Title

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1 Summary

- Limits
  - L'Hopital's
- Continuity
- The Derivative
  - Definitions
    - Computing using limit definition
    - Interpretations as slope of tangent line / rate of change
  - Computing with rules
    - Elementary functions:
      - $f(x) = c$ a constant
      - $f(x) = x^n$

            - $f(x) = e^x$
            - $f(x) = \ln(x)$
        - Combining with rules
            - Product Rule
            - Quotient Rule
            - Chain Rule
        - Implicit differentiation
        - Related Rates
- Extrema
    - Maximization/minimization
    - Finding critical points
    - 2nd Derivative test
- Approximations
- The Intermediate Value Theorem
- The Extreme Value Theorem
- The Mean Value Theorem
- Antiderivatives/Differential Equations

2 Limits

Always try to just "plug in" first:

(a) $$\lim_{x\to 4} \frac{x^2+14}{x^3-24} = \frac{16+14}{64-24} = \frac{30}{40} = \frac{3}{4}$$

This may yield an indeterminate form:

$$(0)\,(\pm\infty) \qquad 1^\infty \qquad 0^0 \qquad \infty^0 \qquad \infty-\infty$$

Note that

$$(0)(0), \quad (\pm\infty)(\pm\infty), \quad \infty^\infty$$

are okay.

> Motto: things are alright if everything gets big or small together. If one thing gets big but another gets small (or one stays the same), these are indeterminate.

First see if Algebra helps:

(b) $\lim_{x\to 6} \dfrac{x^2-8x+12}{2x^2-7x-30} = \dfrac{36-48+12}{72-42-30} = \dfrac{0}{0}$

$$= \lim_{x\to 6} \frac{(x-6)(x-2)}{(x-6)(2x+5)} = \lim_{x\to 6} \frac{x-2}{2x+5} = \frac{4}{17}$$

If square roots, also try conjugation.

2.1 L'Hopital's

Always check for indeterminate forms.

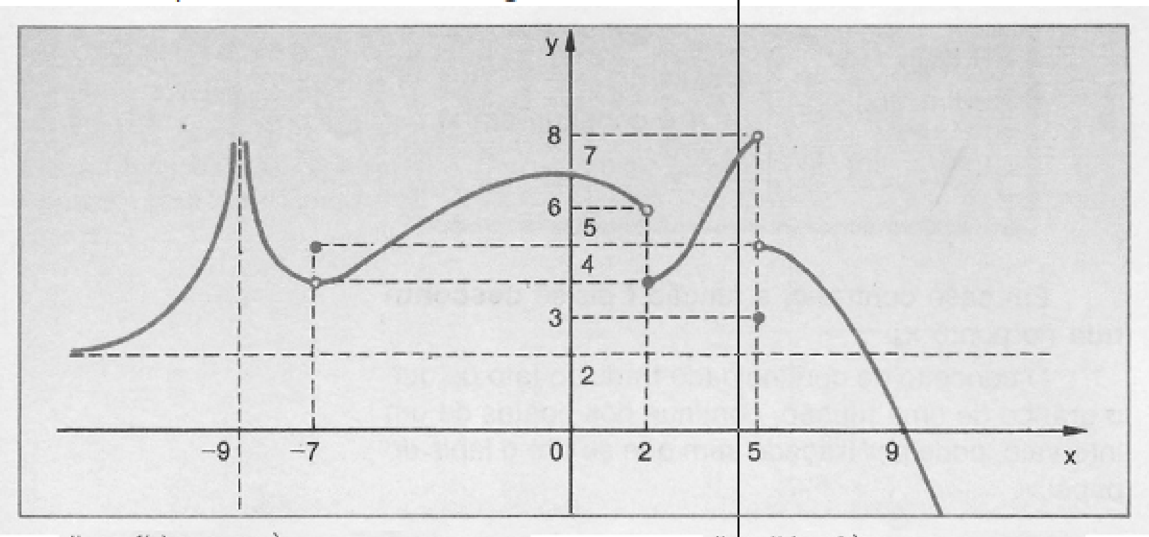
"Bald" answers and "plug & chug" / "guess & check" will receive little or no credit.
If a limit Does Not Exist, mark it **DNE** and explain WHY the limit does not exist.

(a) $\lim_{x\to\infty} \dfrac{\ln(9x)}{\sqrt[3]{x}} = \dfrac{\infty}{\infty}$

$$\overset{LH}{=} \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim_{x\to\infty} \frac{3x^{2/3}}{x} = \lim_{x\to\infty} \frac{3}{x^{1/3}} = 0$$

2.2 Limits on Graphs

Here, to recapitulate, some interesting cases:



$$\left.\begin{array}{l}\lim_{x\to -9^-} f(x) = +\infty \\ \lim_{x\to -9^+} f(x) = +\infty\end{array}\right\} \Rightarrow \lim_{x\to -9} f(x) = +\infty$$

$$\left.\begin{array}{l}\lim_{x\to -7^-} f(x) = 4 \\ \lim_{x\to -7^+} f(x) = 4\end{array}\right\} \Rightarrow \lim_{x\to -7} f(x) = 4$$

$$\left.\begin{array}{l}\lim_{x\to 0^-} f(x) = 7 \\ \lim_{x\to 0^+} f(x) = 7\end{array}\right\} \Rightarrow \lim_{x\to 0} f(x) = 7$$

$$\left.\begin{array}{l}\lim_{x\to 2^-} f(x) = 6 \\ \lim_{x\to 2^+} f(x) = 4\end{array}\right\} \Rightarrow \nexists \lim_{x\to 2} f(x)$$

$$\left.\begin{array}{l}\lim_{x\to 5^-} f(x) = 8 \\ \lim_{x\to 5^+} f(x) = 5\end{array}\right\} \Rightarrow \nexists \lim_{x\to 5} f(x)$$

$$\left.\begin{array}{l}\lim_{x\to 9^-} f(x) = 0 \\ \lim_{x\to 9^+} f(x) = 0\end{array}\right\} \Rightarrow \lim_{x\to 9} f(x) = 0$$

$$\lim_{x\to -\infty} f(x) = 2 \text{ e } \lim_{x\to +\infty} f(x) = -\infty$$

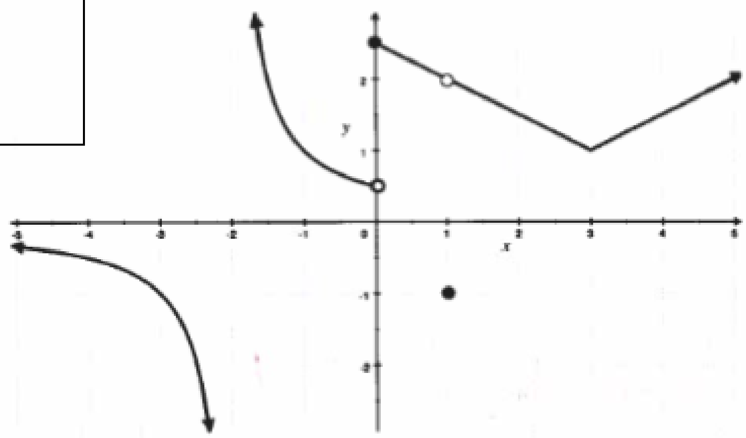
3 Continuity

What does it mean to be continuous?

Statement: $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point p if

1. $\lim_{x\to p} f(x)$ *exists*, (no jumps) and
2. $\lim_{x\to p} f(x) = f(p)$ (no holes)

2. Consider the graph $y = g(x)$ below.



For which values of x does $g(x)$ fail to be continuous?
Justify your answers by citing the definition of continuity and noting how it fails at each x.

$x=-2$ $\quad g(-2)$ DNE $\neq \lim_{x\to -2} g(x) = \pm\infty$

$x=0$ $\quad \lim_{x\to 0^-} g(x) = \frac{1}{2} \neq \frac{5}{2} = \lim_{x\to 0^+} g(x)$ $\quad$ $g(0)$ irrelevant since limit DNE

$x=1$ $\quad \lim_{x\to 1} g(x) = 2 \neq -1 = g(-1)$

In above image: where does f fail to be differentiable?

4 The Derivative

4.1 Definition

Limit definition:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

- RHS: interpreted as slopes of secant lines
- LHS: interpret as slope of tangent line at x

4.2 Applications

Finding the equation of the tangent line at a point p:

$$T(x) : y - f(p) = f'(p)(x - p)$$

Can use $m = \frac{\Delta y}{\Delta x}$ to help remember this, and just recall that $m = f'(p)$!

(c) Find an equation of the line tangent to $f(x)$ at $x = 1$

$$y - f(1) = f'(1)(x-1)$$

$$y - 1 = -3(x-1)$$

4.3 Computing with Rules

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$(f(x)/g(x))' = (g(x)f'(x) - f(x)g'(x))/g^2(x)$$

$$(f(g(x))' = f'(g(x))\ g'(x).$$

Product rule example:

(b) $g(x) = \left(5x^2 - \frac{4}{x} + 2\right) e^x$

$$g'(x) = \left(10x + \frac{4}{x^2}\right) e^x + \left(5x^2 - \frac{4}{x} + 2\right) e^x$$

Chain rule example:

(b) $g(x) = e^{2e^{3x^4}}$

$$e^{2e^{3x^4}} \cdot 2e^{3x^4} \cdot 12x^3$$

4.4 Related Rates

7. A spherical balloon is inflated with helium at a rate of $100\pi\ ft^3/min$.

Volume of a Sphere $V = \frac{4}{3}\pi r^3$ Surface Area of a Sphere $SA = 4\pi r^2$

(a) How fast is the balloon's radius increasing when the volume is $36\pi\ ft^3$?

$\frac{dV}{dt} = 100\pi$ $\frac{dr}{dt} = ?$ when $V = 36\pi = \frac{4}{3}\pi r^3 \implies r^3 = \frac{36 \cdot 3}{4} = 27 \implies r = 3$ ft.

$V = \frac{4}{3}\pi r^3$ $\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$ $100\pi = 4\pi(3^2) \cdot \frac{dr}{dt} \implies \frac{dr}{dt} = \frac{25}{9}$ ft/min

(b) How fast is the balloon's surface area increasing when the volume is $36\pi\ ft^3$?

$\frac{dSA}{dt} = ?$ when $V = 36\pi$ ft.^{2} i.e. when $r = 3$ ft.

$SA = 4\pi r^2$ $\frac{dSA}{dt} = 4\pi \cdot 2r\frac{dr}{dt} = 4\pi(2 \cdot 3)\left(\frac{25}{9}\right) = \frac{200}{3}\pi$ ft.^{2}/min

- Draw a picture!!
- Label what you're looking for (here $\frac{\partial r}{\partial t}$)
- Label what you know ($\frac{\partial V}{\partial t}$)
- Will always take implicit derivative, so come up with an equation that **relates** the rate you *know* (from step 3) to the one you *want* (step 2).

4.5 Extrema

The first derivative test:

$$f(c) \text{ is an extrema} \iff f'(c) = 0$$

Always the same procedure:

- Compute a formula for $f'(x)$
- Set $f'(x) = 0$ and solve for x; add these to list of critical points
- Also check where $f'(x)$ does not exist; also add these points.
- If on a interval, add the boundary points
- Classifying which are maxes/mins:
  - Check sign of f' near the critical points
    - Increasing then decreasing: local max
    - Decreasing then increasing: local min
  - Alternatively: 2nd derivative test
    - $f'' < 0$: local max, concave down
    - $f'' > 0$: local min, concave up
    - $f'' = 0$: inconclusive
- Determining if it's a *global* extrema:
  - If only checking on an interval, nothing else to do
  - Otherwise, may need more analysis: look at $\lim_{x\to p} f(x)$ for places where f isn't defined, e.g. $\pm\infty$ or vertical asymptotes

Note that inflection points are where $f''(x) = 0$, i.e. where slopes go from increasing to decreasing (or vice-versa).

(a) $g(x) = x^{2/3}(2x-1)^2$ on $[-1, 1]$

$$g'(x) = \frac{2}{3}x^{-1/3}(2x-1)^2 + x^{2/3}\,2(2x-1)\cdot 2 = \frac{2}{3}x^{-1/3}(2x-1)\left[(2x-1)+6x\right]$$

$$= \frac{2}{3}x^{-1/3}(2x-1)(8x-1)$$

$g'(0)$ DNE $g(-1) = 9$ $g(0) = 0$

$g'(\frac{1}{2}) = 0$ $g(1) = 1$ $g(\frac{1}{2}) = 0$

$g'(\frac{1}{8}) = 0$ $g(\frac{1}{8}) = \frac{1}{4}(\frac{1}{4}-1)^2 = \frac{9}{64}$

Min Value: 0   Max Value: 9

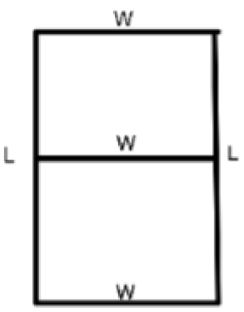
4.6 Maximization

Always the same procedure:

- Draw a picture!! Label relevant quantities
- Write a formula for the function f you want to maximize in terms of labels (note: f may depend on multiple variables at this point)

- Write a *constraint equation* that relates some of the variables appearing in f, rewrite f in terms of *one single unknown*
- Apply the known critical point procedure

1. A farmer wishes to build a 216 m^2 rectangular corral. The corral needs an outer fence and another piece of fence parallel to one of the sides to divide the pasture into two even parts (one for sheep & one for pigs). What is the smallest amount of fence required?



Constraint: $A = lw = 216 \implies w = \frac{216}{l}$

Minimize: $F = 2l + 3w = 2l + 3\left(\frac{216}{l}\right)$

$F'(l) = 2 - \frac{648}{l^2}$ $\quad F'(0)$ DNE $\quad F'(l) = 0 \implies l = 18$

$-\quad +$

$0 \quad 18$

$l = 18$ is the only minimum

$l = 18 \implies w = \frac{216}{18} = 12 \implies 72m$ of fence

5 Intermediate Value Theorem

Statement: if $f : \mathbb{R} \to \mathbb{R}$ is

- Continuous on the interval $[a, b]$,

Then for *any* value $y \in [f(a), f(b)]$, there is some x such that $f(x) = y$.

Special case: if we know $f(a) < 0$ and $f(b) > 0$ for some pair a, b, then we know $f(x) = 0$ for some $x \in (a, b)$.

5. Use the *Intermediate Value Theorem* (if applicable) to show that there is a solution on $[0, 1]$ to

$$x^5 + 2x = 1$$

Be sure to justify that you are allowed to use IVT by checking the necessary hypotheses.

$f(x) = x^5 + 2x - 1$ is CTS on $[0,1]$

$f(0) = -1 < 0 < 2 = f(1)$

IVT guarantees there is a solution to $f(x) = 0$ on $[0,1]$

6 Mean Value Theorem

Statement: If $f : \mathbb{R} \to \mathbb{R}$ is

- Continuous on a closed interval $[a, b]$, and

- Differentiable on the open interval (a, b),

then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- RHS: "Average rate of change", or slope of secant line between $(a, f(a))$ and $(b, f(b))$
- LHS: The derivative evaluated at *some* point c. We just know it exists.

Note: if $f(b) = f(a)$, this means $f'(c) = 0$ for some c – i.e., there's a critical point in (a, b).

2. What is the average rate of change of $F(x) = \sqrt{3x^2 - 2x + 4}$ on the interval $[-1, 3]$

$$\frac{F(3) - F(-1)}{3 - (-1)} = \frac{\sqrt{27-6+4} - \sqrt{3+2+4}}{4} = \frac{\sqrt{25} - \sqrt{9}}{4} = \frac{5-3}{4} = \frac{1}{2}$$

Example:

7 Curve Sketching

7. Let $f(x) = 4x^{1/3} - x^{4/3}$

(a) Domain of $f(x)$ $(-\infty, \infty)$

(b) Roots of $f(x)$ $x = 0$ & $x = 4$

(c) $\lim_{x \to -\infty} f(x) = -\infty$

(d) $\lim_{x \to \infty} f(x) = -\infty$

(e) Critical values of $f(x)$ $x = 0$ & $x = 1$

(f) Inflection values of $f(x)$ $x = -2$ & $x = 0$

(g) Interval $f(x)$ is increasing $(-\infty, 1)$

(h) Intervals $f(x)$ is concave down $(-\infty, -2) \cup (0, \infty)$

2

(i) Sketch the curve $y = f(x)$ [graph paper to be provided on test]
Label all critical points and inflection points (both x & y coordinates are required for full credit).