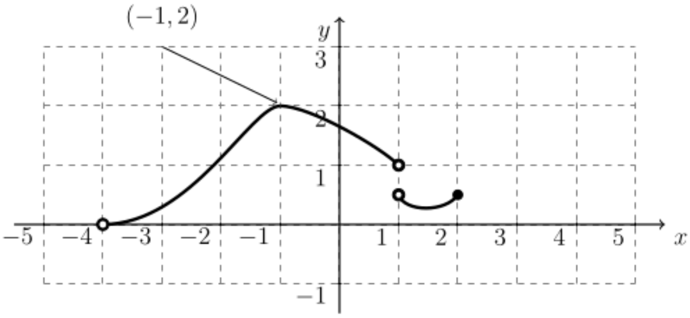
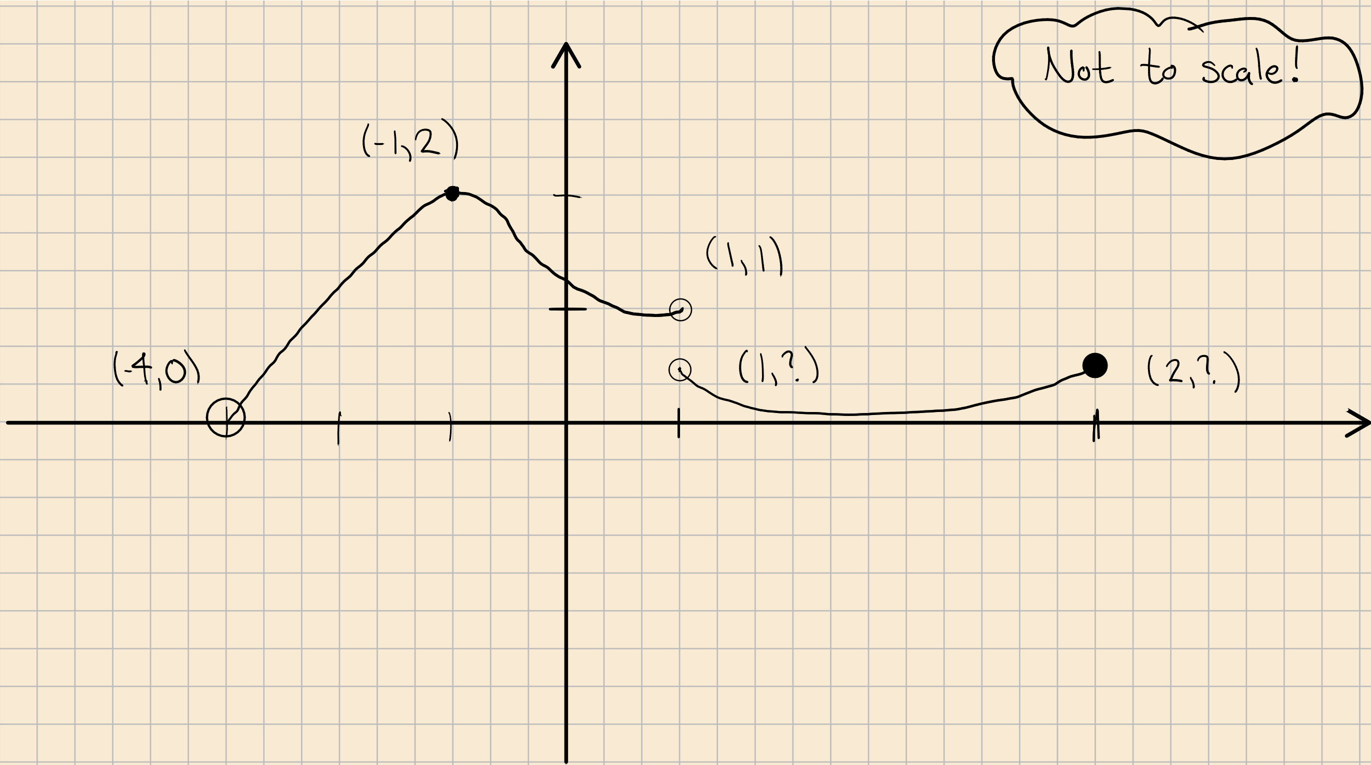
1. (5 points) A function f is specified by the following graph:

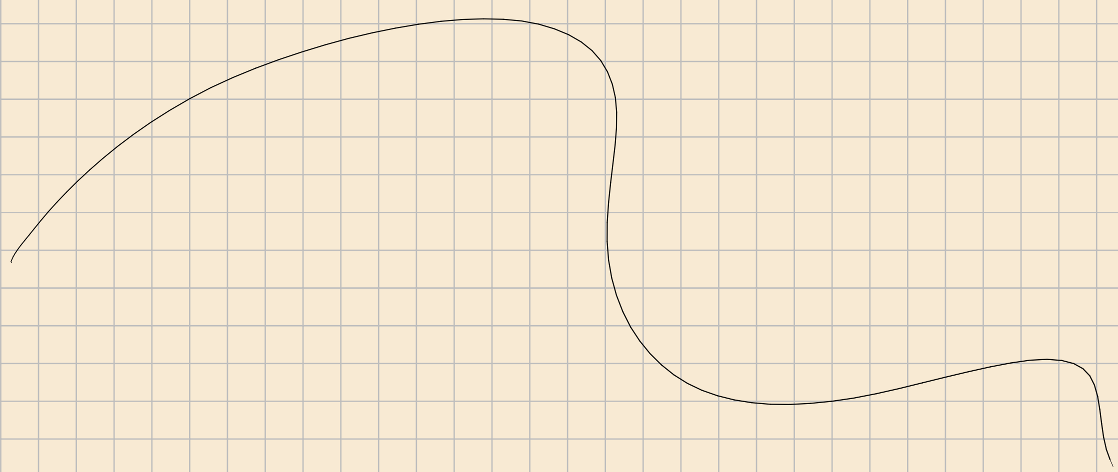


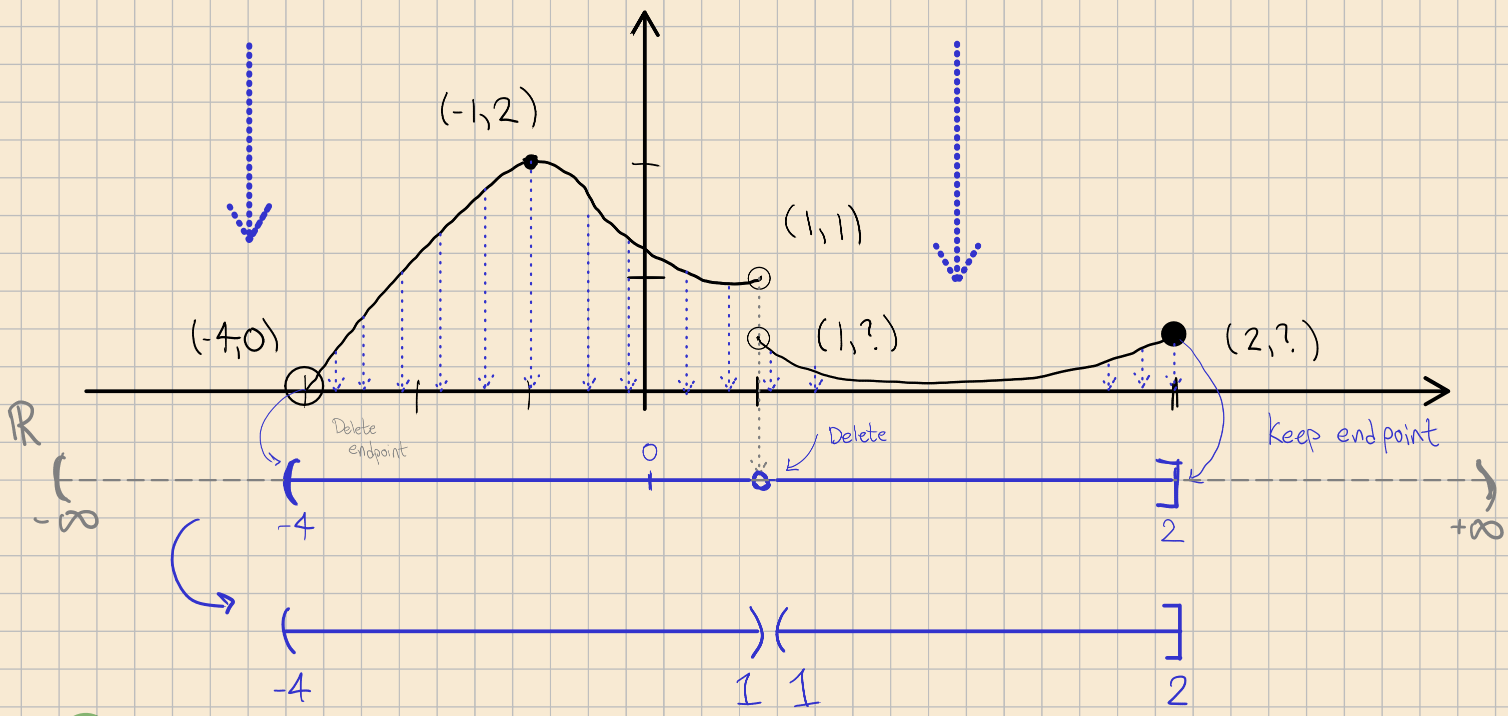
Determine the domain and range of f in interval notation, and write at least one sentence explaining and/or justifying your answer.

Hint: consider the extreme x and y values of this graph, along with points that should be deleted from the domain or range. Be careful! Are there other points on the graph that "add numbers back in" to the domain or range?



Domain: Project onto x-axis





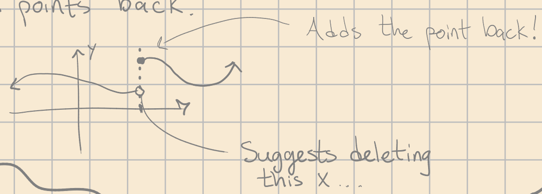
So domain(f) = $(-4,1) \cup (1,2]$.

Can also think about "deleting" regions:

- Hope: All of $\mathbb{R}$!
- Graph doesn't exist for
  - $x < -4$ ⟶ delete $(-\infty, -4)$
  - $x > 2$ ⟶ delete $(2, \infty)$
- Open circles correspond to deleted points
  - At $x = 1$ ⟶ delete $\{1\}$
  - At $x = -4$ ⟶ delete $\{-4\}$

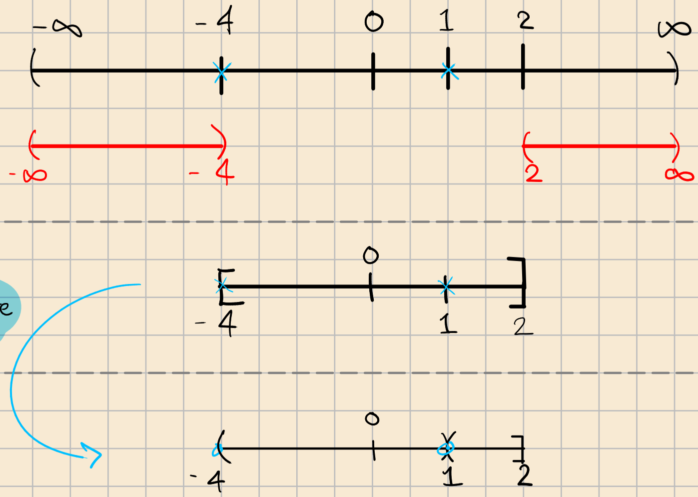
⚠ Careful! Manually check if some other part of the graph exists above these points, which would add these deleted points back.

Example



Take $\mathbb{R}$ ...

Then delete $(-\infty,-4)$ & $(2,\infty)$



$\mathbb{R} \setminus ((-\infty,-4) \cup (2,\infty)) \setminus (\{-4,1\})$

$= [-4,2] \setminus \{-4,1\}$

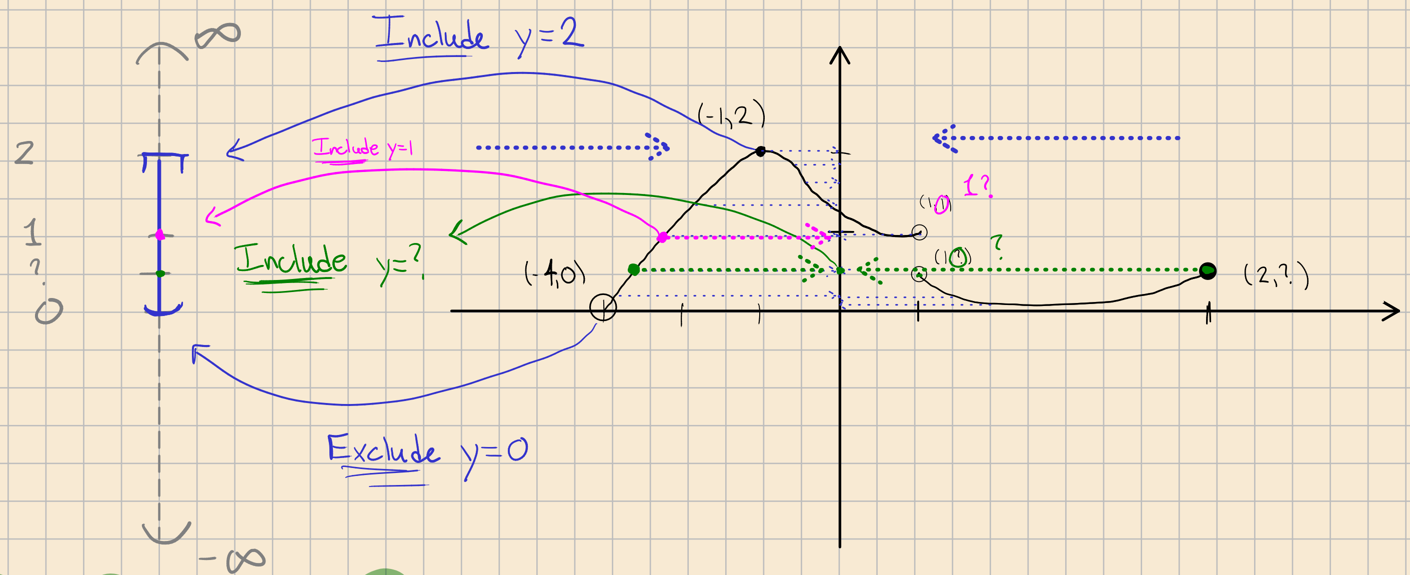
Then delete $\{-4\}, \{1\}$

$= (-4,2] \setminus \{1\}$

$= (-4,1) \cup (1,2].$

$\boxed{\Rightarrow \text{domain}(f) = (-4,1) \cup (1,2].}$

Range: Project onto y-axis

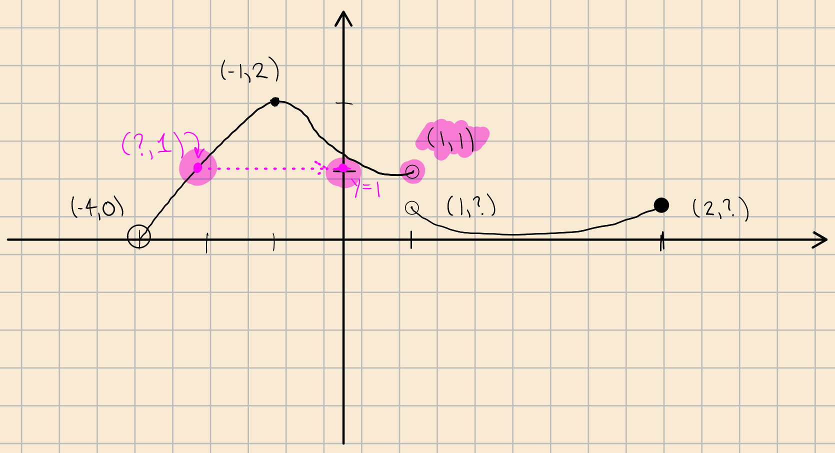


$\boxed{\Rightarrow \text{range}(f) = (0,2].}$

Tempting to exclude $y=?$ & $y=1$, but there are other points on the graph that add them back in!

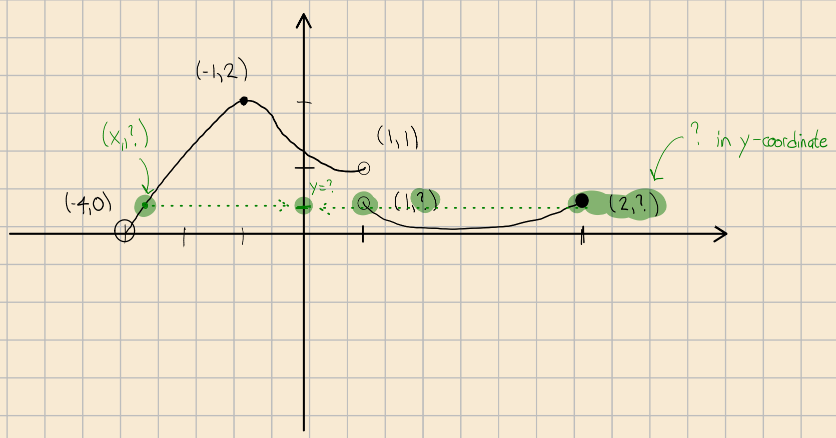
Alterntatively:

- Hope it's all of $\mathbb{R}$
- Doesn't exist outside of $(0,2)$, so delete $(-\infty,0)$ & $(2,\infty)$
- Open circle at $(-4,0) \rightsquigarrow$ delete $\{0\}$?
  - ↳ Check that $y=0$ doesn't occur elsewhere: okay!
- Open circle at $(1,1) \rightsquigarrow$ delete $\{1\}$?
  - ↳ Check if $y=1$ occurs elsewhere on the graph:
    it does, so don't delete!



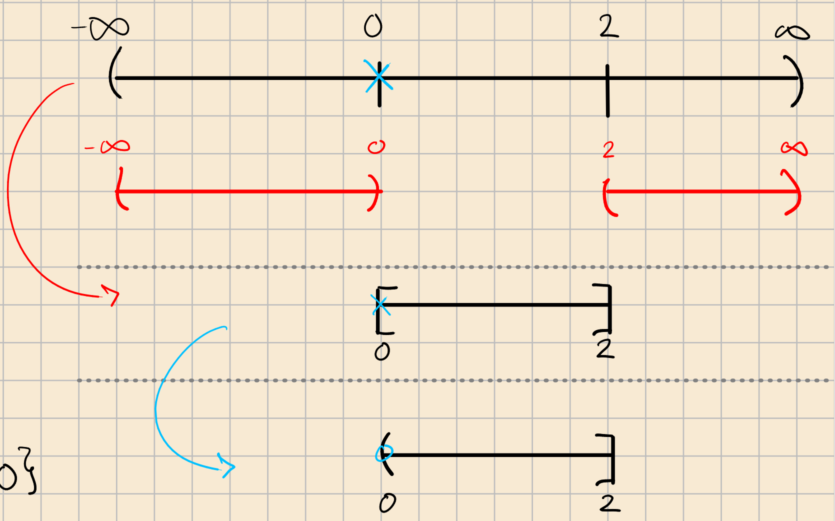
• Open circle at $(1,?)$ ⤳ delete $\{?\}$

↳ Check if $y=?$ occurs elsewhere on the graph: it <u>does</u> (multiple times), so <u>don't</u> delete!



So we want

- All of $\mathbb{R}$
- Delete $(-\infty,0)$ & $(2,\infty)$
- Delete $\{0\}$.



$$\Rightarrow \text{range}(f) = \mathbb{R} \setminus \big((-\infty,0)\cup(2,\infty)\big)\setminus\{0\}$$

$$= [0,2]\setminus\{0\}$$

$$\boxed{= (0,2].}$$

2. (5 points) Let $\alpha \in \mathbb{R}$ be some unknown **positive** real number, so $\alpha > 0$, or in other words α is contained in the open interval $(0, \infty)$.

Consider the function specified by the following formula:

$$g(x) = \frac{1}{(x-\alpha)(x+\alpha)}.$$

Determine the domain of g in interval notation, and write at least one sentence explaining and/or justifying your answer.

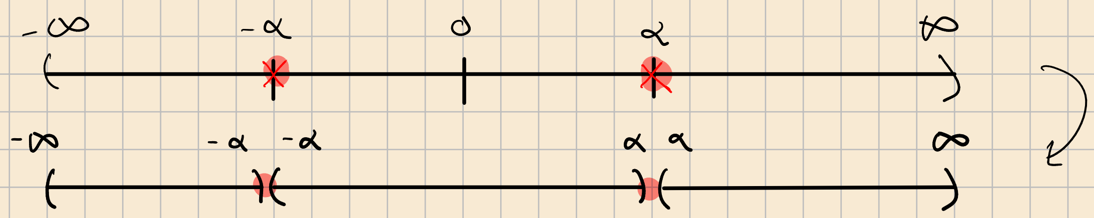
Hint: For which values of x is $g(x)$ undefined? Your answer should depend on the unknown α. Drawing a number line may help.

Math goggles: Something of the form $1/?$, has issues when $? = 0$.

Here $? = (x-\alpha)(x+\alpha)$, so

$? = 0 \Rightarrow (x-\alpha)(x+\alpha) = 0 \Rightarrow x = \alpha$ or $x = -\alpha$.

We know $\alpha > 0$, so what's the picture?



Delete $\alpha, -\alpha$.

$$\boxed{\begin{aligned} \Rightarrow \text{domain}(f) &= \mathbb{R} \setminus \{\alpha, -\alpha\} \\ &= (-\infty, -\alpha) \cup (-\alpha, \alpha) \cup (\alpha, \infty). \end{aligned}}$$

3. (5 points) Let $\Lambda \in \mathbb{R}$ be some unknown real number. A function h is specified by the following formula:

$$h(x) = \frac{\pi}{\sqrt{2x - \Lambda}}.$$

Determine the domain and range of h in terms of the unknown Λ, and write at least one sentence explaining and/or justifying your answer.

Hint: Break this problem into two smaller pieces. What points or regions of the number line are "problematic" for each piece?

Math goggles: there are two things happenning:

#1) $1/?_1$ where $?_1 = \sqrt{2x-\Lambda}$

↳ problems when $?_1 = 0$.

#2) $\sqrt{?_2}$ where $?_2 = 2x - \Lambda$

↳ problems when $?_2 < 0$.

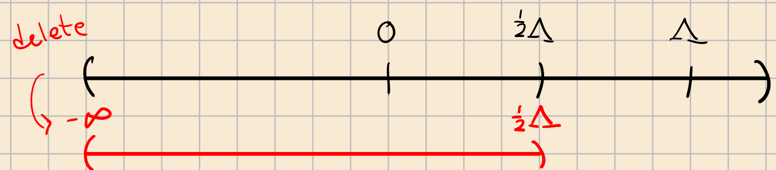
Handling #2:

$?_2 < 0 \Rightarrow 2x - \Lambda < 0$

$\Rightarrow 2x < \Lambda$

$\Rightarrow x < \frac{1}{2}\Lambda$ $\rightsquigarrow$ So we need to delete

$\{x \in \mathbb{R} \mid x < \frac{1}{2}\Lambda\} = (-\infty, \frac{1}{2}\Lambda)$



Note: the picture can now be inaccurate, since we don't know if Λ is positive! But it can still help us reason about this problem.

Handling #1:

$$\textcircled{?}_1 = 0 \Rightarrow 2x - \Lambda = 0$$
$$\Rightarrow 2x = \Lambda$$
$$\Rightarrow x = \tfrac{1}{2}\Lambda,$$ so we need to delete $\{\tfrac{1}{2}\Lambda\}$

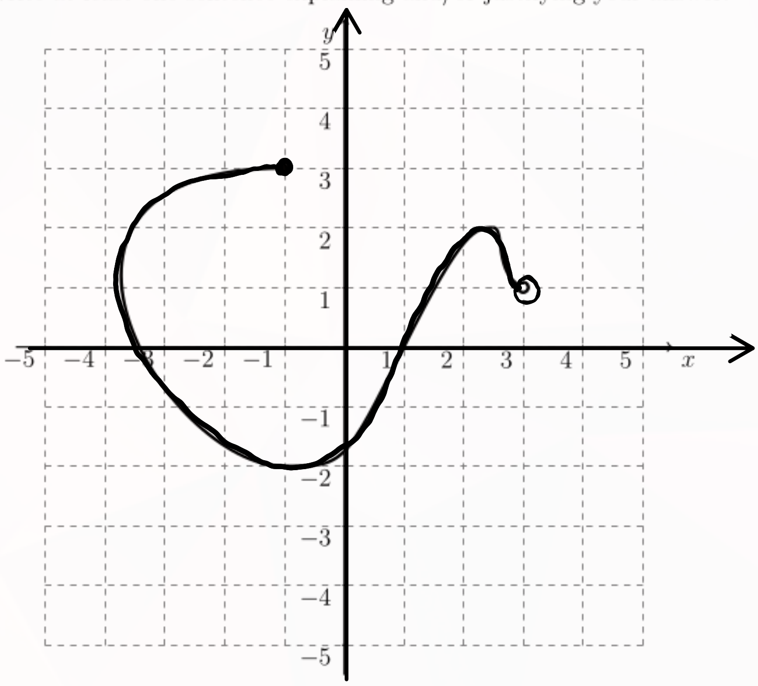
Summary

- Hope: All of $\mathbb{R}$
- Delete $(-\infty, \tfrac{1}{2}\Lambda)$ due to problems in #2
  $\mathbb{R} \setminus (-\infty, \tfrac{1}{2}\Lambda) = [\tfrac{1}{2}\Lambda, \infty)$
- Delete $\{\tfrac{1}{2}\Lambda\}$ due to problems in #1
  $[\tfrac{1}{2}\Lambda, \infty) \setminus \{\tfrac{1}{2}\Lambda\} = (\tfrac{1}{2}\Lambda, \infty)$.

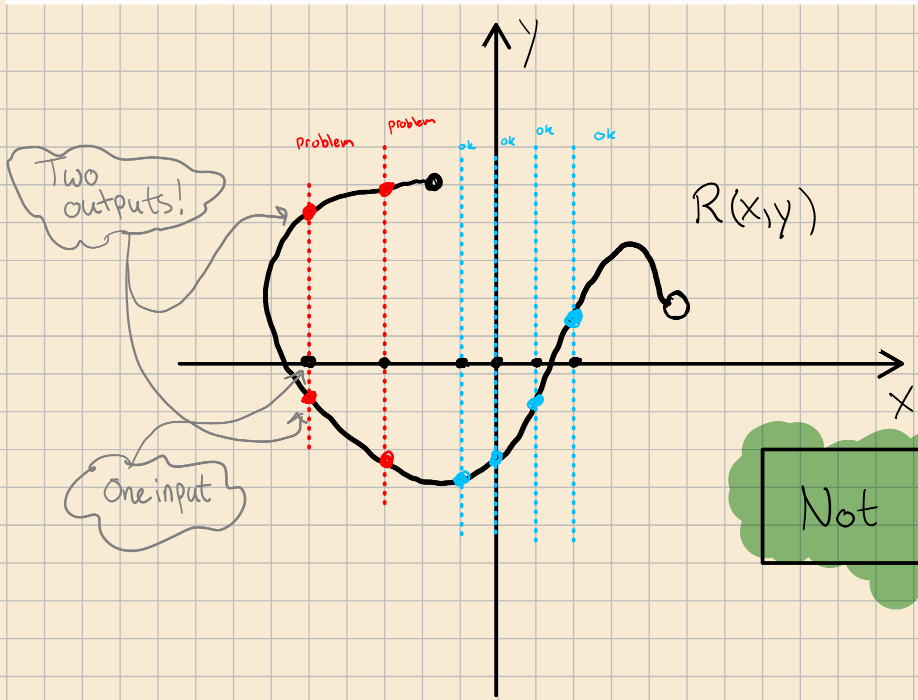
$$\boxed{\Rightarrow \text{domain}(f) = (\tfrac{1}{2}\Lambda, \infty).}$$

4. (5 points) The graph of a relation between x and y is shown below. Does this relation determine a function? Why or why not?

Write at least one sentence explaining and/or justifying your answer.



Hint: you may use a "line test", but in any case, your answer should include what it means for a relation to be a function in terms of inputs x and outputs y.



$\boxed{\text{Not a function.}}$

Mnemonic: Vertical line test.

If there is even <u>one</u> place where this test fails, it is not a function.

What's going wrong? Functions should be "deterministic": every input determines a unique output. I.e., for every x in the domain, there is <u>exactly</u> one corresponding y-value.