Talbot 2022: Scissors Congruence and Algebraic K-theory

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1 Talk 8: Annihilator of the Lefschetz Motive (D. Zack Garza)

Reference: [Zak17]

1.1 Preliminaries

Remark 1.1. Let's begin by getting a sense of where we are now and where we are headed:

- Yesterday we discussed classical scissors congruence.
- The main theme of today is going from scissors congruence to $\mathsf{K}-$theory; that is, how can we encode and detect scissors congruence in the language of $\mathsf{K}-$theory? One approach we've seen uses assemblers to enrich the classical Grothendieck group to a spectrum, and we've seen how classical motivic measures can be formulated in this setting.
- Tomorrow and for the next few days, we'll be studying how to go from $\mathsf{K}-$theory back to scissors congruence; that is, what kind of cut-and-paste information is encoded in K_0 and higher K_i? We will discus enriching motivic measures, generalizing assemblers to other cut-and-paste problems, and working towards topological approaches to a generalized variant of Hilbert's 3rd problem.

Although we are now likely familiar with most of the objects that will appear here, there are some subtle differences in conventions that are worth highlighting:

Definition 1.2 (Varieties)**.** Let k be a field and $\mathsf{Var}_{/k}$ be the category of **varieties** over k, which we will take to mean reduced separated schemes of finite-type over the point $\operatorname{Spec} k$. We will say two varieties X, Y are **isomorphic** if and only if they are isomorphic in $\mathsf{Sch}_{/k}$, and will denote this by $X \cong Y$.

Warning 1.3. There is a subtlety in the definition of the category of schemes: a morphism (and hence an isomorphism) of schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is *not* simply a morphism of arbitrary ringed spaces, which would be a pair (F, ϕ) where $F : X \to Y$ is a morphism of spaces and $\phi : \mathcal{O}_Y \to F_* \mathcal{O}_X$ is a morphism of sheaves, where F_* denotes the direct image. Instead, they are defined as maps $f_i : U_i \to V_i$ defined on open affine covers $\{U_i = \operatorname{Spec} R_i\}, \{V_i = \operatorname{Spec} S_i\}$ of X and Y respectively where each f_i is induced by a morphism of rings $S_i \to R_i$. Equivalently, morphisms of schemes can be characterized as morphisms of *locally* ringed spaces.

Definition 1.4 (Stratified spaces)**.** Let X be a topological space, and for $U, V \subseteq X$, write $Y = U \biguplus V$ for the *internal disjoint union*, which indicates that U and V may not necessarily be disjoint but that their intersection $U \cap V$ is measure zero (which for example occurs if the intersection is lower-dimensional). A

stratification of X is the data of a (internally disjoint) partition of X into locally closed subspaces $X = \biguplus_{i \in I} X_i$ indexed by a poset $(I, \leq)$. The subspaces X_i are referred to as **strata**, and we additionally require that for each $j \in I$,

$$\overline{X_j} \subset \biguplus_{i \leq j} X_i,$$

i.e. the closure of X_j in X is contained in the union of lower-index strata.

Definition 1.5 (The Grothendieck ring of varieties)**.** Let Sp be a category of spectra – concretely, one can take the category of symmetric spectra of simplicial sets along with its stable model structure with levelwise cofibrations. Let $\mathcal{V}_k$ to be the assembler whose objects are the objects of $\mathsf{Var}_{/k}$ and whose morphisms are closed inclusions of varieties, or equivalently locally closed embeddings of schemes. Since the field k will be fixed in the statements of most theorems, we will suppress the base field and write $\mathcal{V}$. Let $\mathsf{K}(\mathcal{V})$ be its associated K-theory spectrum. The group $\mathsf{K}_0(\mathcal{V}) := \pi_0 \mathsf{K}(\mathcal{V})$ has a ring structure and can be shown to coincide with the **Grothendieck ring of varieties** as in Michael's talk. We will write elements in this ring using square brackets, so if X is a variety, $[X]$ denotes its equivalence class in $\mathsf{K}_0(\mathcal{V})$.

Definition 1.6 (The Lefschetz motive and its annihilator)**.** The class of the affine line $\mathbb{A}^1 := \mathbb{A}^1_{/k}$ in $\mathsf{K}_0(\mathcal{V})$ is referred to as **the Lefschetz motive** and denoted

$$\mathbb{L} := [\mathbb{A}^1_{/k}] \in \mathsf{K}_0(\mathcal{V}),$$

where we suppress the dependence on the base field k. Since this is simply an element of a ring, we can define its annihilator in the usual way as

$$\mathrm{Ann}(\mathbb{L}) := \ker(\mathsf{K}_0(\mathcal{V}) \xrightarrow{\cdot \mathbb{L}} \mathsf{K}_0(\mathcal{V})),$$

where $\cdot\mathbb{L}$ is the map induced by the morphism of assemblers

$$\begin{aligned} F : \mathcal{V} &\to \mathcal{V} \\ X &\mapsto X \underset{k}{\times} \mathbb{A}^1_{/k} \end{aligned}$$

Fact 1.7. It is an exercise in commutative algebra that $\mathbb{L}$ is a ring-theoretic zero divisor in $\mathsf{K}_0(\mathcal{V})$ if and only if $\mathrm{Ann}(\mathbb{L}) = 0$. A first step toward understanding equations in a ring might be understanding its zero divisors, and several motivating problems and conjectures concern whether or not $\mathbb{L}$ in particular is a zero divisor. As a convention we will frame questions about zero divisors in this section as questions about triviality of annihilators, and in particular we will study when $\mathrm{Ann}(\mathbb{L})$ is trivial.

Example 1.8 (Working with $\mathbb{L}$)**.** We saw in talk 7 some ways to work with elements in $\mathsf{K}_0(\mathcal{V})$ and in particular how to work with formulas involving $\mathbb{L}$. One can show the following identities:

- $[\mathbb{G}_m] := [\mathbb{A}^n \setminus \{0\}] = \mathbb{L} - [\mathrm{pt}]$,

- $[\mathbb{P}^1] = \mathbb{L} + [\mathrm{pt}]$,
- For $\mathcal{E} \to X$ a rank n vector bundle[1], $[\mathcal{E}] = [X] \cdot [\mathbb{A}^n] = [X] \cdot \mathbb{L}^n$.

The last example shows that $\mathsf{K}_0(\mathcal{V})$ does not distinguish between trivial and nontrivial bundles. [Bor15] profitably uses this fact and similar computations to prove that a cut-and-paste conjecture of Larsen-Lunts fails, which conjecturally has applications to rationality of motivic zeta functions.

Definition 1.9 (Birational varieties)**.** Two varieties X, Y are **birational** if and only there is an isomorphism of $\varphi : U \xrightarrow{\sim} V$ of nonempty dense[2] open subschemes. Note that φ need not extend to a well-defined function on all of X and Y, and does not generally imply $X \cong Y$.

Remark 1.10. It is a standard convention to denote such a birational morphism defined on $U \subseteq X$ and $V \subseteq Y$ as $X \dashrightarrow Y$; here I will use the suggestive notation $X \overset{\sim}{\dashrightarrow} Y$ as a reminder that birational varieties are meant to be "almost" isomorphic. Why is this? In equations, a birational morphism φ is given not by polynomial equations but rather by rational functions, which allows denominators and introduces poles or a branch locus – generally in the complements $X \setminus U$ and $Y \setminus V$ respectively. These exceptional singular loci are meant to be "small" in some sense.

This weakening of the notion of isomorphism turns out to be the right way to study the **minimal model program**, an active area of current research which aims for a full classification of varieties up to some notion of equivalence, along with an understanding of particularly nice[3] "minimal" representatives in each class. This is of course an extremely difficult problem, but moving into the world of birational morphisms yields a much more tractable problem since the exceptional loci can often be stratified and cut into smaller pieces to study.

Definition 1.11 (Stably birational varieties)**.** Two varieties X, Y are **stably birational** if and only if there is a birational isomorphism

$$X \times \mathbb{P}^N \overset{\sim}{\dashrightarrow} Y \times \mathbb{P}^M$$

for some N, M large enough.

Remark 1.12. Many interesting invariants of birational geometry are in fact *stable* birational invariants. Some examples include:

- The Hodge number

$$h^{0,1}(X) = \dim_{\mathbb{C}} H^{0,1}(X^{\mathrm{an}})$$

where $H^{p,q}(X^{\mathrm{an}}) := H^0(X^{\mathrm{an}}; \Omega^1_{X^{\mathrm{an}}})$,

[1] Here, a vector bundle over a variety X means a Zariski-locally trivial fibration over X with fibers isomorphic to $\mathbb{A}^n$.

[2] In fact, any nonempty open subset $U \subseteq X$ is automatically dense in X in the Zariski topology.

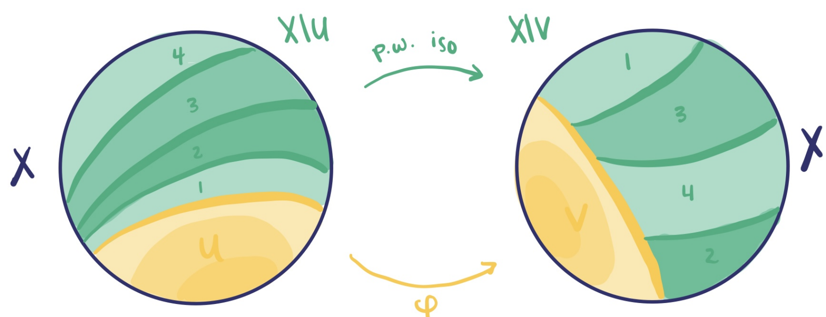
[3] Smooth, or singular with very well-understood singularities.

- the (analytic) fundamental group $\pi_1(X^{\mathrm{an}})$, and
- the zeroth Chow group $\mathrm{CH}_0(X)$.

A recent exposition of other applications of stable birationality is given in [Voi16].

Definition 1.13 (Piecewise isomorphisms)**.** Two varieties X, Y are **piecewise isomorphic** if and only if there exist stratifications $X = \biguplus_{i \in I} X_i$ and $Y = \biguplus_{i \in I} Y_i$ with each $X_i \cong Y_i$. Since we will be working with several notions of isomorphism, we will denote piecewise isomorphisms by $X \underset{\mathrm{pw}}{\cong} Y$.

Remark 1.14. This definition of a piecewise isomorphism is meant to capture the notion of cut-and-paste equivalence of varieties. To see how this relates to K-theory, note that if X and Y are piecewise isomorphism, then their classes are equal in $\mathsf{K}_0(\mathcal{V})$. On the other hand, if X and Y are birational, it is not generally the case that their classes are equal in $\mathsf{K}_0(\mathcal{V})$. However, if there is a birational morphism $X \dashrightarrow Y$ defined on $U \subseteq X$ and $V \subseteq Y$ and one *additionally* requires that $X \setminus U \cong Y \setminus V$, then X and Y are in fact piecewise isomorphic and thus have equal classes in $\mathsf{K}_0(\mathcal{V})$.



1.1.1 Motivation and Main Questions

There are two broad questions we would like to consider:

Question 1.15. When is the canonical ring localization morphism $\mathsf{K}_0(\mathcal{V}) \to \mathsf{K}_0(\mathcal{V})[1/\mathbb{L}]$ injective? In particular, when can equations in the localization be pulled back to valid equations in the original ring?

Question 1.16. What does equality in $\mathsf{K}_0(\mathcal{V})$ actually *mean* geometrically? What geometric information is the Grothendieck ring capturing, and what conclusions can be drawn from equations in this ring?

Remark 1.17. [Zak17] poses and answers two primary structural questions as a way to shed light on this:

1.1.2 Question 1: Does $\mathsf{K}_0(\mathcal{V}_k)$ detect either birationality or piecewise isomorphisms?

Fact 1.18. There is a filtration on $\mathsf{K}_0(\mathcal{V}_k)$ where gr_n is induced by the image of

$$\mathsf{gr}_n\mathsf{K}_0(\mathcal{V}) = \operatorname{im}\left(\frac{\mathbb{Z}\Big[X \;\Big|\; \dim X \leq n\Big]}{([X]=[Y]+[X\backslash Y])} \xrightarrow{\psi_n} \mathsf{K}_0(\mathcal{V})\right)$$

Question 1.19 (Gromov)**.** If $U, V \hookrightarrow X$ with $X \setminus U \cong X \setminus V$, how far are U and V from being birational? If $X = Y$, can every birational automorphism $\phi : X \xrightarrow{\sim} X$ be extended to a piecewise isomorphism $\tilde{\phi} : X \underset{\text{pw}}{\cong} X$?

This can equivalently be restated as a question about injectivity of the maps ψ_n, where failure of injectivity at a particular n indicates extra relations in $\mathsf{K}_0(\mathcal{V})$ coming from classes of higher-dimensional varieties.

Conjecture 1.20 (A cut-and-paste conjecture of Larsen-Lunts)**.** *If $[X] = [Y]$ is an equality the Grothendieck ring K_0, then there is a piecewise isomorphism $X \underset{\text{pw}}{\cong} Y$.*

Answer 1.21. This conjecture is now known to be false – Borisov and Karzhemanov construct counterexamples for fields k that embed in $\mathbb{C}$, and [Zak17] shows that this additionally fails for a wider class of *convenient*[4] fields.

Conjecture 1.22. *This is almost true, and the only obstructions come from* $\mathrm{Ann}(\mathbb{L})$.

Conjecture 1.23. *For certain varieties, equality $[X] = [Y]$ in the Grothendieck ring implies that X, Y are* ***stably birational****.*

1.1.3 Question 2: When is $\mathrm{Ann}(\mathbb{L})$ nonzero?

Remark 1.24. Why might one care about this *particular* ring-theoretic property? Recall that this condition is equivalent to the injectivity of the map $\cdot\mathbb{L}$, and thus one answer is that having a nonzero annihilator allows cancellation by $\mathbb{L}$ in equations. So computations like the following can be carried out:

$$[X]\cdot\mathbb{L} = [Y]\cdot\mathbb{L} \implies ([X]-[Y])\cdot\mathbb{L} = 0 \overset{\mathrm{Ann}(\mathbb{L})=0}{\implies} [X]-[Y] = 0 \implies [X] = [Y],$$

and so equality "up to a power of $\mathbb{L}$" implies honest equality. A separate motivation comes from the purely algebraic fact that the localization morphism $R \to S^{-1}R$ for a multiplicative set S is injective precisely when S does not contain zero divisors, and so if $\mathrm{Ann}(\mathbb{L}) = 0$ then $\mathsf{K}_0(\mathcal{V}) \hookrightarrow \mathsf{K}_0(\mathcal{V})[1/\mathbb{L}]$ is injective.

The latter ring appears in conjectures concerning rationality of motivic zeta functions $\zeta_X(t)$. Larsen-Lunts have recently exhibited a K3 surface X in [LL20] such that $\zeta_X(t)$ is *not* rational over $\mathsf{K}_0(\mathcal{V})$, and discuss the possibility rationality as formal power series in $\mathsf{K}_0(\mathcal{V})[1/\mathbb{L}]$ instead.

[4]This is a technical condition to be described later.

Answer 1.25. [Bor15] and [Kar14] partially answer this question by showing that $\mathbb{L}$ generally **is** a zero divisor, witnessed by explicit constructions producing elements that are equal in $\mathsf{K}_0(\mathcal{V})$ that are not piecewise isomorphic, thus producing elements in $\mathrm{Ann}(\mathbb{L})$. Seemingly coincidentally, their construction also produces elements in $\ker \psi_n$, and so a natural question is whether or not this is actually a coincidence at all.

Proposition 1.1 (Borisov)**.** *The cut-and-paste conjecture of Larsen and Lunts is false.*

Proof. This is proved in [Bor15, Theorem 2.13]. There is a certain pair "mirror" varieties X_W and Y_W[5] which are provably **not** birational and for which stable birationality would imply birationality. One starts with an equality in $\mathsf{K}_0(\mathcal{V})$, and toward a contradiction supposes that equality in the Grothendieck ring implies piecewise-isomorphism. Several properties of bundles over these varieties are used to make the following series of computations:

$$\begin{aligned}
&[X_W]\left(\mathbb{L}^2-1\right)(\mathbb{L}-1)\mathbb{L}^7 = [Y_W]\left(\mathbb{L}^2-1\right)(\mathbb{L}-1)\mathbb{L}^7\\
\implies &[\mathrm{GL}_2(\mathbb{C})\times\mathbb{C}^6\times X_W] = [\mathrm{GL}_2(\mathbb{C})\times\mathbb{C}^6\times Y_W]\\
\implies &\mathrm{GL}_2(\mathbb{C})\times\mathbb{C}^6\times X_W \underset{\text{pw}}{\cong} \mathrm{GL}_2(\mathbb{C})\times\mathbb{C}^6\times Y_W \qquad \text{if Larsen-Lunts is true}\\
\implies &X_W\times\mathrm{GL}_2(\mathbb{C})\times\mathbb{C}^6 \overset{\sim}{\dashrightarrow} Y_W\times\mathrm{GL}_2(\mathbb{C})\times\mathbb{C}^6\\
&\implies X_W \overset{\sim_{\text{Stab}}}{\dashrightarrow} Y_W \qquad \text{i.e. } X_W, Y_W \text{ are stably birational}\\
&\implies X_W \overset{\sim}{\dashrightarrow} Y_W,
\end{aligned}$$

thus concluding that X_W and Y_W are birational and reaching the desired contradiction. □

Question 1.26. How and why are $\mathrm{Ann}(\mathbb{L})$ and $\ker \psi_n$ related? [Zak17] gives a precise answer.

1.1.4 Outline of Results

Slogan 1.27. The following are some slogans for what's shown in [Zak17], to give you some feeling for what might be true:

- **Theorem A:** There is a stable (filtered) homotopy type $\mathsf{K}(\mathcal{V})$ whose associated graded *spectrum* $\mathsf{gr}\,\mathsf{K}(\mathcal{V})$ is simpler than the the associated graded *ring* $\mathsf{gr}\,\mathsf{K}_0(\mathcal{V})$.
- **Theorem B**: The associated spectral sequence[6] is an obstruction theory for birational automorphisms extending to piecewise isomorphisms. Thus the spectral sequence detects $\ker \psi_n$ for various n.

[5] Roughly speaking, these are smooth derived-equivalent Calabi-Yau threefolds, see the *Pfaffian-Grassmannian correspondence*.

[6] That is, the spectral sequence naturally associated to a filtered spectrum. How exactly this is constructed is spelled out in [Zak17, Section 2].

- **Theorem C:** Questions 1 and 2 are precisely linked: elements in $\mathrm{Ann}(\mathbb{L})$ yield elements in $\ker(\psi_n)$.
- **Theorem D**: There is a partial characterization of $\mathrm{Ann}(\mathbb{L})$ in terms of varieties satisfying certain equations in $\mathsf{K}_0(\mathcal{V})$ which are not piecewise isomorphic.
- **Theorem E**: $\mathsf{K}_0(\mathcal{V}) \mod \mathbb{L}$ completely captures stable birational geometry: there is an isomorphism of abelian groups[7]

  $$\mathsf{K}_0(\mathcal{V})/\left\langle \mathbb{L} \right\rangle \cong \mathbb{Z}[\mathrm{SB}],$$

  where SB is the set of stable birational equivalence classes of varieties.

Moreover, a main conclusion is that elements in $\mathrm{Ann}(\mathbb{L})$ *always* produce elements in $\ker \psi_n$. We'll now look at these theorems in more detail.

1.2 Theorems

1.2.1 Theorem A: There is a homotopical enrichment of $\mathsf{K}_0(\mathcal{V})$ with a simple associated graded

Theorem 1.28 ([Zak17] Theorem A)**.** *There is a homotopical enrichment of $\mathsf{K}_0(\mathcal{V})$ with a simple associated graded. Let*

- *$\mathcal{V}^{(n)}$ be the nth filtered assembler of $\mathcal{V}$ generated by varieties of dimension $d \leq n$,*
- *$\mathrm{Aut}_k\, k(X)$ be the group of birational automorphisms of the variety X,*
- *B_n be the set of birational isomorphism classes of varieties of dimension $d = n$.*

There is a spectrum $\mathsf{K}(\mathcal{V})$ such that $\mathsf{K}_0(\mathcal{V}) \coloneqq \pi_0 \mathsf{K}(\mathcal{V})$ coincides with the previously defined Grothendieck group of varieties, and $\mathcal{V}^{(n)}$ induces a filtration on $\mathsf{K}(\mathcal{V})$ such that

$$\mathsf{gr}_{\,n}\mathsf{K}(\mathcal{V}) = \bigvee_{[X]\in B_n} \Sigma^\infty_+ \mathbf{B}\mathrm{Aut}_k\, k(X),$$

with an associated spectral sequence

$$E^1_{p,q} = \bigvee_{[X]\in B_n} (\pi_p \Sigma^\infty \mathbf{B}\mathrm{Aut}_k\, k(X) \oplus \pi_p \mathbb{S}) \Rightarrow \mathsf{K}_p(\mathcal{V})$$

Remark 1.29. Note that the $p = 0$ column converges to $\mathsf{K}_0(\mathcal{V})$.

Proof.

[7] This result was previously known, and the significance is that this can now be proved using homotopy-theoretic techniques.

- Define $\mathcal{V}^{(n.n-1)} = \mathsf{Var}_{/k}^{\dim = n} \cup \{\emptyset\}$, the varieties of dimension *exactly* n.
- Use [Zak17, Theorem 1.8] to extract cofibers in the filtration and identify the associated graded:

$$
\begin{array}{ccc}
\mathsf{K}(\mathcal{V}) & & \\
\uparrow & & \\
\vdots & & \\
\uparrow & & \\
\mathsf{K}(\mathcal{V}^{(n)}) & \twoheadrightarrow & \mathsf{K}(\mathcal{V}^{(n-1)}) \\
\uparrow & & \\
\mathsf{K}(\mathcal{V}^{(n-1)}) & \twoheadrightarrow & \mathsf{K}(\mathcal{V}^{(n-1,n-2)}) \\
\uparrow & & \\
\vdots & & \\
\uparrow & & \\
\mathsf{K}(\mathcal{V}^{(2)}) & \twoheadrightarrow & \mathsf{K}(\mathcal{V}^{(2,1)}) \\
\uparrow & & \\
\mathsf{K}(\mathcal{V}^{(1)}) & & \\
 & & \\
\text{Fil} & & \text{gr}
\end{array}
$$

- Finish by a magic computation:

$$
\begin{aligned}
\mathsf{K}(\mathcal{V}^{(n,n-1)}) &\simeq \tilde{\mathsf{K}}(\mathcal{V}^{(n,n-1)}) \\
&\simeq \mathsf{K}(\mathsf{C}) \\
&\simeq \mathsf{K}\left(\bigvee_{\alpha \in B_n} \mathsf{C}_{X_\alpha} \right) \\
&\simeq \bigvee_{\alpha \in B_n} \mathsf{K}(\mathsf{C}_{X_\alpha}) \\
&\cong \bigvee_{\alpha \in B_n} \Sigma^\infty_+ \mathbf{B}\mathrm{Aut}_k k(X_\alpha) \qquad \text{Zak17a} \\
&\coloneqq \bigvee_{\alpha \in B_n} \Sigma^\infty_+ \mathbf{B}\,\mathrm{Aut}(\alpha),
\end{aligned}
$$

where

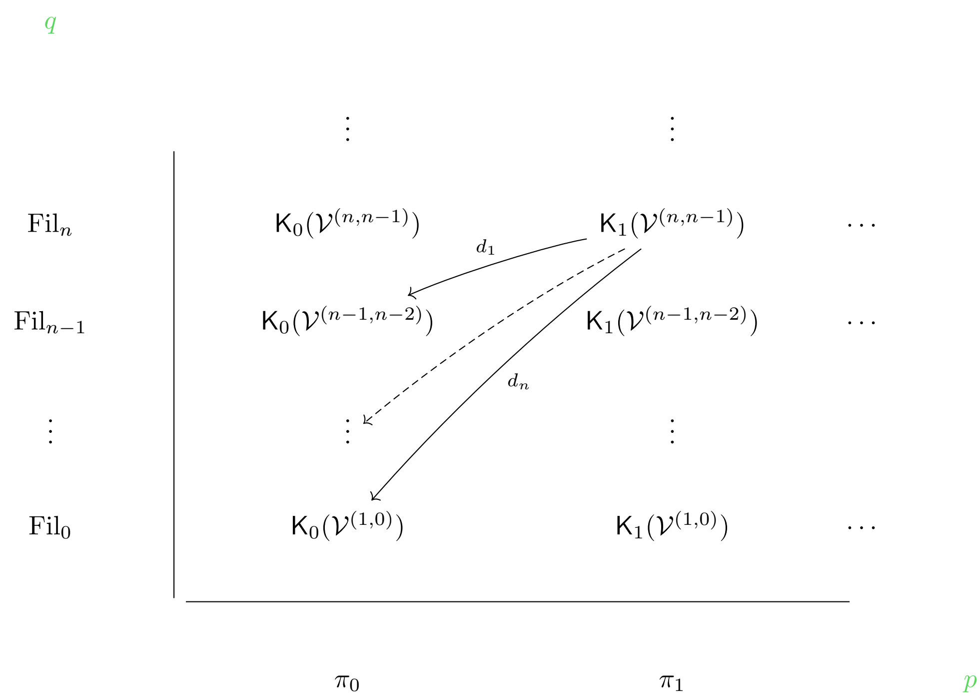
- $\tilde{\mathsf{K}}(\mathcal{V}^{(n,n-1)})$: the full subassembler of irreducible varieties.
  - **Why the reduction works:** general theorem [Zak17, Theorem 1.9] on subassemblers with enough disjoint open covers
- $\mathsf{C} \leq \mathcal{V}^{(n,n-1)}$: subvarieties of some X_α representing some α, as α ranges over B_n.
  - **Why the reduction works:** apply [Zak17, Theorem 1.9] again
- C_{X_α} is the subassembler of only those varieties admitting a (unique) morphism to X_α for a fixed α.
  - **Why the reduction works:** each nonempty variety admits a morphism to exactly one X_α representing some α – otherwise, if $X \mapsto X_\alpha, X_\beta$ then X_α and X_β are forced to be birational (the morphisms are inclusions of dense opens) implying $\alpha = \beta$
- $\mathrm{Aut}(\alpha) \coloneqq \mathrm{Aut}_k\, k(X)$ for any X representing $\alpha \in B_n$.

Note that much of this proof amounts to repeated application of dévissage. □

1.2.2 Theorem B: the spectral sequence measures $\ker \psi_n$ and how birational morphisms can fail to extend to piecewise isomorphisms

Theorem 1.30 ([Zak17] Theorem B)**.** *There exists nontrivial differentials d_r from column 1 to column 0 in some page of E^* $\iff$ $\cup_n \ker \psi_n \neq 0$ (ψ_n has a nonzero kernel for some n). More precisely, $\varphi \in \mathrm{Aut}_k k(X)$ extends to a piecewise automorphism if and only if $d_r[\varphi] = 0 \quad \forall r \geq 1$.*

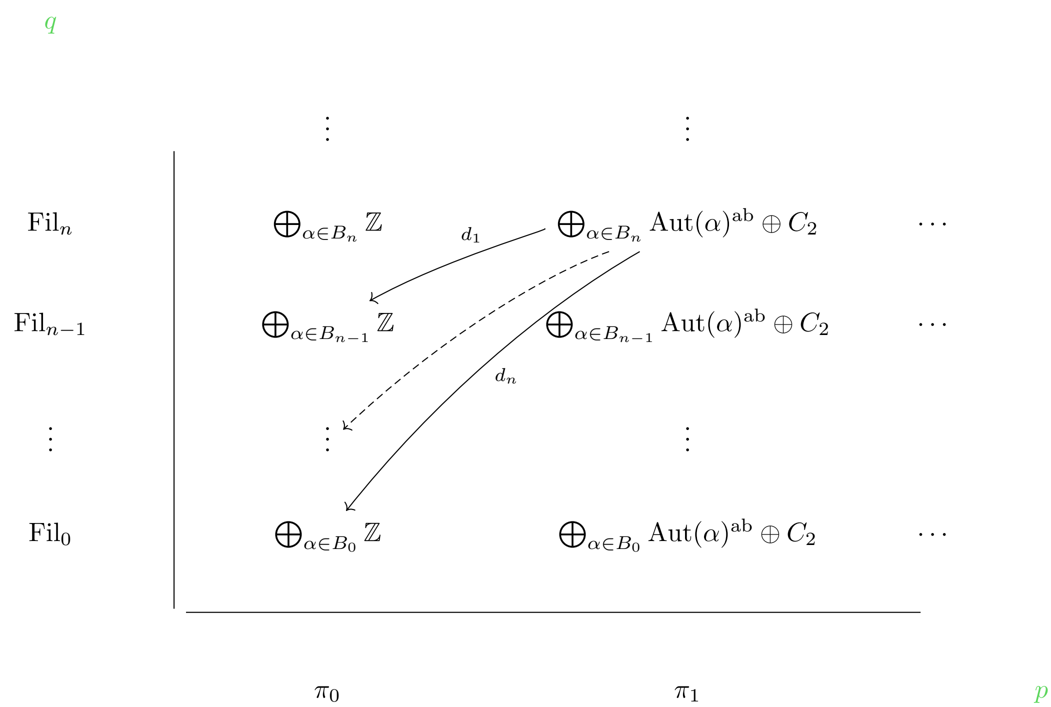
Remark 1.31. Before proving this result, it is helpful to look at the actual spectral sequence. The following is the the E^1 page:



To identify the terms, one carries out a short computation:

$$\begin{aligned}
\mathsf{K}_p(\mathcal{V}^{(n,n-1)}) &\coloneqq \pi_p \mathsf{K}(\mathcal{V}^{(n,n-1)}) \\
&\simeq \pi_p \bigvee_{\alpha \in B_n} \Sigma^\infty_+ \mathbf{B}\,\mathrm{Aut}(\alpha) \\
&\cong \bigoplus_{\alpha \in B_n} \pi_p \Sigma^\infty_+ \mathbf{B}\,\mathrm{Aut}(\alpha).
\end{aligned}$$

Now using that $\pi_p \Sigma^\infty_+ \mathbf{B}G$ is $\mathbb{Z}$ for $p = 0$ and $G^{\mathrm{ab}} \oplus C_2$ for $p = 2$, we have the following:

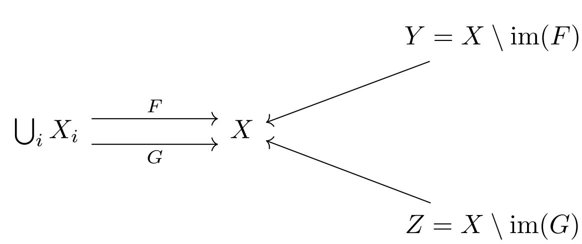


Lemma 1.32 ([Zak17] Lemma 3.2)**.** *Note that there is a boundary map ∂ coming from the connecting map in the LES in homotopy of a pair for the filtration. If $\varphi \in \mathrm{Aut}(\alpha)$ for $\alpha \in B_q$ is represented by $\varphi : U \to V$ then*

$$\partial[\varphi] = [X \setminus V] - [X \setminus U] \quad \in \mathsf{K}_0(\mathcal{V}^{(q-1)})$$

Proof of lemma.

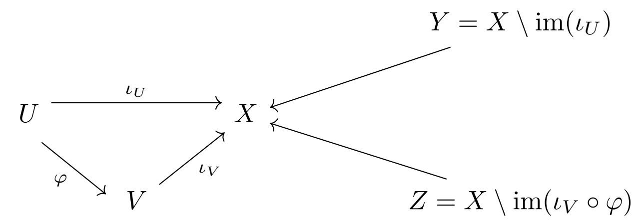
- In general, $x \in \mathsf{K}_1(\mathcal{V}^{(q,q-1)})$ corresponds to the following data: X a variety, a dense open subset with two embeddings F and G, the two possible complements, where $\{X_i\}$ is a covering family over X where $\bigcup_i X_i$ is a dense open subset of X, and the complements are of dimension at most $q-1$:



- [Zak17, Prop. 3.13] shows that for this data,

$$\partial[x] = [Z] - [Y] \in \mathsf{K}_0(\mathcal{V}^{(q-1)})$$

- For φ, we can represent it with the data:



- One can then conclude

$$\partial[\varphi] = [Z] - [Y] = [X \setminus V] - [X \setminus U].$$

□

Proof of theorem B ($\Longrightarrow$). Suppose φ extends to a piecewise automorphism.

- Then $[X \setminus U] = [X \setminus V] \in \mathsf{K}_0(\mathcal{V}^{q-1})$ since $X \setminus U \xrightarrow{\sim} X \setminus V$ by assumption
- By lemma 3.2 above,

$$\partial[\varphi] = [X \setminus V] - [X \setminus U] = 0$$

- [Zak17, Lemma 2.1] shows that d_1 and higher d_r are built using ∂, so $\partial(x) = 0 \implies d_r(x) = 0$ for all $r \geq 1$, yielding a permanent boundary.

□

Proof of theorem B, ($\Longleftarrow$). Suppose $d_r[\varphi] = 0$ for all $r \geq 1$.

- Since in particular $d_1[\varphi] = 0$, we have

$$[X \setminus U] = [X \setminus V] \in \mathsf{K}_0(\mathcal{V}^{(q,q-1)}),$$

since $d_1 = \partial \circ p$ for some map p.

- An inductive argument allows one to write $X = U_r \uplus X'_r = V_r \uplus Y'_r$ where

$$U_r \underset{\text{pw}}{\cong} V_r, \quad \dim X'_r, \dim Y'_r < n - r, \quad \partial[\varphi] = [Y'_r] - [X'_r]$$

- Take $r = n$ to get

$$\dim X'_n, \dim Y'_n < 0 \implies X'_n = Y'_n = \emptyset \quad \text{and} \quad X = U_n = V_n$$

- Then

$$\partial[\varphi] = [\emptyset] - [\emptyset] = 0 \implies \varphi \text{ extends}.$$

□

Remark 1.33. A general remark on why $\partial[\varphi] = 0$ implies it extends:

- $\partial[\varphi]$ measures the failure of φ to extend to a piecewise isomorphism:

$$\partial[\varphi] = 0 \implies [X \setminus V] = [X \setminus U] \implies \exists \psi : X \setminus V \underset{\text{pw}}{\cong} X \setminus U$$

- If additionally $U \cong V$ then $\varphi \uplus \psi$ assemble to a piecewise automorphism of X.

1.2.3 Theorem C: There is a direct link between $\bigcup_{n \geq 0} \ker \psi_n$ and $\mathrm{Ann}(\mathbb{L})$

Theorem 1.34 ([Zak17] Theorem C)**.** *Let k be a **convenient field**, e.g. $\operatorname{ch} k = 0$. Then $\mathbb{L}$ is a zero divisor in $\mathsf{K}_0(\mathcal{V}) \implies \psi_n$ is not injective for some n. In other words, for k convenient,*

$$\mathrm{Ann}(\mathbb{L}) \neq 0 \implies \bigcup_n \ker \psi_n \neq \emptyset.$$

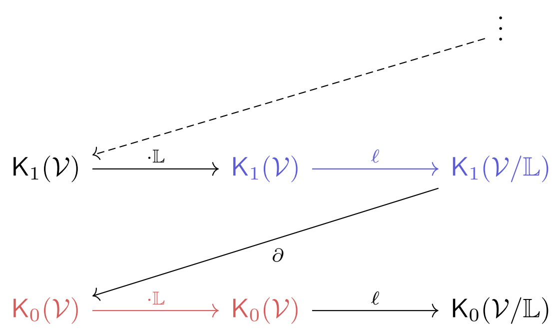
Proof of theorem C.

- Strategy: contrapositive. Suppose $\ker \psi_n = 0$ for all n. There is a cofiber sequence

$$\mathsf{K}(\mathcal{V}) \overset{\cdot \mathbb{L}}{\hookrightarrow} \mathsf{K}(\mathcal{V}) \overset{\ell}{\twoheadrightarrow} \mathsf{K}(\mathcal{V}/\mathbb{L})$$

  where $\mathcal{V}/\mathbb{L}$ is a "cofiber assembler" [Zak17, Def 1.11].

- Take the associated lon exact sequence to identify $\ker(\cdot\mathbb{L})$ with $\operatorname{coker}(\ell)$:



- Reduce to analyzing

$$\operatorname{coker}(E^{\infty}_{1,q} \to \tilde{E}^{\infty}_{1,q})$$

  where $\tilde{E}$ is an auxiliary spectral sequence.

- Suppose all α extend, then all differentials from column 1 to column 0 are zero.
- The map $E^r \to \tilde{E}^r$ is surjective for all r on all components that survive to E^∞.
- All differentials out of these components are zero, so $E^\infty \twoheadrightarrow \tilde{E}^\infty$.
- Then $\mathsf{K}_1(\mathcal{V}) \xrightarrow{\ell} \!\!\!\rightarrow \mathsf{K}_1(\mathcal{V}/\mathbb{L})$, making $0 = \operatorname{coker}(\ell) = \ker(\cdot\mathbb{L})$ so $\mathbb{L}$ is not a zero divisor.

□

1.2.4 Theorem D: Equality in K_0 doesn't imply PW iso and elements in $\operatorname{Ann}(\mathbb{L})$ give rise to elements in $\bigcup \ker \psi_n$.

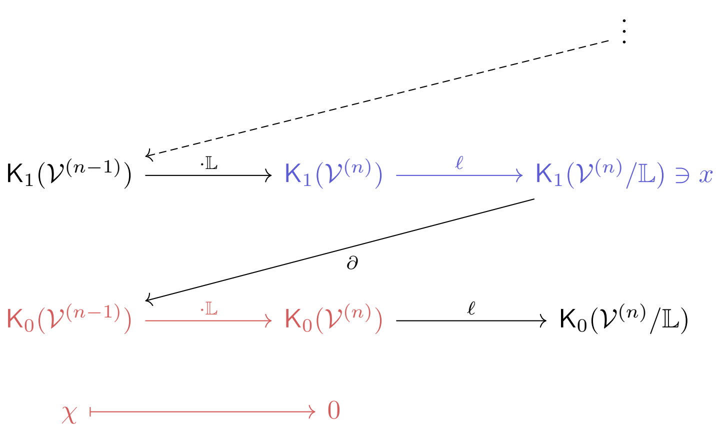
Theorem 1.35 ([Zak17] Theorem D)**.** *Suppose that k is a **convenient** field. If $\chi \in \operatorname{Ann}(\mathbb{L})$ then $\chi = [X] - [Y]$ where*

$$\left[X \times \mathbb{A}^1\right] = \left[Y \times \mathbb{A}^1\right] \quad \text{but } X \times \mathbb{A}^1 \underset{\text{pw}}{\not\cong} Y \times \mathbb{A}^1.$$

Thus elements in $\operatorname{Ann}(\mathbb{L})$ give rise to elements in $\bigcup_{n \geq 0} \ker \psi_n$.

Proof of theorem D.

- Let $\chi \in \ker(\cdot\mathbb{L})$ and pullback in the LES to $x \in \mathsf{K}(\mathcal{V}^{(n)}/\mathbb{L})$ where n is minimal among filtration degrees:



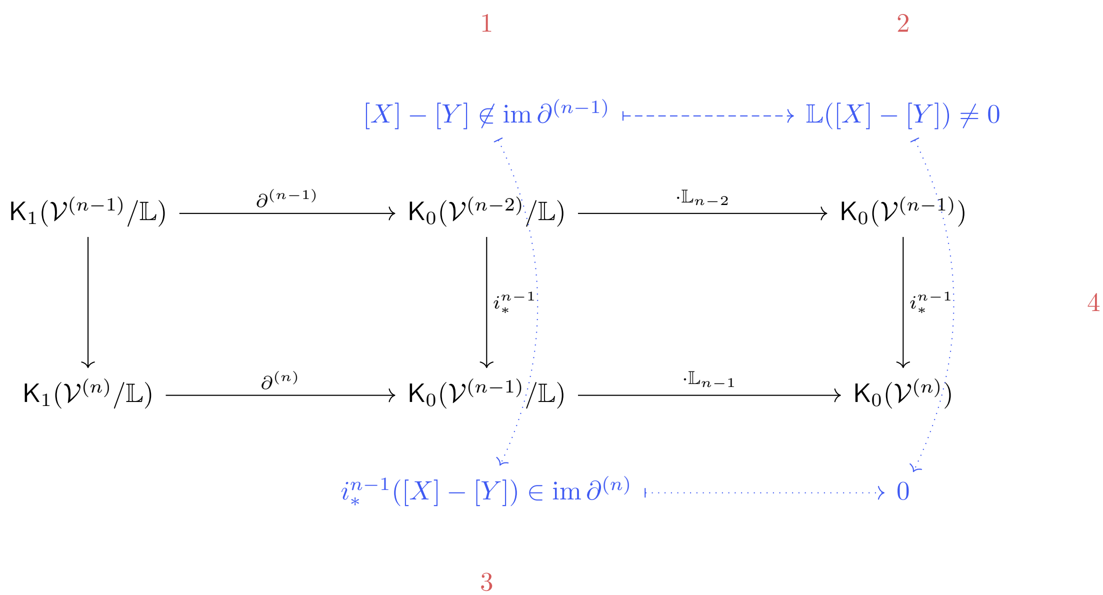
- Write $\partial[x] = [X] - [Y]$ with X, Y of minimal dimension.
- By [**LS10**],

$$\begin{aligned}[X \times \mathbb{A}^1] = [Y \times \mathbb{A}^1] &\implies \dim X + 1 = \dim Y + 1 \\ &\implies \dim X = \dim Y = d\end{aligned}$$

Claim 1.36. d is small: $d < n - 1$.

Note that we're done if this claim is true: proceed by showing X and Y are not piecewise isomorphic by showing $\ker \psi_n$ is nontrivial by a diagram chase. □

Proof of claim. The proof boils down to a diagram chase, which roughly goes as follows:



1. $[X] - [Y] \notin \operatorname{im}(\partial)$ by the minimality of n for x, noting $\partial[x] = [X] - [Y]$.
2. By exactness $\operatorname{im} \partial = \ker(\cdot\mathbb{L})$, so $\mathbb{L}([X] - [Y]) \neq 0$.
3. By choice of n, $i_*(\mathbb{L}([X] - [Y])) \in \operatorname{im} \partial = \ker(\cdot\mathbb{L})$ in bottom row, so $\mathbb{L}([X] - [Y]) = 0$ in bottom-right.
4. Commutativity forces $\mathbb{L}([X] - [Y]) \in \ker i_*^{n-1}$.

Thus $\mathbb{L}([X] - [Y])$ corresponds to an element in $\ker \psi_n$.

□

1.2.5 Theorem E: $\mathsf{K}_0 \mod \mathbb{L}$ models stable birational geometry

Theorem 1.37 ([Zak17] Theorem E)**.** *There is an isomorphism*

$$\mathsf{K}_0(\mathcal{V}_{\mathbb{C}})/\langle \mathbb{L} \rangle \xrightarrow{\sim} \mathbb{Z}[\mathsf{SB}_{\mathbb{C}}] \qquad \in \mathbb{Z}\text{-Mod}.$$

Remark 1.38. Proof: omitted.

1.3 Closing Remarks

Remark 1.39. What we've accomplished: establishing a precise relationship between questions 1 and 2.

Question 1.40. Some currently open questions:

- What fields are convenient?
- What is the associated graded for the filtration induced by ψ_n?
- Is there a characterization of $\mathrm{Ann}(\mathbb{L})$?
- (Interesting) What is the kernel of the localization $\mathsf{K}_0(\mathcal{V}) \to \mathsf{K}_0(\mathcal{V})[\frac{1}{\mathbb{L}}]$?
- Does ψ_n fail to be injective over *every* field k?

Conjecture 1.41. *There is a correction to Question 1 cpncerning* $\ker \psi_n$ *which may be true: let* X, Y *be varieties over a convenient field with* $[X] = [Y$*. Then there exist varieties* X', Y' *such that*

- $[X'] \neq [Y']$
- $[X' \times \mathbb{A}^1] = [X'] \cdot \mathbb{L} = [Y'] \cdot \mathbb{L} = [Y' \times \mathbb{A}^1]$
- $X \coprod X' \times \mathbb{A}^1 \underset{\text{pw}}{\cong} Y \coprod Y' \times \mathbb{A}^1$

Remark 1.42. If the conjecture holds, if X, Y are not birational but are *stably* birational, then the error of birationality is measured by a power of $\mathbb{L}$.

Contingent upon this conjecture, one might hope to show

$$[X] \equiv [Y] \bmod \mathbb{L} \implies X \overset{\sim_{\text{Stab}}}{\dashrightarrow} Y,$$

so that the equality in the quotient ring completely captures stable birational geometry.

References

[Bor15] Lev Borisov. *Class of the Affine Line Is a Zero Divisor in the Grothendieck Ring*. Mar. 2015. DOI: 10.48550/arXiv.1412.6194. arXiv: 1412.6194 [math]. URL: http://arxiv.org/abs/1412.6194 (visited on 06/06/2022).

[Kar14] Ilya Karzhemanov. *On the cut-and-paste property of algebraic varieties*. 2014. DOI: 10.48550/ARXIV.1411.6084. URL: https://arxiv.org/abs/1411.6084.

[LL20] Michael J. Larsen and Valery A. Lunts. "Irrationality of motivic zeta functions". In: *Duke Mathematical Journal* 169.1 (Jan. 2020). DOI: 10.1215/00127094-2019-0035. URL: https://doi.org/10.1215%2F00127094-2019-0035.

[Voi16] Claire Voisin. "Stable birational invariants and the lüroth problem". In: *Surveys in Differential Geometry* 21.1 (2016), pp. 313–342. DOI: 10.4310/sdg.2016.v21.n1.a8.

[Zak17] Inna Zakharevich. "The Annihilator of the Lefschetz Motive". In: *Duke Mathematical Journal* 166.11 (Aug. 2017). ISSN: 0012-7094. DOI: 10.1215/00127094-0000016X. arXiv: 1506.06200 [math]. URL: http://arxiv.org/abs/1506.06200 (visited on 06/06/2022).