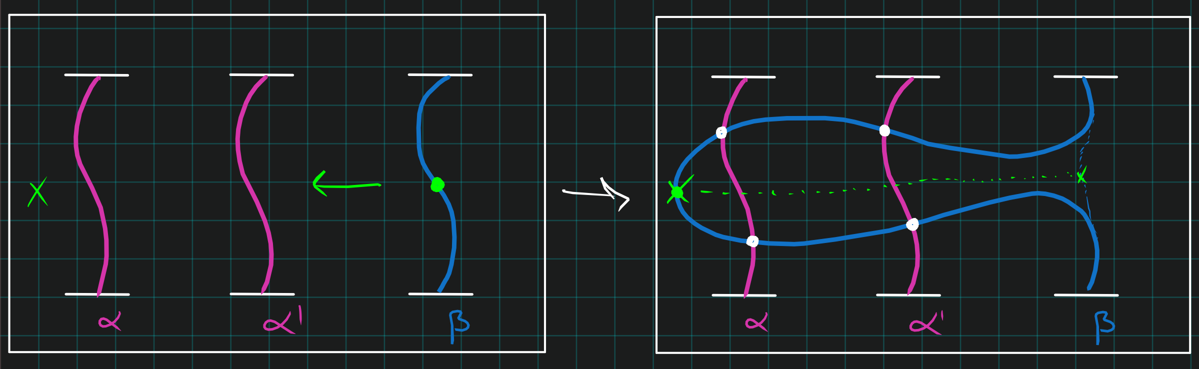
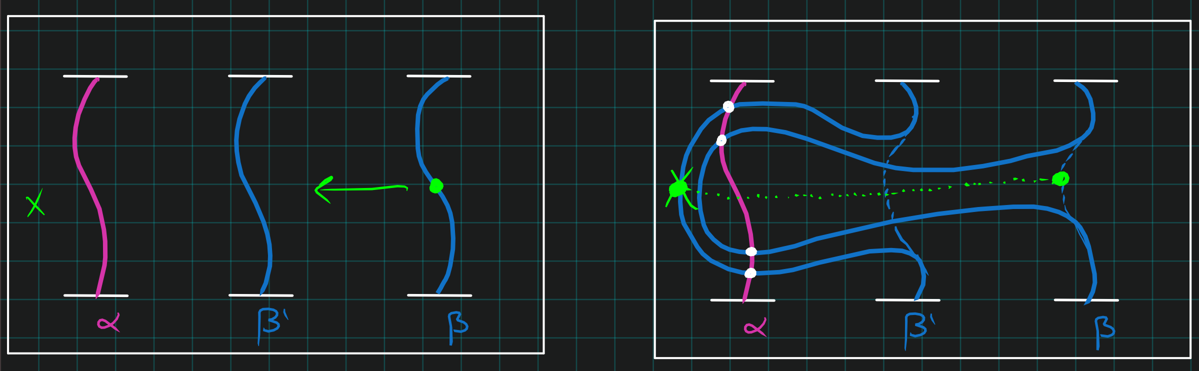
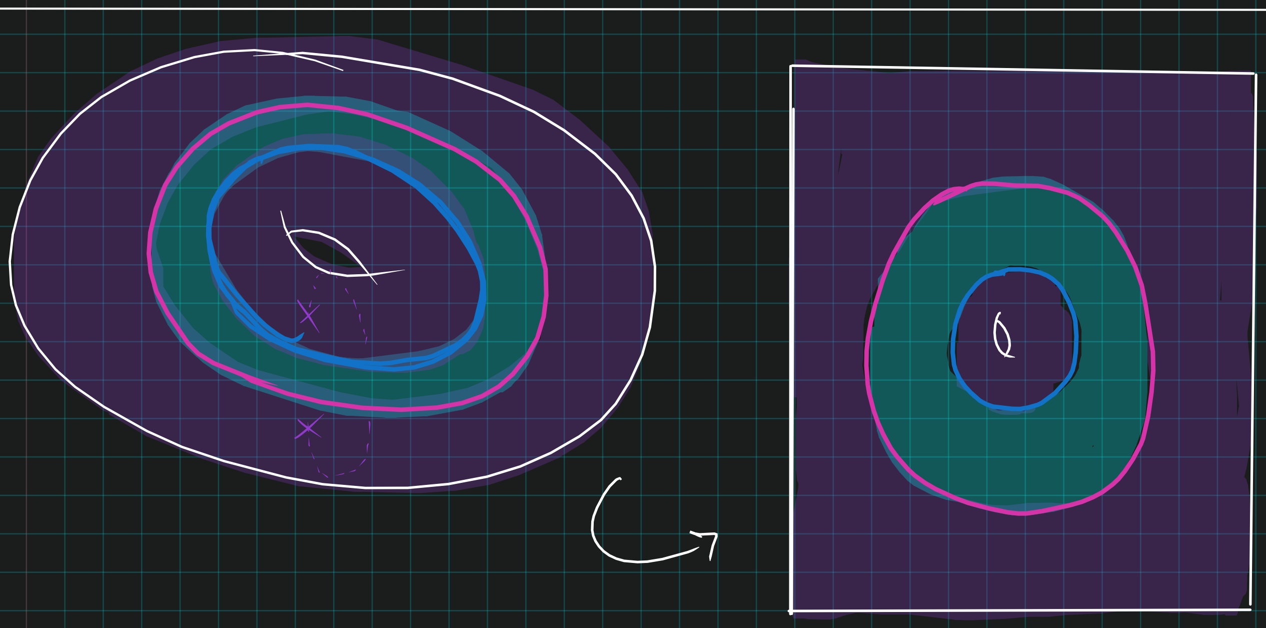
Making every α curve intersect some β curve



Making every α curve intersect some β curve



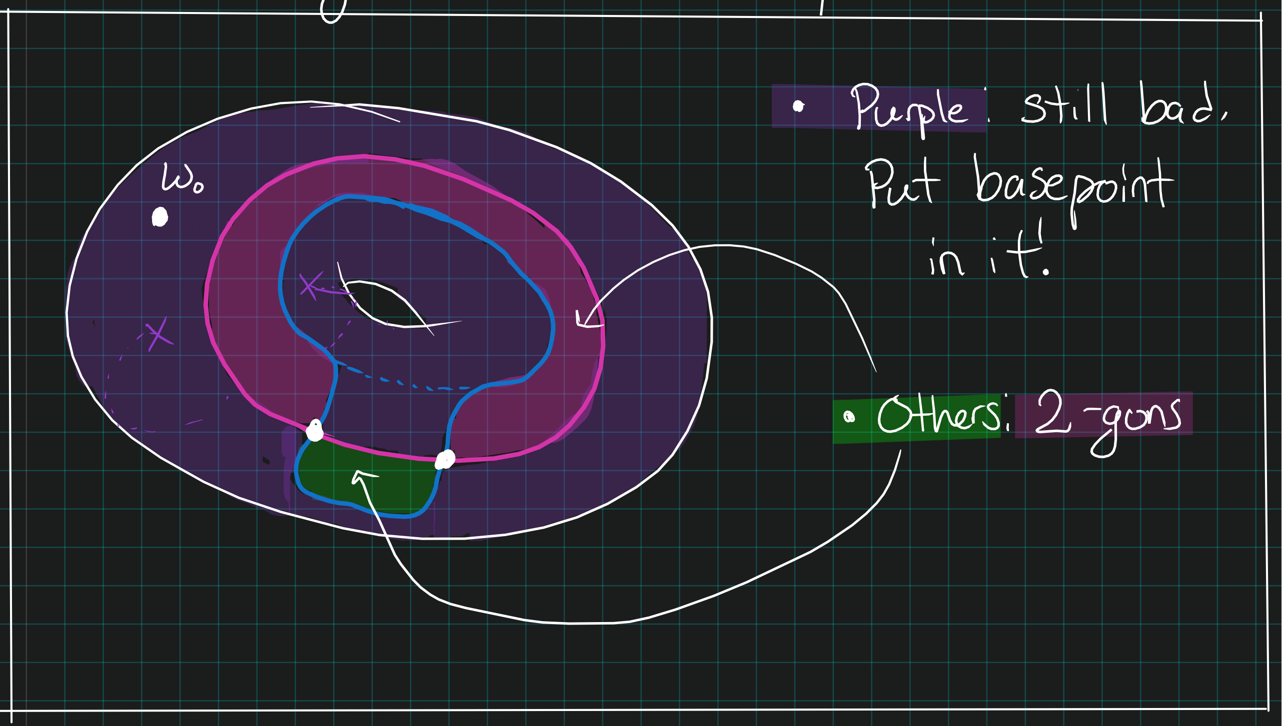
Problem: non-disc regions ("poly-annuli")



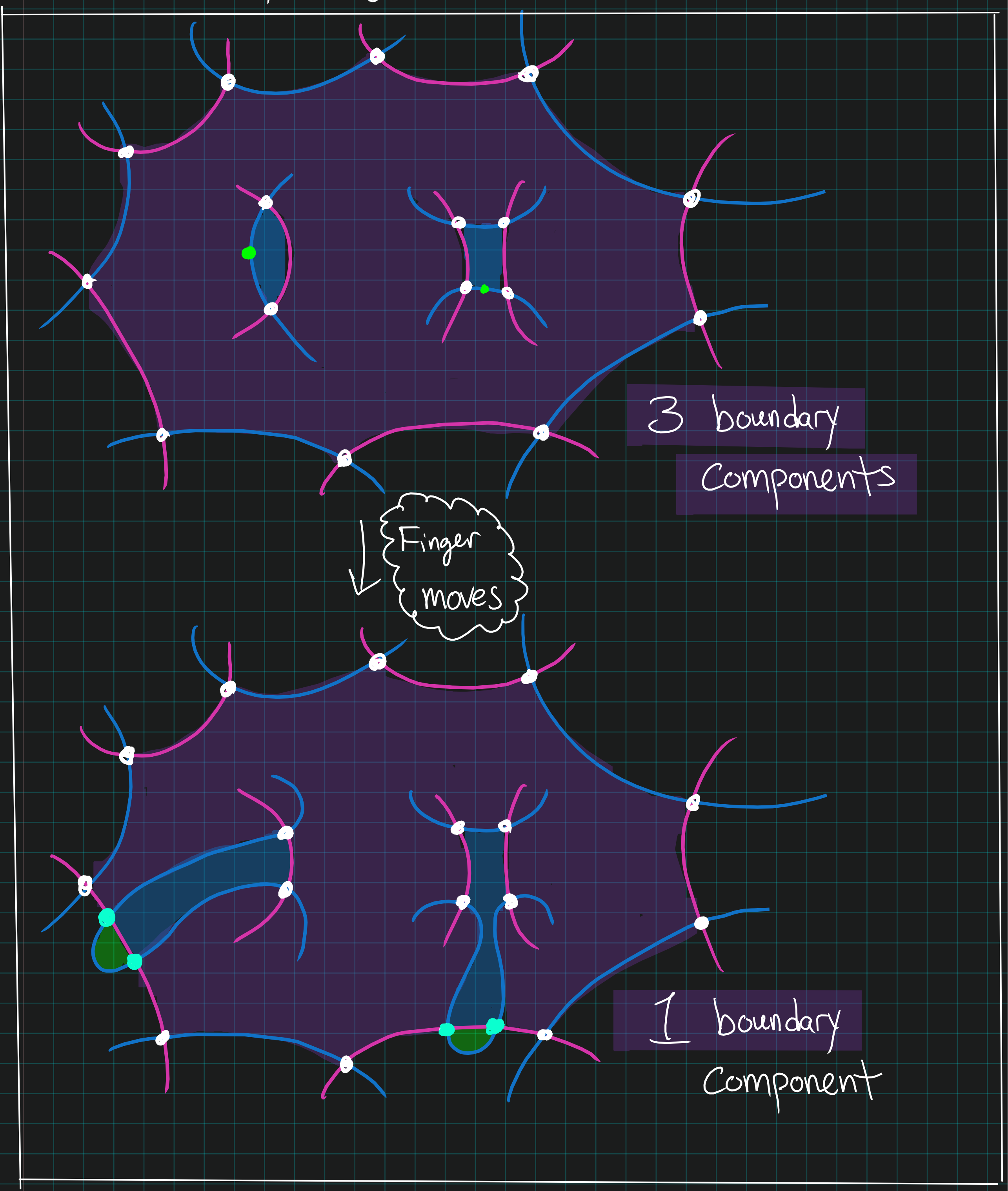
Annulus region (non-disc)

2 boundary components

Fix: finger moves on β curves

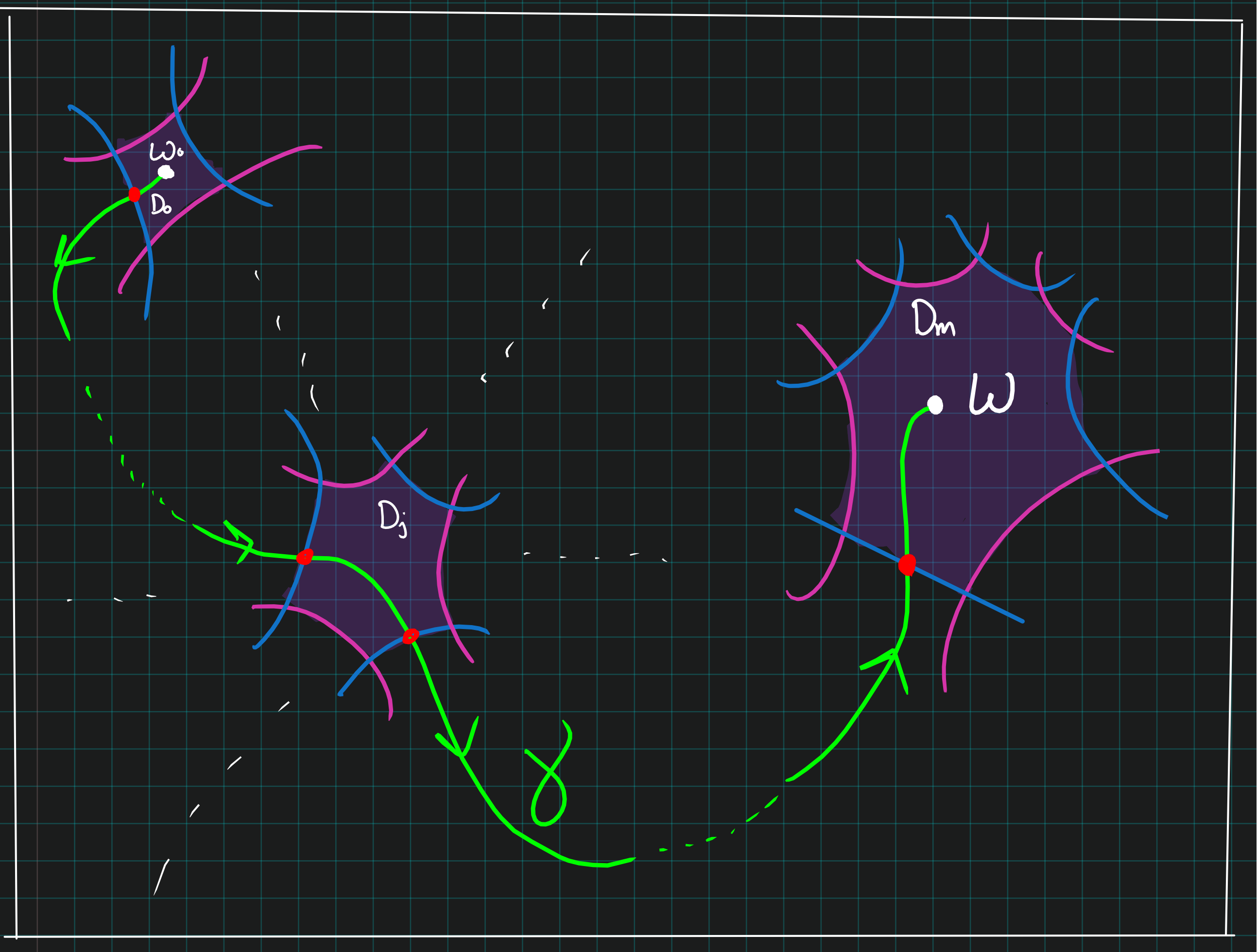


More generally: regions with $\# \pi_0 \partial R \geq 2$



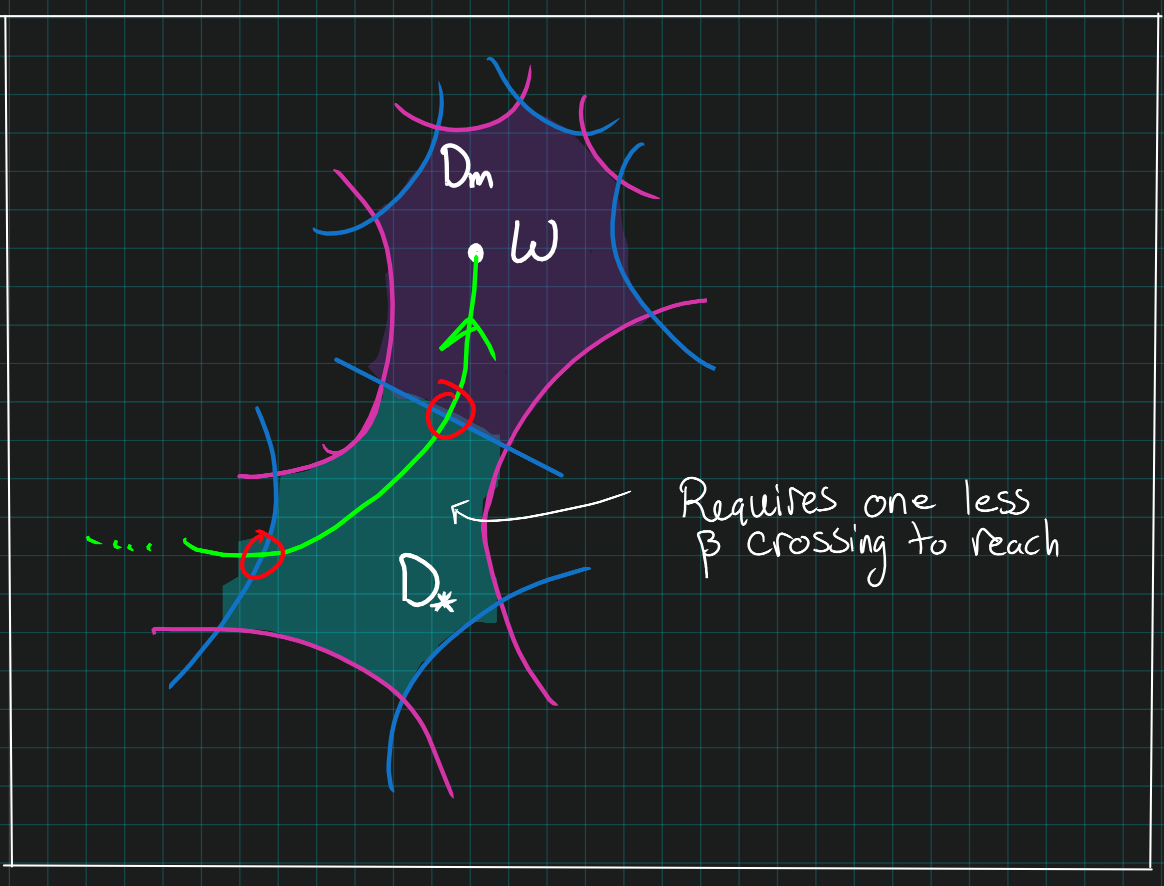
Distance

$d(D_m) := \min \#(\gamma \cap \vec{\beta})$ where $\gamma : D_0 \to D_m$, $\omega_0 \to \omega$

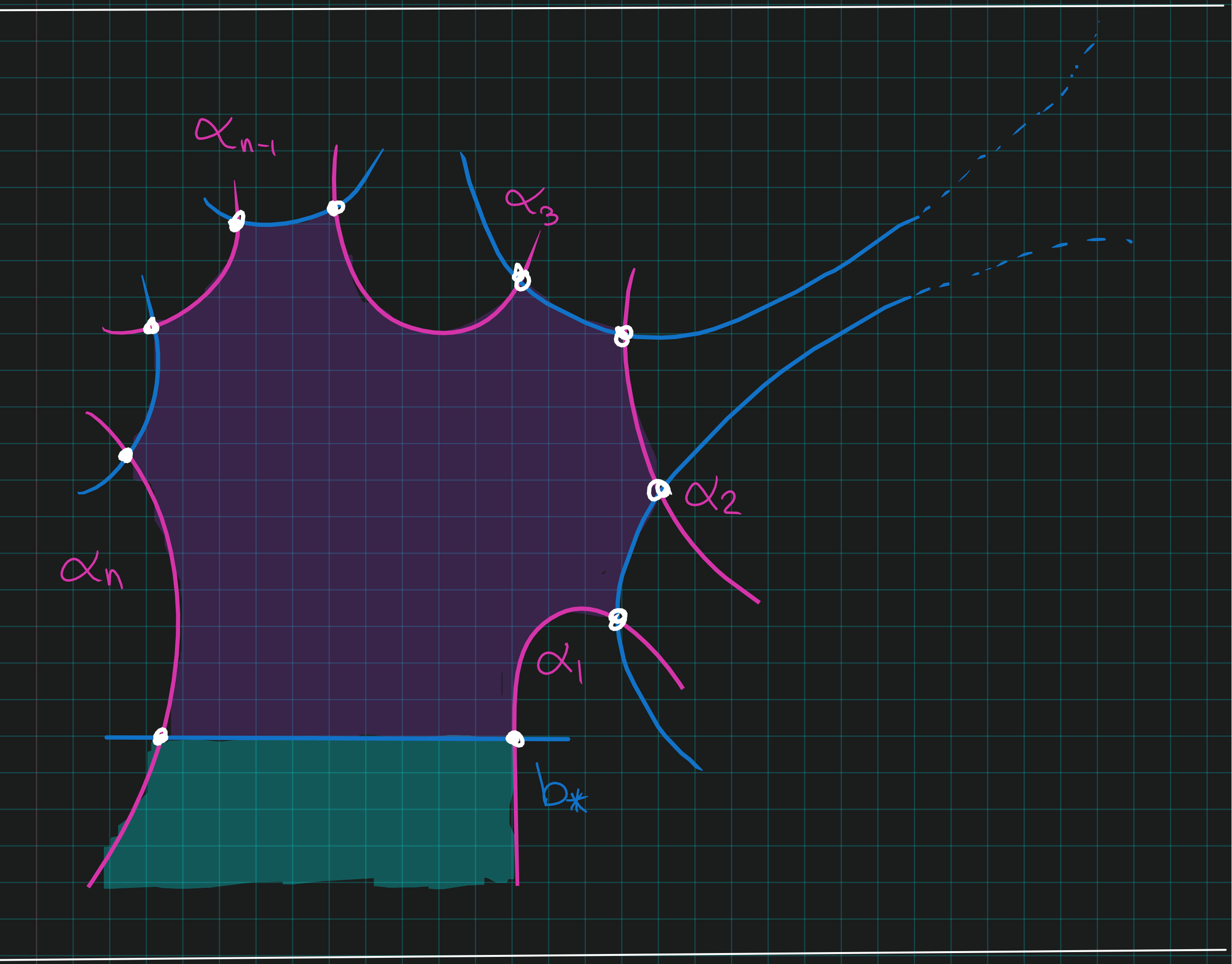


There is a region D_* adjacent to D_m where

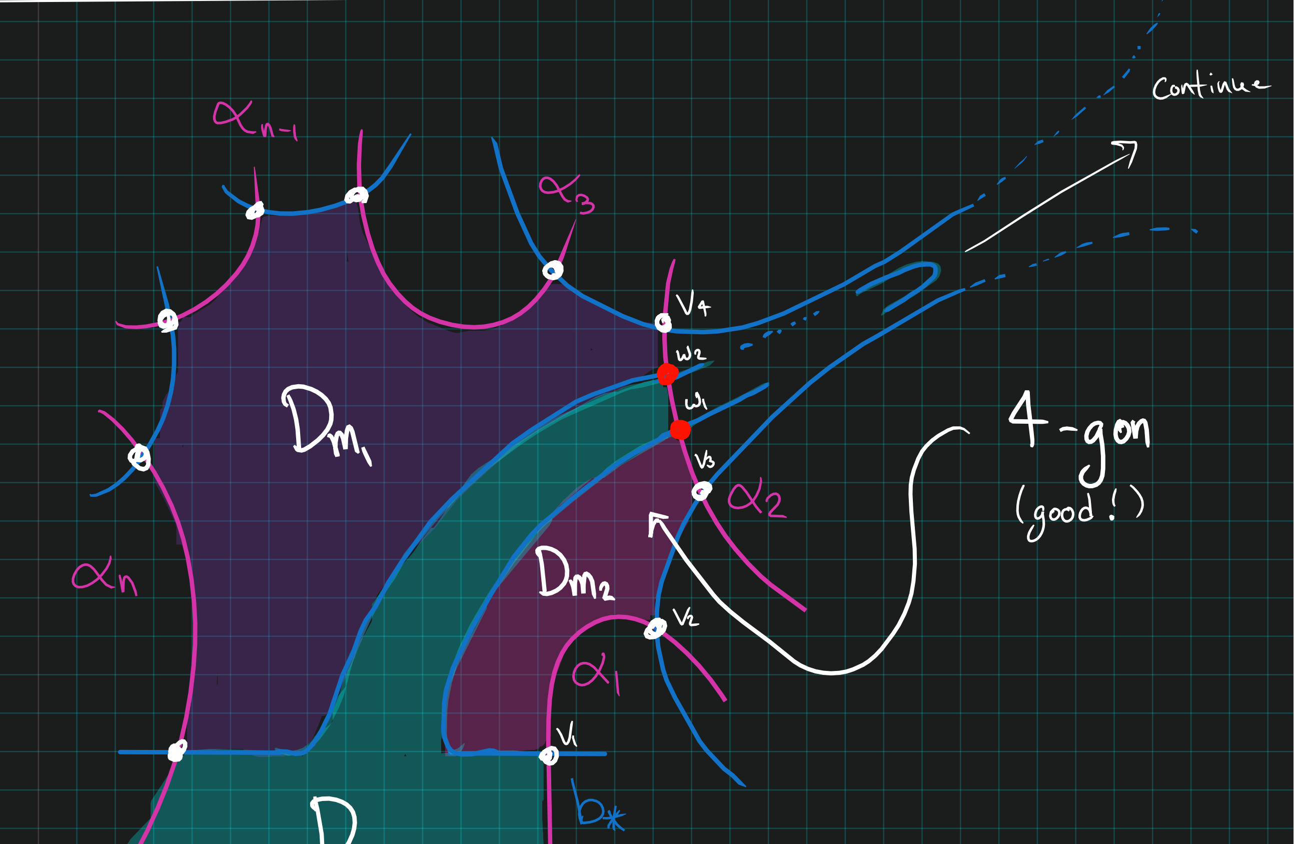
$d(D_*) = d(D_m) - 1$



Zoom in on situation in D_m:



Idea: finger move b_* through D_m and exit at α_2.



New badness of D_m:

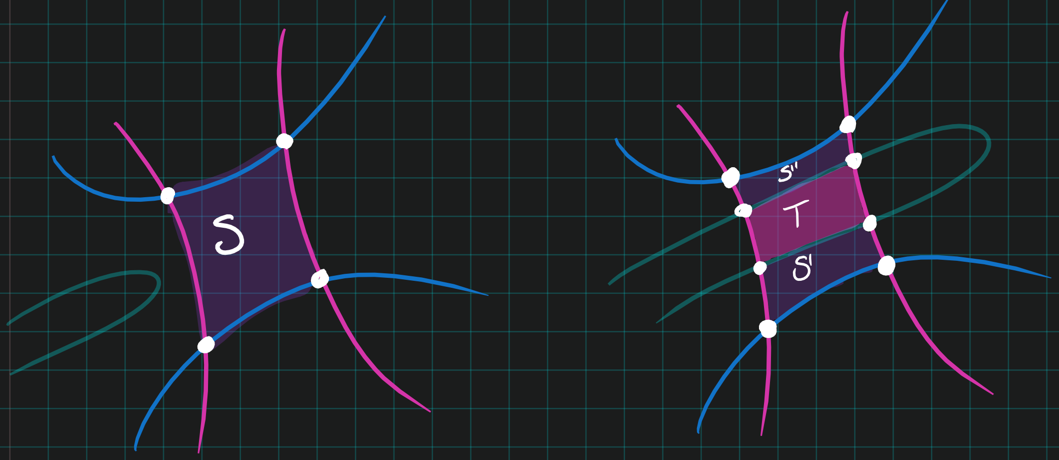
- (−3) vertices $\{v_i\}_{i=1}^{3}$
- (+1) vertex w_2

−2 vertices

$\Rightarrow$ −1 badness

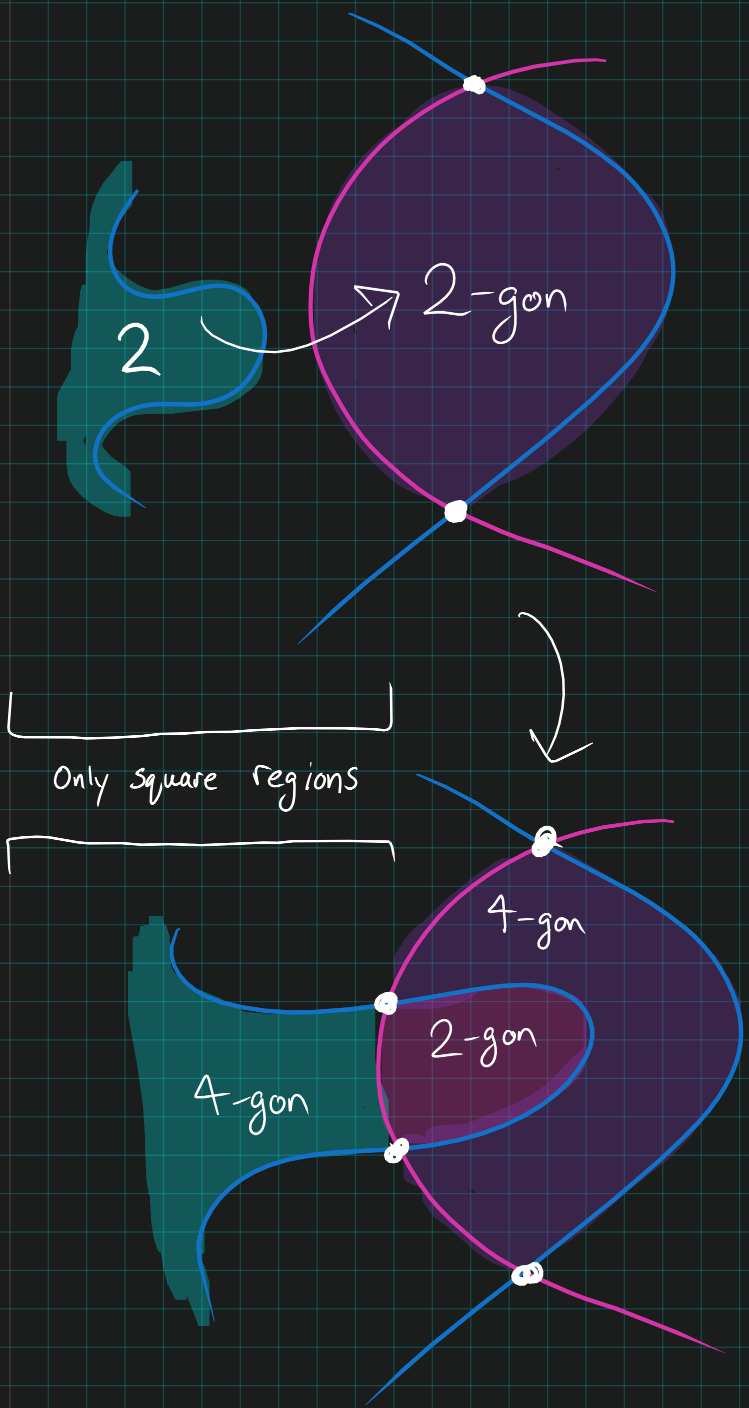
$b(D_{m_1}) = b(D_m) - 1$

Passing through square regions



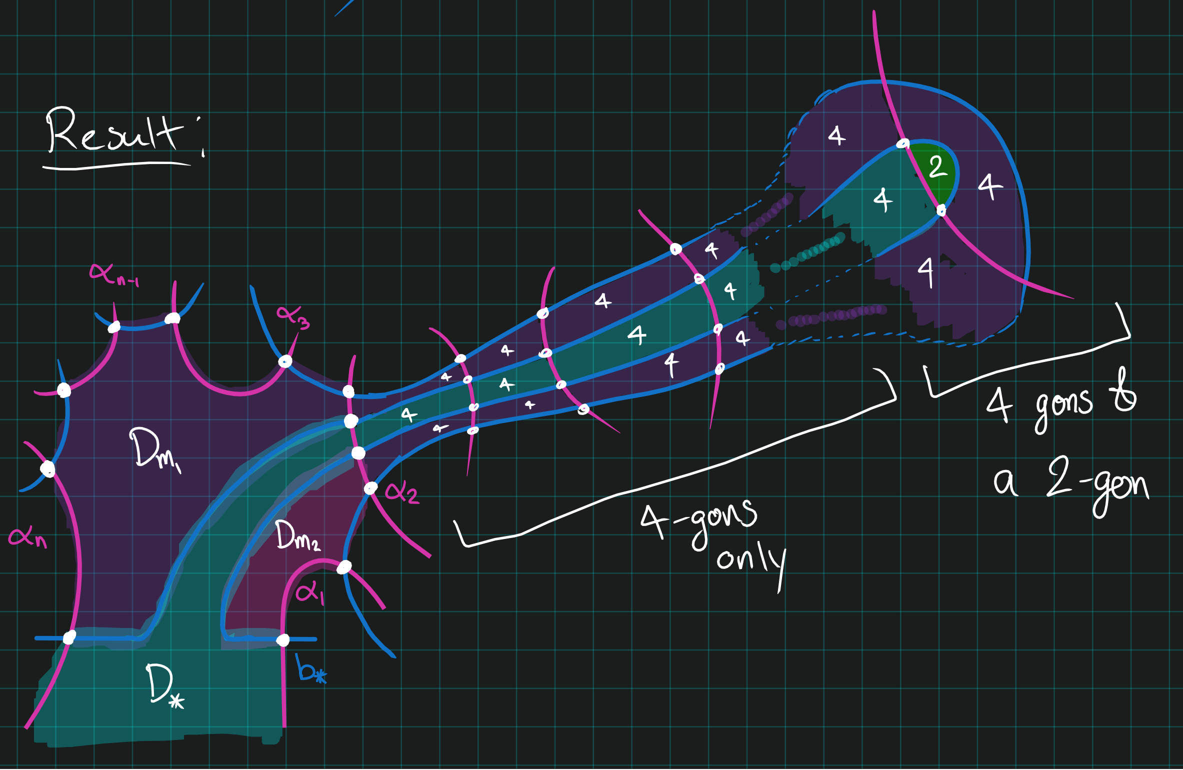
1 4-gon $\rightsquigarrow$ 3 $\underbrace{\text{4-gons}}_{\text{good!}}$

Case ①: Reaching a bigon (Dead end!)

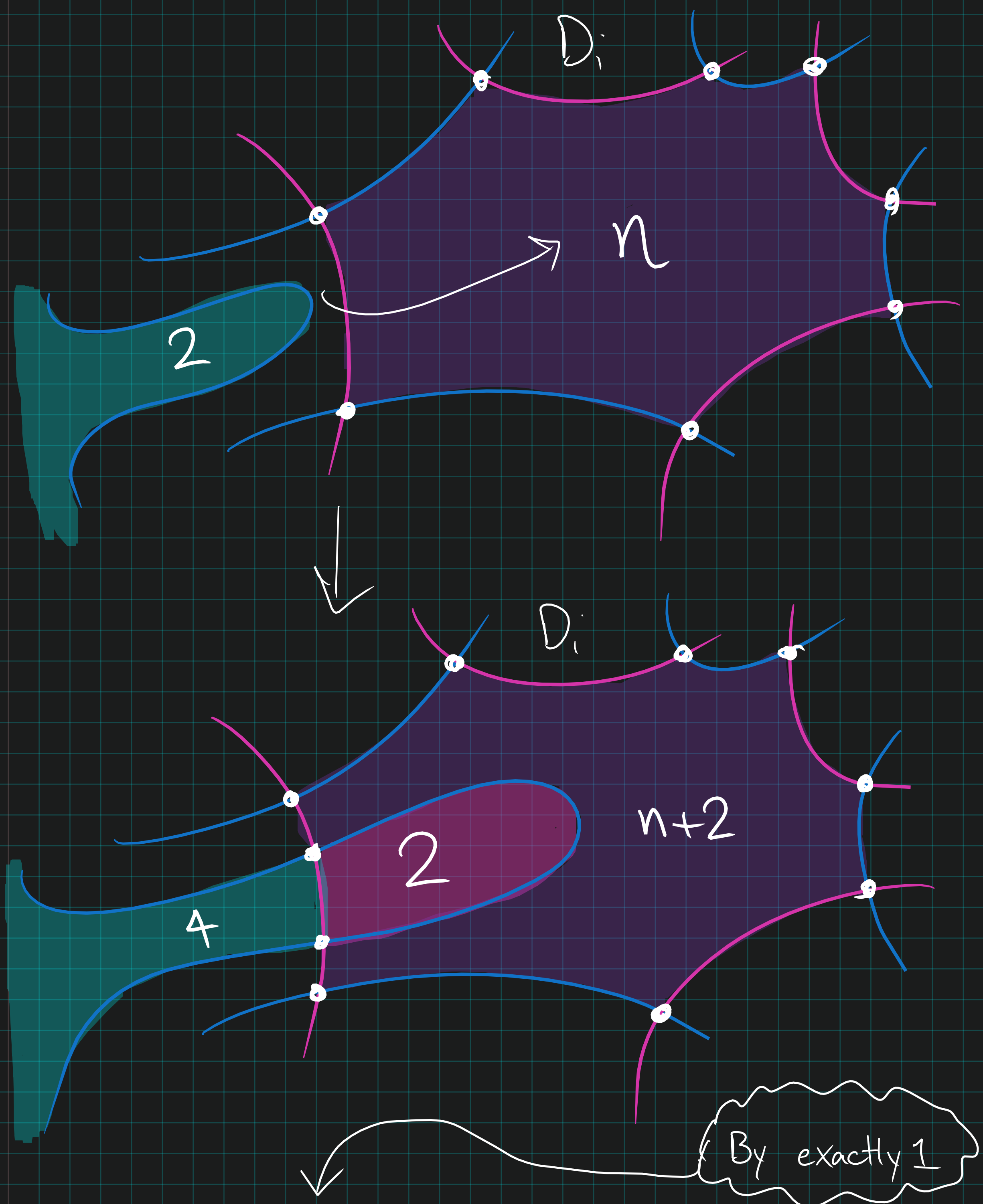


Doesn't increase badness.

Result:



Case ② : A region (possibly a 2-gon) with a smaller distance.

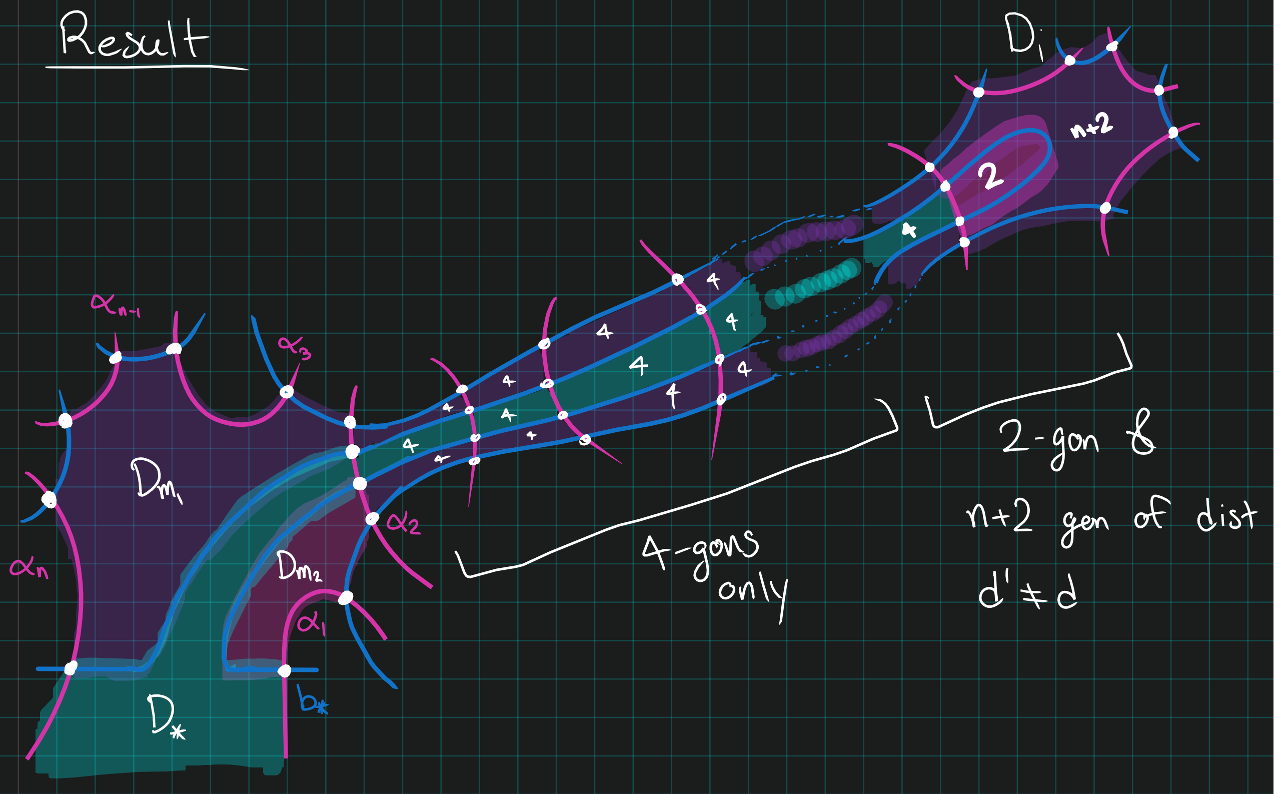


Badness worse, but occurs in

distance d' complexity with $d' < d$

$\Rightarrow C_d$ decreases

Result



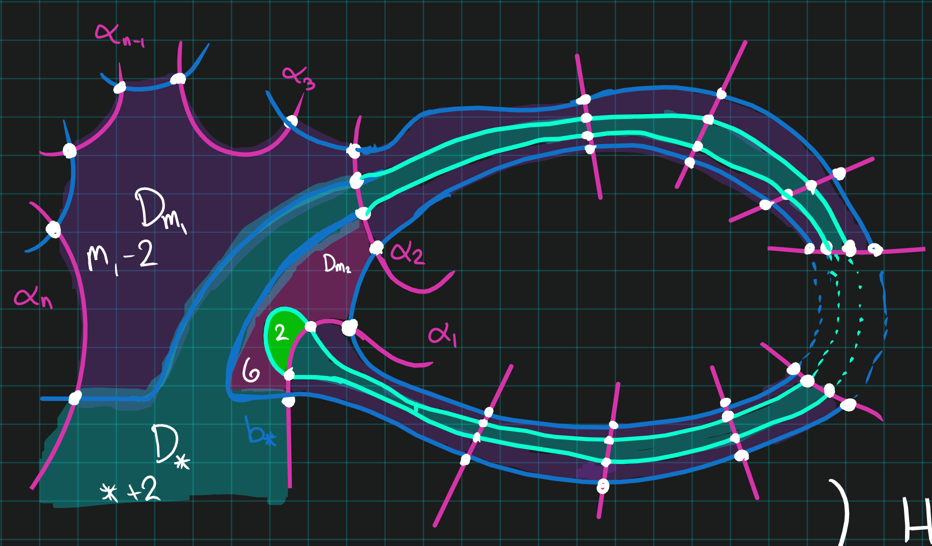
Case 3 of same distance: Same picture

- (-1) badness from original region
- (+1) badness from new region
- Lower C_d in lex order:

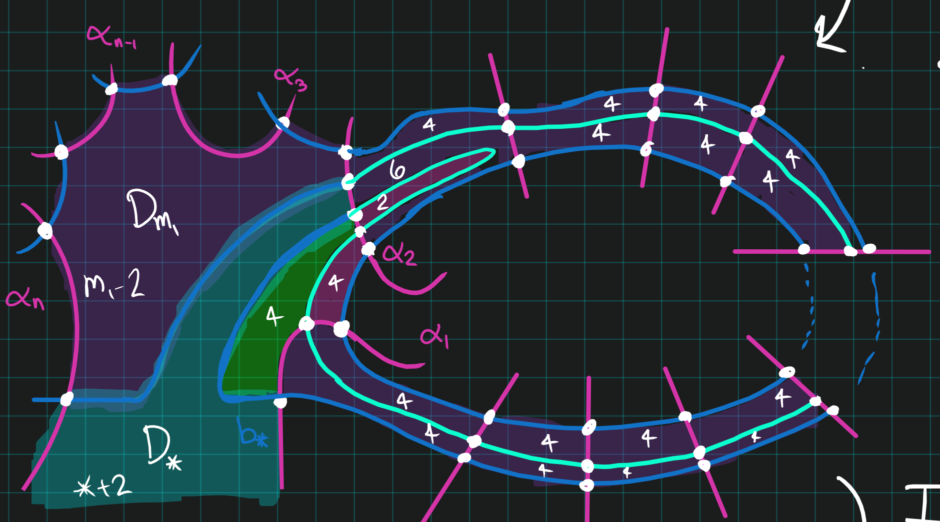
$$C_d(\mathcal{H}) = \left[\sum b(D_i), -b(D), \overset{\geq}{\cdots}, \underset{-(+1)}{-b(D_i)}, \overset{\geq}{\cdots}, \underset{-(-1)}{-b(D_m)}, \overset{\geq}{\cdots}\right]$$

Case ④: Return to D_m

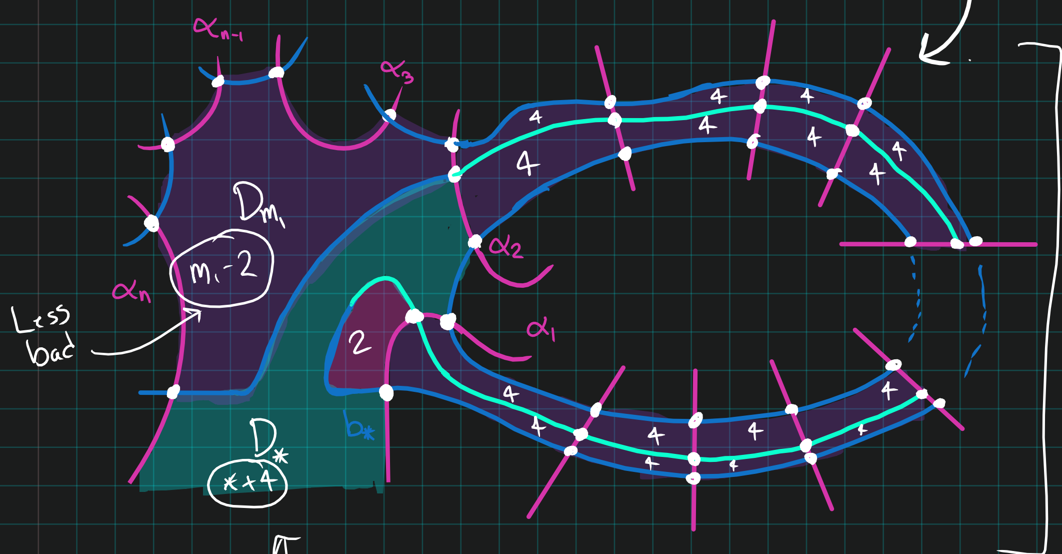
Subcase A: Return via an α adjacent to α_2 (i.e. α_1 or α_2).



Handle Slide



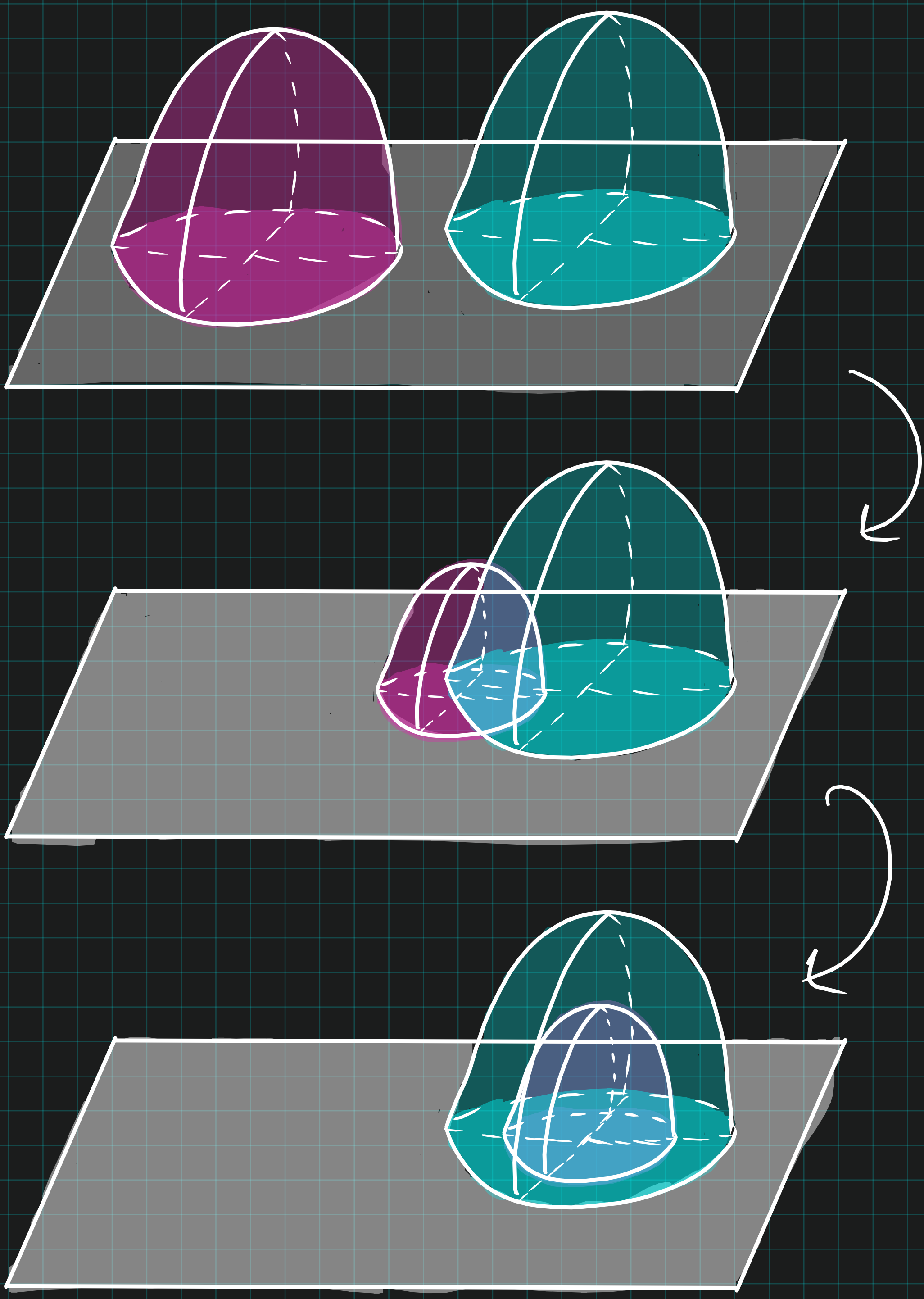
Isotope



All 2,4-gons (good!)

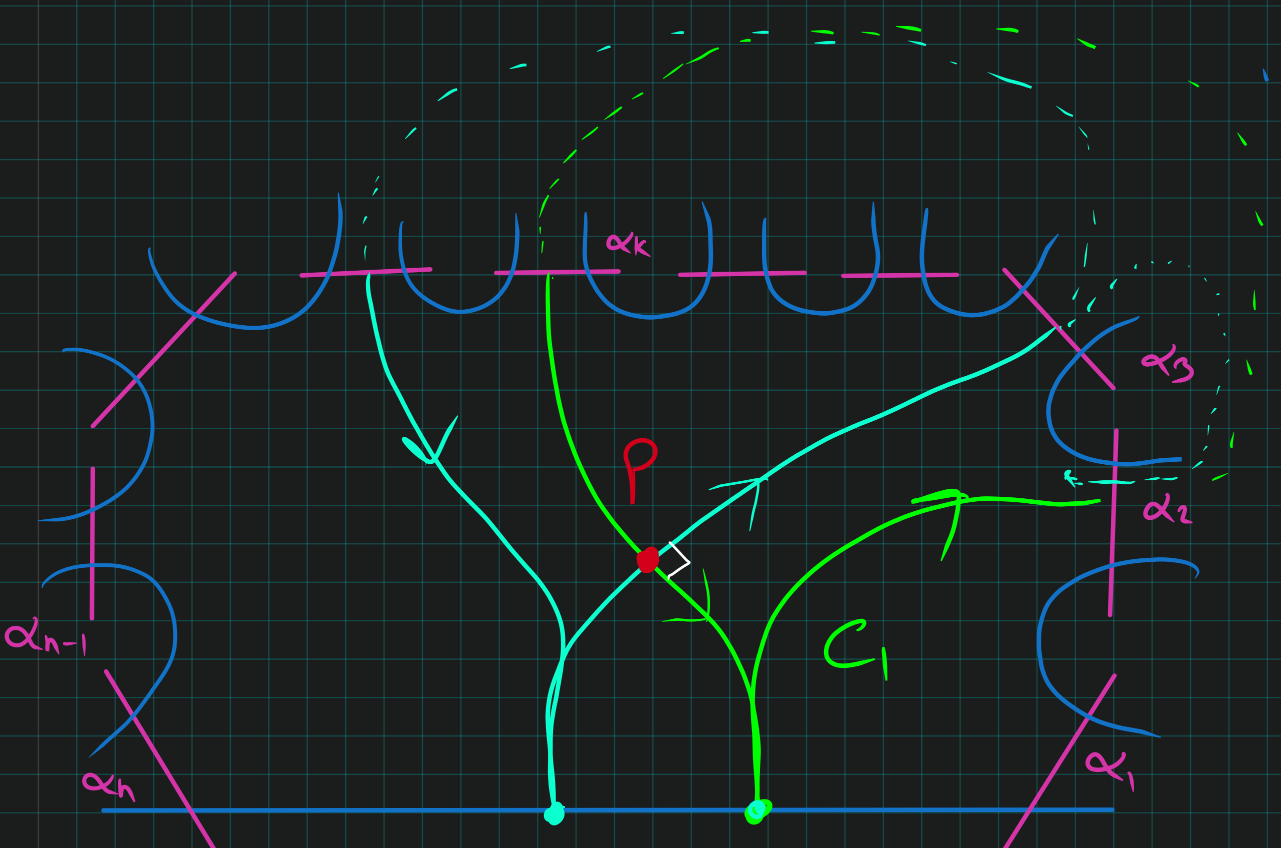
More badness, but in smaller distance region

Why "handle slide"?



Subcase 2: Return via some α_j not adjacent to α_2

Consider cores of finger moves



Can't return through $i>k$:

- $C_1 \pitchfork C_2 = \{p\}$, $C_1, C_2 \in \vec{\beta}^c$
- $\Sigma \setminus \vec{\beta} \cong S^2 - \coprod_{i=1}^{\#\vec{\beta}} \{q_i\}$
  - ↳ Attach $\#\vec{\beta}$ discs to get a sphere
  - ↳ Then $[C_1]\cdot[C_2] = \pm 1 \in H^1(S^2) = 0$ ↯

Can't return through k:

The minimal chain of squares out of α_k leads to α_2

Can't return through 3:

Adjacent to α_2!

Conclusion

- If we exit @ α_3, we either
  - Return to D_m via α_i, $3<i<k$
  - Else, no return to D_m (prev cases)
- Run same argument on $\alpha_4, \alpha_5, \ldots$
  - $4<i<k$
  - $5<i<k$
  - $\vdots$
  - Finitely many!

$\Rightarrow$ Finger doesn't come back

or

Finger leaves @ α_j, returns @ α_{j+1} (?)

↳ This case: adjacent regions, so do similar handleslide