Homotopy Groups of Spheres

Graduate Student Seminar

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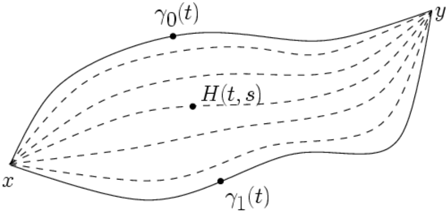
Introduction

Outline

- Homotopy as a means of classification somewhere between homeomorphism and cobordism
- Comparison to homology
- Higher homotopy groups of spheres exist
- Homotopy groups of spheres govern gluing of CW complexes
- CW complexes fully capture that homotopy category of spaces
- There are concrete topological constructions of many important algebraic operations at the level of spaces (quotients, tensor products)
- Relation to framed cobordism?
- "Measuring stick" for current tools, similar to special values of L-functions
- Serre's computation

Intuition

Homotopies of paths:



- Regard paths γ in X and homotopies of paths H as morphisms

$$\gamma \in \hom_{\text{Top}}(I, X)$$
$$H \in \hom_{\text{Top}}(I \times I, X).$$

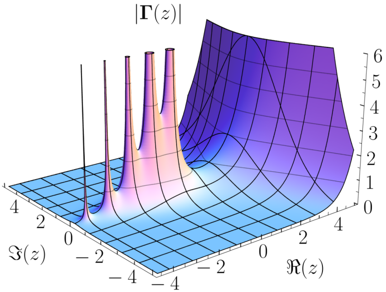
- Yields an equivalence relation: write

$$\gamma_0 \sim \gamma_1 \iff \exists H \text{ with } H(0) = \gamma_0, H(1) = \gamma(1)$$

- Write $[\gamma]$ to denote a homotopy class of paths.

Intuition

- Why care about path homotopies? Historically: contour integrals in $\mathbb{C}$



- By the residue theorem, for a meromorphic function f with simple poles $P = \{p_i\}$ we know that

$$\oint_\gamma f(z)\ dz \text{ is determined by } [\gamma] \in \pi_1(\mathbb{C} \setminus P)$$

Definitions

- Generalize to a homotopy of *morphisms*:

$$f, g \in \hom_{\mathrm{Top}}(X, Y) \quad f \sim g \iff \exists F \in \hom_{\mathrm{Top}}(X \times I, Y)$$

  such that $F(0) = f, F(1) = g$.
- This yields an equivalence relation on morphisms, *homotopy classes of maps*

$$[X, Y] \coloneqq \hom_{\mathrm{Top}}(X, Y)/\sim$$

- Definition of homotopy equivalence:

$$X \sim Y \iff \exists \begin{cases} f \in \hom(X, Y) \\ g \in \hom(Y, X) \end{cases} \text{such that} \begin{cases} f \circ g \sim \mathrm{id}_Y \\ g \circ f \sim \mathrm{id}_X \end{cases}$$

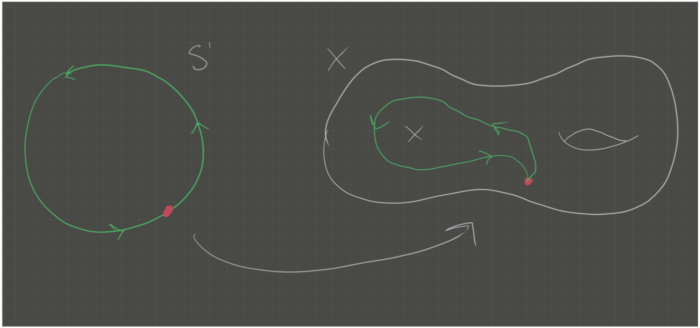
- Similarly write

$$[X] = \left\{ Y \in \mathrm{Top} \;\middle|\; Y \sim X \right\}.$$

The Fundamental Group

- $\pi_1(X)$ is the group of homotopy classes of loops:
- Can recover this definition by finding a (co)representing object:

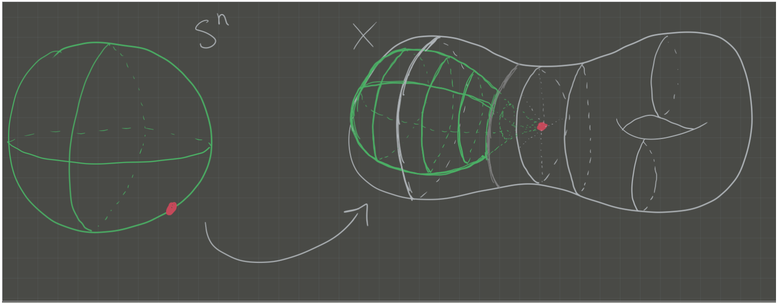
$$\pi_1(X) = [S^1, X]$$



Higher Homotopy Groups

- Can now generalize to define

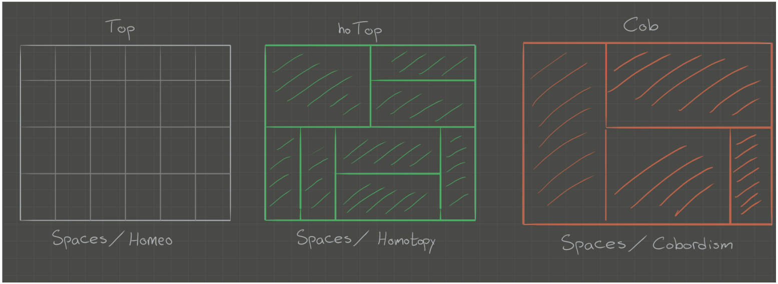
$$\pi_k(X) := [S^k, X]$$



Fun side note: this kind of definition generalizes to AG, see Motivic Homotopy Theory – the (co)representing objects look $\mathbb{A}^1$ *or* $\mathbb{P}^1$.

Classification

- Holy grail: understand the topological category completely
  - I.e. have a well-understood geometric model one space of each homeomorphism type



Also have the derived category DTop, its interplay with hoTop is the subject of e.g. the Poincare conjecture(s).

- Any representative from a green box: a *homotopy type*.

Example: Homotopy Equivalence is Useful

Proposition: Let B be a CW complex; then isomorphism classes of $\mathbb{R}^1$-bundles over B are given by $H^1(X, \mathbb{Z}/2\mathbb{Z})$.

- Use the fact that for any fixed group G, the functor

$$\begin{aligned} h_G(\,\cdot\,) : \mathrm{hoTop}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ X &\mapsto \{G\text{-bundles over } X\} \end{aligned}$$

  is representable by a space called BG (Brown's representability theorem).
- I.e., let $\mathrm{Bun}_G(X) = \{G\text{-bundles}/B\} / \sim$, there is an isomorphism

$$\mathrm{Bun}_G(X) \cong [X, BG]$$

- In general, identify $G = \mathrm{Aut}(F)$ the automorphism group of the fibers – for vector bundles of rank n, take $G = GL(n, \mathbb{R})$.

Example: Homotopy Equivalence is Useful

Note that for a poset of spaces $(M_i, \hookrightarrow)$, the space $M^\infty := \varinjlim M_i$. These are infinite dimensional "Hilbert manifolds".

Proof:

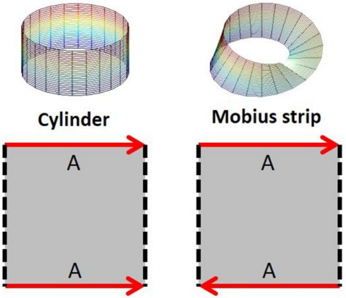
$$\begin{aligned}
\mathrm{Bun}_{\mathbb{R}^1}(X) &= [X, B\mathrm{GL}(1, \mathbb{R})] \\
&= [X, \mathrm{Gr}(1, \mathbb{R}^\infty)] \\
&= [X, \mathbb{RP}^\infty] \\
&= [X, K(\mathbb{Z}/2\mathbb{Z}, 1)] \\
&= H^1(X; \mathbb{Z}/2\mathbb{Z})
\end{aligned}$$

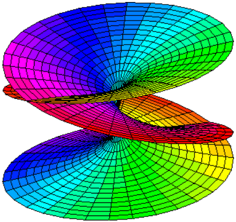
Work being swept under the rug: identifying the homotopy type of the representing object.

Example: Homotopy Equivalence is Useful

Corollary: There are 2 distinct line bundles over $X = S^1$ (the cylinder and the mobius strip), since $H^1(S^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Corollary: A Riemann surface Σ_g satisfies $H^1(\Sigma_g; \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{2g}$ and thus there are 2^{2g} distinct real line bundles over it.





Example: Higher Homotopy Groups are Useful

- Application: computing $\pi_1(\mathrm{SO}(n, \mathbb{R})$ (rigid rotations in $\mathbb{R}^n$).
- The fibration

$$\mathrm{SO}(n, \mathbb{R}) \longrightarrow \mathrm{SO}(n+1, \mathbb{R}) \longrightarrow S^n$$

yields a LES in homotopy:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_2(SO(n, \mathbb{R})) \arrow[r] & \pi_2(SO(n, \mathbb{R})) \arrow[r] & \pi_2(S^n) \arrow[dll] \\
\pi_1(SO(n, \mathbb{R})) \arrow[rr] & & \pi_1(SO(n, \mathbb{R})) \arrow[r] & \pi_1(S^n)
\end{tikzcd}$$

Uses of Higher Homotopy

Knowing $\pi_k S^n$, this reduces to

$$\begin{tikzcd}
\cdots 0 \arrow[r] & \pi_2(SO(n,\mathbb{R})) \arrow[r] & \pi_2(SO(n,\mathbb{R})) \arrow[r] \arrow[dll] & 0 \\
\pi_1(SO(n,\mathbb{R})) \arrow[rr] & & \pi_1(SO(n,\mathbb{R})) \arrow[r] & 0
\end{tikzcd}$$

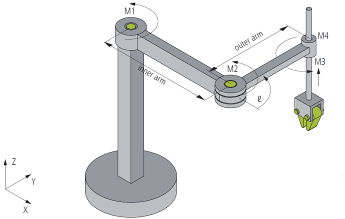
- Thus $\pi_1(\mathrm{SO}(3,\mathbb{R})) \cong \pi_1(\mathrm{SO}(4,\mathbb{R})) \cong \cdots$ and it suffices to compute $\pi_1(\mathrm{SO}(3,\mathbb{R}))$ (stabilization)
- Use the fact that "accidental" homeomorphism in low dimension $\mathrm{SO}(3,\mathbb{R}) \cong_{\mathrm{Top}} \mathbb{RP}^3$, and algebraic topology I yields $\pi_1 \mathbb{RP}^3 \cong \mathbb{Z}/2\mathbb{Z}$.

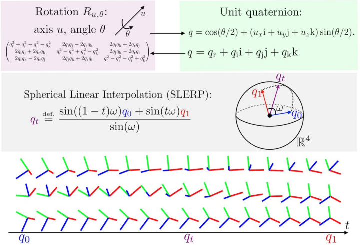
  Can also use the fact that $SU(2,\mathbb{R}) \longrightarrow SO(3,\mathbb{R})$ is a double cover from the universal cover.

Uses of Higher Homotopy

- Important consequence: $SO(3, \mathbb{R})$ is not simply connected!
- See "plate trick": non-contractible loop of rotations that squares to the identity.
- Robotics: paths in configuration spaces with singularities

- Computer graphics: smoothly interpolating between quaternions for rotated camera views





Spheres

Setup

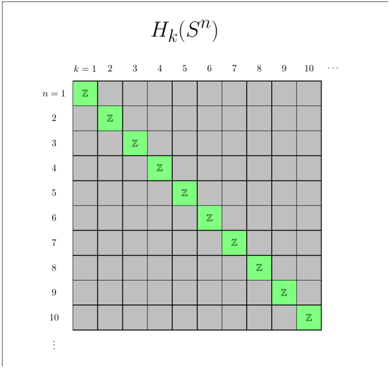
- Defining $\pi_k(X) = [S^k, X]$, the simplest objects to investigate: $X = S^n$
- Can consider the bigraded group $\pi_S := [S^k, S^n]$:

$$\pi_k(S^n)$$

	$k=1$	2	3	4	5	6	7	8	9	10	$\cdots$
$n=1$											
2											
3											
4											
5											
6											
7											
8											
9											
10											
$\vdots$											

But Wait!

The corresponding picture in homology is very easy:



Slogan: "conservation/duality of complexity"

History

- 1895: Poincare, *Analysis situs* ("the analysis of position") in analogy to Euler *Geometria situs* in 1865 on the Kongisberg bridge problem
  - Studies spaces arising from gluing polygons, polyhedra, etc (surfaces!), first use of "algebraic invariant theory" for spaces by introducing π_1 and homology.
- 1920s: Rigorous proof of classification of surfaces (Klein, Möbius, Clifford, Dehn, Heegard)
  - Captured entirely by π_1 (equivalently, by genus and orientability).
- **1931: Hopf discovers a nontrivial (not homotopic to identity) map $S^3 \longrightarrow S^2$**

History

- 1932/1935: Cech (indep. Hurewicz) introduce higher homotopy groups, gives map relating $\pi_* \longrightarrow H_*$, shows $\pi_n X$ are **abelian** groups for $n \geq 2$.
  - Withdrew his paper because of this theorem!
- 1951: Serre uses spectral sequences to show that **all groups** $\pi_k S^n$ **are torsion except**,
  - $k = n$, since $\pi_n S^n = \mathbb{Z}$
  - $k \equiv 3 \mod 4$, $n \equiv 0 \mod 2$, then $\mathbb{Z} \oplus T$
  - Tight bounds on where p-torsion can occur.
- 1953: Whitehead shows the homotopy groups of spheres split into stable and unstable ranges.

Today: We know $\pi_{n+k} S^n$ for

- $k \leq 64$ when $n \geq k+2$ (stable range)
- $k \leq 19$ when $n < k+2$ (unstable range)
- We *only* have a complete list for S^0 and S^1, and know *no* patterns beyond this!
  - Open for ~ 80 years.

Spheres

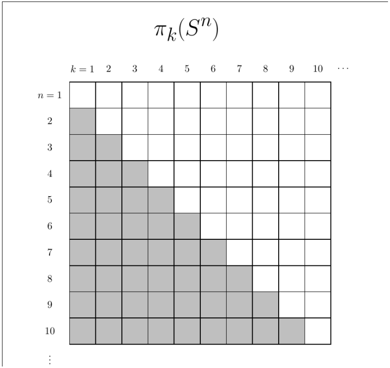
We'll fill out as much of this table as is easily known:

$$\pi_k(S^n)$$

	$k=1$	2	3	4	5	6	7	8	9	10	$\cdots$
$n=1$											
2											
3											
4											
5											
6											
7											
8											
9											
10											
$\vdots$											

$k < n$

Claim: $[S^k, S^n] = 0$ for $k < n$.

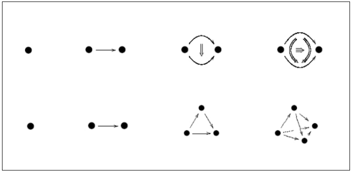


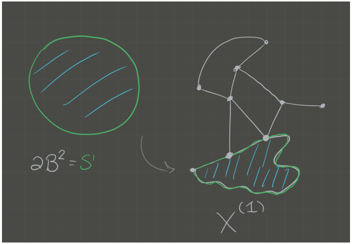
This follows easily from CW approximation:

> *Any map $X \xrightarrow{f} Y$ between CW complexes is homotopic to a cellular map.*

k < n: CW Complexes

- Analogy from analysis: C^1 functions dense in L^2.
  - If you're just computing homotopy groups, *any* space can be replaced with a *weakly equivalent* CW complex.



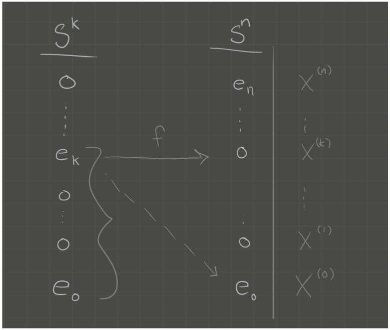


k < n: CW Complexes

AT1 can show that spheres have a simple cell decomposition

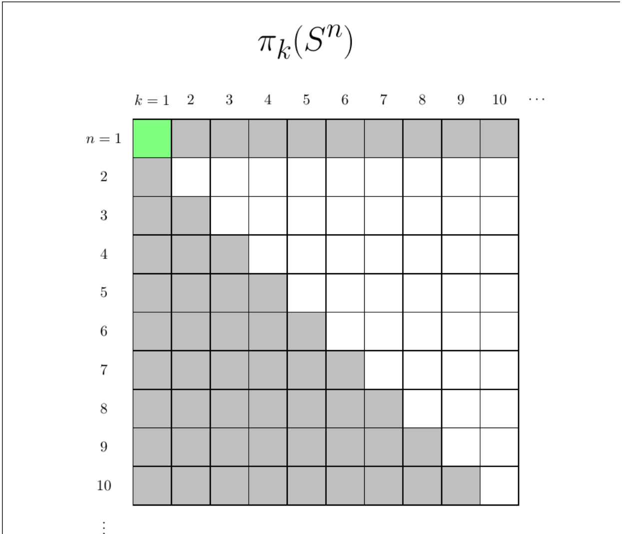
$$S^k = e_0 \coprod_f e_k$$

Thus any map $f : S^k \longrightarrow S^n$ must send the k-skeleton of S^k to the k-skeleton of S^n, which is just a point:



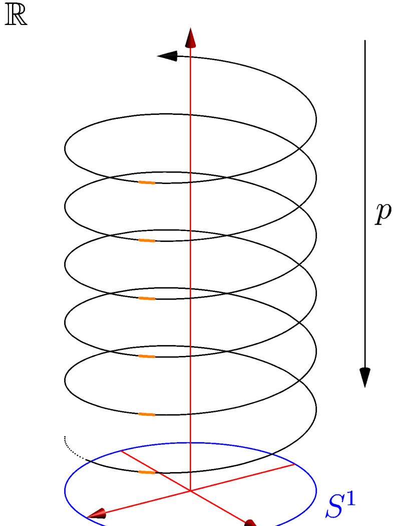
$k \geq 1, n = 1$: Covering Space Theory

Claim: $\pi_1 S^1 = \mathbb{Z}$ and $\pi_{\geq 2} S^1 = 0$.



$k \geq 1, n = 1$: Covering Space Theory

- Use the fact that $\mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow S^1$ is a covering space and $\mathbb{Z} \curvearrowright \mathbb{R}$ freely.



$k \geq 1, n = 1$: Covering Space Theory

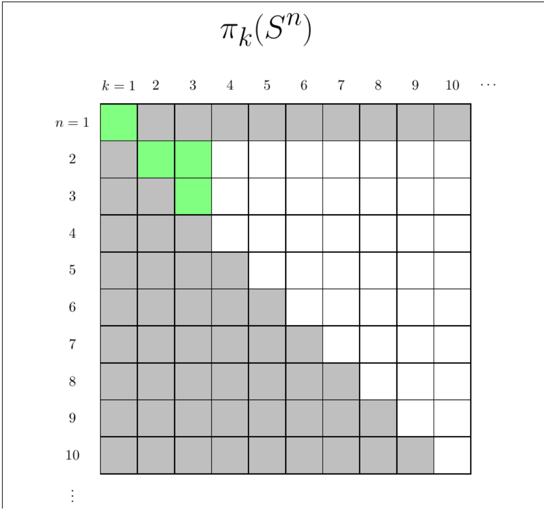
Theorem: If $F \longrightarrow E \longrightarrow B$ is a *Serre Fibration* then there is a LES in homotopy

$$
\begin{tikzcd}
\cdots & \cdots \arrow[r] & \pi_{k+1}(B) \arrow[dll] \\
\pi_k(F) \arrow[r] & \pi_k(E) \arrow[r] & \pi_k(B) \arrow[dll] \\
\pi_{k-1}(F) \arrow[r] & \cdots & \cdots
\end{tikzcd}
$$

- If $\tilde{X} \longrightarrow X$ is a universal cover then $\pi_{\geq 2}(X) \cong \pi_{\geq 2}\tilde{X}$.
  - Proof coming up!

Misc: Serre Fibrations

Claim: $\pi_2 S^2 = \pi_3 S^2 = \pi_3 S^3 = \mathbb{Z}$.



Misc: Serre Fibrations

Use the Hopf fibration: $S^1 \longrightarrow S^3 \longrightarrow S^2$ and the fact that $\pi_{\geq 2} S^1 = 0$:

$$
\begin{tikzcd}
\pi_3(S^1) \arrow[r] & \pi_3(S^3) \arrow[r] & \pi_3(S^2) \arrow[dll] \\
\pi_2(S^1) \arrow[r] & \pi_2(S^3) \arrow[r] & \pi_2(S^2) \arrow[dll] \\
\pi_1(S^1) \arrow[r] & \pi_1(S^3) \arrow[r] & \pi_1(S^2)
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & \pi_4(S^3) \arrow[r] & \pi_4(S^2) \arrow[dll] \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \pi_3(S^2) \arrow[dll] \\
0 \arrow[r] & 0 \arrow[r] & \pi_2(S^2) \arrow[dll] \\
\mathbb{Z} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}
$$

Note that this works whenever the fiber is contractible (e.g. universal covers, fibers are discrete)

– Hopf Fibration Visualizer

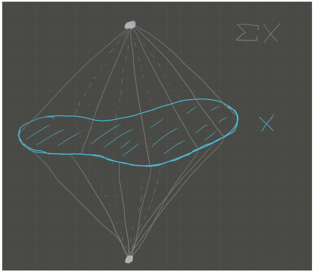
$n = k$: Stabilization

- Theorem (1937, Freudenthal): For $k \gg 0$,

$$[\Sigma^k X, \Sigma^k Y] \cong [\Sigma^{k+1} X, \Sigma^{k+1} Y]$$

- Use the fact that $\Sigma S^k \cong S^{k+1}$, then in some *stable range*

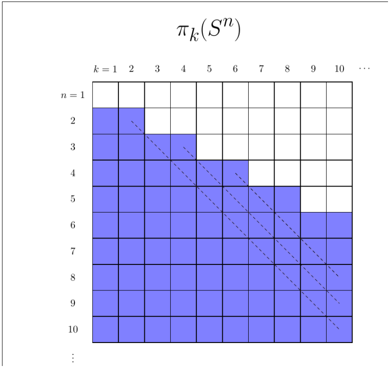
$$\pi_{n+k} S^n \cong \pi_{n+k+1} S^{n+1}$$



Fun note: corresponds to "smash with a sphere"

n=k: Stabilization

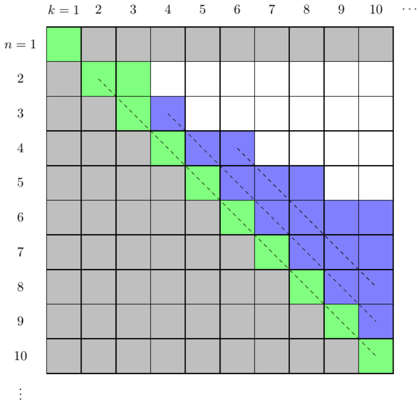
Stable range:



n = k: Stabilization

We can thus suspend things we already know:

$$\pi_k(S^n)$$



k = 4, n = 3

- Construct a map $S^3 \longrightarrow K(\mathbb{Z},2)$ by "killing off homotopy" (identify K)
- Convert to a fibration and take the homotopy fiber to get $F \longrightarrow S^2 \longrightarrow \mathbb{CP}^\infty$
- By LES, $\pi_{\geq 3} F \cong \pi_{\geq 3} S^2$, by Hurewicz $\pi_3 F \cong H_3 F$, which we can compute
- Kill homotopy again and *iterated* homotopy fiber to get $G \longrightarrow F \longrightarrow K(\mathbb{Z}, 3)$
- By LES, $\pi \geq 4G \cong \pi_{\geq 4} S^2$, by Hurewicz $\pi_4 G \cong H_4 G$.

k=4, n=3

- Look at Serre's spectral sequence for $G \longrightarrow F \longrightarrow K(\mathbb{Z}, 3)$.

$$
\begin{tikzcd}
H_5 G & & & & & \\
H_4 G & & & & & \\
0 & & & & & \\
0 & & & & & \\
0 & & & & & \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} & ? & ? \arrow[uuuulllll]
\end{tikzcd}
$$

k=4, n=3

- Want target of arrow. Need to know
  - $H^4 F, H^5 F$ (total cohomology)
  - $H^5 K(\mathbb{Z}, 3)$ (source of arrow)
- Use Serre SS on $F \longrightarrow S^2 \longrightarrow K(\mathbb{Z}, 2)$ to deduce $H^5 F = H^6 F = \mathbb{Z}$
- Use Serre SS on $\Omega K(\mathbb{Z}, 3) \longrightarrow \{\text{pt}\} \longrightarrow K(\mathbb{Z}, 3)$:

k=4, n=3

Relevant quantity will be C term in SSS:

$$
\begin{tikzcd}
\alpha^3 & & & & & & \\
0 & & & & & & \\
\alpha^2 \arrow[ddrrr] \arrow[ddddrrrrr] & & & & & & \\
0 & & & & & & \\
\alpha \arrow[ddrrr] & & & \alpha\otimes\beta \arrow[ddrrr] & & & \\
0 & & & & & & \\
1 & 0 & 0 & \beta & A & B & C=\beta^2
\end{tikzcd}
$$

- Everything must go! Converges to homology of a point.
- Work out $d\alpha = \beta$
- Work out $d(\alpha^2) = 2\alpha\otimes\beta$
- Work out $\beta^2 \neq 0$
- Work out $\ker(\alpha^2 \longrightarrow \alpha\otimes\beta) = 2\langle\alpha\otimes\beta\rangle$
- Conclude $A = B = 0$ and $C \cong \mathbb{Z}/2\mathbb{Z}$.

What is Known

Homotopy Groups of Spheres

D. Zack Garza

Introduction

Spheres

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	${\mathbb{Z}_2}^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$