A History of Algebraic Geometry

D. Zack Garza

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Abstract

The primary purpose of this paper is to discuss Alexandre Grothendieck, who is often cited as one of the most influential mathematical thinkers of the 20$^{\text{th}}$ century. In order to fully appreciate the impact of his contributions, it is necessary to provide some historical context in which to place his work, and to this end, this paper will also discuss some the history, and mathematical content of Algebraic Geometry, as well as several key figures in the field.

This topic was chosen because of the significance of Grothendieck's accomplishments and the legacy he has left on modern mathematical theory, but also because it allows place the field of Geometry in a wider context, with a narrative that spans from the time of the Greeks to the proof of the infamous last theorem of Fermat near the turn of the 20$^{\text{th}}$ century. In particular, the interplay between Algebra and Geometry has a rich and storied history, including a unification between the two fields that has advanced rapidly in the past two to three centuries.

Treating this subject cohesively will require not only examining the historical context, however – in order to fully appreciate the impact of some modern results, it will be necessary to cover introduce some mathematical content as well. To this end, this paper does not seek to provide a rigorous treatment of Algebraic Geometry – many references are provided in the reference section that serve this purpose quite nicely.

Instead, such mathematical inclusions are meant to inform the historical narrative, and so attention will be restricted to only those definitions that provide a common and unifying language in which to frame the results that are mentioned. It is often touted that the field of Algebraic Geometry is somewhat obtuse and rife with "heavy mathematical machinery" – it is for this reason that a secondary goal of this paper is to help demystify the subject, and perhaps provide some motivations for why such mathematical machinery would be invented, and why it has earned its place as a rich and distinguished field of mathematical inquiry.

Ultimately, the goal of this paper is to discuss Grothendieck's pioneering use of *schemes*, a construct introduced in his well-known 1957 'Tohoku' paper, which helped lay a new framework for Algebraic Geometry and has driven advances in the field ever since. In order to understand the significance of schemes, however, one must first understand *sheaves*, the construct that schemes are meant to generalize and extend. Sheaves, in turn, are in many ways defined analogously to *manifolds*, which are often thought of as spaces that locally resemble standard Euclidean space, and it is from this construct that a great deal of geometric intuition can be derived and used to guide powerful algebraic generalizations.

1 A Historical Perspective

1.1 The Greeks

Algebraic Geometry is among the oldest branches of mathematics, which was studied as early as 400 BCE by the Greeks in the form of conic sections. For the Greeks, in fact, the division between Algebra and Geometry was perhaps indistinguishable in either direction. Without the benefits of a numeral system amenable to calculations, nor the convenient symbolic shorthand used in modern mathematics, the very problems they studied were inextricably tied to the geometric situations from which they were born.

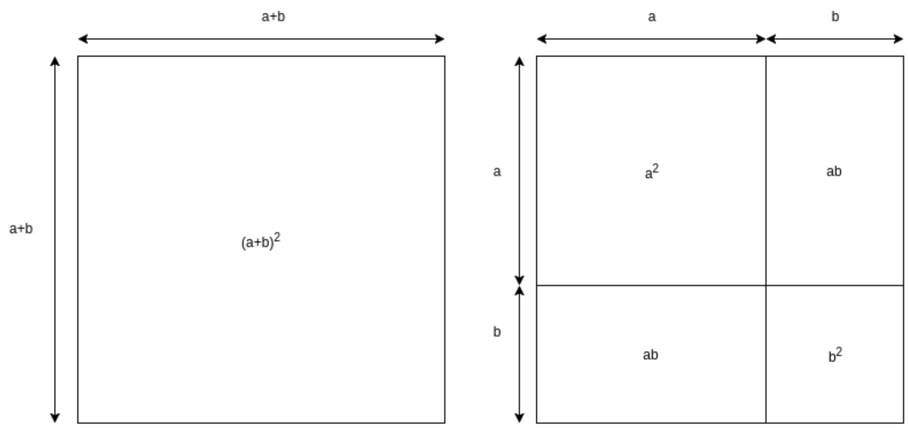


Figure 1: An Algebraic Identity Expressed Geometrically

Because the Greeks worked not primarily with individual number, but instead *magnitudes*: lengths, areas, volumes, and ratios thereof. Because of this, algebraic identities such as

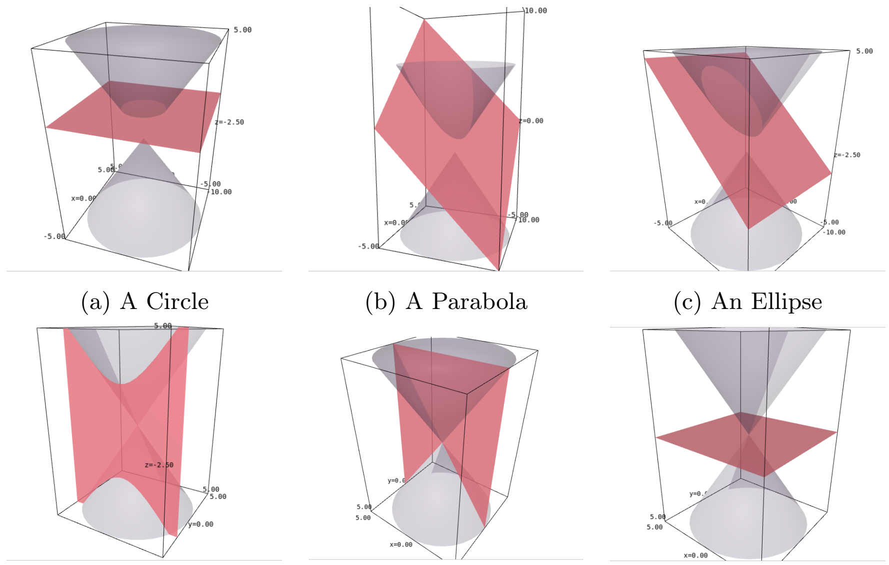
$$(a+b)^2 = a^2 + 2ab + b^2$$

would have been expressed in terms of relationships between geometric figures – in this case, perhaps as rearrangements of certain squares and rectangles, as shown in the figure above.

Of primary interest to the Greeks was the solution of algebraic equations by means of examining the intersection of algebraic curves. In this way, they were led to study the *conic sections*, those curves which can be obtained by intersecting a plane with the surface of a cone. In modern parlance, one might refer to such sections as those traced out by real numbers x and y that satisfy a relationship of the form

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0,$$

where the coefficients A through F are also taken to be real. Despite not quite having the algebraic notation to state the problem in this form nor the benefit of modern analytic and algebraic tools, the Greeks were able to characterize all such sections, yielding circles, ellipses, hyperbolas, parabolas, straight lines, and the degenerate case of a single point.



(a) A Circle (b) A Parabola (c) An Ellipse

(d) A Hyperbola (e) Two Lines (f) A Single Point

Figure 2: The Conic Sections

Accordingly, the Greeks were aware of methods (for example) to solve problems such as "squaring the rectangle", to which one might associate equations of the form $x^2 = ab$, regarding the quantity x^2 as the area of a unknown square and both a and b as known side lengths. In their mathematical framework, all such constructions were necessarily geometric in nature, and were thus restricted to those which could be obtained with use of a compass and straightedge. In these terms, such a problem might be cast as finding the solutions x that instead satisfy the cross-ratio

$$\frac{x}{b} = \frac{a}{x}$$

A similar example is the problem of "doubling the cube" – that is, given a cube with known side lengths, constructing a second cube with exactly twice the volume of the first. This amounts to finding x that satisfy a similar ratio,

$$\frac{x^3}{a^3} = \frac{b}{a}$$

Considering the simplest case of the unit cube led to questions concerning whether number such as $\sqrt[3]{2}$ were constructible in the geometric sense described above, a question that would remain unsolved until the advent of the algebraic tools of Galois Theory in the 19^{th} century.

However, although in many cases the Greeks made use of coordinates, their study of geometry did not have the same analytic flavor that it takes on today – the modern notion of a coordinate system would not enter the mathematical zeitgeist until nearly the 17^{th} century, with the work of Rene Descartes. Moreover, the classical study was restricted to algebraic curves in at most 3 dimensions, and usually over fields such as $\mathbb{R}$ or $\mathbb{Q}$, and so it remained until the "Geometric Renaissance" of the 18^{th} and 19^{th} centuries.

1.2 Modern Times

Due to reverence of the infamous fifth postulate of Euclid, the study of *synthetic geometry* – that built on a collection axiomatic foundations – remained the dominant mode of Geometric thought. With the advent of Cartesian coordinates, however, a separate study of *analytic* (or coordinate) geometry began, coinciding with the beginning of the study of alternative geometries based upon negating the fifth postulate. However, the unreasonable effectiveness of coordinates in applications to science and engineering endeavors, along with the invention of Calculus in the mid 17^{th} century, led to the flourishing of the analytic branch in favor of synthetic approaches.

And so it remained until roughly the 19^{th} century – it was at this point in time that it began to become apparent that alternative and equally valid non-Euclidean geometries could arise from the negation of the fifth postulate, leading to the study of elliptic and hyperbolic geometries. In particular, during this period there was a resurgence of the field of *projective geometry*, which was originally studied by Brunelleschi as the "geometry of perspective" in the 15^{th} century, and reformulated in terms of "points at infinity" around the 17^{th} century by Kepler and Desargues.

Around this time, a notion of *affine geometry* had also come into the picture, which is roughly characterized as a generalization of Euclidean geometry in which the notion of absolute distance is forgotten, and the concepts that remain meaningful are instead the notions of parallel lines, collinearity, and the preservation of certain ratios. It was known early on that Euclidean geometry could be recovered as a special case of projective geometry, and as affine geometry and new non-Euclidean geometries were discovered, it was soon thought that *all* such geometries could in fact be recovered in such a way, making projective geometry a more general and universal theory.

A major proponent of this point of view in the 19^{th} century was Felix Klein (of Klein Bottle fame), who sought to classify all geometries with his *Erlangen Program.* He believed that projective geometry would provide a unifying framework under which all other geometries could be united, and introduced the idea that such geometries could be characterized using the theory of groups. It was with the advent of Klein's program that traditionally synthetic approaches found rigorous footing in analytic and algebraic notions.

2 The Erlangen Program

2.1 Euclidean Geometries

In particular, the *Erlangen* program laid out a framework in which any given geometry could be categorized as a group of symmetries acting on a vector space, and it is of some interest to see how such constructions can yield familiar geometries.

For example, fix some field k – in the familiar setting, one might choose $\mathbb{R}$, but many interesting ideas can be brought to light by considering the general case. One can choose a coordinate system and consider sets of ordered n-tuples

$$k^n := \{(k_1, k_2, \cdots k_n) \mid k_i \in k\}.$$

Equipping this set with the usual point-wise operations of addition and scalar multiplication, a vector space V_k of n dimensions over the base field k can be obtained. With a vector space in hand, one can consider linear maps from V_k to itself (sometimes referred to as *operators*), and if n is finite, these are entirely characterized by $n \times n$ square matrices. In particular, one can form groups of such matrices by equipping them with the usual notion of matrix multiplication, and examine the actions of these groups on V_k.

Since several such groups will be useful in recovering familiar geometries, a few definitions are in order. In each situation, it will be assumed that k is a field, and V_k is a vector space of finite dimension n over the field k.

Definition 2.1. The set of $n \times m$ matrices with entries in k will be denoted $M_{n,m}(k)$. If $n = m$, this will simply be abbreviated to $M_n(k)$, which denotes the set of $n \times n$ (or *square*) matrices over k.

In general, a square matrix is equivalently a mapping from V_k to itself, which can be realized via matrix-vector multiplication. One is often interested in such mappings that are invertible, which prompts the next definition.

Definition 2.2. The *general linear group of dimension n,* $GL_n(K)$, is the group defined by the set

$$GL_n(K) := \{M \in M_n(k) \mid \det(M) \neq 0\},$$

equipped with matrix multiplication. Equivalently, this is the set of invertible (and necessarily square) matrices with entries in k.

The general linear group can alternatively be characterized as a representation of the set of invertible linear operators on V_k, equipped with function composition. For the purpose of this discussion, this group is quite large, so we will be interested in certain subgroups.

Definition 2.3. The *orthogonal group of dimension* n, $O_n(k)$, is defined as the set

$$O_n(k) := \{M \in GL_n(k) \mid MM^T = I\},$$

equipped with matrix multiplication, where M^T denotes the transpose of a matrix and I denotes the unique $n \times n$ matrix that satisfies $AI = IA = A$ for every $A \in M_n(K)$.

It can be shown $O_n(k)$ is in fact a subgroup of $GL_n(k)$. Equivalently, it can be characterized by those matrices M for which $M^{-1} = M^T$, or equivalently those for which $\det M = \pm 1$. If one takes $k = \mathbb{R}$, these are exactly the distance-preserving transformations (often referred to as *isometries*) of $\mathbb{R}^n$ that preserve the origin. Taking the dimension to be either 2 or 3 reveals that this produces a group consisting of both rotations around the origin, and reflections *through* the origin.

In Euclidean geometry, one is also interested in translating and transporting one figure to another point in space, which motivates the next definition.

Definition 2.4. The *group of translations* in vector space V_k of dimension n is denoted $T_n(k)$, and is equivalent to k^n, the set of all n-tuples with entries in k.

The above definition is a consequence of the fact that any translation can be identified with a vector along which the translation occurs, which does not depend on the point being considered.

Equipped with these definitions, we can finally define one of the main groups of interest that will help characterize Euclidean geometry:

Definition 2.5. The *Euclidean group of dimension n* is the group defined by $E_n(k) \coloneqq T_n(k) \rtimes O_n(k)$, the semi-direct group product of the group of translations with the orthogonal group.

The purpose of this definition is to capture some of what is already known about Euclidean geometry – it is invariant under rigid motions (or isometries), which can, in turn, be characterized by combinations of translations, rotations, and reflections about lines. It is these exact types of motions that preserve the essential elements of classical Euclidean geometry – length, angle, ratios, parallel lines, and intersections of lines.

While defining the notion of a semidirect product is perhaps outside the scope of this paper, it happens to be the exact algebraic tool that describes how such isometries can be constructed. In this case, it captures the notion the if one first performs a translation, followed by a rotation or reflection, this motion can equivalently be carried out by first performing the rotation or reflection, and then translating by the new rotated or reflected image of the original translation vector.

Taking a field such as $\mathbb{R}$ and the dimension of $n = 2$, we find that the vector space $\mathbb{R}^2$ along with the group $E_2(\mathbb{R})$ provides enough information to recover classical Euclidean geometry in the plane.

2.2 Affine Geometries

In the affine case, one is often interested in maps $T : k^n \to k^n$ of the form $v \mapsto Mv + b$ where M is a linear translation and b is another vector in k^n.

These maps are often called *affine transformations*, and spaces that result from quotienting by this action are *affine spaces*, which are often described as "vector spaces in which the origin is forgotten." Another way of stating this is that allowing the morphisms to be combinations of both linear maps *and* translations obviates the need for a distinguished zero vector, and allows one to equivalently treat similar figures without reference to an absolute coordinate system.

While this may seem like an abstract notion, it is, in fact, one that is commonly and implicitly used by anyone who has worked with vector spaces in any capacity: vector spaces are in fact affine spaces over themselves. It is this fact that allows one to denote a *vector* in this space by an n-tuple of points, and to interchangeably use the same notation for a *point* of that space. Similarly, one commonly uses the affine structure of a vector space when transporting a vector to a chosen origin and treating it equivalently to the original vector.

Affine transformations are themselves more general than linear transformations. For one, linear maps must preserve the origin, sending zero to zero, while affine maps may not. In addition to the rotations and translations present in the Euclidean case, elements of the affine group additionally may induce both uniform and non-uniform scaling, as well as shearing of figures. And in contrast to the Euclidean case, affine transformations do not in general preserve angles and distances – they do, however, still preserve straight lines, and send parallel lines to parallel lines.

In order to obtain this type of geometry in the framework of the Erlangen program, one can carry out similar constructions to arrive at the following result:

Definition 2.6. The *affine group of dimension* n is defined as $Aff_n(k) := T_n(k) \rtimes GL_n(k)$.

As noted previously, the group $O_n(k)$ is a subgroup of $GL_n(k)$, and so this result suggested to Klein and his contemporaries that Euclidean geometry was, in fact, a restriction, or special case, of affine geometry.

2.3 Projective Geometries

In the case of projective geometry, one wants to introduce a notion of "perspective projection" in addition to the isometries obtained in the Euclidean and Affine cases. However, in order to do so, one must forego the preservation of parallel lines. This is a consequence of the early study of the subject, with respect to capturing 3-dimensional images on a 2-dimensional medium. This required the introduction of a point at which all parallel lines in the image would meet, which is now referred to as a *vanishing point* or a *point at infinity.*

To see why Klein considered Projective geometry to be the most general and universal among the geometries known at his time, it suffices to carry out a similar construction in the projective case. One first takes a vector space V_k, constructs a group that will act as the symmetries that preserve the desired properties, and identifies the geometry as the space that remains invariant under such a group action. In the projective case, the group itself is constructed in a slightly different way, and so a definition from group theory is needed.

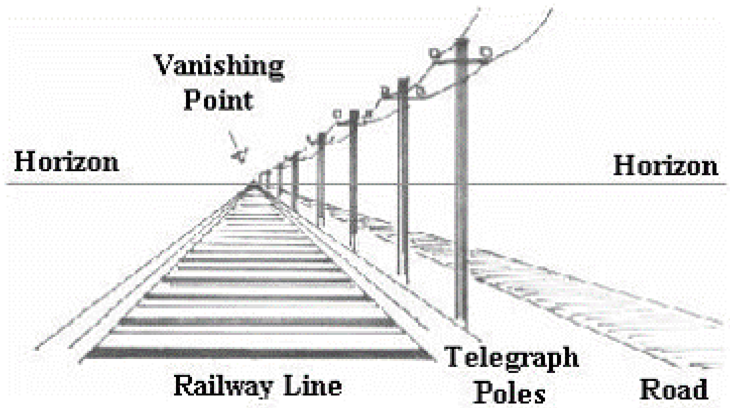


Figure 3: Using a Vanishing Point for Perspective Projection

Definition 2.7. Given a group G, the *center* of a group $Z(G)$ is defined as $Z(G) \coloneqq \{g \in G \mid gh = hg \; \forall h \in G\}$, the set of elements that commute with every other element of the group.

Definition 2.8. The *projective linear group of dimension* $n-1$ is the group defined by $PGL_{n-1}(k) = GL_n(k)/Z(GL_n(k))$, the general linear group quotiented by its center.

The $n+1$ condition is the first noticeable difference, which arises from the fact that a projective space of dimension n is obtained from a vector space of dimension $n+1$, and the projective transformations (often called *homographies*) are induced by the linear transformations in that vector space. Such projective transformations play an important role in complex analysis, where one can realize the *Mobius group* $PGL_2(\mathbb{C})$ as fractional linear transformations on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, which could equivalently be denoted the *complex projective line*.

It can then be shown that, for fixed k and n, both the Euclidean and Affine groups are isomorphic to subgroups of the Projective group, and it is in this way that those geometries can be recovered as special cases of projective geometry.

3 Into the 20$^{\text{th}}$ Century

3.1 Post-Erlangen Program

Armed with the tools of the Erlangen program, geometry once again become an active mathematical research topic and its impact rippled throughout the 19$^{\text{th}}$ and 20$^{\text{th}}$ centuries. It was one of the first moderately successful modern attempts to provide a unifying framework under which many disparate parts of mathematics could be connected and derived from one another, revealing new connections and insights.

It is a fact that the program was not all-encompassing - for example, the recently developed Riemannian geometry did not easily fit into this framework. But perhaps its most significant and lasting impact on Mathematics lies in its cultural effects, and in particular, in the way it affected how mathematical knowledge was organized and synthesized. It proved the usefulness of having such unifying frameworks, a theme that would become prominent in the 20$^{\text{th}}$ century, and laid the groundwork for the philosophy that would drive the development of category theory in the 1940s.

3.2 The Italian School

Meanwhile, the field of Algebraic Geometry progressed somewhat independently, but it became clear that affine and projective spaces were fundamental to the subject, as were functions (and particularly polynomials) on those spaces. Major work was done in this area in the late 19$^{\text{th}}$ and early 20$^{\text{th}}$ centuries by a group referred to as "The Italian School" in Rome, primarily driven by the work of Severi, Castelnuovo, and Enriques. In order to discuss the significance of their work, however, it is necessary to introduce several more definitions.

Definition 3.1. *Affine n-space* over a field k, denoted $A^n(k)$ is defined as the set of n-tuples $(k_1, k_2, \cdots k_n)$, along with the information of the vector spaces V_k of dimension n over k and the action of the affine group $Aff_n(k)$ on V_k.

In many ways, this is very similar to the vector space k^n defined previously, and in fact, there is a map $k^n \to A^n(k)$ that is colloquially referred to as "forgetting the origin", and a reverse map $A^n(k) \to k^n$ that amounts to choosing a coordinate system. An affine space is chosen generally when one doesn't need the full structure of a vector space, but would still like to consider notions such as points, relative distances, collinearity, and the other isometries preserved by the affine group.

If one then fixes some multivariate polynomial p in the polynomial ring $k[x_1, x_2, \cdots, x_n]$ over n indeterminates with coefficients in k, one can examine points of the form

$$\bar{a} = (a_1, a_2, \cdots a_n)$$

in $A^n(k)$ such that $p(a_1, a_2, \cdots, a_n) = 0$. The collection of such points is referred to as the *zero locus* of such a polynomial, or equivalently an *algebraic hypersurface*.

For example, take $k = \mathbb{R}$ and consider polynomials of degree 2 in $k[x, y]$. In generality, the hypersurface of such a polynomial $p(x, y)$ will be defined by the relation

$$p(x, y) = Ax^2 + By^2 + Cxy + Dx + Ey + F = 0,$$

where the coefficients $A - F$ are taken to be real numbers. But this exactly describes the general equation of a conic section, as studied by the Greeks, and so we find that this new notion of a hypersurface perfectly generalizes algebraic surfaces such as conic sections, but allows for variation in both dimension and base field.

This prompts the following definition:

Definition 3.2. Given a set $S \subseteq k[x_1, x_2, \cdots x_n]$ of the form $S = \{p_1, p_2, \cdots p_m\}$, the *affine algebraic variety of* S is defined as

$$V(S) := \{\bar{a} \mid \bar{a} \in A^n(k),\ p_i(\bar{a}) = 0\ \forall p_i \in S\}.$$

(Note that the set S is sometimes referred to as a *system* of polynomials.)

In other words, one can take a number of polynomials and consider the points where they *simultaneously* vanish, which is equivalent to looking at the intersections of their associated hypersurfaces, and combine all of this information into a structure called a *variety.* It is these objects – roughly speaking, the zero loci of polynomials – that form one of the fundamental structures upon which much Algebraic Geometry is based.

To see why such an object may be interesting, take $k = \mathbb{R}$, the ring $k[x, y]$, and consider the polynomial

$$p_1(x, y) = y^2 - x^3.$$

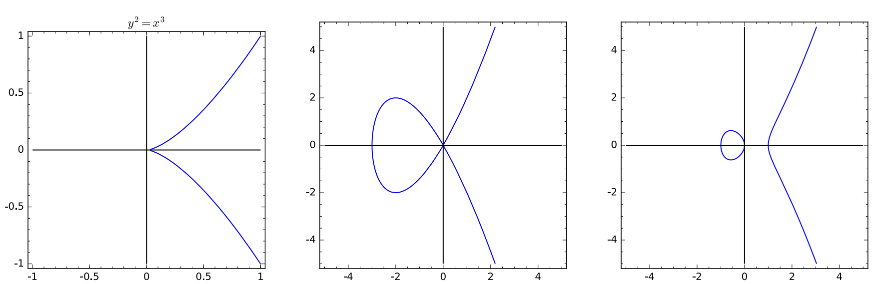
The variety $V(\{p1\})$ traces out an *algebraic curve* in the x-y plane, shown in the first figure below. In contrast, consider also the polynomial

$$p_2(x, y) = y^2 - x^3 + x,$$

then $V(\{p_2\})$ is also an algebraic curve, as shown in the second figure. Finally, consider

$$p_3(x, y) = y^2 - x^3 - 3x^2,$$

shown in the third figure.



(a) $y^2 - x^3 = 0$ (b) $y^2 - x^3 + x = 0$ (c) $y^2 - x^3 - 3x^2 = 0$

Figure 4: Examples of Algebraic Curves

An immediate qualitative difference is that p_1 has a "cusp" near 0, where it may be unclear what the correct assignment of a tangent line should be, while the second is "smooth" but also has a "double point" at which there are two possible choices for a tangent line. The third, on the other hand, does not seem to exhibit any such irregularities, and is in fact an *elliptic curve*, an object with a rich analytic and algebraic structure.

The types of irregularities present in $p1$ and $p2$ are in general referred to as *singularities* of the curve, and so one might be inclined to wonder what causes such singularities to occur, and if there are perhaps conditions on the polynomials themselves that might guarantee that the curves generated contain no such singularities.

Such questions drove many results in the classification of algebraic curves and it was found that the *genus* associated to the curve, an essentially topological property, was a powerful tool in such classifications. It was the extension of these ideas to the classification algebraic *surfaces* in higher dimensions that the Italian school focused their efforts on. To this end, many results were produced, but their results and methods of proof have remained contentious throughout the 20$^{\text{th}}$ century.

3.3 The Weil Conjectures and Grothendieck

Various progress in other areas of geometry was made throughout the first half of the 20$^{\text{th}}$ century, most notably the advances of Hilbert and his proof of the *Nullstellensatz* (roughly translated as "theorem of zero loci"). Recalling that a variety was defined over system S of polynomials, where S is contained in some polynomial ring $k[x_1, x_2, \cdots, x_n]$, one form of the *Nullstellensatz* shows that if one wants to check whether the variety $V(S)$ is nonempty, one can equivalently check whether S is a (proper) ideal in this ring. This firmly cemented the link between Algebra and Geometry, providing a bridge that allowed many of the tools from either one of these fields to be used in the other.

Around 1950, a mathematician named Andre Weil set forth "The Weil Conjectures", three proposals that stem from setting k to be a finite field, considering the resulting varieties, and defining analogs of the Riemann-Zeta function in order to count the number of rational points on the resulting algebraic curves. Weil conjectured that certain properties similar to the traditional Zeta function should hold, particularly with respect to the locations of their zeros.

In doing so, he was able to formulate an analog of the Riemann Hypothesis for these situations, and further conjectured that many of the newly formed tools of homological algebra could be brought to bear on such a problem. Homological tools had proved to be both powerful and successful in topology in the first half of the century, and it was at this point that many researchers were beginning to generalize such theories and apply them to other areas. In particular, Weil suspected that applying homological methods from algebraic topology to these conjectures would have significant number-theoretic ramifications.

It is here that Alexandre Grothendieck enters the picture, as well as his contemporary John-Pierre Serre, for within 15 years Grothendieck had published solutions to two out of three of these conjectures. In the process, he introduced many of the tools that would become standard in modern Algebraic Geometry – his major contributions being in the development of the necessary homological tools, a cohomology theory for varieties. Such contributions were framed within the recently developed language of category theory, and included the definitions of abelian categories, derived functors, injective resolutions, schemes, and sheaves.

Grothendieck's general approach, which has gained traction widely across mathematics in the resulting years, was noticing that certain classes of geometric objects could be characterized by instead considering all of the maps *into* that object from other similar objects. In particular, this applies to algebraic variety – one can associate to a given variety a collection of particularly well-behaved functions on that variety, the so-called "regular" functions, and in this way study the variety itself.

This collection of functions constitutes the simplest example of a *sheaf*, modulo a number of technical conditions. Perhaps more surprisingly, this collection forms a ring. Given any ring R, one can form the set of prime ideals of R, denoted $\mathrm{Spec}(R)$, and even equip with with a topology (what is usually referred to as the *Zariski topology.* If one then introduces a topological space into the picture, there is a construction entitled the *structure sheaf* on the space which contains information about the functions on that space, and taking a topological space along with such a structure sheaf produces what is known as a *scheme*.

Although the construction of scheme itself is quite a bit more detailed and nuanced than what has been described here, its power lies in its similarity to notions in differential geometry, where homological methods had been successfully applied prior to Grothendieck. It is in this way that many powerful tools from the study of manifolds can be generalized to work on algebraic objects, and similarly to allow advances in the algebraic theory of rings to be brought to bear on geometric problems.

However, such constructions proved to not just be limited to making connections between geometry, topology, and algebra, but as Weil had hoped, laid the groundwork for many of these tools to be applied to number theory. This culminated in the celebrated proof of Fermat's Last Theorem by Andrew Wiles in the 1990s, which built upon on many of the tools developed during this period.

4 Conclusion

In summary, perhaps the most notable influence of Algebraic Geometry on the face of mathematics has been the bridges it has created between different fields. In its study, many mathematical "Rosetta Stones" have been built, allowing deeper connections to be built between seemingly disparate fields of mathematics, and moreover making possible the transport of powerful techniques across these domains. In the latter half of the 20^{th} century, Grothendieck left the world of Mathematics, and spent most of his remaining days in solitude, living in the Pyrenees until his death in 2014. And although mathematics is not a lone endeavor, it is particularly hard to overstate the singular contributions of Grothendieck, for his ideas continue to shape how modern mathematics is formulated.

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