Compact Moduli of Enriques Surfaces

(with a numerical deg 2 polarization)

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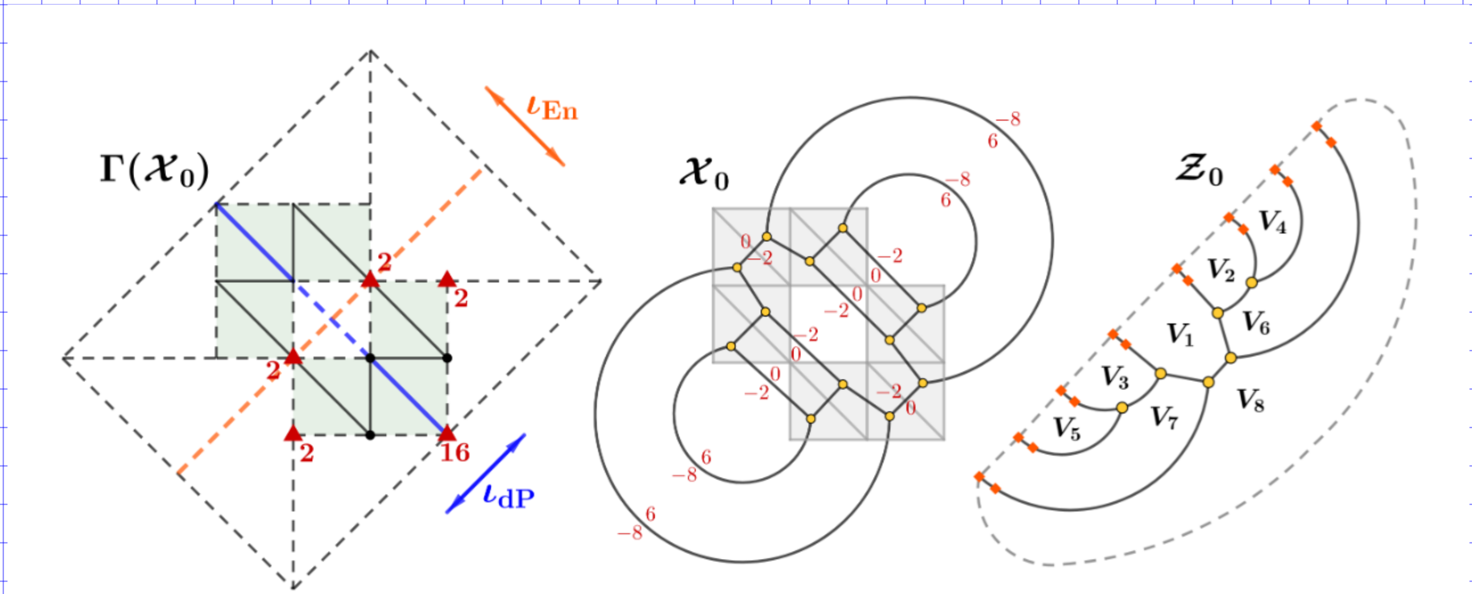
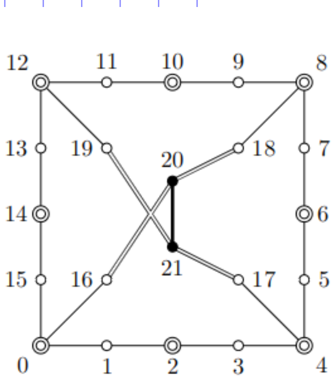
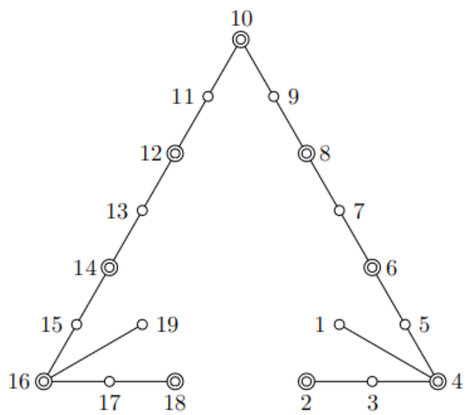
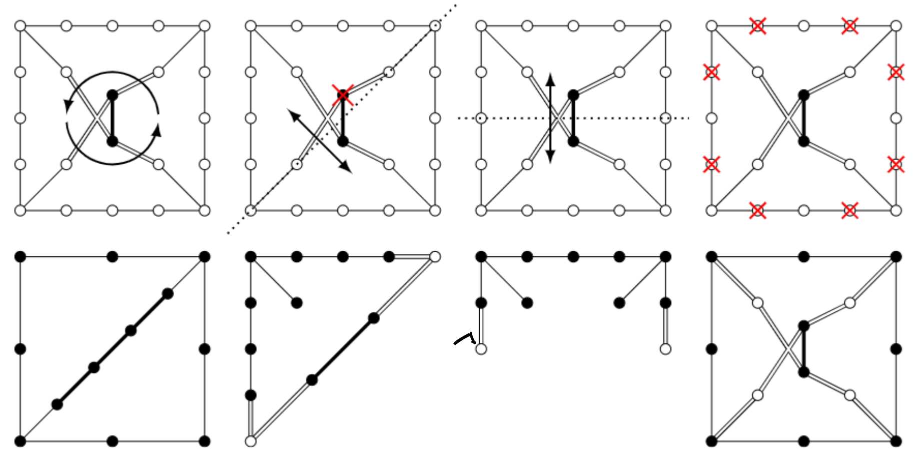
(Arxiv 2312.03638)





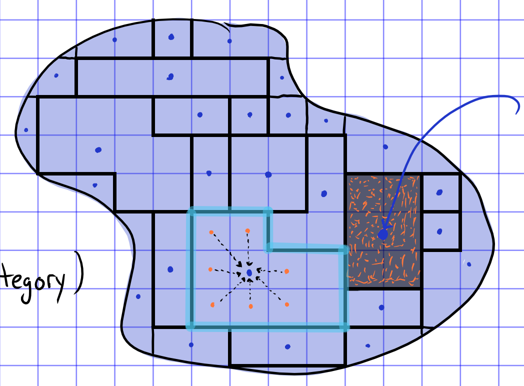




Moduli Spaces

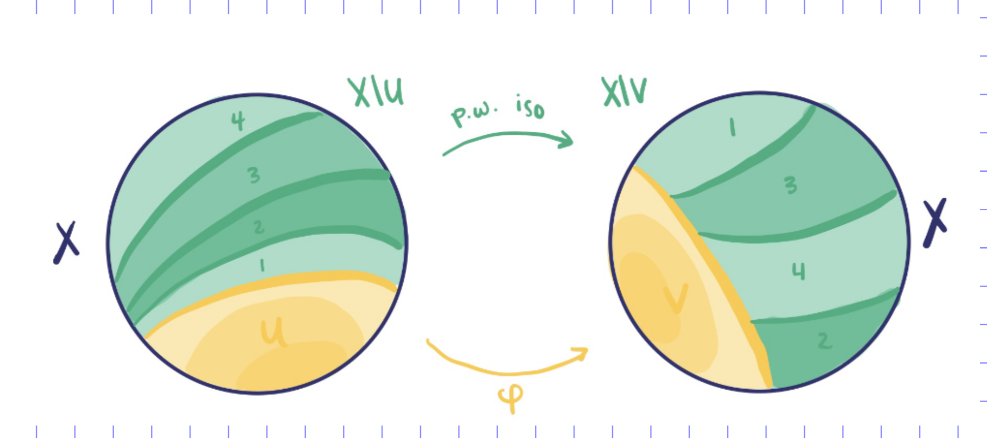
- Throughout this talk: $K = \overline{K} = \mathbb{C}$, $\mathcal{M}$ is a moduli space, $\overline{\mathcal{M}}$ is the stable pair (or KSBA) compactification.
- **Impossible goal**: classify $\pi_0(\mathrm{Var}/\mathbb{C})$ where $X_1 \simeq X_2 \Leftrightarrow \exists\, X_1 \overset{\sim}{\dashrightarrow} X_2$ a birational isomorphism.

$\mathrm{Var}/\mathbb{C}$ (as an ∞-category)



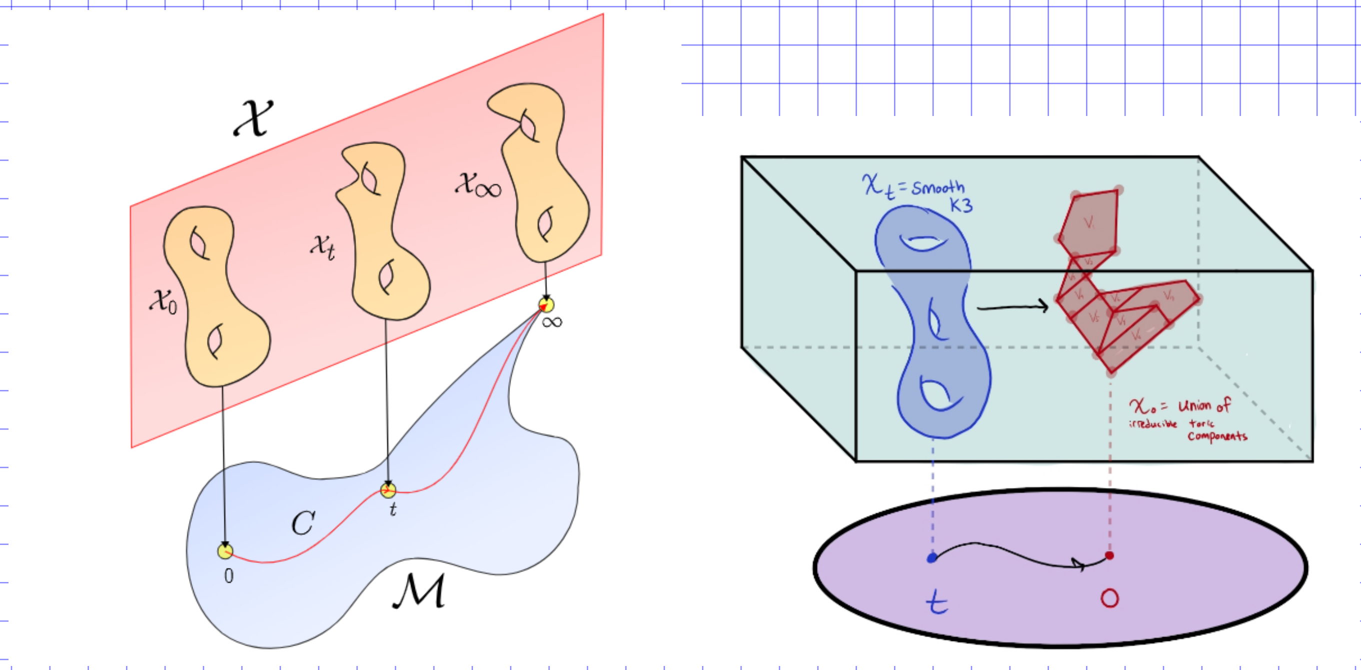
Can we find a canonical model for every birational equivalence class? Minimal? Smooth? Geometrically tractable?

$\rightsquigarrow$ Minimal model program (MMP)



- A first step: Can we enumerate & classify the ways in which smooth varieties degenerate into (mildly) singular ones?

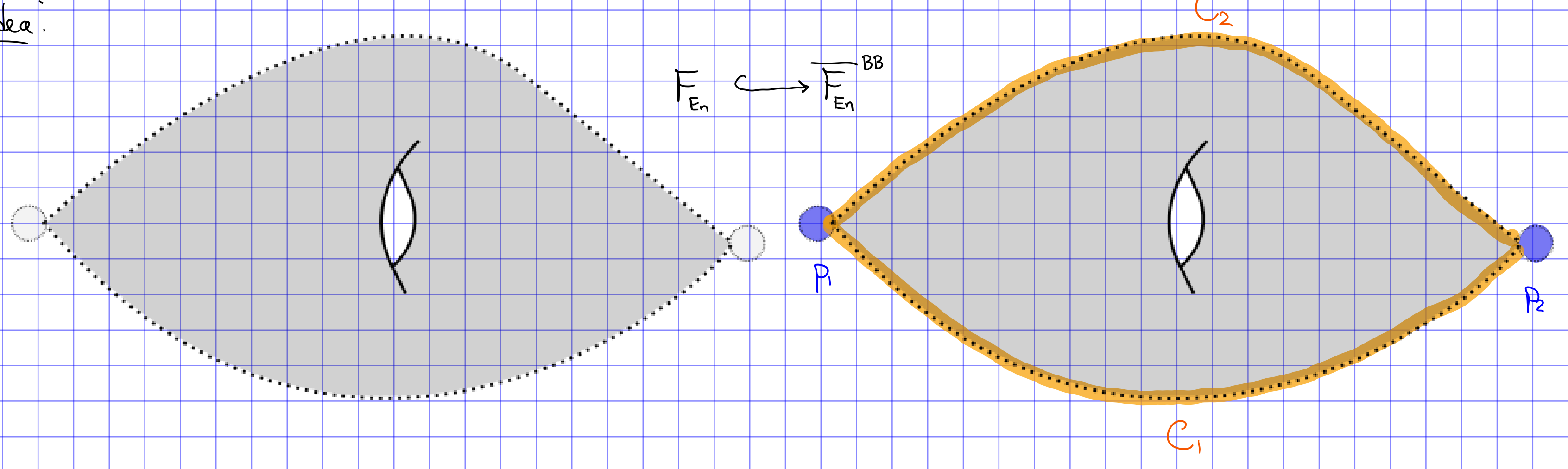
- Strategy: Construct moduli spaces of varieties $\mathcal{M}$, find geometrically meaningful compactifications $\overline{\mathcal{M}}$, & stratify/classify $\partial\overline{\mathcal{M}}$ combinatorially.



- Many choices: $\overline{\mathcal{M}}$, $\overline{\mathcal{M}}^{GIT}$, $\overline{\mathcal{M}}^{BB}$, $\overline{\mathcal{M}}^{Tor}$, $\overline{\mathcal{M}}^{\mathcal{F}}$, ... how do they compare?

Stable pairs — GIT quotients — Baily-Borel — Toroidal — Semi-toroidal

Idea:



- Inspiration: $K(X) = 0 \Rightarrow$
  - Abelian $\to A_2$ [$\overline{A_g}$: Ale02]
  - K3 $\to F_{2d}$ [$\overline{F_{2d}}$: AE23]
  - Enriques $\to F_{En,d}$ [$\overline{F_{En,2}}$: AEGS24]
  - Bielliptic $\to$? Open?

  surface

Previous work

- DM69: $\partial\overline{\mathcal{M}_{g,n}}$ = pointed stable curves

- Hor78: studied possibilities for f & maps $F_{En,2} \to \Gamma_L \backslash D_L$.

- Ste 91: $\partial\overline{F_{En,2}}^{BB}$ using lattice theory, Coxeter diagrams, cusp diagrams.

- Ale02, Annals: $\partial\overline{A_g}$ = Semi-abelic pairs ("tropical PPAVs"), 2nd Voronoi fan $\rightleftarrows$ Delaunay (Toroidal)

- Sch17: Describes $\overline{U} \subseteq F_{En,6}$ a 4-dim family

- AE23, Annals: Fully general theory for $\overline{F_{2d}}$, recognisable divisors

$\hookrightarrow$ NB: hard to compare $\overline{\mathcal{M}}^{GIT}$ & $\overline{\mathcal{M}}^{BB}$ in general, viz. Loo86, LO21.

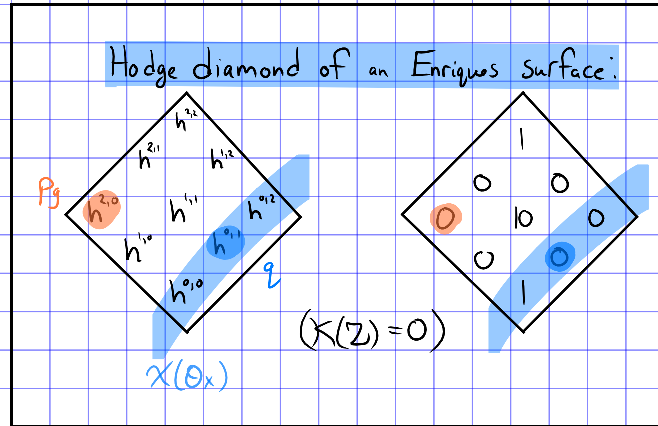
$\mathcal{M} = F_2$, $\mathcal{M} = F_{(2,2,0)}$, K3's w/ $\mathrm{Pic}(\mathcal{X}_\eta) = (2,2,0)$

See also:
- AE22
- ABE22
- GH17

Enriques Surfaces

Def: An Enriques surface $Z \in \mathrm{Var}/_{\mathbb{C}}$ is a smooth, projective variety w/ $\dim_{\mathbb{C}} Z = 2$ satisfying:

- $\omega_Z \not\cong \mathcal{O}_Z$ but $\omega_Z^{\otimes 2} \cong \mathcal{O}_Z$ $\Leftrightarrow$ $K_Z \not\sim 0$ but $2K_Z \sim 0$
- $h^{0,1}(Z) = 0$ $\quad\left(\Leftrightarrow h^1(\mathcal{O}_X) = q(Z) = p_g(Z) - p_a(Z) = 0 \right.$ (irregularity zero) $\left.\right)$



Rmk (Why study Enriques sfcs?)

- Story recounted by Dolgachev: late 19th century, Enriques & Castelnuovo would walk through the arcades in Bologna discussing

$$q(X) = p_g(X) = P_2(X) = 0 \Rightarrow X \overset{\sim}{\dashrightarrow} \mathbb{P}^2.$$

(N.B. $P_n(X) := h^0(\omega_X^{\otimes n})$)

Can $P_2(X) = 0$ be eliminated?

(N.B: True for curves)

- Answer: no, viz. the Enriques sextic w/ $p_g(X) = q(X) = 0$, $P_2(X) = 1$, $K(X) = 0$.
- Both found examples w/ $p_g(X) = q(X) = 0$ and $K_X^{\otimes m}$ effective for some m

$\Rightarrow$ no termination of adjunction $\Rightarrow$ not rational
$\left[h^0(\mathcal{L} + nK_X) \xrightarrow[n\to\infty]{} 0 \right]$

E.g.

- Define $\tau : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$

$$([x_0 : y_0], [x_1 : y_1]) \mapsto ([x_0 : -y_0], [x_1 : -y_1])$$

- Check $\mathrm{Fix}(\tau) = \left\{ \begin{matrix} (0,0), \\ (0,\infty), \\ (\infty, 0), \\ (\infty,\infty) \end{matrix} \right\} = \left\{ \begin{matrix} ([0:1],[0:1]), \\ ([0:1],[1:0]), \\ ([1:0],[0:1]), \\ ([1:0],[1:0]) \end{matrix} \right\}$

- Let $B \in |-2K_{\mathbb{P}^1\times\mathbb{P}^1}|^{\tau} = |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4)|^{\tau}$ be a reduced bidegree $(4,4)$ τ-invariant curve

E.g. $f(x_0, y_0, x_1, y_1) = x_0^2 y_0^2 x_1^4 + x_0^4 y_1^4 + x_0^2 y_0^2 x_1^2 y_1^2 + \cdots$

- Take the 2-to-1 branched cover $X \to \mathbb{P}^1 \times \mathbb{P}^1$ branched over B;

Hor78a shows that X is a (quartic hyperelliptic) K3 surface.

- There are <u>two</u> lifts of $\tau : \mathbb{P}^1\times\mathbb{P}^1 \circlearrowleft$ to involutions on X:

$$(x,y) \overset{\tau}{\mapsto} (-x,-y) \quad \begin{matrix} \nearrow & (x,y,z) \overset{\iota_{En}}{\mapsto} (-x,-y,-z) = \iota_{Nik} \circ \iota_{dP}, & \text{fixed point free} \\ \searrow & (x,y,z) \underset{\iota_{Nik}}{\mapsto} (-x,-y,z) = \iota_{En} \circ \iota_{dP}, & \text{fixed locus} = \{(0,0,z)\} \end{matrix}$$

Rmk: Since ω_Z is 2-torsion, there is an étale 2-to-1 cover $X \to Z$.

$$\Rightarrow \quad \boxed{\begin{tikzcd} \iota_{En} \circlearrowright X \circlearrowleft \iota_{Nik} \arrow[rr, "\bullet/\iota_{En}"] \arrow[d, "/\iota_{dP}"'] & & Z = X/\iota_{En} \\ \tau \circlearrowright \mathbb{P}^1\times\mathbb{P}^1 = X/\iota_{dP} \\ \cup\!\!\mid \; B \end{tikzcd}}$$

The main geometric construction

$$\begin{tikzcd} R \subseteq \underset{}{\overset{K3}{X}} \arrow[rr, "/\iota_{En}"] \arrow[d, "/\iota_{dP}"] \arrow[drr, phantom, "\lrcorner"] & & Z = \text{Enriques} \arrow[d, "/\tau_{dP}"] \\ |-2K_Y| \ni B \subseteq \underset{\mathbb{P}^1\times\mathbb{P}^1}{Y} \arrow[rr, "/\tau"] & & W = \text{Quartic del Pezzo} \end{tikzcd}$$

($R \overset{2:1}{\dashrightarrow} B$; $K3 = X$, $Y = \mathbb{P}^1\times\mathbb{P}^1$)

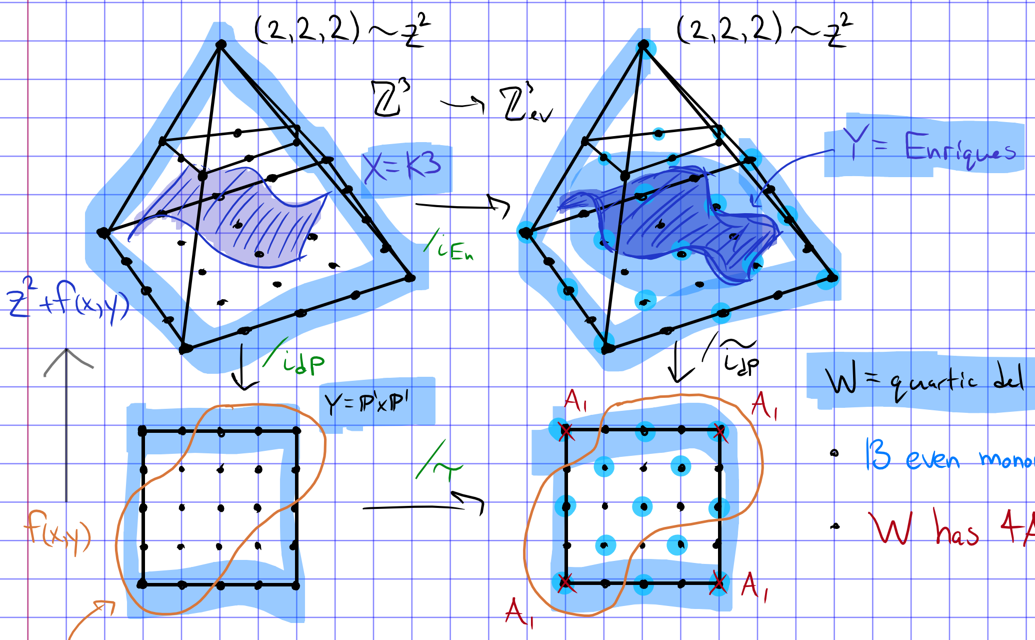
$\boxed{\begin{matrix} \text{Cos 83: this diagram} \\ \text{Hor 78a: birational to above} \\ \text{Enr 06: double plane model} \end{matrix}}$

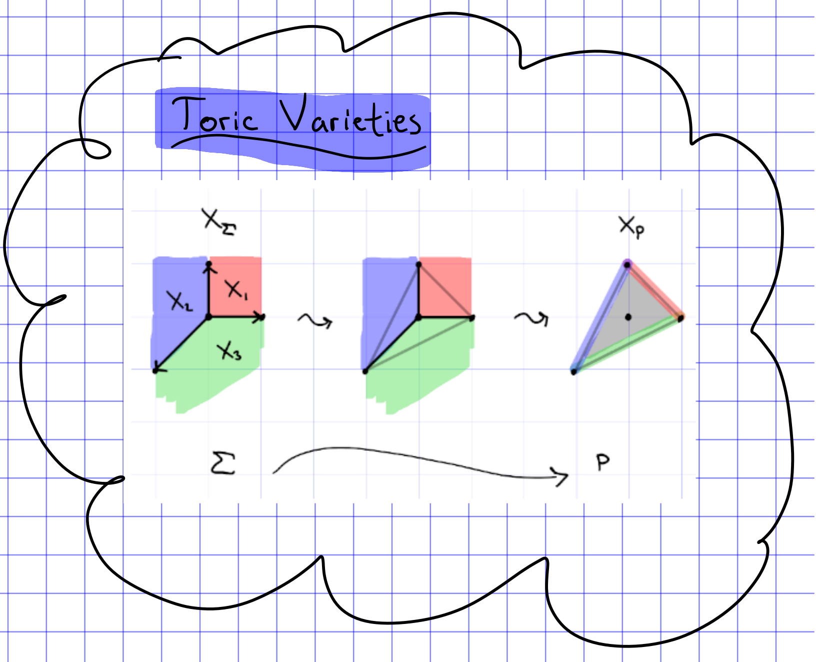
where...

- $\tau(x,y) = (-x,-y)$
- $\iota_{dP}(x,y,z) = (x,y,-z)$, $\quad \iota_{dP}(\omega) = -\omega$
- $\iota_{En}(x,y,z) = (-x,-y,-z)$, $\quad \underbrace{\iota_{En}(\omega) = -\omega}_{\text{Nonsymplectic.}}$

$H^{2,0}(X) = \mathbb{C}\omega, \quad \omega = \dfrac{dx \wedge dy \wedge dz}{z^2 + f(x,y)}$

A toric picture





$W =$ quartic del Pezzo

- 13 even monomials
- W has $4A_1$ singularities

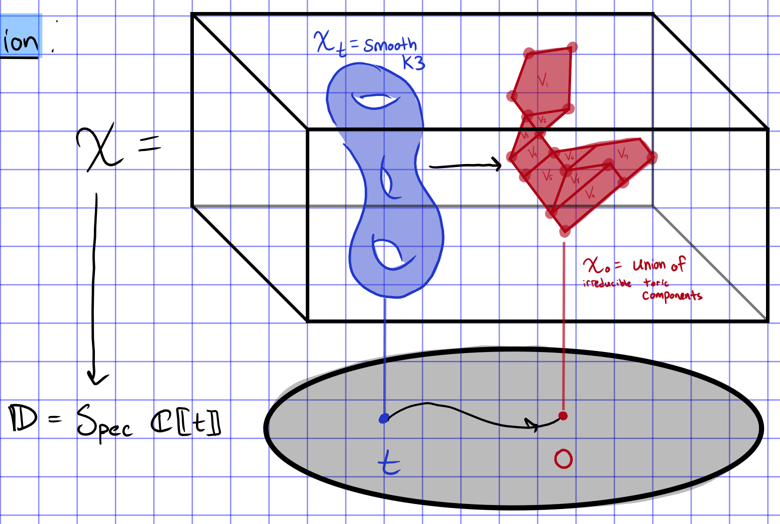
$B \in |-2K_{\mathbb{P}^1\times\mathbb{P}^1}| = |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4)|$, homogeneous bidegree $(4,4)$ curve.

$\Rightarrow B = V(f) \Rightarrow f \in U \subseteq \mathbb{P}^{12}$, so we can form an Enriques moduli space

$$F_{E_n,2} \subsetneq \frac{U}{D_4 \ltimes (\mathbb{C}^\times)^2}$$

Goal: Construct $\overline{F_{E_n,2}}$ & classify $\partial F_{E_n,2}$

Expectation:



NB. Generally expect something similar for a maximally unipotent degeneration of CY n-folds.
("Large complex structure limit", "maximally unipotent degeneration")

Strategy: Use the K3 cover $X \to Z$ to leverage AE22 work on compact moduli $\overline{F_L}$ of K3s with non-symplectic involutions.

↳ **Rmk:** $i_{E_n} \curvearrowright X$ is nonsymplectic but fixed point free, whilst AE22 requires $R := \mathrm{Ram}(i \curvearrowright X) \supseteq C$ a curve, $g(C) \geq 2$, to form stable pairs $(X, \varepsilon R)$.
So one must find another involution on X — we use $i_{dP} \curvearrowright X$, $R := \mathrm{Ram}(i_{dP})$.

Theorem (Alexeev–Engel–G–Schaffler, '24)

$$\overline{F_{E_n,2}} \xrightarrow{\sim} \overline{F_{E_n,2}}^{\mathcal{F}}$$

modular, geometric — combinatorial

where $\mathcal{F} = \{\mathcal{F}^1, \mathcal{F}^2, \mathcal{F}^3, \mathcal{F}^4, \mathcal{F}^5\}$ is an explicit collection of semifans. Moreover,

fan — strict semifan

$$\overline{F_{E_n,2}}^{\mathcal{F}_{can}} \xrightarrow{\text{dominant}} \overline{F_{E_n,2}}^{\mathcal{F}}$$

where $\mathcal{F}_{can} = \{\mathcal{F}^1_{can}, \mathcal{F}^2_{can}, \mathcal{F}^3_{can}, \mathcal{F}^4_{can}, \mathcal{F}^5_{can}\}$ are canonical Coxeter–Vinberg fans.

all fans

Rmk: We use this result to explicitly combinatorially classify $\partial \overline{F_{E_n,2}}$ in terms of folded ADE + BC surfaces & find dlt models for degenerations.

Rmk (on proof strategy)

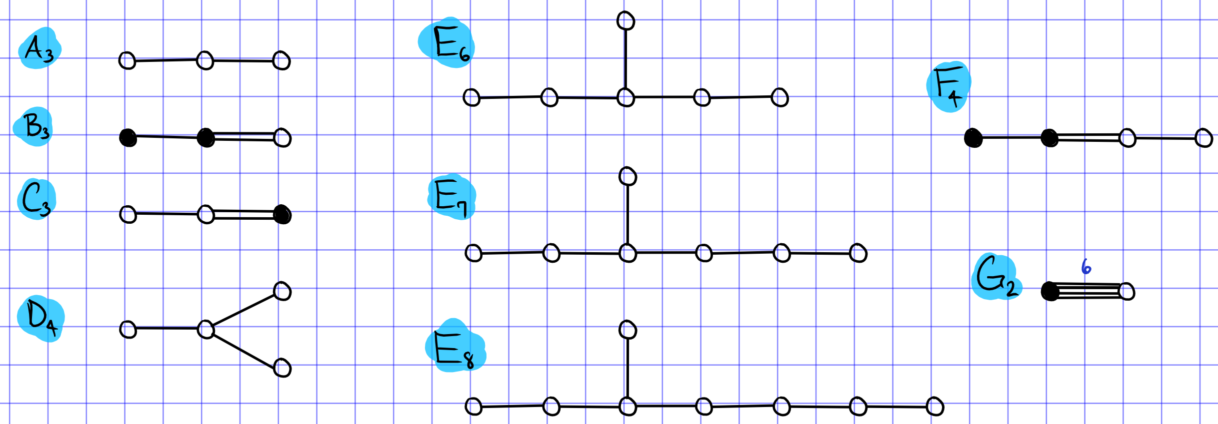
- Embed $F_{E_n,2} \hookrightarrow \overline{F_{(2,2,0)}}$ & take closure.
- Restrict/intersect/"fold" boundary data for $\partial \overline{F_{(2,2,0)}}$

Dynkin Diagrams & Folding

• We study & classify $\partial \overline{F}_{E_n,2}$ using a comparison morphism to $\overline{F}^{t}_{E_n,2}$ and classify degenerations using folding of Dynkin diagrams:

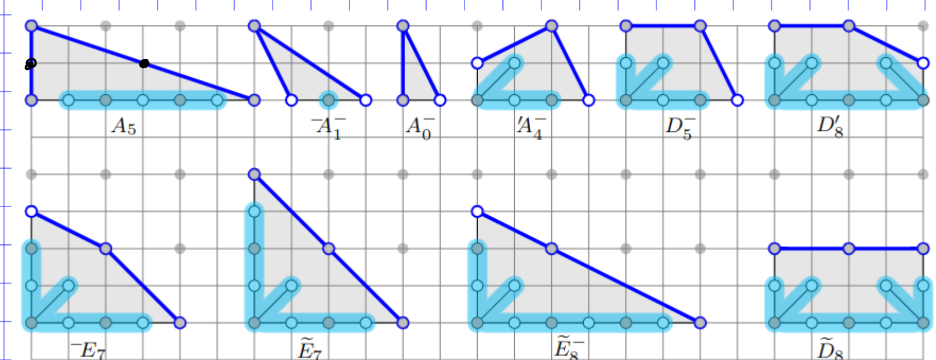
Convention:

○ : $V^2 = -2$
● : $V^2 = -4$



→ ADE Surfaces

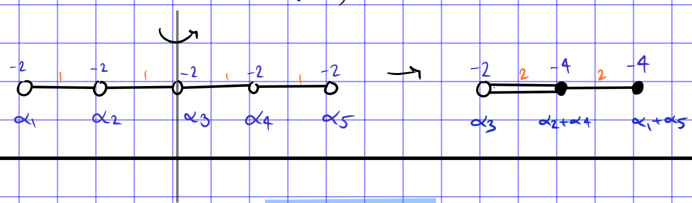
[AE22]



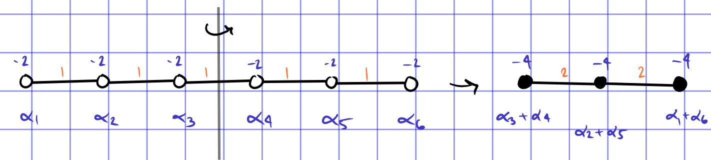
Folding of simply laced Dynkin diagrams:

$A_{2n-1} \longrightarrow B_n$

$(n=3)$

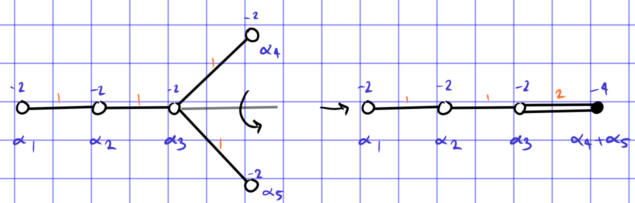


$A_n \longrightarrow A_n(2)$

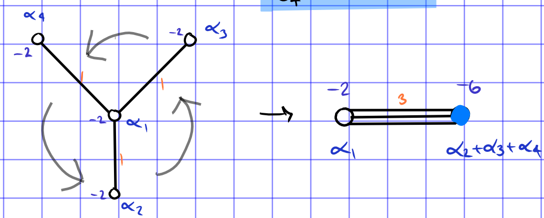


$D_n \longrightarrow C_{n-1}$

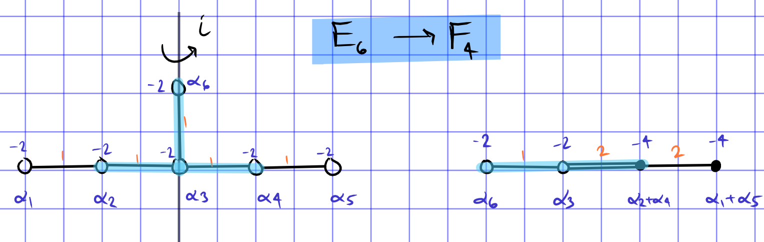
$(n=5)$



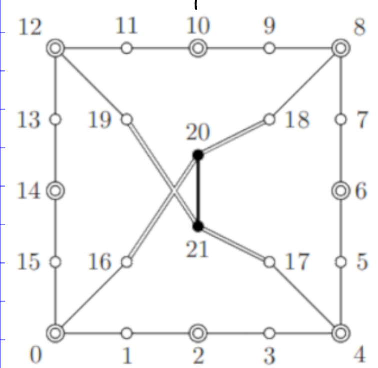
$D_4 \longrightarrow G_2$

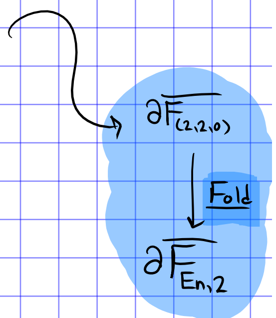


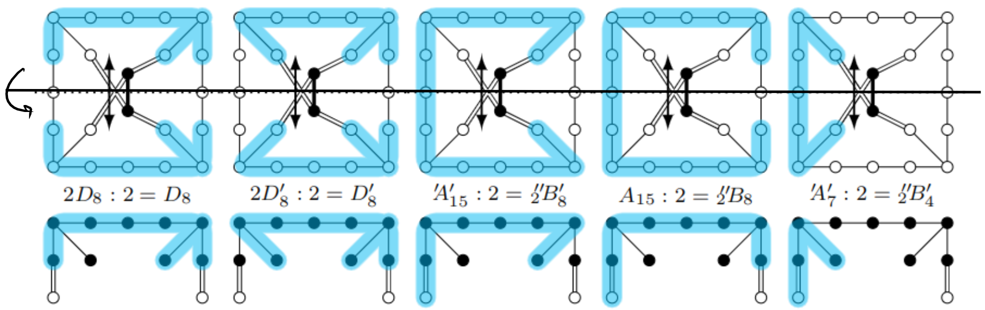
$E_6 \longrightarrow F_4$



At a cusp in $\partial \overline{F_{(2,2,0)}}$:







(0-Cusp #4 in $\partial \overline{F_{E_n}}$)

A general strategy for describing $\partial\overline{\mathcal{M}}$:

1) Find L & $\Gamma_L \leq O(L)$ s.t. $M \xrightarrow{\sim} \Gamma_L \backslash \Omega_L$ a Hodge-theoretic period domain (Type IV Hermitian symmetric)

Γ_L: neat, arithmetic subgroup
M: coarse space

2) Prove $\exists \mathcal{F}$ a semifan s.t.

$$\exists f : (\overline{M})^{\eta} \xrightarrow{\sim} \overline{\left(\Gamma_L \backslash \Omega_L\right)}^{\mathcal{F}} \quad \text{(on coarse spaces)}$$

3) Use the correspondence:

$$\partial\overline{M}_{\bullet}^{BB} \rightleftharpoons \Gamma_L \backslash OGr_{\bullet+1}(L) \quad \text{where } OGr_k(L) = \left\{ \begin{array}{l} \text{isotropic subspaces of} \\ L \otimes_{\mathbb{Z}} \mathbb{R} \text{ of dimension } k \end{array} \right\}$$

$B_L|_V \equiv 0$

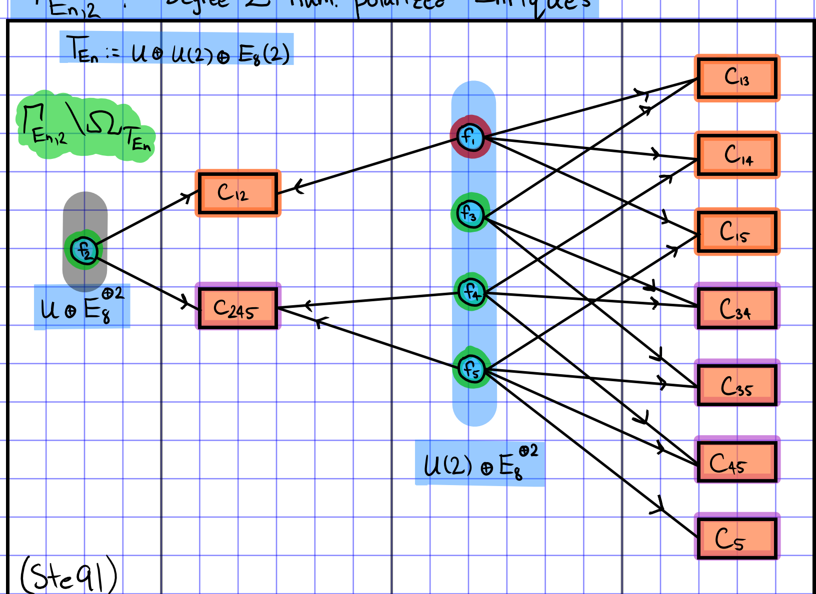
$\{e_i\}_{i\in I}$ = 0-cusps $\rightleftharpoons$ (orbits of) isotropic vectors $v^2 = 0$

$\{C_j\}_{j\in J}$ = 1-cusps $\rightleftharpoons$ (orbits of) isotropic planes $H = \langle v_1, v_2\rangle_{\mathbb{Z}}$, $v_i v_j = 0 \quad \forall i,j$.

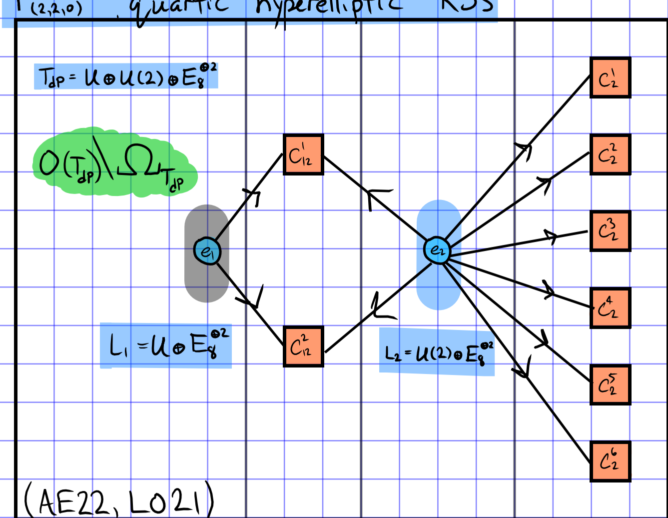
etc.

$\rightsquigarrow$ Enumerate all cusps to build cusp diagram, and compare to those of other moduli spaces:

$\overline{F}_{En,2}^{BB}$: Degree 2 num. polarized Enriques



$\overline{F}_{(2,2,0)}^{BB}$: quartic hyperelliptic K3s

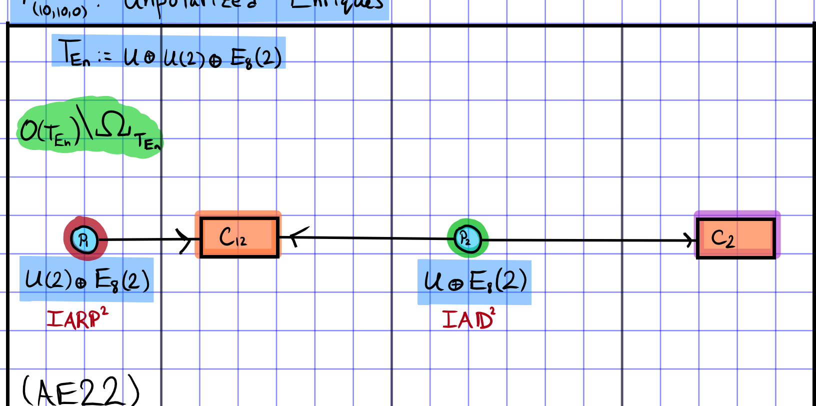


- Blue: 0-cusps (points)
- Orange: 1-cusps (modular curves with level structure)

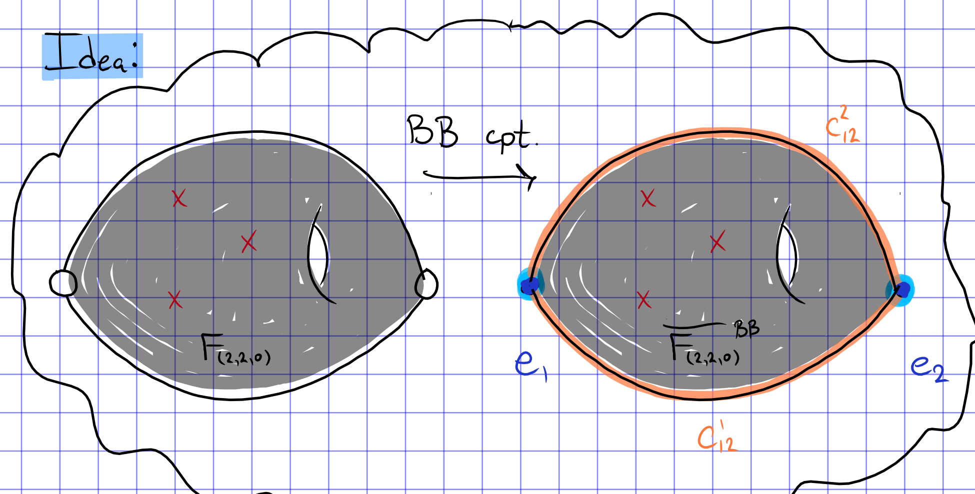
[nonsymplectic involution, generic $\mathrm{Pic}(X_\eta) = (2,2,0)$.]

$\downarrow$ cusp correspondence

$\overline{F}_{(10,10,0)}^{BB}$: unpolarized Enriques

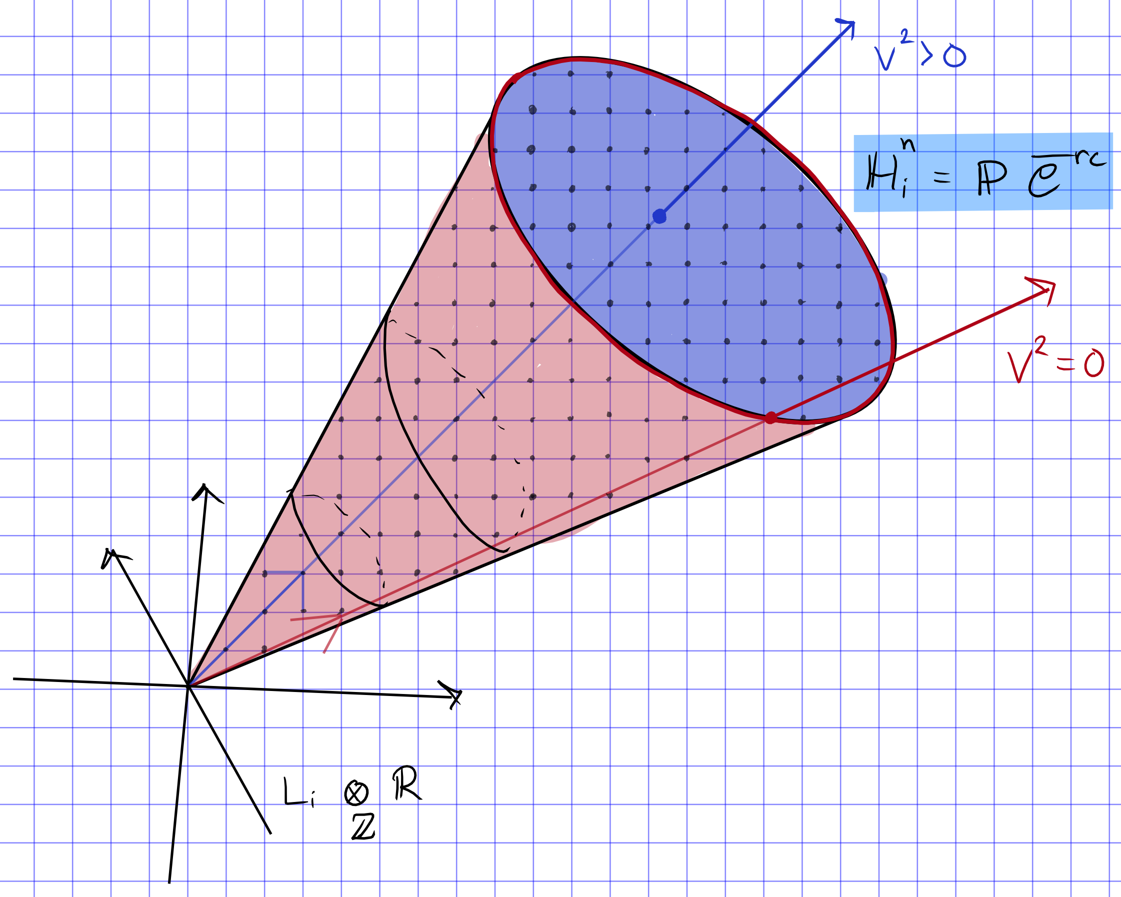


Idea:



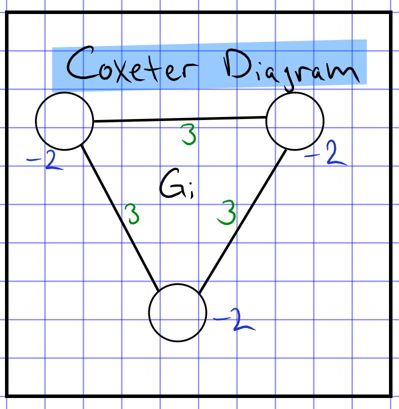
At each 0-cusp e_i:

- Construct a hyperbolic lattice $L_i := e_i^{\perp L_i}/e_i$ & a hyperbolic space $\mathbb{H}_i^n$ using the light cone $\overline{\mathcal{C}}^{rc}$ of L_i:
  $\mathrm{sgn}(L_i) = (1, r-1)$.

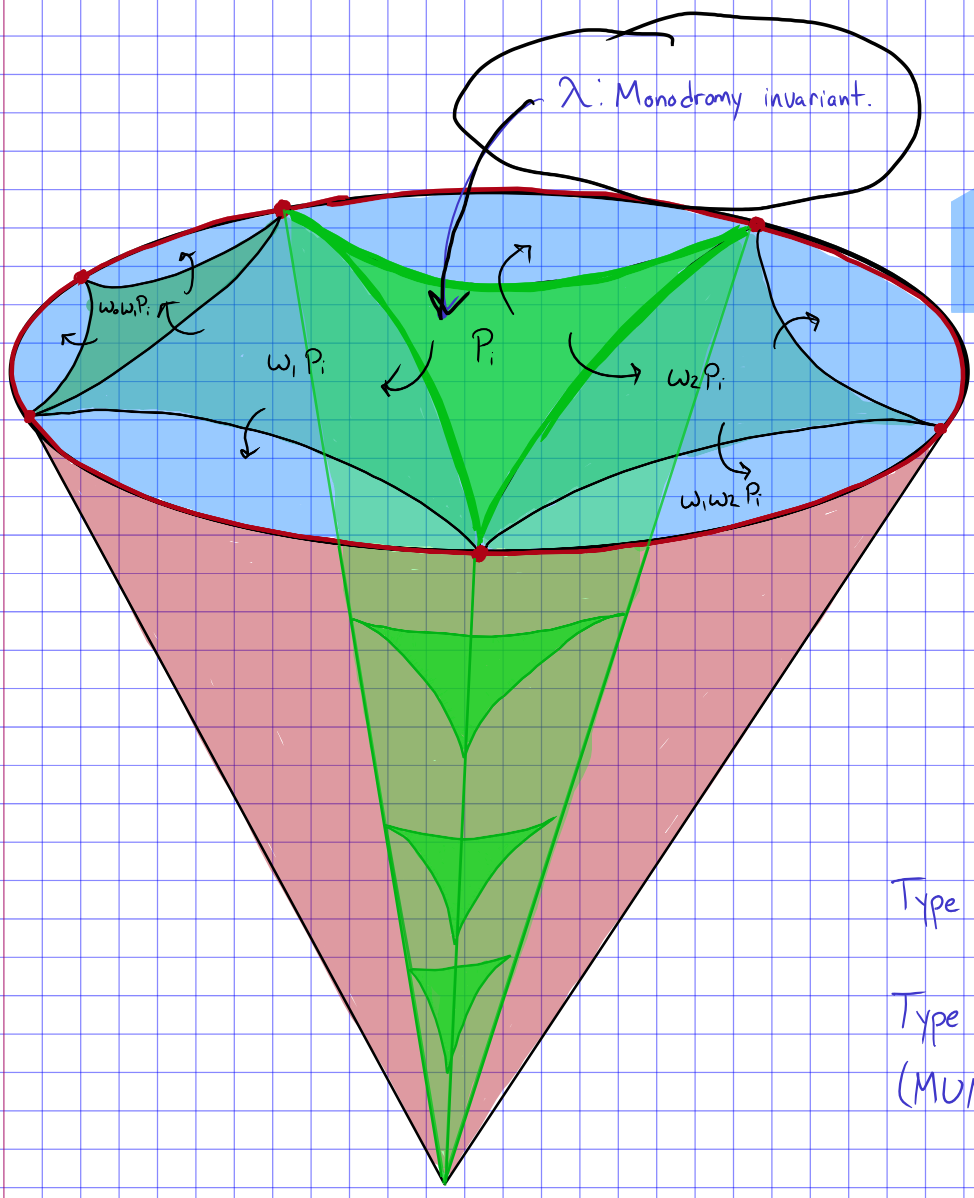


$$\overline{\mathcal{C}}^{rc} := \{\mathbb{R}_+ v \in L \otimes_{\mathbb{Z}} \mathbb{R} \mid v^2 > 0\} \cup \{\mathbb{R}_+ \omega \in L \otimes_{\mathbb{Z}} \mathbb{R} \mid \omega^2 = 0 \text{ \& } \omega \text{ is } \underline{\text{rational}}\}$$

- Find a Weyl group $W_i = W(L_i)$ with Coxeter diagram G_i
- Find a hyperbolic polytope P_i, e.g. $P_i^{cv} :=$ fundamental domain of $W_i \curvearrowright L_i \otimes_{\mathbb{Z}} \mathbb{R}$.
- Act by reflections to obtain a tiling of $\mathbb{H}_i^n$ by polytopes



Vin85



Hyperbolic polytope / tiling

$\lambda \in P_i \rightsquigarrow B(\lambda)$ an IAS^2

$\rightsquigarrow B(\lambda) = \Gamma(\mathcal{X}_0)$, $\mathcal{X}_0$ a degeneration

$\rightsquigarrow \mathcal{X}_0$ smooths to $\mathcal{X}$

a Kulikov model

Type II $\rightleftarrows \lambda \in \partial P_i \rightleftarrows$ Parabolic subdiagrams

Type III $\rightleftarrows \lambda \in P_i^{\circ} \rightleftarrows$ Elliptic subdiagrams
(MUM)

- Cone off to get a semifan $\mathcal{F}_i$ and set $\mathcal{F} = \bigcup_i \{\mathcal{F}_i\}$.
- Choose P_i (& thus $\mathcal{F}_i$) s.t. $(\overline{M})^n \xrightarrow{\sim} \overline{M}^{\mathcal{F}}$

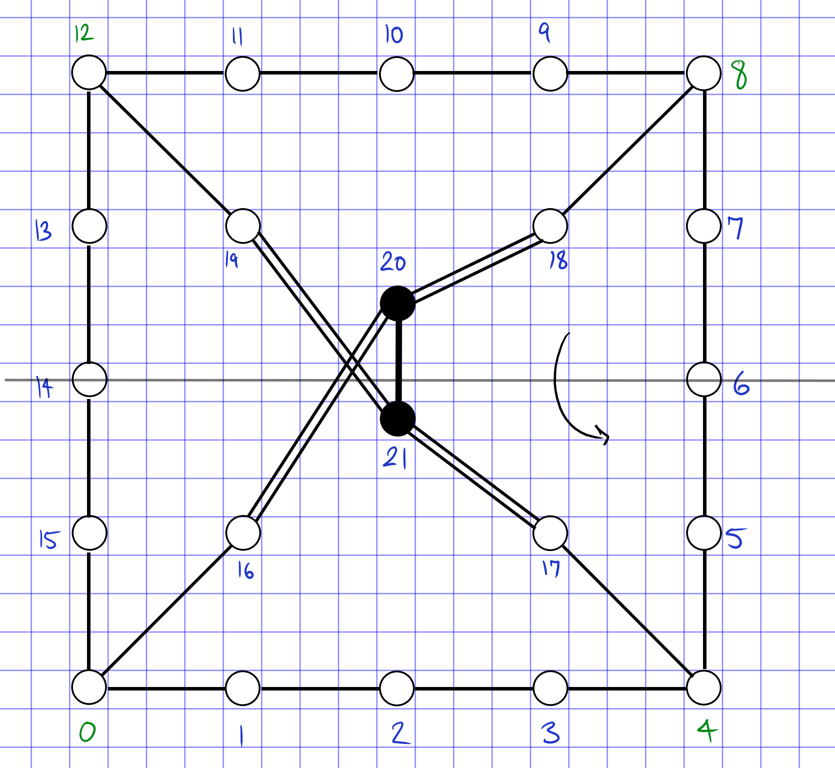
- Use elliptic/parabolic subdiagram poset of G_i to describe stable degenerations at $\partial(\overline{M})_i^n$, e.g. by building an IAS^2 $B(\lambda_i)$ from a monodromy invariant $\lambda_i \in L_i$ at this cusp.
  - ↳ Empty diagram $\rightsquigarrow$ Maximal degeneration
  - ↳ Subdiagrams $\rightsquigarrow$ Set corresponding $\ell_j := \lambda_i \cdot \alpha_j$ to zero (used in construction of $B(\lambda_i)$)

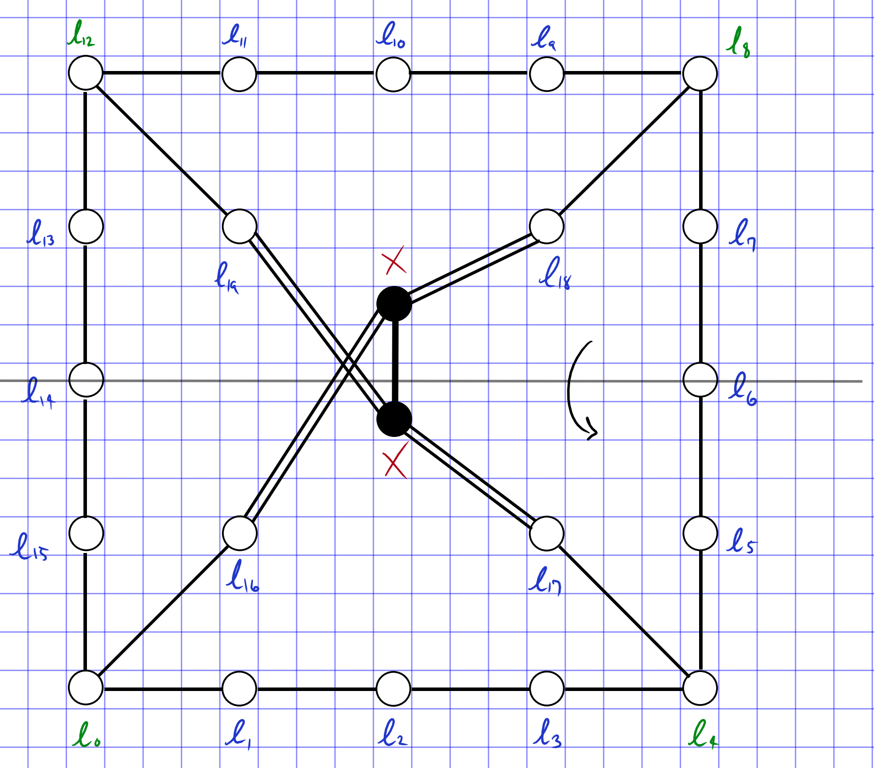
- Classify all ways to degenerate $B(\lambda_i)$ to produce stable pair degenerations & read off (ADE + BC surfaces). ▨

Constructing an IAS^2 $B(\lambda_i)$

$D_{(16,2,0)_1}$: 22 roots, 16 outer $\rightsquigarrow$ F_{2, E_n} cusp #4.

Assign lengths ℓ_i:





$\ell_0 = \ell_{12}$ $\quad$ $\ell_5 = \ell_7$ $\quad$ $\ell_{16} = \ell_{19}$ $\quad$ ℓ_{14} free

$\ell_1 = \ell_{11}$ $\quad$ $\ell_{15} = \ell_{13}$ $\quad$ $\ell_{17} = \ell_{18}$ $\quad$ ℓ_{16} free $\quad \Rightarrow$ 11 parameters

$\ell_2 = \ell_{10}$

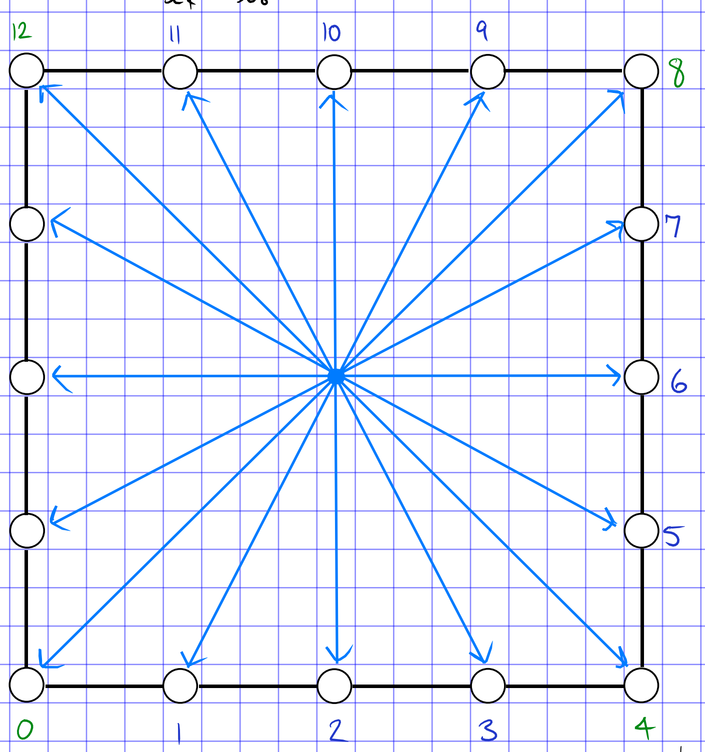
$\ell_3 = \ell_9$

$\ell_4 = \ell_8$

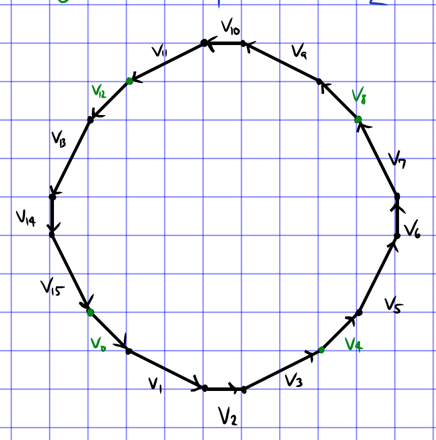
$\ell_i = \lambda \cdot \alpha_i$

monodromy invt.

roots



$\rightsquigarrow$ Defines a toric variety w/ 16 rays, whose dual polytope is a 16-gon



16-gon

$v_1 = \frac{1}{2}\ell_1\,[2,-1]$

$v_2 = \ell_2\,[1,0]$

$v_3 = \frac{1}{2}\ell_3\,[2,1]$

$v_5 = \frac{1}{2}\ell_5\,[1,2]$

$v_6 = \ell_6\,[0,1]$

$v_7 = \frac{1}{2}\ell_7\,[-1,2]$

$v_9 = \frac{1}{2}\ell_9\,[-2,1]$

$v_{10} = \ell_{10}\,[-1,0]$

$v_{11} = \frac{1}{2}\ell_{11}\,[-2,-1]$

$v_{13} = \frac{1}{2}\ell_{13}\,[-1,-2]$

$v_{14} = \ell_{14}\,[0,-1]$

$v_{15} = \frac{1}{2}\ell_{15}\,[1,-2]$

$v_0 = (\ell_0 + \frac{1}{2}\ell_{16})\,[1,-1]$

$v_4 = (\ell_4 + \frac{1}{2}\ell_{17})\,[1,1]$

$v_8 = (\ell_8 + \frac{1}{2}\ell_{18})\,[-1,1]$

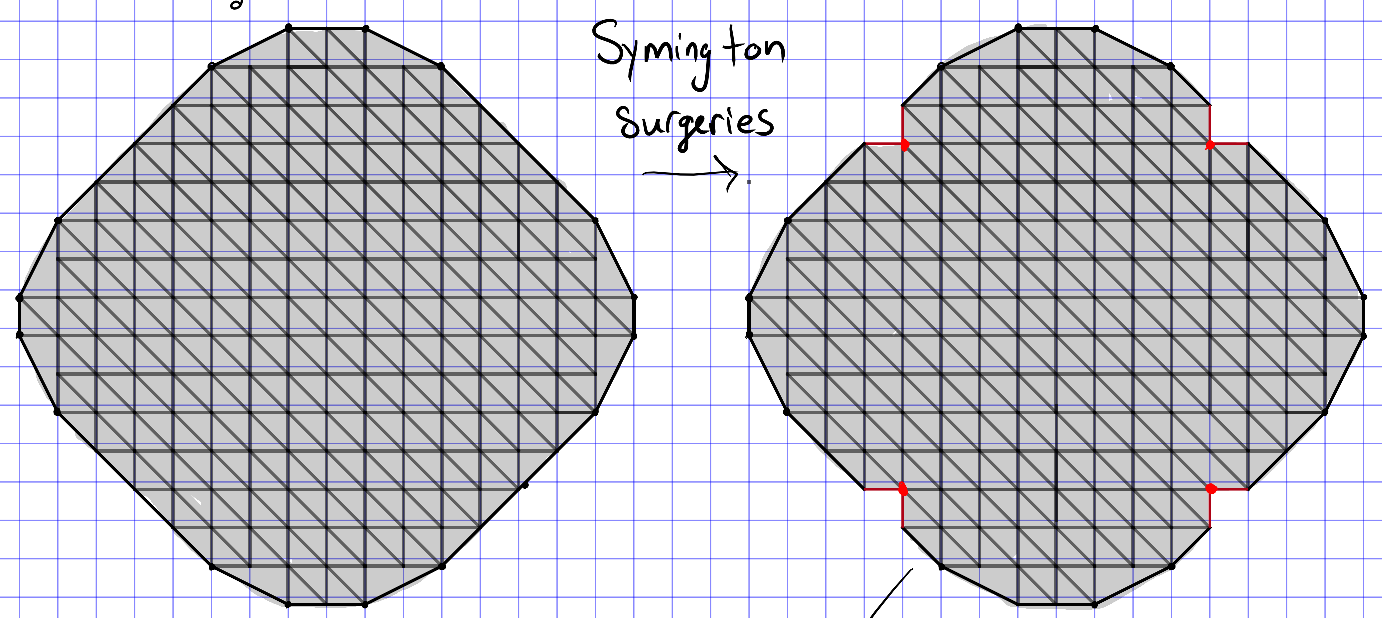
$v_{12} = (\ell_{12} + \frac{1}{2}\ell_{19})\,[-1,-1]$

String the v_i together to form a polytope P — then P "closes" $\Leftrightarrow \sum v_i = 0$

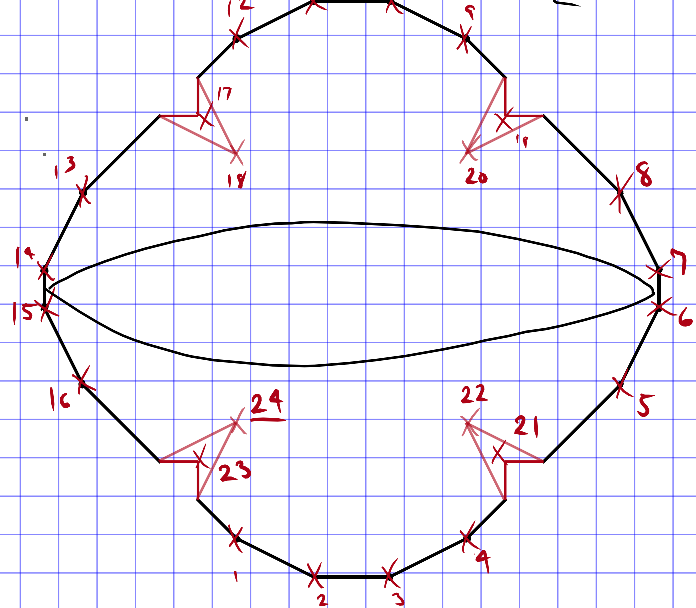
One condition, so $\vec{\ell} \in \mathbb{Z}^{10} \rightsquigarrow$ 10 dim family

The above 16-gon corresponds to the empty subdiagram of $D_{(16,2,0)_1}$

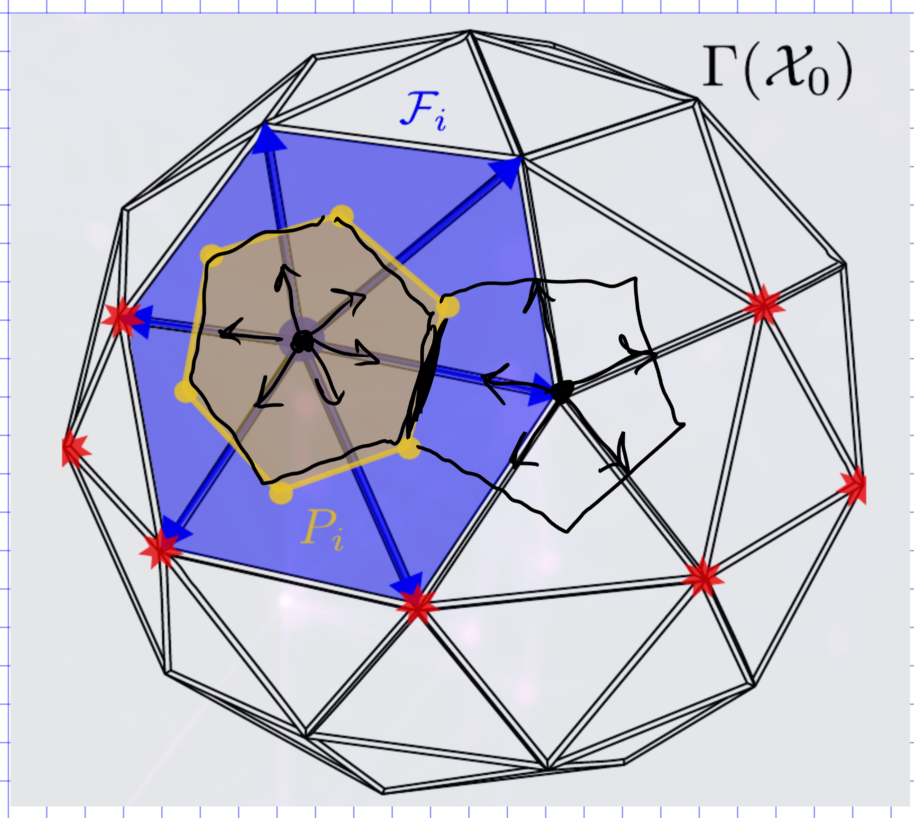
$B(\ell) := P \sqcup_\partial P^{op}$



$P \cup P^{op}$



IAS^2 w/ charge 24

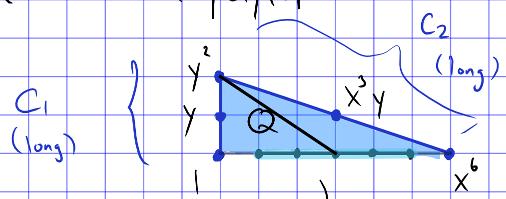


E.g. (of ADE surfaces)

Recall A_5:

$(1,0)\to(1,0)$
$(0,1)\to(3,1)$
$\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\3x+y\end{bmatrix}$

- Associate a polytope:



$\longrightarrow (V_Q, \mathcal{L}_Q)$ a toric variety.

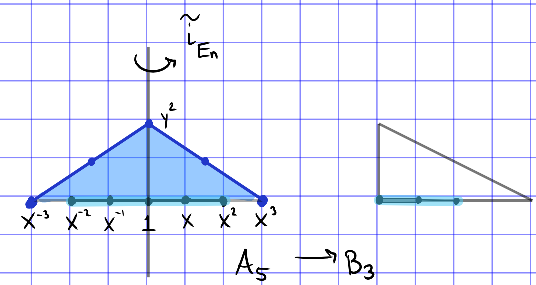
$a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$

$\Rightarrow f(x,y) = (1 + y^2 + x^6) + (a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5) \in H^0(\mathcal{L}_Q)$.

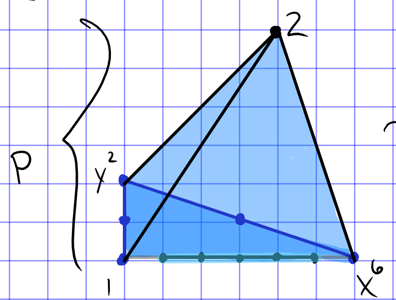
$\rightsquigarrow$ Yields a del Pezzo ADE sfc. $(V_Q, C_1 + C_2)$.

- Strategy: Understand ADE + BC sfcs as quotients of ADE sfcs. by this involution.

Type B Surfaces



- Cone



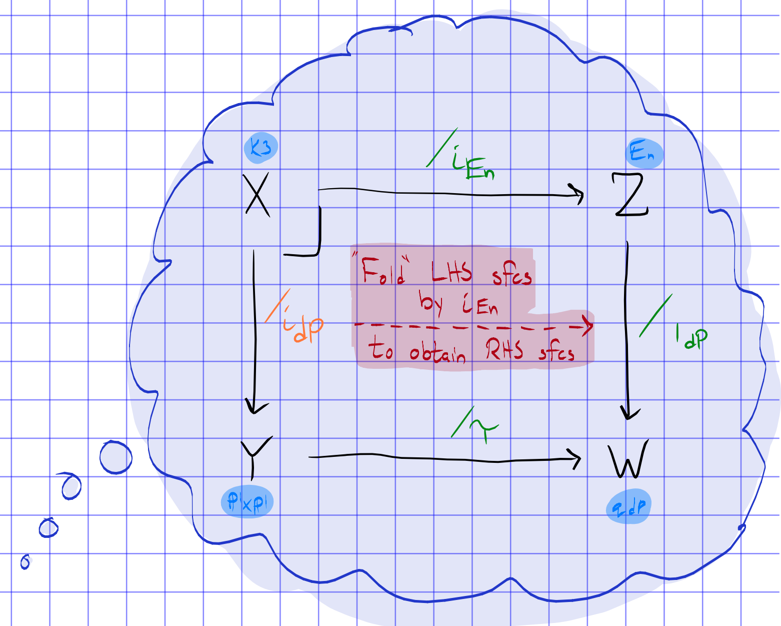
$\rightsquigarrow (V_P, \mathcal{L}_P)$ a toric 3-fold.

$\Rightarrow z^2 + f(x,y) \in H^0(\mathcal{L}_P)$.

- Take zero set of this section to produce an ADE sfc. X

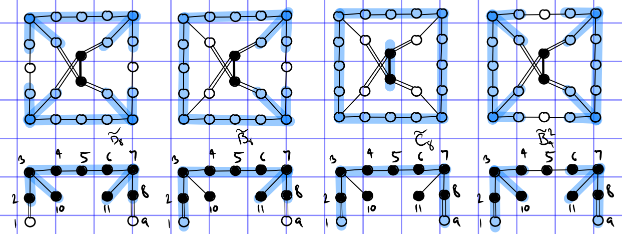
- Quotient by ι_{dP} to get Y (ramified).

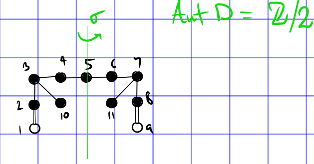
- Insight: $X = K3 \downarrow Y = \mathbb{P}^1 \times \mathbb{P}^1 \rightsquigarrow$ (fold ADE sfcs. by ι_{E_n}) Z = Enriques $\downarrow W$ = quartic del Pezzo

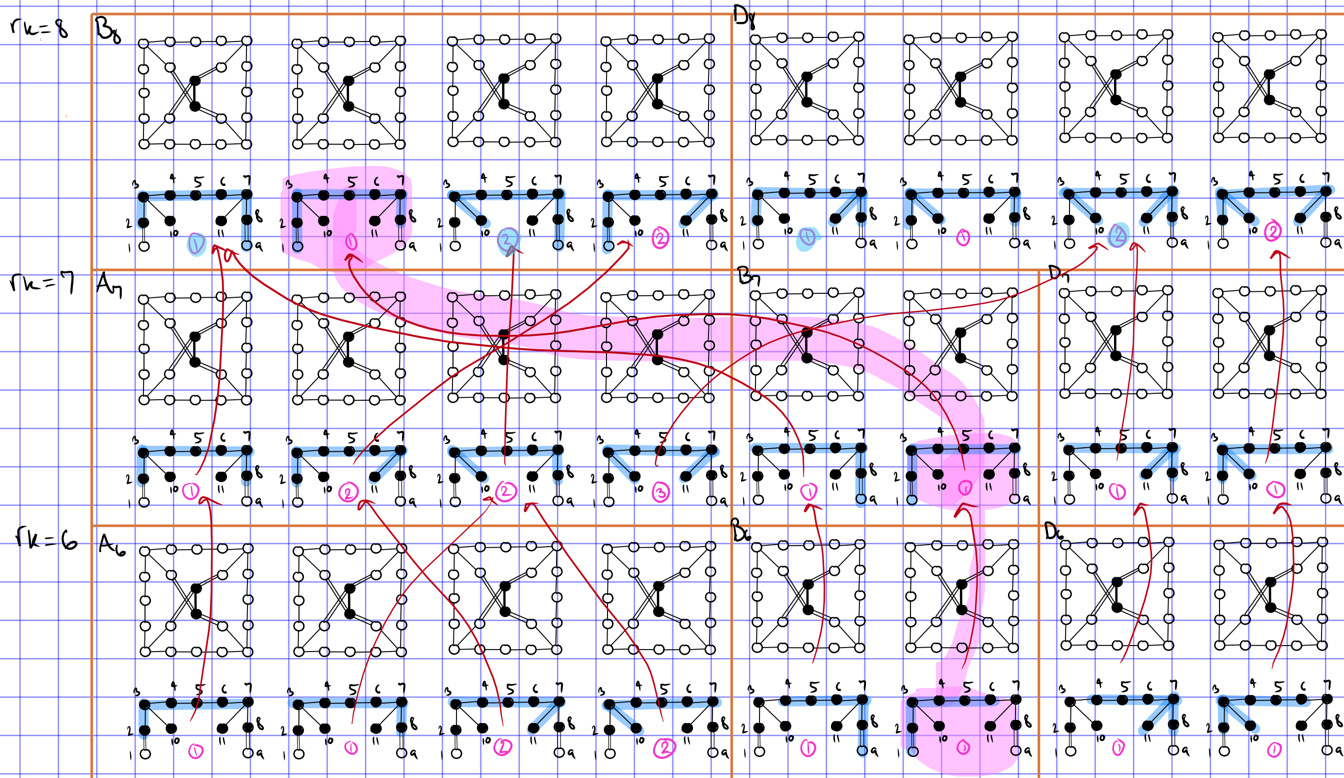


A Glimpse of the Classification Work

Sterk 4







Future Work

- $K=0$: F_{2d}, $\mathcal{M}_{B:Ell}$, & $K=1,2$ to complete classification for sfcs.
- Current project:
  - Techniques for $D \in \mathrm{Div}(\mathcal{M})$
  - More generally families $S \subseteq \mathcal{M}$.
  - Coble & Halphen surfaces

  } Describe KSBA compactifications
- Ideas:
  - Similar cyclic G-Galois ramified covers
  - $\overline{\mathcal{M}}_g(X;\beta)$ {"relative" moduli}
  - Passing to "level structure" covers: $\begin{array}{c} F_{2d}(\Gamma) \\ \downarrow \\ F_{2d} \end{array}$ (possibly ramified or birational)
  - Moduli of
    - CY 3folds
    - Kählers
    - Holomorphic symplectic mfds
    - Bielliptic sfcs
    - $K=0$ n-folds
    - Tropical varieties
    - Fanos / K-moduli

    } Can we understand $\partial\mathcal{M}$ combinatorially?
  - Combinatorial techniques for derived moduli stacks
  - Generate accessible, interesting problems for graduate students & undergrads
  - Sage functions for combinatorial & lattice-theoretic AG
  - Hyperbolic volumes $\rightsquigarrow$ special values of L-functions, ζ-values.

Thank you!