UGA GSS

3/30/2021

Stable Homotopy Theory

Motivation: Some Impossible Goals

· Attempt 1: Classify Top

$X = \mathrm{Ob}(\mathrm{Top})$, $Y = \mathrm{Mor}(\mathrm{Top})$ — Fully Understand these as ....

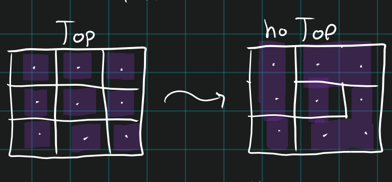
(spaces? stacks? alg structs?)

↳ Hard! Enriched over itself
$\mathrm{Top}(\mathbb{R}^2, \mathbb{R}^2)$ not contractible

$\mathrm{Emb}(X,Y)$, $\mathrm{Homeo}(X)$, $\mathrm{MCG}(X)$, ..

↳ Weird! Space-filling curves

$\mathrm{Top}(A,B) \longrightarrow [A,B]$



Homeo Types $\cong$ $\rightsquigarrow$ Homotopy Types $\simeq$

Eg. $\mathbb{R}^n \not\cong \mathbb{R}^m$ when $m \neq n$
("Invariance of domain", Brouwer 1912)

But $\mathbb{R}^n \simeq \mathbb{R}^m \simeq \mathbb{R}^0 := \{*\}$.

Q: When are hty-equiv spaces homeo?

· Attempt 2: Classify hoTop

↳ Still hard!

Simplest case is wide open:

$\pi_* S^* := \{[S^n, S^m] \mid n, m \geq \mathbb{Z}^{\geq 0}\}$ only barely known!

n,m≥2 Simply connected Compact Smooth mflds (nice)

Freudenthal Suspension Theorem (Cor)

$$\begin{tikzcd}
{[S^{n+k}, S^n]} \arrow[r, "\Sigma^*"] \arrow[d, "\wr"] & {[S^{n+k+1}, S^{n+1}]} \arrow[d, "\wr"] \\
\pi_{n+k} S^n \arrow[r, "\sim"] & \pi_{n+k+1} S^{n+1}
\end{tikzcd}$$

for $n \geq k+2$

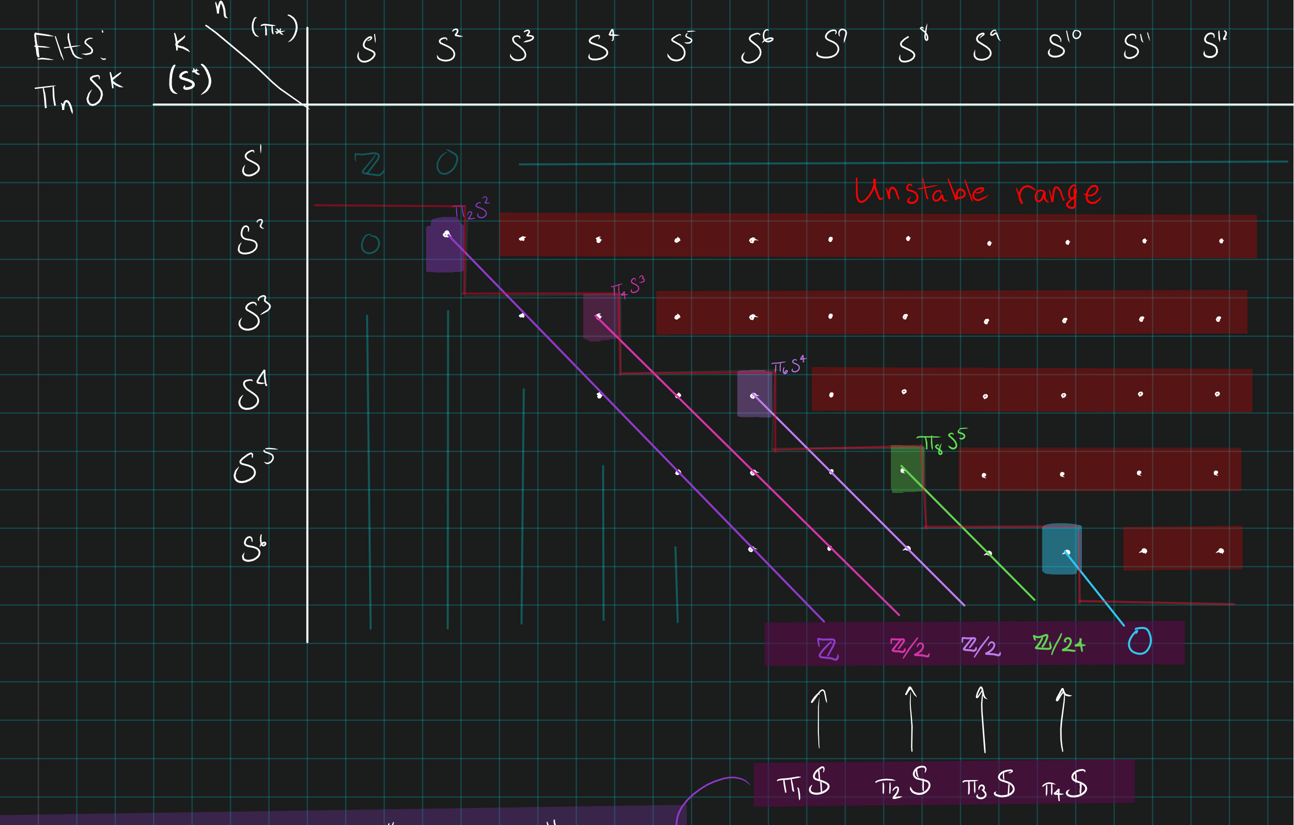
Def: $\pi_k^{st} S^n := \operatorname{colim}_{k} \pi_{n+k} S^{n+k}$

↳ Take $n = k+2$

k	$\pi_{2k+2} S^{k+2}$
0	$\pi_2 S^2$
1	$\pi_4 S^3$
2	$\pi_6 S^4$
3	$\pi_8 S^5$
⋮	⋮

bidegree $(2,1)$

Homotopy Groups of Spheres



· Attempt 3: Classify "Stab(hoTop)" ≈ $\mathcal{S}p$.

↳ Make Ω, Σ endofunctors inducing self-equivs

$\bigoplus_{i,j} [S^i, S^j]^{st}$ (hoTop) $\xrightarrow{\text{package up}}$ $\boxed{\pi_* \mathbb{S}}$ ("Spectra")

Many models

Rmk: Can localize $\mathbb{S} \longrightarrow \underbrace{\mathbb{S}_{(p)}}_{p\text{-local sphere spectrum}}$

Hopkins–Ravenel "Chromatic convergence thm"

$$\operatorname{holim}(\cdots \to \overbrace{L_{E(1)}}^{\text{Bousfield loc wrt Morava E}} \mathbb{S}_{(p)} \to L_{E(0)} \mathbb{S}_{(p)}) \simeq \mathbb{S}_{(p)}$$

$\rightsquigarrow$ Chromatic hty theory!

Try to compute
- One prime at a time
- One "level" at a time

Serre, 1951

- All torsion, except with a free $\mathrm{rank}_{\mathbb{Z}}$-1 summand when $\begin{cases} k \equiv 3 \bmod 4 \ \& \\ n \text{ even} \end{cases}$

(We don't even know all p-torsion for any p!)

Why Care About $\pi_* \mathbb{S}$?

↳ Building CW complexes

- Hatcher 0.18: For $A \subseteq X$ CW

  $[f]=[g] : A \to Y$ an attaching map,

  $X \sqcup_f Y \simeq X \sqcup_g Y$ (rel Y)

$\Rightarrow$ Need gluing data to find the homotopy type!

Attaching maps: $\left[\{ \phi_n^i \in Top(\partial B^n, X^{n-1}) \}_{i \in I} \right]$

S^{n-1}

$\leq n$ cells $\simeq S^1, S^2, \ldots$

Iterated pushouts:

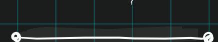
$$\begin{tikzcd} S^{n-1} \arrow[r, "\phi_n^i"] \arrow[d, hook, "\partial"'] & X^{n-1} \arrow[d] \\ D^n \arrow[r] & X^n \end{tikzcd}$$

Ex: Where do $\pi_k S^n$ show up?

Art break!

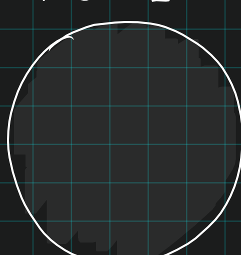
$\pi_0 S^0$

B^1



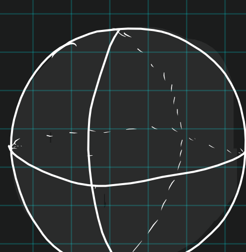
$\partial B^1 = S^0 = * \sqcup *$

B^2



$\partial B^2 = S^1$

B^3



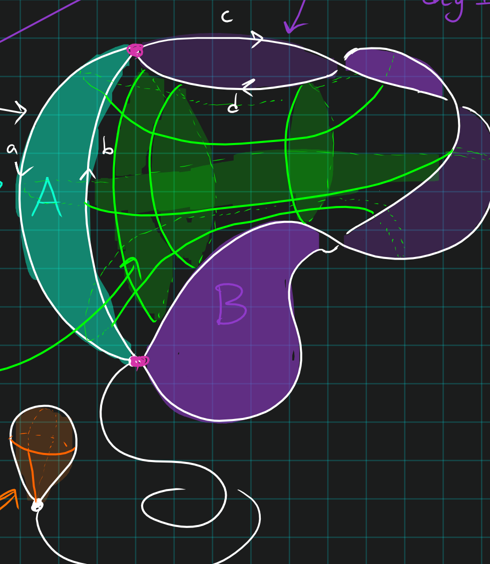
$\partial B^3 = S^2$

$\pi_1 S^1$

deg 1: cd

$\pi_1 S^1$

deg 2: abab



$\pi_2 S^2$

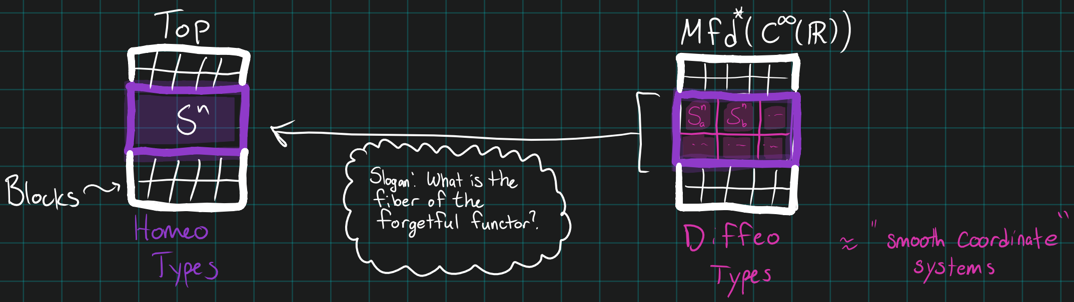
$A \mapsto B$, $b \mapsto c$, $a \mapsto d$

$\pi_2 S^0$

Why Care About $\pi_* \mathbb{S}$?

Question: For which n are there exotic S^n?

I.e.



- (Milnor, '56) $\exists$ exotic S^7's $\quad \Theta_n$
- $\{\text{Smooth structures on } S^n\} \rightleftarrows \overbrace{\{\text{Oriented hty } S^n\}/_{h\text{-cobordism}}}$

Finite abelian group under # for $n \neq 4$.

For $n = 4$, very open (Poincaré!!!)

- There is a map

$$\Theta_n / bP_{n+1} \hookrightarrow \pi_n \mathbb{S} / J$$

"Cokernel of the J-homomorphism"

$J : \pi_k SO_n(\mathbb{R}) \to \pi_{n+k} S^n$

The image is either

- Index 2 — Kervaire invariant one problem
- Index 1, an iso if $\exists\, M \in Mfd^n_{Fr}(C^\infty)$ s.t. $\Phi(M) = 1$

Φ: Kervaire Invariant

Thm (Hill-Hopkins-Ravenel): These can only exist in dimensions

$n = 2, 6, 14, 30, 62, \Big] \; 2^k - 2, \quad \Phi \neq 0$

$\boxed{\text{Only open case: } n = 126!}$

(Atiyah had some conjectures!)

Crash Course: Spaces

For Spaces

- $\mathrm{Top} := \mathrm{Top}_*^{CW}$, pointed CGWH ("Spaces")

- $A \wedge B := \dfrac{A \times B}{A \vee B} \in \mathrm{Top}$

- $\Sigma A := S^1 \wedge A \in \mathrm{Top}_*$, $\Sigma^k A := (S^1)^{\wedge k} \wedge A$ k-fold Smash

  $\cong S^k \wedge A$

- $\Omega A := \mathrm{Top}(S^1, A)$ w/ compact-open topology

- $[A, B] := \mathrm{Top}(A, B) / \text{homotopy}$

- $\pi_k A := [S^k, A] \cong [\Sigma^k S^0, A]$

  $\cong [S^0, \Omega^k A]$

  $= \pi_0 \Omega^k A$ Loop spaces!

- $\underset{\in \mathrm{Top}}{[A,B]^{st}} := \mathrm{colim}\left([A,B] \to [\Sigma A, \Sigma B] \xrightarrow{\Sigma} [\Sigma^2 A, \Sigma^2 B] \underbrace{\xrightarrow{\Sigma} \cdots}_{\text{Stabilizes by Freudenthal.}} \right)$

  $= \underset{j}{\mathrm{colim}}\, [\Sigma^j A, \Sigma^j B]$

- $\underset{\in \mathrm{Top}}{\pi_k^{st} A} := [S^0, A]^{st}$ using $\Sigma S^k \cong S^{k+1}$

Tools from Top that we want in Sp

(CGWH, Classical model struct)

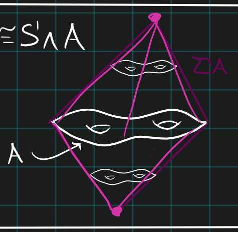
• Homotopy gps

$$\pi_\bullet : Top \to Gr^{\mathbb{Z}} Ab$$
$$X \mapsto \pi_\bullet X := \bigoplus_{i \geq 1} [S^i, X]$$

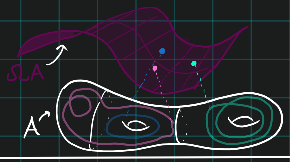
A corepresentable functor

• Suspension - Loop Adjunction

$\Sigma A \cong S^1 \wedge A$



$\Omega A = Top(S^1, A)$



$$[\Sigma A, B] \cong [A, \Omega B]$$

• Puppe (Cofibre)

1) Resolve

$$\left(X \xrightarrow{f} Y\right) \rightsquigarrow \left(X \xrightarrow{f} Y \to C_f\right) \rightsquigarrow$$

Cofib. replacement

Extract hocofibers

$$\boxed{\begin{array}{l} X \xrightarrow{f} Y \to C_f \\ \Sigma X \to \Sigma Y \to \Sigma C_f \\ \Sigma^2 X \to \cdots\cdots \end{array}}$$

LES^{cof}

BTW:

$$C_f = Y \coprod_f CX$$

2) Apply a functor
$F(\cdot) = [\cdot, K(G,1)]$
("Takes homology")

$H^j(\Sigma X) \cong H^{j-1}(X)$

$$\boxed{\begin{array}{l} H_n X \xrightarrow{f} H_n Y \to H_n C_f \\ H_{n-1} X \to H_{n-1} Y \to H_{n-1} C_f \\ \cdots\cdots \end{array}}$$

$\rightsquigarrow$ LES in homology.

$$H_n C_f = H_n(Y \coprod_f CX)$$
$$= H_n(Y/X)$$
$$= H_n(Y, X)$$

(LES of a pair.)

• Puppe (fiber)

1) Resolve

$$X \xrightarrow{f} Y \rightsquigarrow \left(M_f \to X \hookrightarrow Y\right) \rightsquigarrow$$

fibration

Extract hofibers

$$\boxed{\begin{array}{l} \cdots\cdots \to \Omega^2 Y \\ \Omega M_f \to \Omega X \to \Omega Y \\ M_f \to X \to Y \end{array}}$$

LES^{fib}

2) Apply a functor
$G(\cdot) = [S^0, \cdot]$

$$\boxed{\begin{array}{l} \cdots\cdots \\ \pi_1 M_f \to \pi_1 X \to \pi_1 Y \\ \pi_0 M_f \to \pi_0 X \to \pi_0 Y \end{array}}$$

LES in homotopy

(Relative hty LES, using
$\pi_0 \Omega^k M_f := [S^0, \Omega^k M_f] = [S^k \wedge S^0, M_f] = \pi_k M_f \xrightarrow{\cong} \pi_k(Y, X)$.)

See Eckmann-Hilton duality!

(Cohomology dual = homotopy)

Tools from Top that we want in Sp:

- **Weak equivalences**

W = Hty equivs

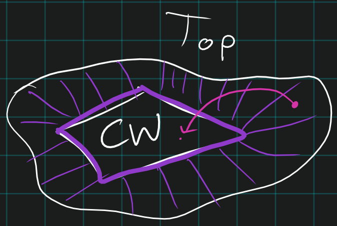
Apply the functor

Induce isos on π_*

$$\mathrm{Top}^W(\cdot,\cdot) \xrightarrow{\pi_*(\,)} \mathrm{GrAb}^{\cong}(\cdot,\cdot)$$

- **Cofibrant objects & replacement**

$\mathrm{Top}^{CW} \hookrightarrow \mathrm{Top}$, every $X \in \mathrm{Top}$ weakly equiv to some $X' \in \mathrm{Top}^{CW}$ by CW approximation



- A functor $\mathrm{Top} \longrightarrow \mathrm{hoTop}$ where $\mathrm{ho}(W)$ are isomorphisms

$\cong \mathrm{Top}[W^{-1}]$

Image of Weak equivalences

and hoTop is <u>triangulated</u>

<u>Eg:</u>

- $K(A)$: $Ob = Ch(A)$, $Mor = Mor(Ch(A))/\text{chain hty}$
- $D(A) = Ch(A)[W^{-1}]$, W = quasi-isomorphisms

- Shift functor $F(\cdot) = (\cdot)[1] = \Sigma(\cdot)$
- Exact Triangles

$$\begin{tikzcd} & A \arrow[dr, "d"] & \\ C \arrow[ur, "{[1]}"] & & B \arrow[ll, "d"] \end{tikzcd}$$

$\Rightarrow$ "$d^2 = 0$"

$$A \to B \to C \xrightarrow{[1]} A[1] \to B[1] \to \cdots$$

- <u>Spectral Sequences</u>

<u>Ex</u> A Serre fibration induces

$$\boxed{\begin{matrix} F \to E \\ \quad\ \downarrow \\ \quad\ B \end{matrix}}$$

$$E_2^{p,q} = H^p(B, H^q F) \Rightarrow H^{p+q} E$$

(Serre Spectral Sequence)

<u>In Sp</u>

<u>Ex</u> $\boxed{Sp(A,B)}$ induces:

Known singular cohomology

p-completion

$$E_2^{p,q} = \mathrm{Ext}_{\mathcal{A}}^{p,q}(H^*A, H^*B) \Rightarrow \underbrace{[A, \hat{B}_p]_{p-q}}_{\text{Get info}}$$

Steenrod

(Adams Spectral Sequence)

<u>Building blocks</u>

- Eilenberg-MacLane Spaces $\quad \pi_* K(G,n) = \begin{cases} G, & * = n \\ 0, & \text{else} \end{cases}$

- Moore Spaces $\quad H_* M(G,n) = \begin{cases} G, & * = n \\ 0, & \text{else} \end{cases}$

Context: A Category of Spectra

$$\text{Notation: } Sp := Sp^{\mathbb{N}}$$

- Def ($Sp^{\mathbb{N}}$, <u>sequential</u> spectra)

  - <u>Objects:</u>

$$\begin{tikzcd}
 & \Sigma X_0 \arrow[dr, "\sigma_0"] & \Sigma X_1 \arrow[dr, "\sigma_1"] & \Sigma X_2 \arrow[dr, "\sigma_2"] & \\
Ob(Sp) = \Big[& X_0 \arrow[u, "\Sigma"] \arrow[d, "\Omega"'] \arrow[dr] , & X_1 \arrow[dr] , & X_2 \arrow[dr] , & \cdots \Big] \\
 & & \Omega X_1 & \Omega X_2 &
\end{tikzcd}$$

$$\sigma_n \in Top_*(\Sigma X_n, X_{n+1})$$

$$\omega_n \in Top^{W}(X_n, \Omega X_{n+1})$$

  - <u>Morphisms:</u>

$$\left\{ \begin{tikzcd}
\Sigma X_n \arrow[r, "\Sigma f_n"] \arrow[d, "\sigma_n^X"'] \arrow[dr, phantom, "\circlearrowright \text{hty}"] & \Sigma Y_n \arrow[d, "\sigma_n^Y"] \\
X_{n+1} \arrow[r, "f_{n+1}"'] & Y_{n+1}
\end{tikzcd} \right\}$$

<u>Note</u>: Not all spectra are in $Sp^{\mathbb{N}}$!

<u>Eg</u>: K-Theory

Examples of Spectra

Suspension Spectrum

• $\Sigma^\infty K := \left[S^0 \wedge K ,\ S^1 \wedge K ,\ S^2 \wedge K ,\ \right]$ $(K \in \mathrm{Top})$

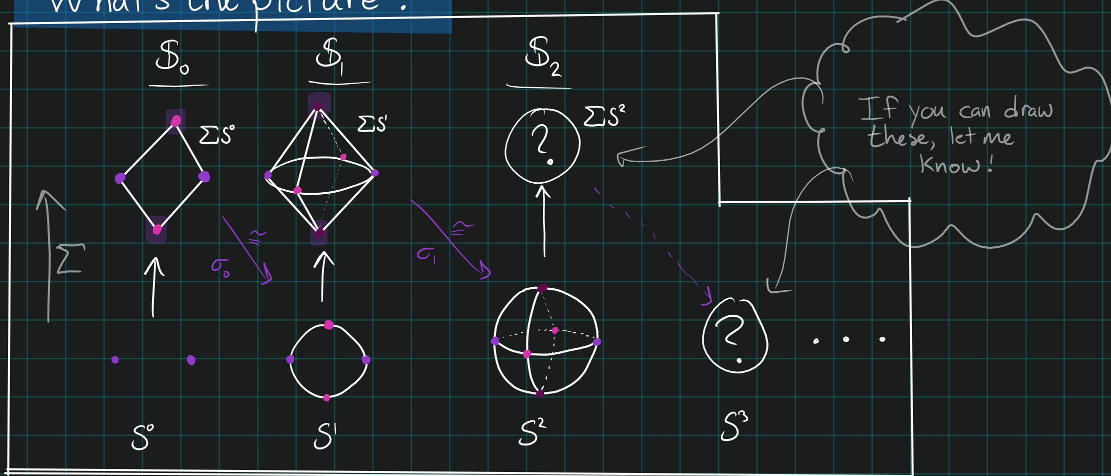
Rmk

Embeds $\mathrm{Top} \hookrightarrow \mathrm{Sp}$.

Sphere Spectrum

• $\mathbb{S} := \Sigma^\infty S^0 = \left[S^0 ,\ S^1 ,\ S^2 ,\ \dots \right]$ (Using $S^1 \wedge S^k \overset{\text{Homeo}}{\cong} S^{k+1}$)

What's the picture?



Moore Spectrum

• $MG :=$ Complicated!

But have $H_* MG = 0$ for $* \neq 0$

& $M\mathbb{Z}/n \approx \mathrm{hocofib}\left(\mathbb{S} \xrightarrow{\cdot n} \mathbb{S} \right)$

Slogans:

• Smash $\approx \otimes_R$

• $\mathbb{S} \approx \mathbb{Z}$

Eg Cohomology Theories

Brown Representability: "nice" functors $F^n : CW_* \to Gr^{\mathbb{Z}}Ab$ (reduced cohomology theories) will be representable in $ho\, CW_*$:

$$F^n(X) \cong [X, E^n] \quad \text{for some } E^n \in CW_*$$

$\rightsquigarrow$ Assemble into a spectrum $E \in Sp$

$\rightsquigarrow$ Recover as $E^n(X) = [(\Sigma^n)^{-1} \Sigma^\infty X, E] \in ho\,Sp.$

Slogan: Spectra functorially represent cohomology theories.

Eilenberg MacLane Spectrum

- $HG := [K(G,0), K(G,1), K(G,2), \ldots]$

Represents $H^*_{Sing}(\cdot\, ; G)$

Lifting Homological Algebra

- For any ring R, can form a category

$$HR\text{-}mod^{Sp} = HR\text{-module spectra}$$

- Use model categories: equip $Ch(R\text{-}mod)$ with a projective model structure

$\approx$ Recipe for computing homology $\approx$ "resolve" by a subcategory of nice objects (projective R-modules)

- Thm (Dold-Kan, roughly!)

There is an equivalence of model categories

$$\underbrace{HR\text{-}mod}_{\text{Homotopy}} \overset{\sim}{\rightleftarrows} \underbrace{Ch(R\text{-}mod)}_{\text{Homology}}$$

Eg

For spaces $\quad H^*_{sing}(X; M) \cong \pi_*(\Sigma^\infty X \wedge HM)$

"Brave new algebra"

$$\begin{tikzcd} \text{Rings} \arrow[r, hook] \arrow[d] & Sp \\ E_\infty\text{-}Ring\,Sp \arrow[ur] & \end{tikzcd}$$

Commutes and associates only up to "coherent homotopy"

$$a \cdot b \xrightarrow{\sim} b \cdot a$$

$$\begin{tikzcd} & ((ab)c)d \arrow[dr, "?"] & \\ (a(bc))d \arrow[ur, "\sim"] & & (ab)(cd) \arrow[dl, "?"] \\ a(bc(d)) \arrow[u, "?"] & a(b(cd)) \arrow[l, "\sim"] & \end{tikzcd}$$

Fill with an "interpolating cell"

$\vdots$ etc for n-fold multiplication!

• K-Theory Spectra

Take the category

Ob = $\left\{ \begin{array}{c} F \hookrightarrow E \\ \downarrow \\ X \end{array} \right\}$

Mor = $\left\{ \begin{tikzcd} E \arrow[rr] \arrow[dr] & & E' \arrow[dl] \\ & X & \end{tikzcd} \right\}$

Mon: Whitney sum $\oplus$

Monoidal structure

Monoidal category: use Grothendieck construction to group complete.

$\rightsquigarrow KU^0(X)$

This yields a homology theory $KU(\cdot) \rightsquigarrow KU \in Sp$

Rmk: Can take similar for coherent sheaves on Noetherian schemes

$\rightsquigarrow$ Intersection theory!

Rmk: Similar story for abelian cats in rep theory:

$$G \xmapsto{\text{semisimplification}} \underbrace{R_G(k)}_{\text{Representation ring}} = \{\text{f.g } k[G]\text{-modules}\} \Big/ \left\{ \begin{array}{l} B = A + C \text{ for} \\ \text{all SES} \\ 0 \to A \to B \to C \to 0 \end{array} \right\}$$

Forces "$V = \underbrace{\oplus V_i}_{\text{irreps}}$"

Formal Group Laws & the Lazard Ring

Def: A *formal group* $\mathcal{F}$ over R is a power series $F(x,y) \in R[[x,y]]$ s.t.

- $x \oplus y = x + y \mod (x,y)$] terms of deg ≥ 2, $O(x,y,xy)$
- $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ (Assoc)
- $x \oplus y = y \oplus x$ (Commut.)
- $x \oplus 0_R = x$, $0_R \oplus y = y$ (identity)
- $\exists g \in R[[t]]$ s.t. $t \oplus g = 0$ (inverse)

Ex

$\hat{G}_a$: $x \oplus y = x + y$

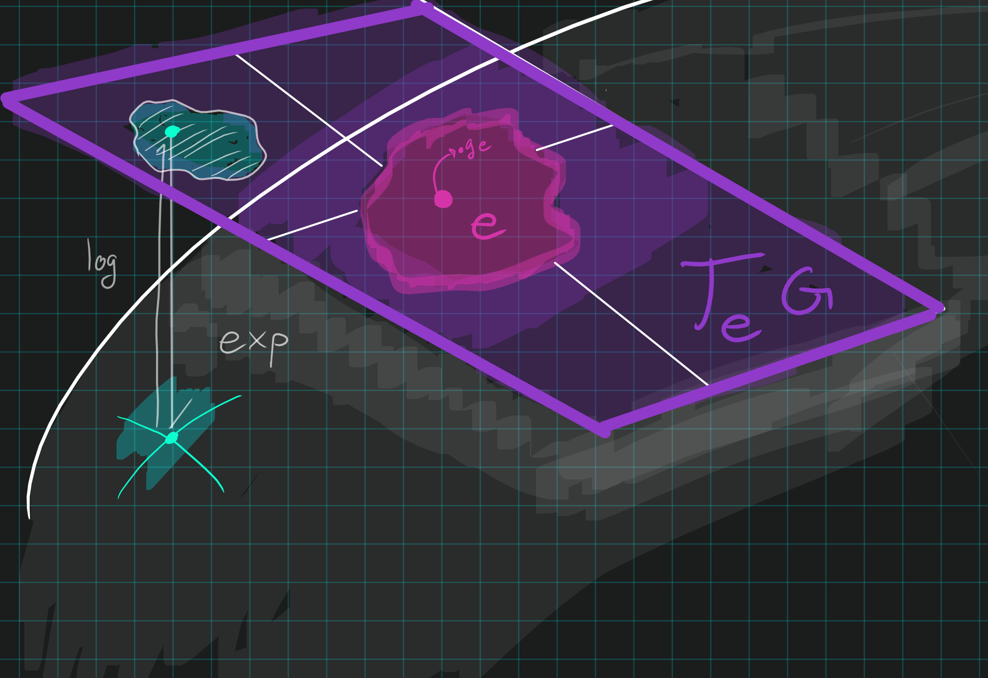
$\hat{G}_m$: $x \oplus y = x + y + xy = (1+x)(1+y) - 1$

Motivation: Analytic Lie Groups in char $p > 2$

$$\begin{tikzcd} & \text{Formal Grp} \arrow[dr, "i"] & \\ \text{Lie Grp} \arrow[ur] \arrow[rr, "T"'] & & \text{Lie Alg} \end{tikzcd}$$

iso in char $= 0$

Where T is the usual process:



Choose coordinates, then $\underbrace{f(x) = gx}_{\hookrightarrow G \times G \to G}$ has a power series expansion in some nbhd of e.

In ch $= 0$, $\hat{G}_m(G) \xrightarrow{\sim} \hat{G}_a(G)$

$$1 + x \mapsto \log(1+x) = x - \tfrac{1}{2}x^2 + \dots$$

Def The *height* of f is $ht(f) := \max \left\{ n \in \mathbb{Z} \mid \exists g \in R[[t]] \text{ s.t. } f(t) = t^{p^n} \right\}$

(i.e. leading term is $a x^{p^n}$)

Prop: $E \in \mathcal{E}ll \Rightarrow E$ has a FGL of ht $= 1$ or 2

Prop (Silverman): Let K be perfect, ch $K = p$, $E/_K \in \mathcal{E}ll$.

Then TFAE:

- $E[p^k] = 0$ for all $k \geq 0$
- The formal group associated to $\hat{E}/_K$ is height 2
- E is supersingular.

If none hold,

- $E[p^k] \cong \mathbb{Z}/p^k$ $\forall k \geq 1$, and
- The formal group has height 1.

Relation to Stable Homotopy

$\pi_* MU$ = Universal Lazard ring $\mathcal{L}$

$\cong \{\text{Almost-complex } C^\infty(\mathbb{R})\text{-mfds}\}/\text{cobordism}$

Surprisingly!

- The universal coefficient ring for all formal groups is given explicitly by

$$\mathcal{L}: \quad F(x,y) = \sum \alpha_{ij} x^i y^j$$

Where if G is any FGL, there is a ring morphism

$$\theta: G \to \mathcal{L}$$

s.t. $G(x,y) = \sum \theta(\alpha_{ij}) x^i y^j$.

- The functor

$$FGL: CRing \to Set$$

$$R \mapsto \left\{ \begin{matrix} \text{Formal group} \\ \text{laws on } R \end{matrix} \right\}$$

is representable by $\mathcal{L}$.

- Can form the moduli stack $\mathcal{M}_{FG}$

Stratified: $\mathcal{M}^n_{FG}(R) = \left\{ \begin{matrix} \text{Formal group laws} \\ \text{of height } n \\ \text{over } R \end{matrix} \right\}$

Landweber

Deformations

Lubin-Tate theory

$\left\{ \begin{matrix} \text{"}\mathbb{C}\text{-oriented"} \\ \text{cohomology theories} \end{matrix} \right\} \subseteq \mathcal{S}pectra$

Induces a filtration

(the chromatic filtration)

Claim (?): There is an "embedding" of stacks

$$\underbrace{\mathcal{M}_g}_{\text{moduli of elliptic curves}} \hookrightarrow \mathcal{M}_{FG}.$$

On Ncatlab but no proof linked!

:(

Derived Algebraic Geometry

(c/o Lurie)

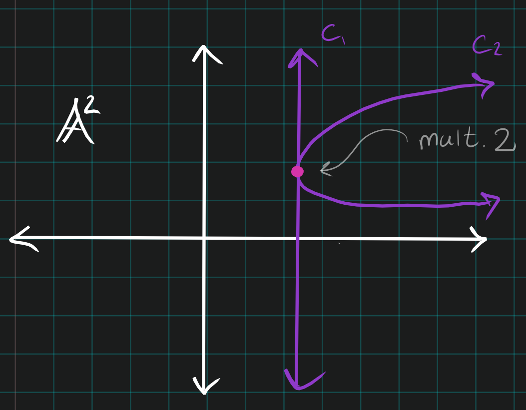
Motivation: Bezout

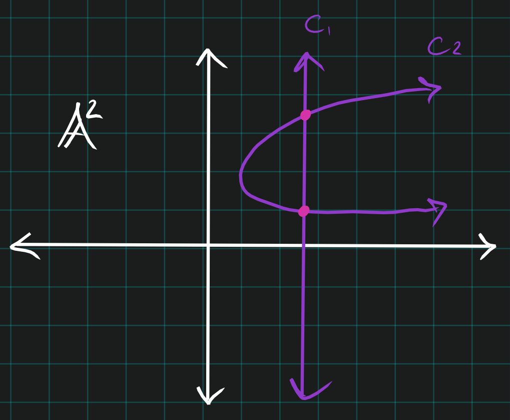
For $C_1, C_2 \subseteq \mathbb{P}^2_{/\mathbb{C}}$ curves, if $C_1 \pitchfork C_2$ then

$$[C_1] \smile [C_2] = [C_1 \frown C_2]$$

Problem

Fails if not transverse: multiplicity!





Fix 1: If $\deg C_i = d_i$, then

$$d_1 \cdot d_2 = \sum_{C_1 \cap C_2} \dim_{\mathbb{C}} \underbrace{\left(\mathcal{O}_{C_1} \otimes_{\mathcal{O}_{\mathbb{P}^2}} \mathcal{O}_{C_2} \right)}_{\text{impose simultaneous equations}}$$

Ringed Spaces
$C_1 = (|C_1|, \mathcal{O}_{C_1})$
$C_2 = (|C_2|, \mathcal{O}_{C_2})$

Problem

Fails in higher dimensions. :(

Fix 2. Serre's intersection formula:

$$d_1 \cdot d_2 = \sum_{p \in C_1 \cap C_2} (-1)^i \dim \operatorname{Tor}_i^{\mathcal{O}_{\mathbb{P}^n}} (\mathcal{O}_{C_1}, \mathcal{O}_{C_2})$$

Idea: Replace schemes $\subseteq$ Top $\times$ Sh(CRing) $\quad A = (|A|, \mathcal{O}_A)$

Sheaves of rings

$$\begin{array}{ccc} A & \in & \text{Top} \times \text{Sh(CRing)} \\ \downarrow & & \downarrow \\ & & \text{Top} \times \text{Sh(Top CRing)} \\ & & \downarrow \\ \hat{A} & \in & \boxed{\text{Top} \times \text{Sh}(\text{Sp})} \end{array}$$

Sheaves of spectra

$\hat{A} = (|\hat{A}|, \hat{\mathcal{O}}_A)$

$\in E_\infty$ Ring Spectra

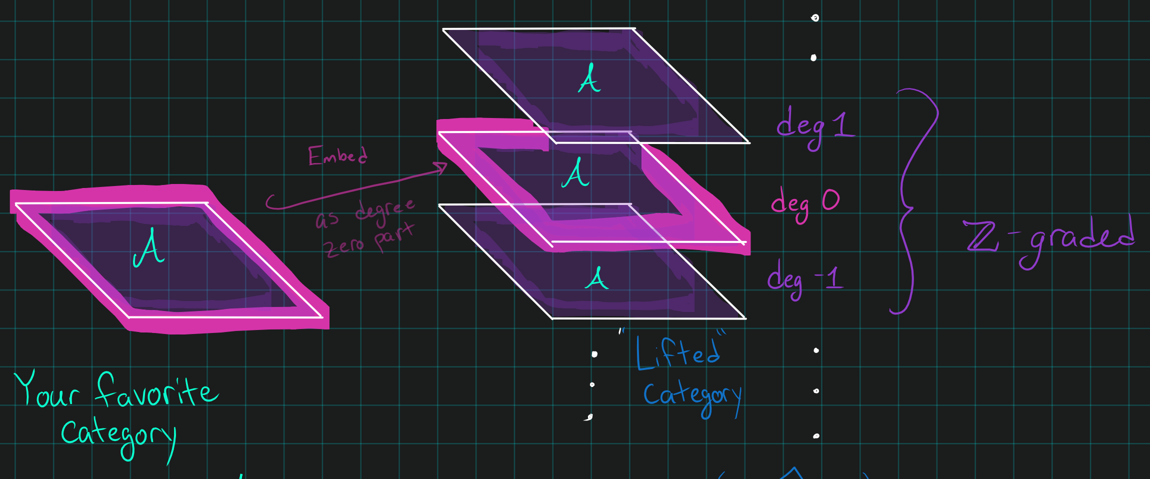
Where locally $\operatorname{Spec} A = \operatorname{Spec} \pi_0 \hat{A}$

& replace: $\boxed{- \otimes_R -} \rightsquigarrow \boxed{- \otimes^{\mathbb{L}}_R -}$] derived tensor product

$\rightsquigarrow$ now carries higher homotopical data!

and $(\hat{A}, \pi_0 \hat{\mathcal{O}}_A)$ is a scheme with $\pi_k \hat{\mathcal{O}}_A \in \pi_0 \hat{\mathcal{O}}_A\text{-mod}^{q\text{coh}}$

The derived yoga:



$\mathcal{A} \hookrightarrow \text{Ch}(\mathcal{A})$ ("$\hat{\mathcal{A}}$")

Top $\hookrightarrow$ Sp

Sch $\hookrightarrow$ DerSch ("$\widehat{\text{Sch}}$")

St $\hookrightarrow$ DerSt

$\underbrace{\qquad}_{\text{Usual world}}$ $\qquad$ $\underbrace{\qquad}_{\text{"Derived" world}}$

Thank You!