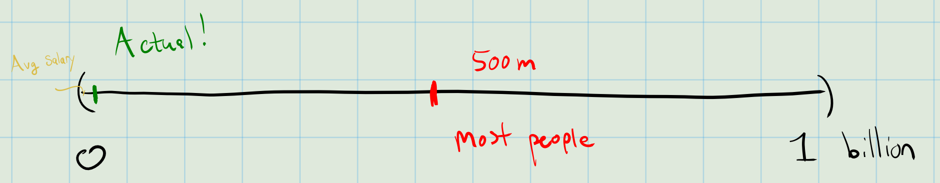
Tuesday

Why care about logs?

1) Humans are bad with big/small #s:

<u>Ex</u> 2018 US Budget $\approx$ 1.068 trillion



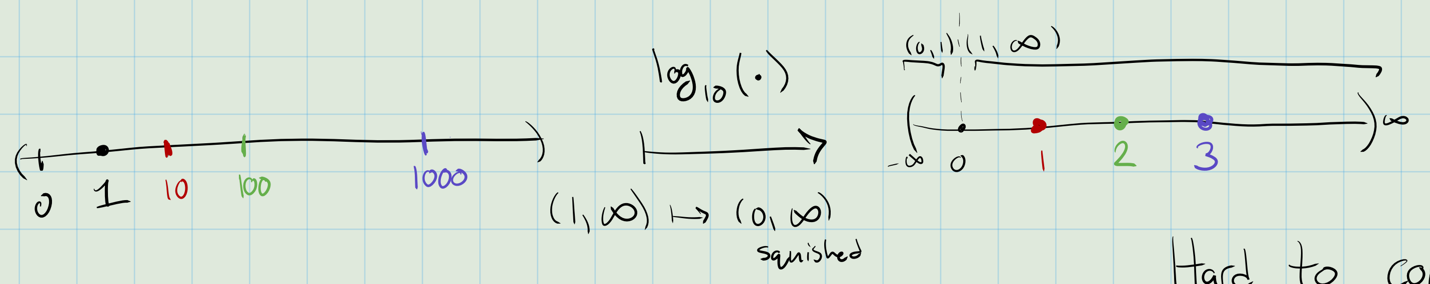
where is 1 million?

If this line is 1 meter, 1 trillion is $\approx$ 1 mile!

But

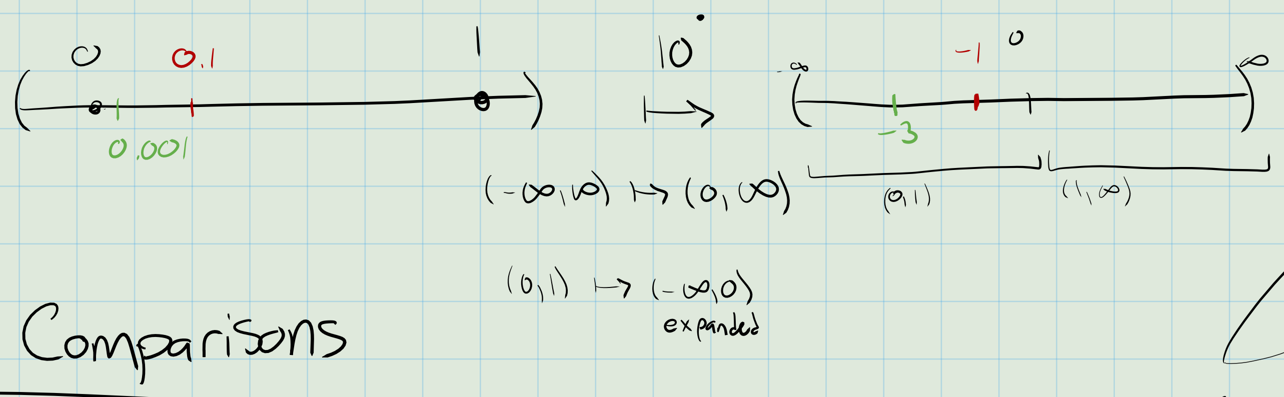
$$\left\{\begin{array}{l} 1M = 10^6 \\ 1B = 10^9 \\ 1T = 10^{12} \end{array}\right. \xrightarrow{\log_{10}(\cdot)} \left.\begin{array}{l} 6 \\ 9 \\ 12 \end{array}\right\}$$

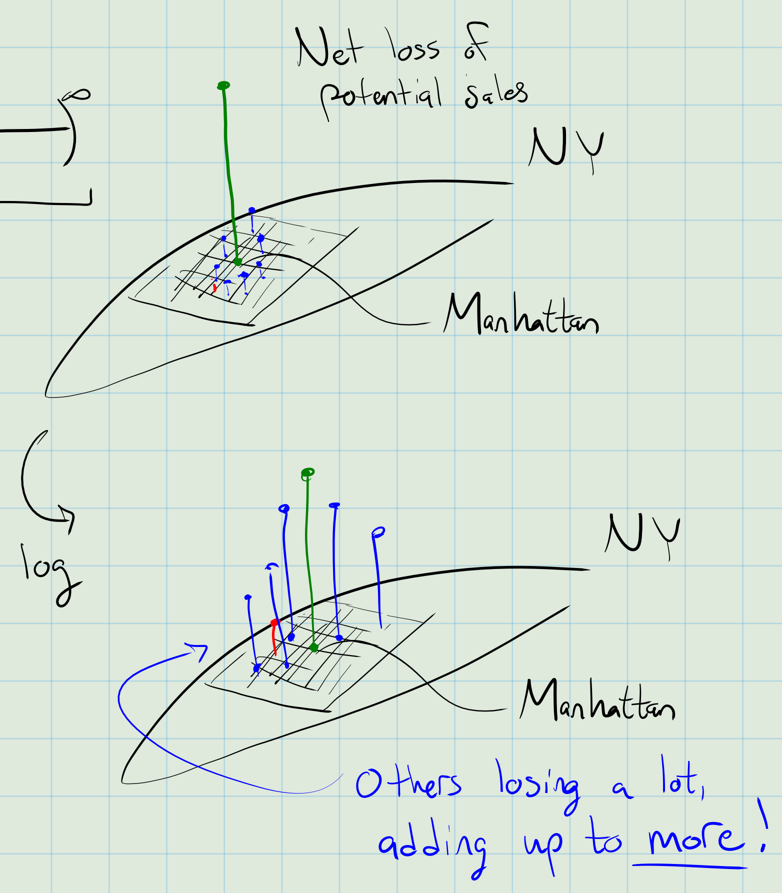
Logs: Telescope $\qquad f(x) = \log_{10}(x)$



Hard to compare big & small #s

Exponentiation: Microscope





2) Sensible Comparisons

- # Grains of sand
  1,000,000,000,000,000,000 $\approx 10^{18}$
- # of stars
  1,000,000,000,000,000,000,000 $\approx 10^{21}$
- # H_2O in 1 cup of water
  1,000,000,000,000,000,000,000,000 $\approx 10^{24}$

3) Multiplying large numbers.

$129 \cdot 505 = ?$

$\log_2(129) \approx 7$
$\log_2(505) \approx 9$

Old days: look up in a log table.

$$\log_2(129 \cdot 505) = \log_2(129) + \log_2(505)$$
$$\approx 7 + 9$$
$$= 16$$
$$\Rightarrow 2^{\log_2(129 \cdot 505)} \approx 2^{16}$$
$$\Rightarrow 129 \cdot 505 \approx 2^{16} = 65{,}536.$$

• Review log properties

- $\log_?(?) = 1$

Ex
$$\left\{\begin{array}{l} \log_\beta(\beta) = 1 \\ \ln(e) = 1 \\ \log_{10}(10) = 1 \end{array}\right.$$

- Modifying in-place & scratch work

<u>Eg</u>: $\log_6(x+1) - \log_6(x-1) = 1/3$

<u>Why?</u>

$\log_\beta(\beta) = \gamma \Rightarrow \beta^{\log_\beta(\beta)} = \beta^\gamma$ (write in exponential form)

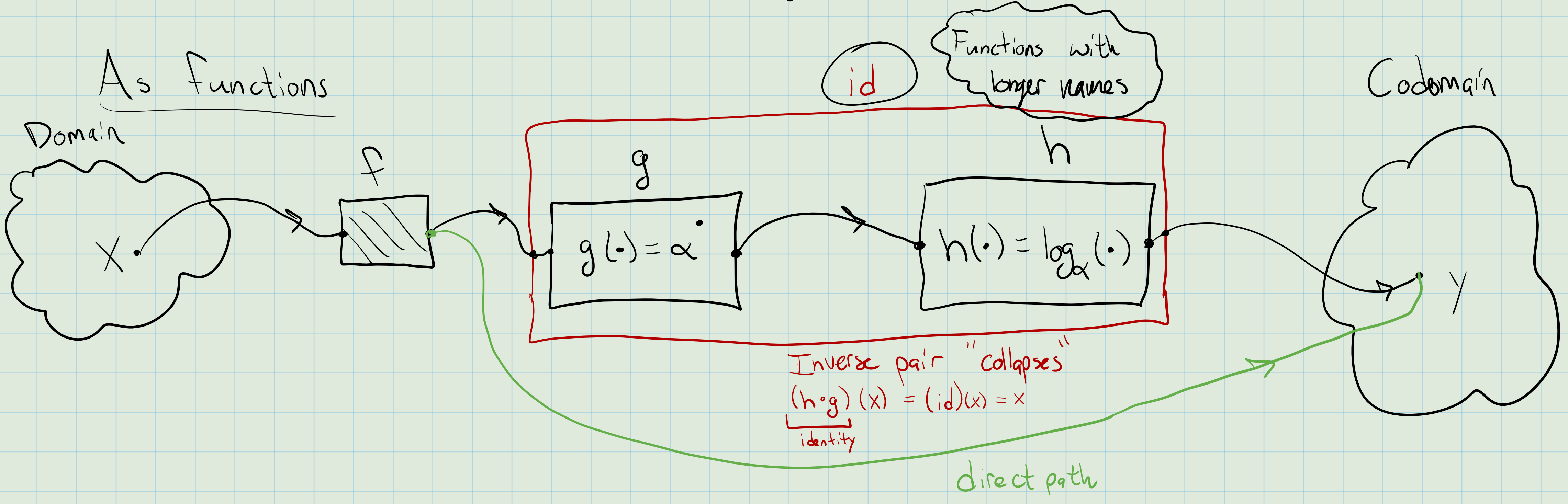
$\Rightarrow \beta = \beta^\gamma$

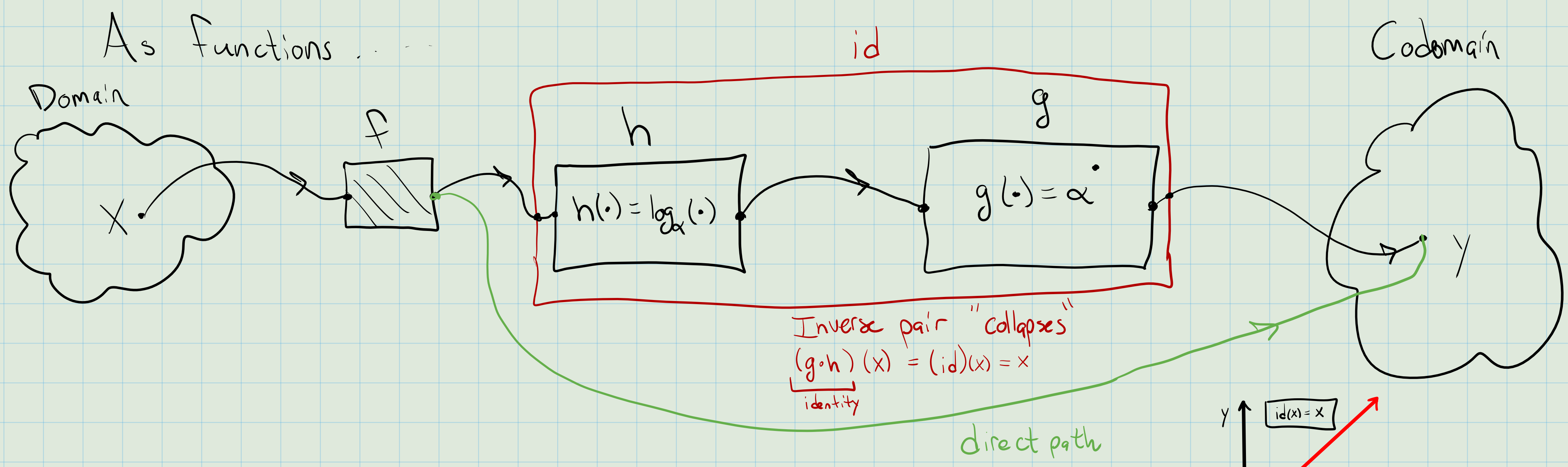
$\Rightarrow \gamma = 1.$

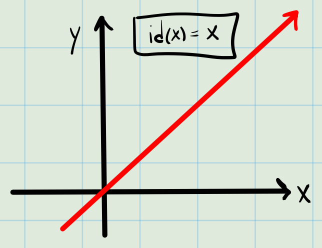
① $\alpha^{\log_\alpha(?)} = ? \quad \rightsquigarrow \quad \alpha^{\log_\alpha(f(x))} = f(x)$

② $\log_\alpha(\alpha^?) = ? \quad \rightsquigarrow \quad \log_\alpha(\alpha^{f(x)}) = f(x)$

As functions







· Word Problems

· Exponential Change: $f(t) = Pe^{rt}$

$$\left\{\begin{array}{l} \overset{\text{known}}{r<0} \Rightarrow \text{decay} \\ r=0 \Rightarrow \text{const} \\ r>0 \Rightarrow \text{growth} \end{array}\right.$$

↳ Can always find P if intial conditions are known

$\alpha = f(0) \Rightarrow \alpha = Pe^{r\cdot 0} = Pe^{0} = P \Rightarrow \boxed{\alpha = P}$

$\Rightarrow f(t) = \alpha e^{rt}$

Reminder on Exponential growth.

$e^x > x^n$ eventually

Making precise $\left\{\begin{array}{l} \dfrac{e^x}{x^n} \xrightarrow{x\to\infty} \infty \\ \dfrac{x^n}{e^x} \xrightarrow{x\to\infty} 0 \end{array}\right.$

↳ Can solve for t when asking for proportion

$f(t) = \alpha e^{rt}$, when does $f(t) = \beta \cdot f(0)$?

$$\left\{\begin{array}{l} 60\% \Rightarrow \beta = 6/10 \\ 25\% \Rightarrow \beta = 1/4 \\ \text{ets} \end{array}\right.$$

Know $\alpha = f(0)$, so solve

$\beta \cdot f(0) = \alpha e^{rt}$ for t

$\Rightarrow \beta \alpha = \alpha e^{rt}$

$\Rightarrow \beta = e^{rt}$

$\Rightarrow \ln(\beta) = rt$

$\Rightarrow t = \frac{1}{r}\ln(\beta)$

Process more important than formula!

$g(t) = 3 + \alpha e^{5t+2}$, $g(0) = 10$

$g(0) = 10 \Rightarrow 10 = 3 + \alpha e^{5\cdot 0+2}$

$\Rightarrow 10 = 3 + \alpha e^{2}$

$\Rightarrow \alpha = 7e^{-2}$

$\Rightarrow g(t) = 3 + (7e^{-2})e^{5t+2}$

$\Rightarrow g(t) = 3 + 7e^{5t}$

When does $g(t) = 0.7\,g(0)$?

$g(0) = 10$, so solve for t in

$(0.7)(10) = 3 + 7e^{5t}$

$\Rightarrow 7 = 3 + 7e^{5t}$

$\Rightarrow 4/7 = e^{5t}$

$\Rightarrow \ln(4/7) = 5t$

$\Rightarrow t = \frac{1}{5}\ln(4/7)$.

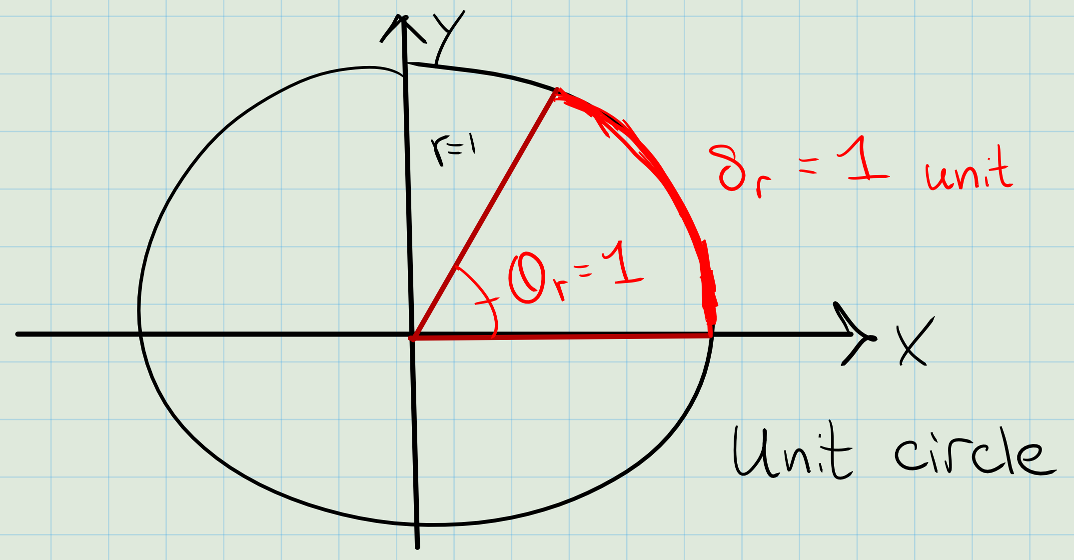
Similar, but not the same.

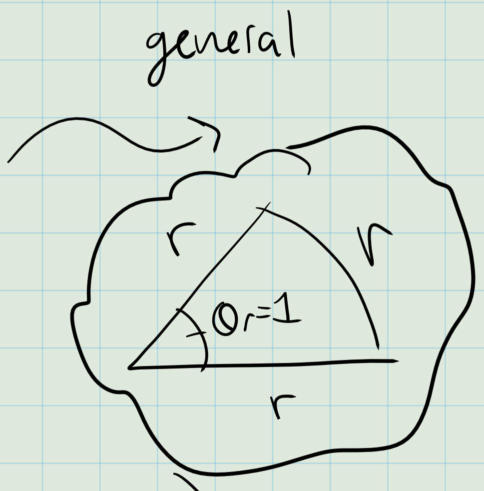
Trigonometry

⚠ Always convert!

- Radians: $\underbrace{360°}_{\theta_d} = \underbrace{2\pi}_{\theta_r}$ rads

  Why? An angle of 1 radian has an arc length of 1.



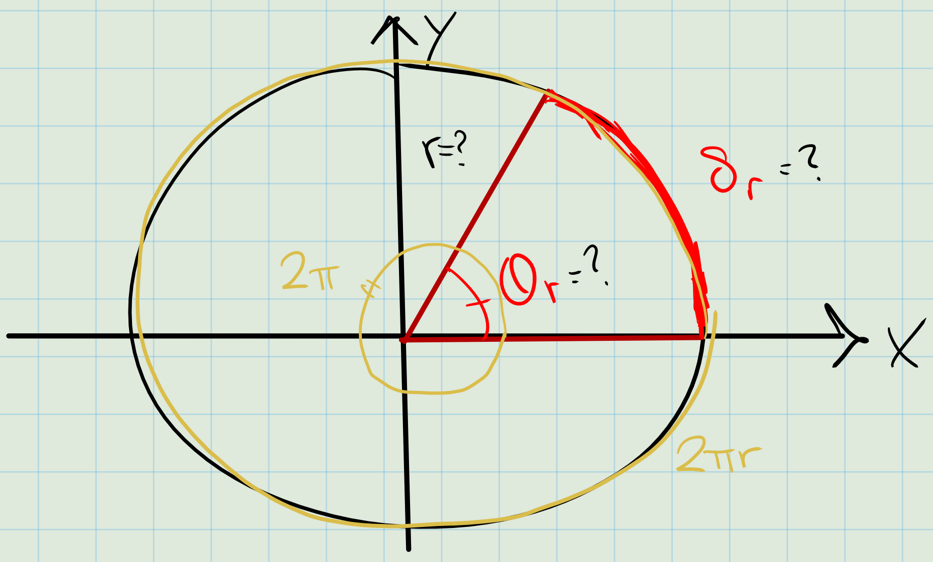


Important equation: $\boxed{\dfrac{\theta_r}{2\pi} = \dfrac{\theta_d}{360}.}$ (Solve for either)

Arc Lengths & Angles

For a general circle:

Big idea: $S_r \propto \theta_r$

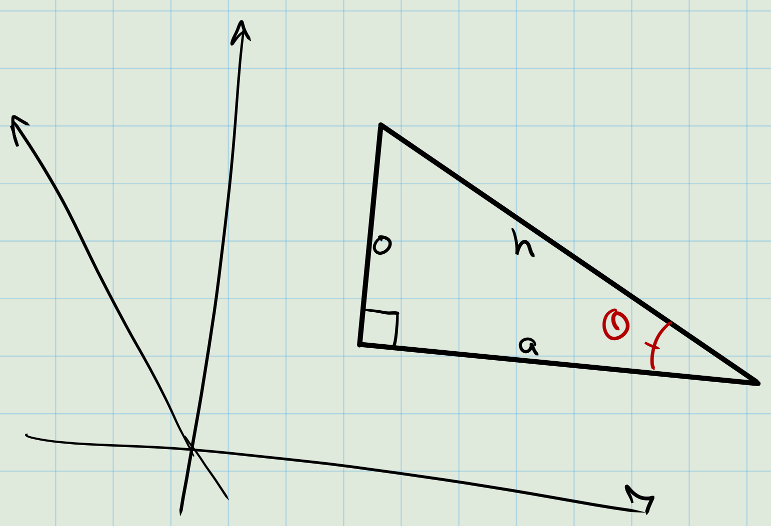


Recall: $\boxed{\begin{array}{l}\text{Area}(r) = \pi r^2, \\ \text{Circumf.}(r) = 2\pi r\end{array}}$

$\dfrac{\text{Angle}}{\text{Total Angle}} = \dfrac{\text{Arc length}}{\text{Circumference}} \Rightarrow \dfrac{\theta_r}{2\pi} = \dfrac{S_r}{2\pi r} \Rightarrow \boxed{S_r = r \cdot \theta_r}$

(Arc length = radis · angle)

- Standard ratios (Planar geometry)



All relative to θ

Pythagoras: $o^2 + a^2 = h^2$

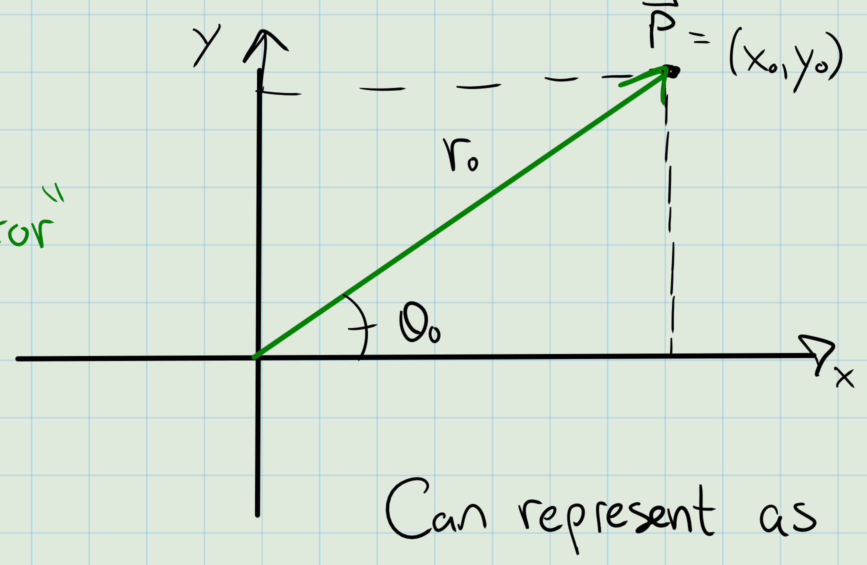
Rmk: Shortest length is straight line

$$\left[\begin{array}{ll} \sin(\theta) = o/h & \csc(\theta) := \frac{1}{\sin(\theta)} = h/o \\ \cos(\theta) = a/h & \sec(\theta) := \frac{1}{\cos(\theta)} = h/a \\ \tan(\theta) = o/a & \cot(\theta) := \frac{1}{\tan(\theta)} = a/o \end{array}\right]$$

- Useful interpretation: polar coordinates

  If $(x_0, y_0) \in \mathbb{R}^2$ (the x-y plane), we can change variables:

"vector"



$\Rightarrow \left\{\begin{array}{l} r_0 = (x_0^2 + y_0^2)^{\frac{1}{2}} \\ \theta_0 = \arctan(y_0/x_0) \end{array}\right.$ magnitude & direction

Can represent as $\vec{p} = (r_0 \cos(\theta_0), r_0 \sin(\theta_0))$

On unit circle, $r_0 = 1$

Useful interpretation:

Cosine ⟶ "horiz component"

Sine ⟶ "vertical component".