finite intersections:

$$U_i \in \mathcal{T} \implies \bigcap_{i=1}^{n} U_i \in \mathcal{T}$$

- In $\mathbb{R}^n$, singletons are closed, and thus so are finite sets of points
  - Useful for constructing counterexamples to statements

Topics: Complex Analysis

General Concepts

- n-th roots:

$$e^{\frac{ki}{2\pi n}}, \qquad k = 1, 2, \cdots n - 1$$

- The Residue theorem:

$$\oint_C f(z)\, dz = 2\pi i \sum_k \mathrm{Res}(f, z_k)$$

  - Exams often include one complex integral
  - Need a number of other theorems for actually computing residues

Topics: Discrete Mathematics + Combinatorics

General Concepts

- Graphs, trees
- Recurrence relations
- Counting problems
  - e.g. number of nonisomorphic structures
- Inclusion-exclusion, etc

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example Problems

Example Problem 1

8. Which of the following is NOT a group?

   (A) The integers under addition

   (B) The nonzero integers under multiplication

   (C) The nonzero real numbers under multiplication

   (D) The complex numbers under addition

   (E) The nonzero complex numbers under multiplication

Example Problem 1

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   (E) The nonzero complex numbers under multiplication

C, because $\mathbb{Z} - \{0\}$ lacks inverses

(Would need to extend to $\mathbb{Q}$)

Example Problem 2

19. If z is a complex variable and $\overline{z}$ denotes the complex conjugate of z, what is $\lim_{z \to 0} \frac{(\overline{z})^2}{z^2}$?

(A) 0 (B) 1 (C) i (D) ∞ (E) The limit does not exist.

Example Problem 2

19. If z is a complex variable and $\overline{z}$ denotes the complex conjugate of z, what is $\lim_{z \to 0} \frac{(\overline{z})^2}{z^2}$?

(A) 0 (B) 1 (C) i (D) ∞ (E) The limit does not exist.

$$L = \lim_{(a,b)\to 0} \frac{(a-bi)^2}{(a+bi)^2} = \lim_{(a,b)\to 0} \frac{a^2 - b^2 - 2abi}{a^2 - b^2 + 2abi}$$

$$a = 0 \implies L = 1$$

$$a = b \implies L = -1$$

So E, because the limit needs to be path-independent.

Example Problem 3

24. Consider the system of linear equations

$$\begin{aligned} w + 3x + 2y + 2z &= 0 \\ w + 4x + y &= 0 \\ 3w + 5x + 10y + 14z &= 0 \\ 2w + 5x + 5y + 6z &= 0 \end{aligned}$$

with solutions of the form (w, x, y, z), where w, x, y, and z are real. Which of the following statements is FALSE?

(A) The system is consistent.

(B) The system has infinitely many solutions.

(C) The sum of any two solutions is a solution.

(D) $(-5, 1, 1, 0)$ is a solution.

(E) Every solution is a scalar multiple of $(-5, 1, 1, 0)$.

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Don't row-reduce or invert! Just one computation

$$\begin{pmatrix} 1 & 3 & 2 & 3 \\ 1 & 4 & 1 & 0 \\ 3 & 5 & 10 & 14 \\ 2 & 5 & 5 & 6 \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \mathbf{0}$$

Example Problem 3

So D, A are true. C is true because it's a homogeneous system. B is true because $A\mathbf{x} = 0 \implies A(t\mathbf{x}) = tA\mathbf{x} = 0$which means $t\mathbf{x}$ is a solution for every t. By process of elimination, E must be false.

Example Problem 4

42. Let $\mathbb{Z}^+$ be the set of positive integers and let d be the metric on $\mathbb{Z}^+$ defined by

$$d(m, n) = \begin{cases} 0 & \text{if } m = n \\ 1 & \text{if } m \neq n \end{cases}$$

for all $m, n \in \mathbb{Z}^+$. Which of the following statements are true about the metric space $(\mathbb{Z}^+, d)$?

I. If $n \in \mathbb{Z}^+$, then $\{n\}$ is an open subset of $\mathbb{Z}^+$.

II. Every subset of $\mathbb{Z}^+$ is closed.

III. Every real-valued function defined on $\mathbb{Z}^+$ is continuous.

(A) None (B) I only (C) III only (D) I and II only (E) I, II, and III

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Note $N_{\frac{1}{2}}(x) = \{x\}$, so every singleton is open. Any subset of $\mathbb{Z}$ is a countable union of its singletons, so every subset of $\mathbb{Z}$ is open. The complement any set is one such subset, so every subset is clopen. The inverse image of any subset of $\mathbb{R}$ under any $f : \mathbb{Z} \to \mathbb{R}$ is a subset of $\mathbb{Z}$, which is open, so every such f is continuous. So E.