UGA '22 Summer AG Reading
Torelli & its Applications

First, some historical context: we love finiteness in NT!

Conj (Mordell, 22)

Let $C/_K$ be a nice* curve with $g(C) \geq 2$ over K a number field. Then $\boxed{\#C(K) < \infty}$.

Mordell

$*$: Smooth, projective

Application: A Fermat curve

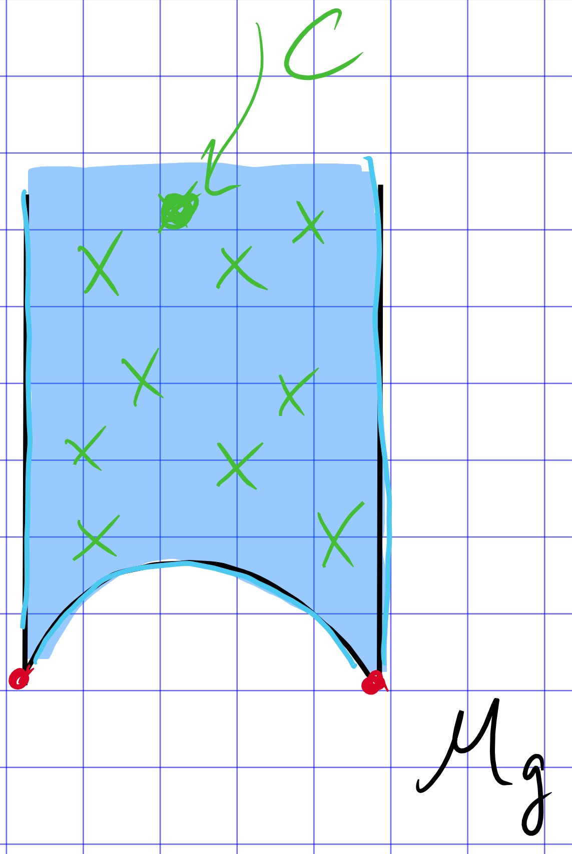
$$x^n + y^n = z^n \quad , \; n \geq 4$$

has only finitely many solutions over any number field!

Conj (Shafarevich, before 68?)

Fix

- $K/\mathbb{Q}$ a number field,
- $g \in \mathbb{Z}_{\geq 0}$ a genus,
- $S \subseteq Pl(K)$ a finite set of places.



Then

$$\underbrace{\# \left\{ [C/_K] \;\middle|\; \begin{array}{l} g(C) = g, \; C \text{ has good} \\ \text{reduction away from } S \end{array} \right\}}_{\# \mathcal{M}_g(\mathcal{O}_{K,S})} < \infty .$$

Rmk: Likely inspired by the following:

Theorem (Hermite, late 1800s?)

For $S \subseteq Pl(\mathbb{Q})$ finite and n fixed,

$$\# \left\{ K/\mathbb{Q} \;\middle|\; \begin{array}{l} [K:\mathbb{Q}] \leq n, \; K \text{ unram.} \\ \text{outside of } S \end{array} \right\} < \infty .$$

which uses...

Theorem (Hermite-Minkowski, late 1800s)

Hermite

Minkowski

For any fixed N,

$$\# \{ K/\mathbb{Q} \mid |\Delta_K| \leq N \} < \infty$$

$\uparrow$

$\mathcal{O}_K = \mathbb{Z}\mathcal{S}$, $\mathcal{S} = \{s_j\}$, $\mathrm{Emb}(K, \mathbb{C}) = \{\sigma_i\}$

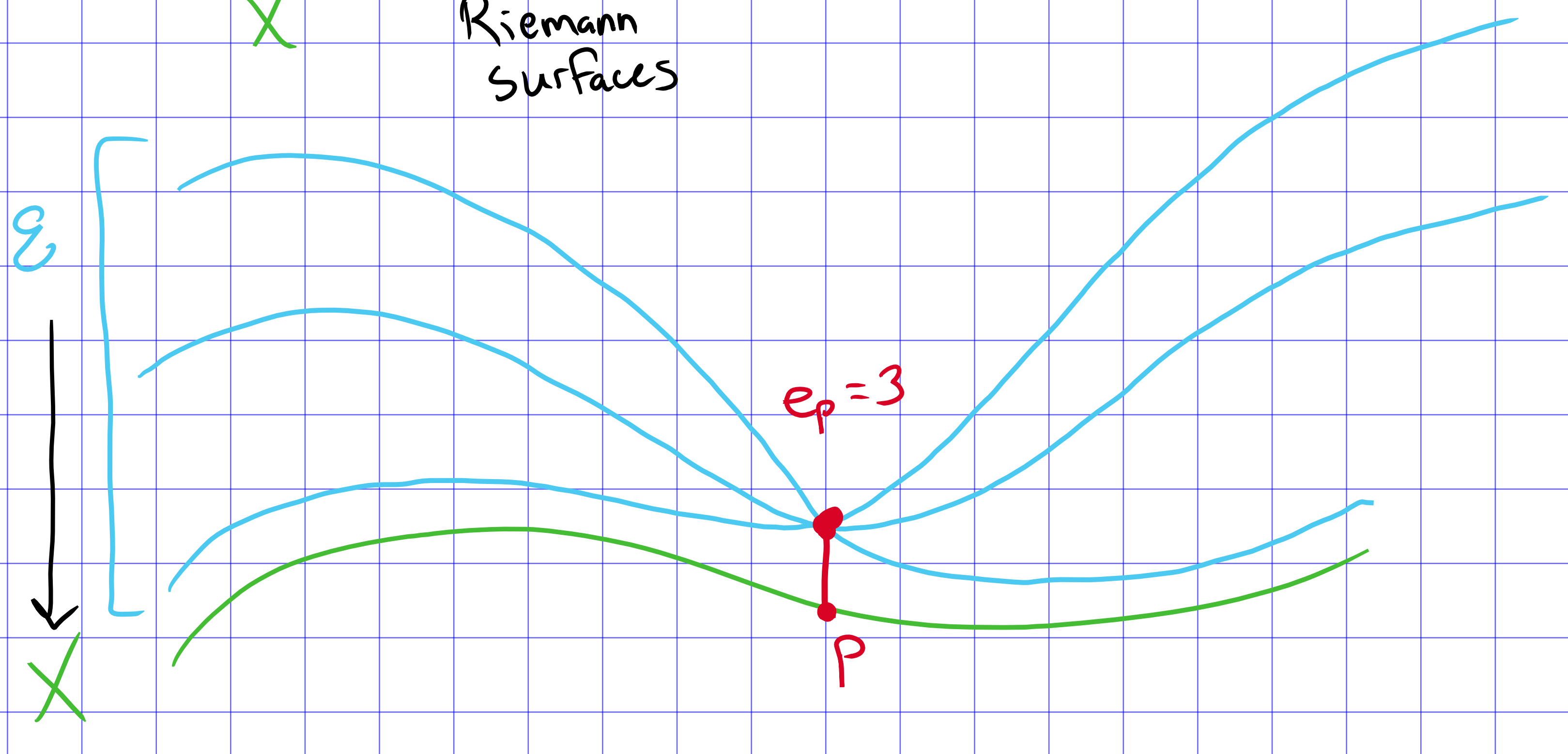
$\Rightarrow \Delta_K = \det\left(\sigma_i(s_j)\right)^2.$

Prop: Shafarevich $\Rightarrow$ Mordell.

Pf:

- Goal: show $\# C(K) < \infty$. Strat: count covers of C.
- Let C have good reduction away from
  $$S = \{ v \in Pl(K) \mid v | 2 \}$$
  ↑ primes lying above 2
- $\exists\, L/K$ containing all extensions of degree $\leq 2^{2g}$ which are unramified away from S.
- Each $p \in C(K)$ yields a cover $f_p : C_p \to C_K$ ramified only over p, of bounded uniform degree $d_p \leq B_g$
- Hurwitz genus formula:

$\begin{array}{c} \mathcal{E} \\ \downarrow \pi \\ X \end{array}$ $\mathbb{C}$-analytic cover of Riemann surfaces $\Rightarrow$ $$\chi(\mathcal{E}) = \deg(\pi) \cdot \chi(X) + \underbrace{\sum_{p \in \mathcal{E}} (e_p - 1)}_{\text{ramification indices}}$$



$$2g(C_p) - 2 \leq B_g \cdot (2g - 2) + (B_g - 1)$$

$$\Rightarrow \quad g(C_p) \leq O(g, B_g) \quad \text{some const.}$$

- Shaf. $\Rightarrow$ only fin. many such curves C_p.
- Apply a result of de Franchis: for a fixed C_p, there are only fin. many maps f_p when $g \geq 2$.
- We win! $\# X(K) < \infty$. ∎

Rmk: Shafarevich conj. proved by Faltings in 83, and thus Mordell's conj. as well. Proof uses a Torelli theorem. Later generalized to K3 surfaces of bounded degree

↑

(N.B., my research topic is on moduli of K3's!)

Jacobians

- Moral: For C a curve, $\mathrm{Bun}_G(C)$ is complicated

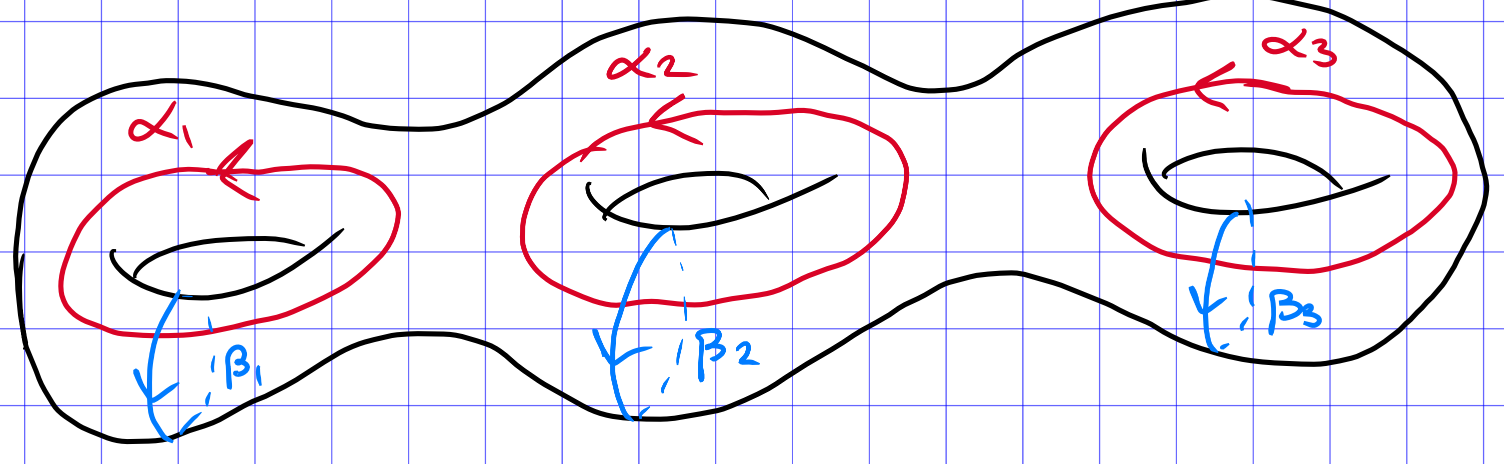
$$\uparrow$$

The functor $\mathrm{Sch} \to \mathrm{Grpd}$

$$C \mapsto \left\{ \begin{array}{c} E \\ \downarrow \\ C \end{array} \ G\text{-torsors} \right\}$$

↳ <u>But</u> degree zero line bundles are much nicer! One gets a moduli space $\mathrm{Pic}^0(C) \underset{\text{Abel-Jacobi thm.}}{\cong} \mathrm{Jac}(C)$ which is a PPAV.

- A heuristic construction for Riemann surfaces:
  - Choose generators $\{\omega_1, \cdots, \omega_g\} \in \Omega^1_C$
  - Choose generators $\{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g\} \in H_1(X;\mathbb{Z})$



$(g=3)$

  - Produce a lattice

$$\Lambda := \left\{ \left(\int_\gamma \omega_1, \cdots, \int_\gamma \omega_g \right) \in \mathbb{C}^g \;\middle|\; \gamma \in \pi_1(C) \right\}$$

which has a $\mathbb{Z}$-basis with $2g$ generators

$$\left\{ \left(\int_{\alpha_i} \omega_1, \cdots \int_{\alpha_i} \omega_g \right) \right\}_{i=1}^{g} \cup \left\{ \left(\int_{\beta_i} \omega_1, \cdots \int_{\beta_i} \omega_g \right) \right\}_{i=1}^{g}$$

  - Set $\mathrm{Jac}(C) := \mathbb{C}^g / \Lambda$.

  - The Abel map takes the form

$$\begin{aligned} C &\longrightarrow \mathrm{Jac}(C) \\ p &\longmapsto \left(\int_{x_0}^{p} \omega_1, \cdots, \int_{x_0}^{p} \omega_g \right) \end{aligned}$$

  - General construction:

$$J(C) := \overbrace{H^0(C;\Omega^1_C)^\vee}^{\cong \mathbb{C}^g} \Big/ \overbrace{H_1(C;\mathbb{Z})}^{\Lambda}.$$

N.B. this generalizes to construct the Albanese variety in higher dimensions, where

$$\mathrm{Alb}(X) \cong \mathrm{Pic}^0(X)^\vee$$

Torelli

Why care about Jacobians?

- Every abelian variety is the surjective image (quotient, in fact) of some Jacobian variety ($K=\overline{K}$ alg. closed)
- **Torelli:** A curve* C is uniquely determined by its (canonically polarized) Jacobian $\mathrm{Jac}(C)$.

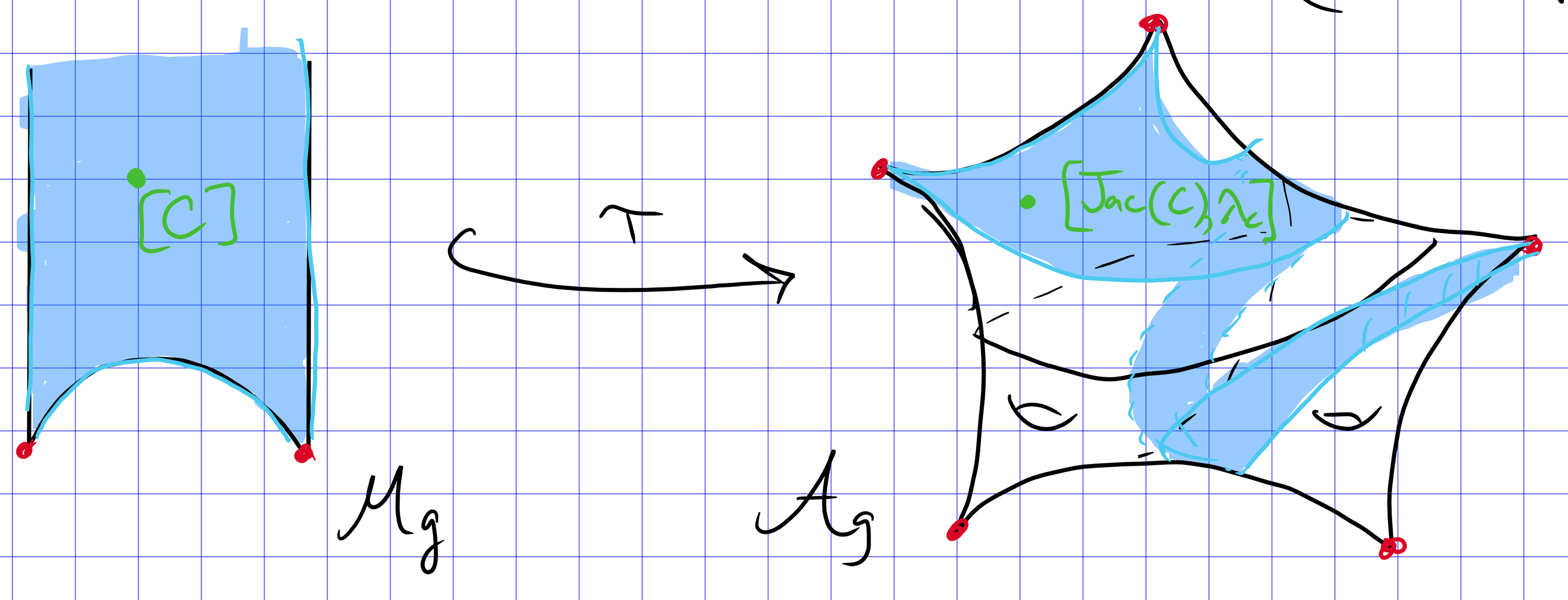
*: Minor issues for hyperelliptic C.

- Transport of structure on moduli stacks: we get a morphism

"period map"
$$\left[\begin{array}{rcl} \tau: \mathcal{M}_g & \longrightarrow & \mathcal{A}_g \\ C & \longmapsto & (\mathrm{Jac}(C), \Theta_C) \end{array}\right.$$

Principally polarized abelian varieties (PPAVs)

which is injective on geometric points (use coarse moduli spaces)



- Can reformulate Torelli: a curve C is determined by its (polarized) Hodge structure: the lattice structure of $\Lambda := H_1(C;\mathbb{Z}) \subseteq H^0(C,\Omega^1_C)^\vee$ with its bilinear form $Q(\omega,\eta) = \int_X \omega\wedge\eta$ (+ Riemann relations)

$\rightsquigarrow$ Can reduce studying curves to lattice theory!

$$H^1_{dR}(C) = H^0(C;\Omega^1_C) \oplus H^1(C;\mathcal{O}_C)$$

A more precise statement:

$\overline{\mathbb{F}_q(t)}$

Thm (Torelli):

Let C, D be complete smooth curves over $K=\overline{K}$.

For any isomorphism of PPAVs

$$\beta: (\mathrm{Jac}(C), \lambda_C) \longrightarrow (\mathrm{Jac}(D), \lambda_D)$$

there is a compatible isomorphism of curves

$$\alpha: C \longrightarrow D$$

in the sense that

$$\begin{tikzcd} \mathcal{M}_g & \mathcal{A}_g \\ C \ni p \arrow[r, "f_p"] \arrow[d, "\alpha"'] & (\mathrm{Jac}(C), \lambda_C) \arrow[d, "\wr\,\beta"] \\ D \ni q \arrow[r, "f_q"] & (\mathrm{Jac}(D), \lambda_D) \end{tikzcd}$$

"Commutes up to a translation"

$$f_q \circ \alpha = (\pm\beta \circ f_p) + c$$

for some $c \in \mathrm{Jac}(D)$

Where f_p, f_q are determined by choices of points $p\in C$, $q\in D$. Moreover, if $g(C)\geq 2$, then α, c, and the choice of sign above are uniquely* determined.

* Minor hyperelliptic issues.

Cor

If K is perfect, $g(C) = g(D) \geq 2$, and $\mathrm{Jac}(C) \cong \mathrm{Jac}(D)$, then $C \cong D$.

Idea of pf.:

- Choose $\beta: \mathrm{Jac}(C) \to \mathrm{Jac}(D)$ defined over K.
- Choosing pts $p\in C(\overline{K})$, $q \in D(\overline{K})$ yields
  $\left.\begin{array}{l} \alpha: C \to D \quad \text{s.t.} \\ f_q\circ\alpha = \pm\beta\circ f_p + c \end{array}\right\}$ defined over $\overline{K}$.
- Show α doesn't depend on choice of (p,q)
- Show $\sigma\alpha = \alpha$ $\forall \sigma \in G_K$ to get α defined over K. ∎

In fact, more is true:

Thm (Local / infinitesimal Torelli)

For $g \geq 2$, $[C] \in \mathcal{M}_g$ is not hyperelliptic $\Leftrightarrow$

$$d\tau^g_{[C]} : T_{[C]}\mathcal{M}_g(\mathbb{C}) \hookrightarrow T_{[\tau(C)]}\mathcal{A}_g(\mathbb{C}) \text{ is injective.}$$

$$T_{[C]}\mathcal{M}_g(\mathbb{C}) \cong H^1(C; T_C) \qquad T_{[\tau(C)]}\mathcal{A}_g(\mathbb{C}) \cong \mathrm{Sym}^2 H^1(C; \mathcal{O}_C)$$

Slogans:

- τ is an immersion away from the hyperelliptic locus
- One can recover C from how $\mathrm{Jac}(C)$ deforms in small neighborhoods
- Hodge-theoretic reformulation: C can be recovered by its infinitesimal variation of (polarized) Hodge structure (VHS).
- If $C \notin \{$trigonal, plane quintic, hyperelliptic$\}$ then C is uniquely determined by $d\tau_{[C]}$.

Where do these results fit into the big picture?

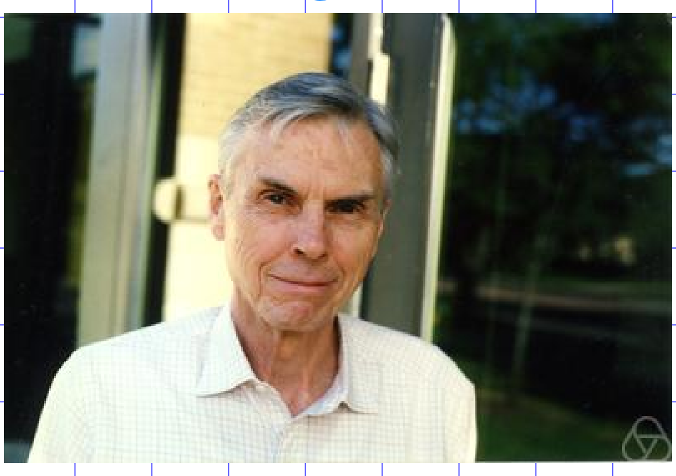
Conjecture (Shafarevich-Tate, 63)

If $A \in \mathrm{AbVar}_{/K}$ for K a global field (e.g. $K_{/\mathbb{Q}}$, $\mathbb{F}_q(t)$),

$$\# Ш(A) < \infty.$$
$$\begin{array}{c} \shortparallel \\ \mathrm{Ker}\left(H^1_{\mathrm{Gal}}(K; A) \longrightarrow \prod_{v \in \mathrm{Pl}(K)} H^1_{\mathrm{Gal}}(K_v; A) \right) \end{array}$$

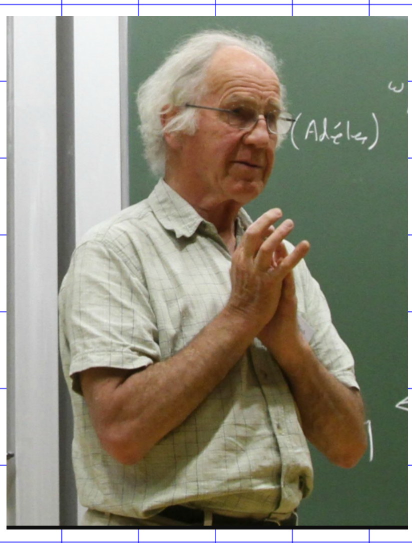
Shafarevich

Tate



Conjecture (Birch & Swinnerton-Dyer, rank version, 65)

Birch



Swinnerton-Dyer

Let $A \in \mathrm{AbVar}_{/K}$ for K a global field, and define

$$L_A(s) := \prod_{\substack{v \in \mathrm{Pl}(A) \\ \text{finite}}} \det\left(1 - q_v^{-s} \cdot \mathrm{Frob}_v \mid (V_\ell A)^{I_v}\right)^{-1} = \underbrace{\prod_v L_{A,v}(s)}_{\text{Local L-fns}}$$

where

- q_v : order of residue field
- $V_\ell A := T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, $T_\ell A := \varprojlim_n A(K^{sep})[\ell^n]$.
- $\mathrm{Frob}_v \in G_K$
- $I_v \leq G_K$ is inertia.

Then

$$\mathrm{rank}_{\mathbb{Z}}\, A(K) = \mathrm{Ord}_{s=1}\, L_A(s).$$

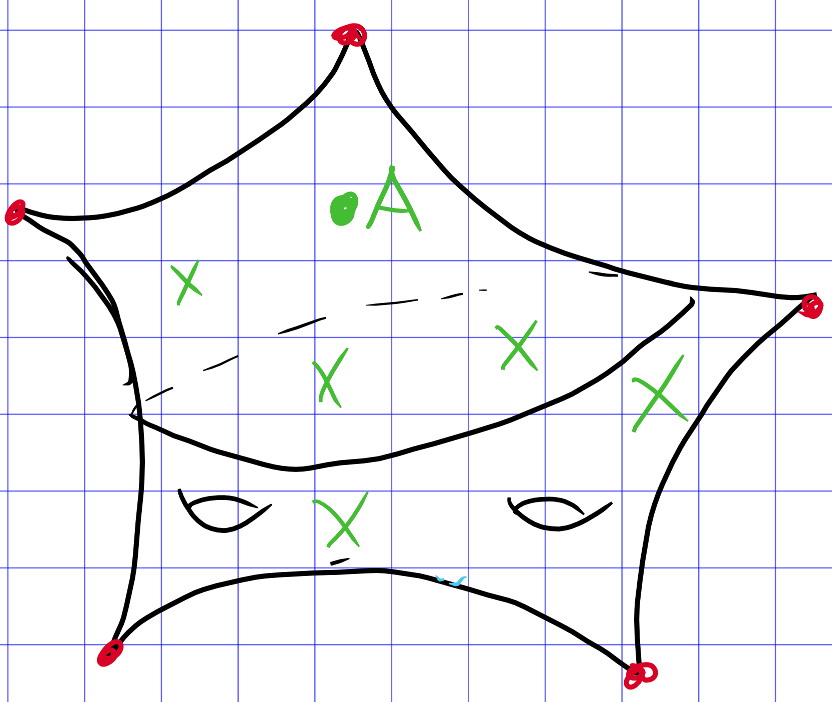
We have made <u>some</u> progress on these, e.g. due to:

Thm (Faltings)

Let $A \in \mathrm{AbVar}_{/K}$ for K a number field. Then

F1: $\#\{A' \in \mathrm{AbVar}_{/K} \mid A' \text{ is isogenous to } A\} < \infty.$

$\mathcal{A}_g(K)$



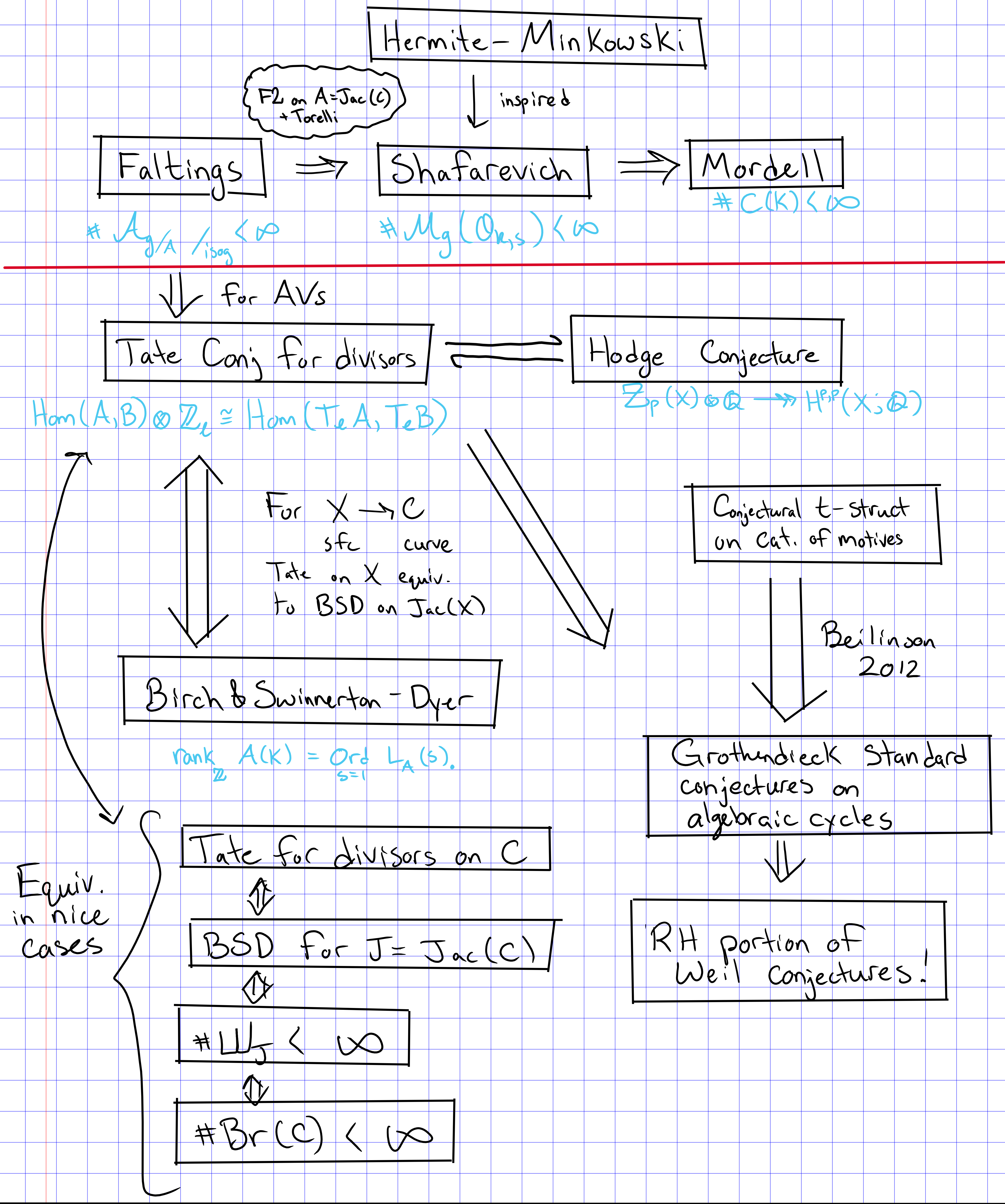
F2: $\# \mathcal{A}_g(\mathcal{O}_{K,S}) < \infty$, i.e. given K a number field, $S \subseteq \mathrm{Pl}(K)$ finite, $g \in \mathbb{Z}_{\geq 0}$, there are only finitely many iso classes of AVs of dim g having good reduction away from S.

__Rmk:__ What does this have to do with Jacobians?

The Weil conjectures!

Weil ('40) proves RH for curves $C/\mathbb{F}_q$ using $\mathrm{Jac}(C)$.

There is a web of conjectures (some now theorems):



__Some refs:__

- Milne's notes on AVs and Jacobians
- Daniel Litt's Class on rational points
  - ↳ Uses parts of Poonen's "Rational Points"
- Adam Dawser's notes on Torelli
- Lots of Wikipedia!

Fin!