YAML

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- Tags:
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Motivation

Goal:
Classify Lagrangians $L \hookrightarrow M$ up to Hamiltonian isotopy.

Potential approach

- Fix (M, ω), define a category $\mathsf{Fuk}(M, \omega)$ whose objects are Lagrangian submanifolds $L \hookrightarrow M$.
- The moduli space $\operatorname{Ob}\mathsf{Fuk}(M, \omega)$ is infinite-dimensional, locally $\ker d^1 : \Omega^1_{/L} \to \mathcal{O}_L$ (closed 1-forms); quotient by Hamiltonian isotopy to get $\beta_1(L)$ (first Betti number).
- Known in special cases, e.g. Lagrangian $S^2 \hookrightarrow S^2 \times S^2$.
- Unknown: Lagrangian $T^2 \hookrightarrow \mathbb{P}^3$?
- Essentially known for X^1, defined by:
  - Let $X \hookrightarrow \mathbb{C}^N$ be a smooth affine algebraic variety
  - Take the projective closure to get a smooth projective $\overline{X} \hookrightarrow \mathbb{CP}^N$
  - Restrict the Fubini-Study-Kahler form ω_{FS} on $\mathbb{CP}^N$ to make $\overline{X}$ an exact symplectic manifold
  - Assume $K_{\overline{X}} \cong \mathcal{O}_{\overline{X}}(-d)$ for some d so X is CY.
  - Take a hyperplane section $X^1 := X \cap \mathbb{C}^{N-1}$.

- **Theorem**: For $d \neq -2, n \geq 1, \operatorname{ch} \mathbb{F} \neq 2$, a certain Fukaya-type category $\Pi\mathsf{TwFuk}(X^1, \omega_{\mathrm{FS}})$ is "computable" from combinatorial data arising from Picard-Lefschetz theory on X, where $\mathsf{Fuk}(X^1, \omega_{\mathrm{FS}})$ is an A_∞ category.
    - Idea: embed $\mathsf{A} \in A_\infty(\mathsf{Cat}) \hookrightarrow \mathsf{TwA} \in A_\infty(\mathsf{Cat})$ into a triangulated A_∞-category, take derived category $D(\mathsf{A}) := H^0\mathsf{TwA}$ to get a usual triangulated category, take a split closure/Karoubi-completion to get $D^\pi(\mathsf{A})$ (close under taking summands of idempotent endomorphisms), lift this to a similar construction at the A_∞ level called $\Pi\mathsf{TwA}$

- **New goal**: find a (hopefully finite) set of Lagrangians $L_i \hookrightarrow M$ which *split-generate* $\Pi\mathsf{TwA}$: every object is obtained by taking iterated mapping cones and direct summands (weak property!).
- **Conjecture (Arnold)**: if M is closed compact and $L \hookrightarrow \mathbf{T}M$ is a closed compact exact Lagrangian with its standard Liouville form, then L is is Hamiltonian isotopic to the zero section of $\mathbf{T}M \to M$.

Defining Fuk

Main characterization of $\mathsf{A} := \mathsf{Fuk}(M, \omega)$: setting $\mathsf{A}(L, L') = \mathrm{CF}(L, L')$ with differential μ_1, composition μ_2, and higher operations μ_k makes $\mathsf{Fuk}(M, \omega)$ into a

- Λ-linear
- $\mathbb{Z}$-graded
- non-unital
    - but cohomologically unital
- A_∞ category.

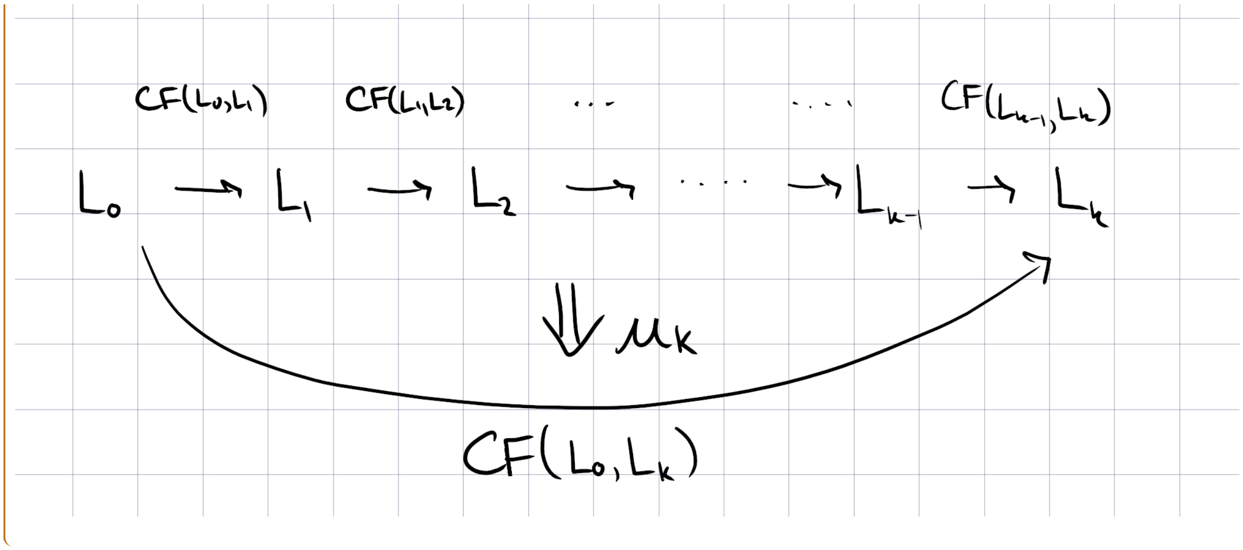
Definition

Definition: the category nu-$A_\infty(\mathsf{Cat})$ of **non-unital A_∞categories:** fix a field $\mathbb{F}$, then A is the data of

- A *set* of objects $\mathrm{Ob}\mathsf{A}$
- A $\mathbb{Z}$-graded vector space $\mathsf{A}(x, y)$ for every two objects
- For every $d \in \mathbb{Z}_{\geq 1}$, a composition map

  $$\mu_{\mathsf{A}}^d : \mathsf{A}(x_{d-1}, x_d) \otimes_{\mathbb{F}} \mathsf{A}(x_{d-2}, x_{d-1}) \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathsf{A}(x_0, x_1) \to \mathsf{A}(x_0, x_d)[2-d],$$

  where $V[n]$ for a graded vector space denotes shifting the grading *down* by n.



- For every such d, A_∞ associativity relations:

$$R_d := \sum_{m=1}^{d} \sum_{n=0}^{d-m} (-1)^{\eta_n} \mu_{\mathsf{A}}^{d-m+1}(a_d,\ a_{d-1}\ \cdots,\ a_{n+m+1},\quad \star(n,m),\quad a_n,\ a_{n-1},\ \cdots,\ a_1) = 0$$

where $\star(n,m) = \mu_{\mathsf{A}}^{m}(a_{n+m},\ a_{n+m-1}, \cdots,\ a_{n+1})$ and $\eta_n := \sum_{i=1}^{n} |a_i| - n$.

> Keep η on board.

NB: non-unital means "not necessarily unital".

Remark: This category carries "higher products" coming from stringing together multiple morphisms. What this looks like in our case, at least when $[\omega].\, \pi_2(M, L_i) = 0$ (zero symplectic area for all spheres):

The map is defined by

$$\mu^k(p_k, \cdots, p_1) = \sum_{q \in L_0 \pitchfork L_k, \left\{[u] \,\middle|\, \mathrm{Ind}\, u = 2-k\right\}} \sharp\mathcal{M}(p_1, \cdots, p_k, q;\ [u], J) T^{\omega([u])} q$$

For $k = 2$:

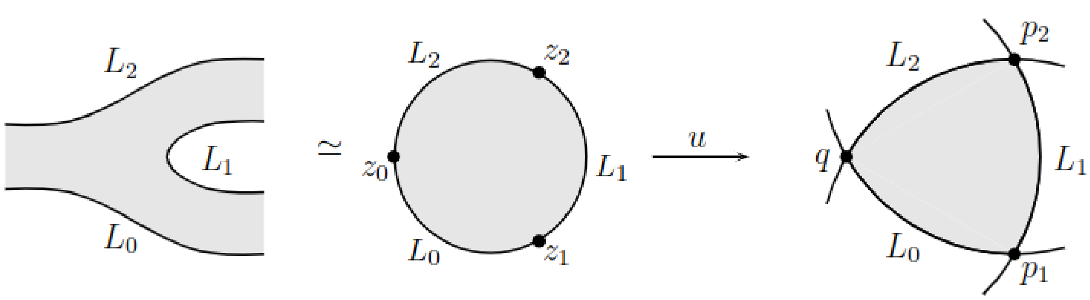
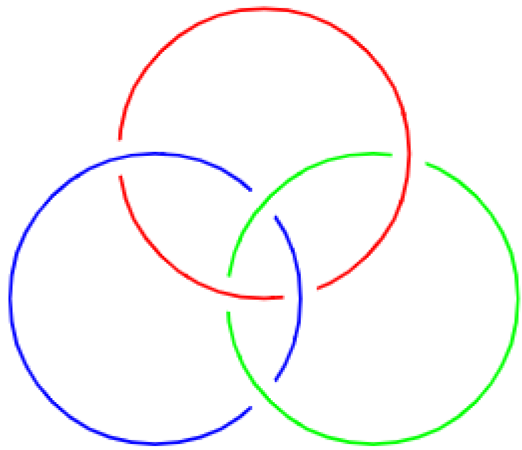


FIGURE 5. A pseudo-holomorphic disc contributing to the product map.

Remark: Why keep track of higher products? Compare cup-products (2-ary) to Massey products (3-ary):

[19] As the conventional example, we have the Borromean Rings, three linked circles embedded into 3-space:



If we embed the borromean rings into S^3, and take the complement $S^3 \setminus B$, we get three generators of the first cohomology group corresponding to the three rings. However, since their pairwise linking numbers are zero, their cup products are as well. However, the third massey product of these three generators is non-zero, and in some sense represents a "three-fold linking number".

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In the usual DGA setting, one can realize the Massey product $\langle x, y, z\rangle_3$ as a "composition" $\mu^3(x, y, z)$.

◐ **Definition**

Definition: For R a ring, a category A is R-**linear** iff it is *enriched* over the monoidal category $(\text{R-Mod}, \otimes_R)$, i.e. $\mathsf{A}(x, y) \in \text{R-Mod}$ and composition $\mathsf{A}(x, y) \otimes_R \mathsf{A}(y, z) \to \mathsf{A}(x, z)$ is a morphism in R-Mod. The category is $\mathbb{Z}$-**graded** if $\mathsf{A}(x, y) = \oplus_{n\in\mathbb{Z}} \mathsf{A}(x, y)_n$, i.e. every hom set decomposes into $\mathbb{Z}$-graded pieces. It is a **differential** $\mathbb{Z}$**-graded category** if it is enriched over $(\text{Ch}(\text{R-Mod}), \otimes_{R,\text{gr}})$, i.e. there are differentials $\partial_{x,y,n} : \mathsf{A}(x, y)_n \to \mathsf{A}(x, y)_{n+1}$ of square zero.

Remark: Typically take $R = \mathbb{F}$ or Λ a field to get vector spaces.

◐ **Definition**

Definition: Let $\mathsf{A} \in$ nu-$A_\infty(\mathsf{Cat})$. Its **cohomological category** $H(\mathsf{A})$ has

- the same objects as A,
- morphisms given by taking cohomology of the morphisms of A, i.e. $H(\mathsf{A})(x,y) = H^*(\mathsf{A}(x,y), \mu^1_{\mathsf{A}})$
- Composition defined by $[g].[f] := (-1)^{|g|}[\mu^2_{\mathsf{A}}(g,f)]$.

Remark: $H(\mathsf{A})$ is generally an (ordinary) R-linear $\mathbb{Z}$-graded category, except it may not have identity morphisms. This the notion of isomorphism is delicate. The A_∞ relations will imply that μ^2_{A} descends to an associative composition on cohomology. If $\mathsf{A} := \mathsf{Fuk}(M,\omega)$, then $H^0(\mathsf{A})$ is sometimes called the **Donaldson-Fukaya category**. However, important information in the higher μ^i is lost.

◐ **Definition**

Definition: For $\mathsf{A}, \mathsf{B} \in$ nu-$A_\infty(\mathsf{Cat})$, define **non-unital A_∞ functors** $F \in$ nu-Fun(A,B) as

- A map $F : \mathrm{Ob}\mathsf{A} \to \mathrm{Ob}\mathsf{B}$
- For every $d \geq 1$,

$$F^d : \mathsf{A}(x_{d-1}, x_d) \otimes_{\mathbb{F}} \cdots \mathsf{A}(x_0, x_1) \to \mathsf{B}(Fx_0, Fx_d)$$

- Relations

$$\sum_{r=1}^{\infty} \sum_{s_1+\cdots+s_r=d} \mu^r_{\mathsf{B}}(\, F^{s_r}(a_d, \cdots, a_{d-s_r+1}),\ \cdots,\ F^{s_1}(a_{s_1}, \cdots, a_1)\,))$$
$$= \sum_{m,n} (-1)^{\eta_n} F^{d-m+1}(a_d, \cdots, a_{n+m+1},\ \mu^m_{\mathsf{A}}(a_{n+m}, \cdots, a_{n+1}),\ \cdots, a_n, \cdots, a_1)$$

$H(F) : H(\mathsf{A}) \to H(\mathsf{B})$ is an ordinary linear graded non-unital functor whose action on morphisms is $[f] \mapsto [F^1(f)]$. We say F is **cohomologically full (resp. faithful)** if $H(F)$ is full (resp. faithful), and F is a **quasi-isomorphism** if $H(F)$ is an isomorphism. Two A_∞ categories are **quasi-isomorphic** iff there exists a quasi-isomorphism.

◐ **Definition**

Definition: the category $\mathsf{Q} := \text{nu-}A_\infty(\mathsf{Cat})$ has objects F as above.
Its morphisms are chain complexes, an element $T \in \mathsf{Q}(F,G)_g$ is a sequence $(T^0, T^1, \cdots)$ where each T^d is a family of multinear maps of degree $(g-d)$:

$$\mathsf{A}(x_{d-1}, x_d) \otimes_{\mathbb{F}} \cdots \mathsf{A}(x_0, x_1) \to \mathsf{B}(Fx, Gx_d)[g-d] \qquad \forall (x_0, \cdots, x_d) \in \mathsf{A}$$

E.g. T^0 is a family of maps in $\mathsf{B}(Fx, Gx)_g$ for each objects $x \in \mathsf{A}$.
We call T a **pre-natural transformation** from F to G.

◐ Definition

Definition: Say $F, G \in \mathrm{Ob}\mathsf{Q}$ are **homotopic** if the following holds: let $D = F - G \in \mathsf{Q}(F,G)_1$ be the pre-natural transformation defined by

- $D^0 = 0$
- $D^d = F_0^d - G_1^d$ for $d > 0$

This yields an ordinary natural transformation where $\mu^1_{\mathsf{Q}}(D) = 0$.
We say that F, G are **homotopic** if $D = \mu^1_{\mathsf{Q}}(T)$ for some $T \in \mathsf{Q}(F,G)_0$. where $T^0 = 0$.

Remark: homotopic functors $F \simeq G$ induce isomorphic functors on homological categories, $H(F) \cong H(G)$.

◐ Definition

Definition: For a fixed $\mathsf{A} \in \text{nu-}A_\infty(\mathsf{Cat})$, the category of **right A_∞-modules over** A is defined as $\text{nu-A-Mod} := \text{nuFun}(A^{\mathrm{op}}, \mathsf{Ch}(\mathbb{F}\text{-Mod}))$. An object $M \in \text{nu-A-Mod}$ is a graded $\mathbb{F}$-modules $M(x)$ for each $x \in \mathsf{A}$, along with maps

$$\mu^d_M : M(x_{d-1}) \otimes_{\mathbb{F}} \mathsf{A}(x_{d-2}, x_{d-1}) \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathsf{A}(x_0, x_1) \to M(x_0)[2-d]$$

This induces $H(M) \in \text{nu-H(A)-Mod}$, i.e. a functor $H(M) \in \text{nu-Fun}(H(A), \mathsf{Ch}(\mathbb{F}\text{-Mod}))$, which for every $x \in A$ is the cohomology of $M(x)$ with respect to the differential $b \xrightarrow{\partial} (-1)^{|b|}\mu^1_M(b)$.

◐ Definition

Definition: A usual category is **unital** if it has identity morphisms for every object. A category $\mathsf{A} \in \text{nu-}A_\infty(\mathsf{Cat})$ is **strictly unital** if for each $x \in \mathsf{A}$ there is a unique

$e_x \in \mathsf{A}(x,x)_0$ such that

- $\mu^1_{\mathsf{A}}(e_x) = 0$
- For every $a \in \mathsf{A}(x_0, x_1)$,

$$(-1)^{|a|}\mu^2_{\mathsf{A}}(e_{x_1}, a) = \mu^2_{\mathsf{A}}(a, e_{x_0}) = a$$

- For $a_k \in \mathsf{A}(x_{k-1}, x_k)$ and any $d > 2$ and $0 \leq n < d$,

$$\mu^d_{\mathsf{A}}(a_{d-1}, \cdots, a_{n+1}, e_{x_n}, a_n, \cdots, a_1) = 0$$

We say A is **cohomologically unital** or c-**unital** iff $H(A)$ is a unital, making it an ordinary graded linear category.

◐ Definition

Definition: A $\mathsf{A} \in$ nu-$A_\infty(\mathsf{Cat})$ is **homotopy unital** if

- $\mathrm{Ob}(\mathsf{A})$ forms a set.
- Homs $\mathsf{A}(x_0, x_1)$ are graded vector spaces,
- There are multilinear maps

$$\mu^{d,(i_d,\cdots,i_0)}_{\mathsf{A}} : \mathsf{A}(x_{d-1}.x_d) \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathsf{A}(x_0, x_1) \to \mathsf{A}(x_0, x_d)[2 - d - 2\sum_k i_k]$$

- Satisfying generalized associativity equations which reduce to the usual ones when $i_1 = \cdots = i_d = 0$:
    - $\mu^1_{\mathsf{A}}(\mu^{0,(1)}_{\mathsf{A}}) = 0$
    - $$(-1)^{|a|-1}\mu^2_{\mathsf{A}}\left(\mu^{0,(1)}_{\mathsf{A}}, a\right) + \mu^1_{\mathsf{A}}\left(\mu^{1,(1,0)}_{\mathsf{A}}(a)\right) + \mu^{1,(1,0)}_{\mathsf{A}}\left(\mu^1_{\mathsf{A}}(a)\right) = a$$
    - $$\mu^2_{\mathsf{A}}\left(a, \mu^{0,(1)}_{\mathsf{A}}\right) + \mu^1_{\mathsf{A}}\left(\mu^{1,(0,1)}_{\mathsf{A}}(a)\right) + (-1)^{|a|-1}\mu^{1,(0,1)}_{\mathsf{A}}\left(\mu^1_{\mathsf{A}}(a)\right) = -a,$$
    - $$\mu^{1,(1,0)}_{\mathsf{A}}\left(\mu^{0,(1)}_{\mathsf{A}}\right) + \mu^{1,(0,1)}_{\mathsf{A}}\left(\mu^{0,(1)}_{\mathsf{A}}\right) + \mu^1_{\mathsf{A}}\left(\mu^{0,(2)}_{\mathsf{A}}\right) = 0.$$

Remark: equations 1 and 2 say that multiplication with the cocycle $e_x = \mu^{0,(1)}_{\mathsf{A}}$ is chain homotopy to the identity. The others say $\mu^2_{\mathsf{A}}(e_x, e_x) = e_x$ up to a coboundary, and the difference of any two such coboundaries is a cohomologically trivial cocycle. Continuing these equations yields higher such coherens.

Remark: $\text{Fuk}(M, \omega)$ will not be strictly unital but will be cohomologically unital and homoopty unital, and homotopy units can be constructed geometrically. Moreover any homotopy unital A_∞ category is quasi-isomorphic to a strictly unital A_∞ category in a canonical way, and there is a general procedure to equip a c-unital category with homotopy units. So we can just work with c-unital categories.