Algebraic Topology 2: Smooth Manifolds

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1 Lecture 1

> This course will use Geometry of differential forms by Shigeyuki Morita, another good reference is Lee's Topological Manifolds.

1.1 Overview

The key point of this class will be a discussion of *smooth structures*. As you may recall, a sensational result of Milnor's exhibited exotic spheres with smooth structures – i.e., a differentiable manifold M which is homeomorphic but *not* diffeomorphic to a sphere.

Summary of this result: Look at bundles $S^3 \to X \to S^4$, then one can construct some $X \cong S^7 \in \mathbf{Top}$ but $X \not\cong S^7 \in \mathbf{Diff}^{\infty}$. There are in fact 7 distinct choices for X.

> It is not known if there are exotic smooth structures on S^4. The Smooth Poincare' conjecture is that these do not exist; this is believed to be false.

The other key point of this course is to show that $X \in \mathbf{Diff}^{\infty} \implies X \hookrightarrow \mathbb{R}^n$ for some n, and is in fact a topological subspace.

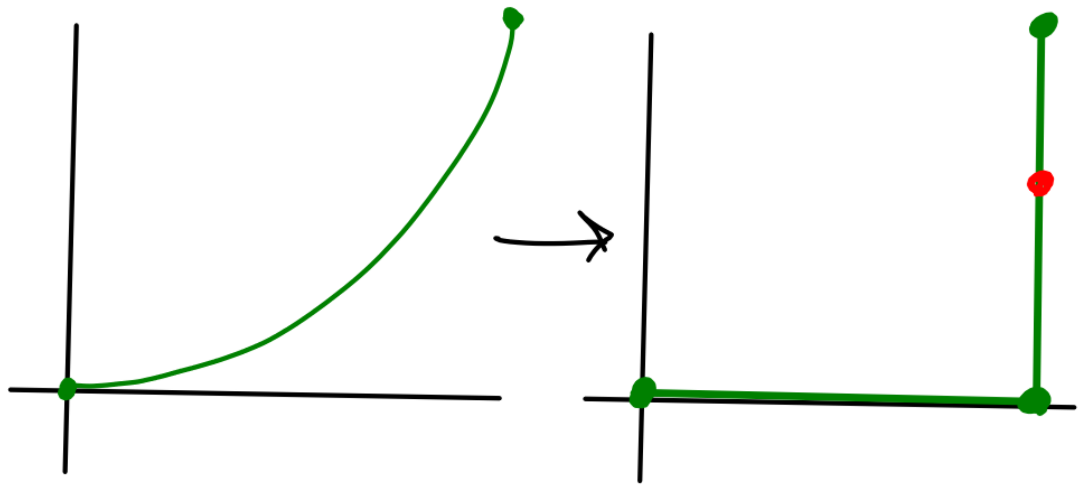
A short list of words/topics we hope to describe:

- Differentiable manifolds
- Local charts
- Submanifolds
- Projective spaces

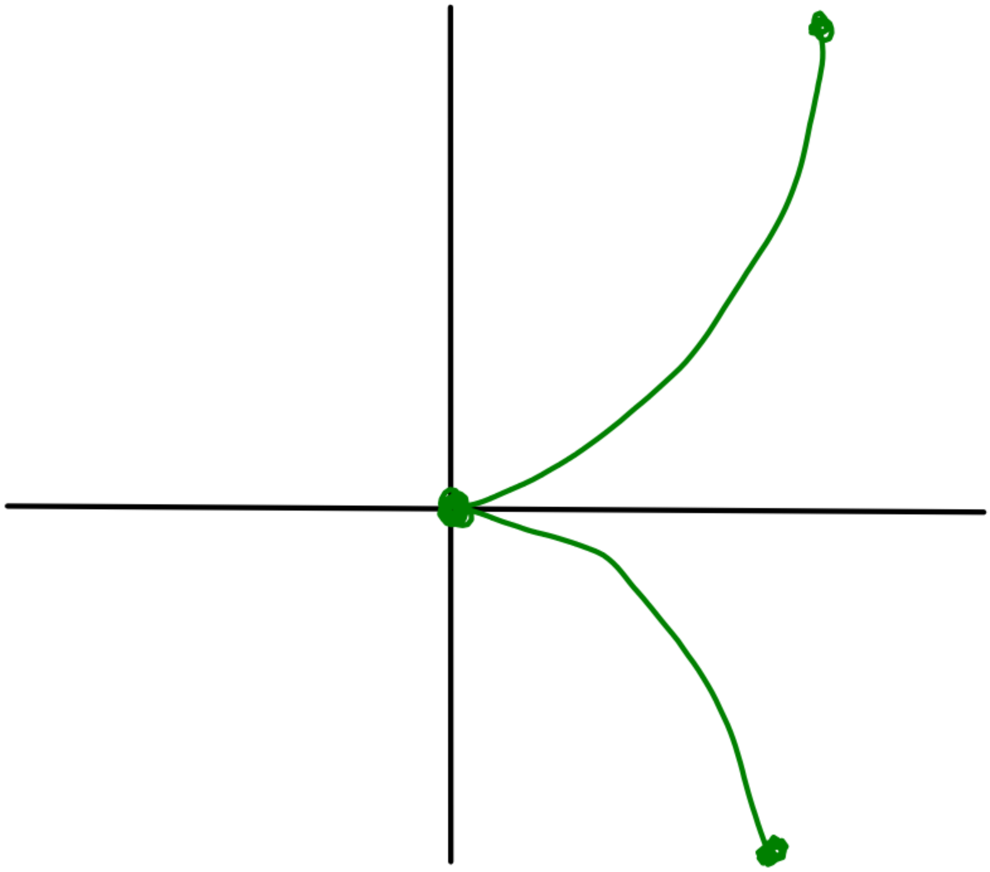
- Lie groups
- Tangent spaces
- Vector fields
- Cotangent spaces
- Differentials of smooth maps G
- Differential forms
- de Rham's theorem

1.2 Motivation

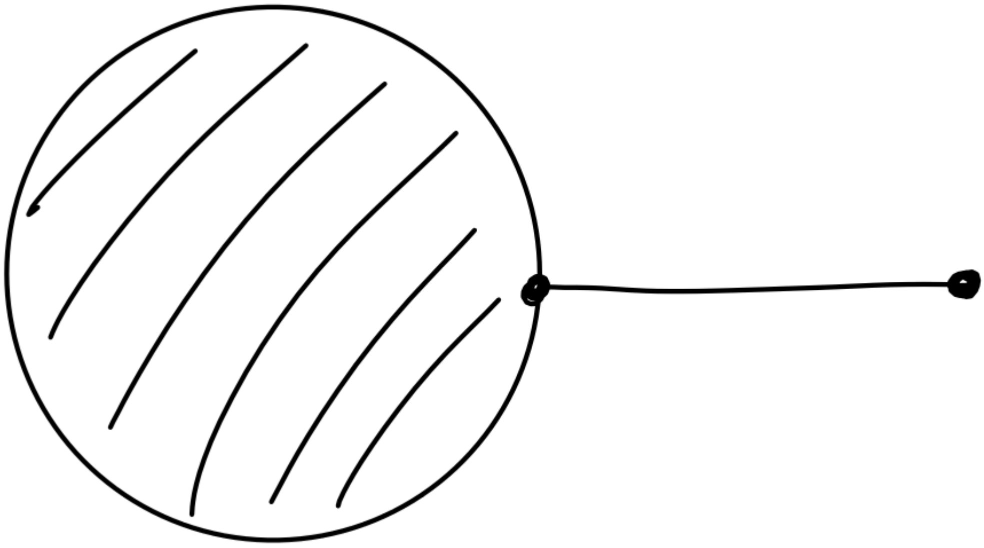
We'd like a notion of "convergence" for, say, curves in $\mathbb{R}^2$. Consider the following examples.



Note the problematic point on the bottom right, as well as the fact that neither of the usual notions of pointwise or uniform convergence will yield a point on the LHS that converges to the red point on the RHS.



Note the problematic point at the origin.



Note the problematic point in the middle, for which all neighborhoods of it are not homeomorphic to either a 2-dimensional nor a 1-dimensional space.

1.3 Defining smooth manifolds

Definition 1. A topological space M is said to be a **topological manifold** when

- M is Hausdorff, so $p \neq q \in M \implies \exists N(p), N(q)$ such that $N(p) \bigcap N(q) = \emptyset$.
- $x \in M \implies$ there exists some $U_x \subseteq M$ and a $\varphi : U_x \to \mathbb{R}^n$ for some n which is a homeomorphism.
- M is 2nd countable

> There are somewhat technical conditions – most of the theory goes through without M being Hausdorff or 2nd countable, but these are needed to construction *partitions of unity* later.

> Also note that these conditions exclude spaces such as the copy of $D_2 \vee I$ from above.

The intuition here is that we'd like spaces that "locally look like $\mathbb{R}^n$", and we introduce the additional structure of smoothness in the following way:

Definition 2. A family of coordinate systems $\{U_\alpha, \varphi_\alpha)\}$ is a **smooth atlas** on M exactly when the change-of-coordinate maps $f_{\alpha,\beta}$ are C^∞.

Exercise 1. Show that S^n is a smooth manifold for every n.

Supposing that $f : M^n \to M^n$ is a map, then locally there is a map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}_n$. Moreover, we can write

$$f(x_1, x_2, \cdots x_n) = [f_1(x_1, x_2, \cdots x_n), f_2(x_1, x_2, \cdots x_n), \cdots f_n(x_1, x_2, \cdots x_n)]$$

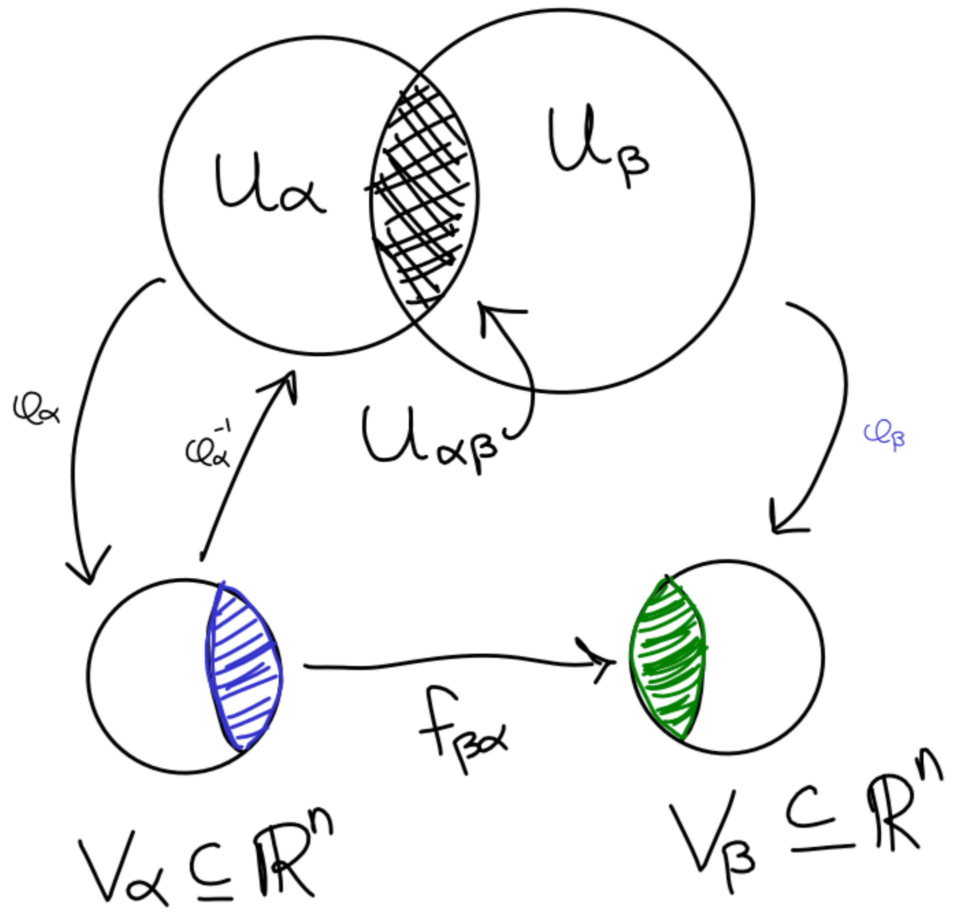


Figure 1: Smooth transition functions

Proposition 1. If M and N are smooth manifolds, then the product $M \times N$ is also a smooth manifolds.

Being Hausdorff and 2nd countable can be checked on the basis elements, and it is indeed true that $\mathcal{B}_1 \times \mathcal{B}_2$ furnishes a basis that satisfies these conditions.

Example 1. The n-fold copy of 1-dimensional sphere is given by

$$(S^1)^n = \prod_n S^1 := \mathbb{T}^n,$$

and is denoted the n-torus.

2 Lecture 2

Recall that last time we gave the definition of a smooth manifolds, discussed examples such as spheres, and saw that this category is closed under products.

Theorem: In $\mathbb{R}^n$, given smooth functions $f_i(x_1, \cdots x_n)$ where $q \leq i \leq n$, the set $Z := \{\mathbf{x} \in \mathbb{R}^n \ni f_i(\mathbf{x}) = \mathbf{0} \quad \forall i\}$, then Z is a smooth manifold if there exist $\{i_1, \cdots, i_m\} \subseteq \{q, \cdots, n\}$ such that the Jacobian $\left(\frac{\partial f_i}{\partial x_{i_j}}\right) \neq 0$.

Without loss of generality, assume $i_j = j$, we can then write this matrix as

$$[\nabla f_1, \nabla f_2, \cdots \nabla f_m]^t,$$

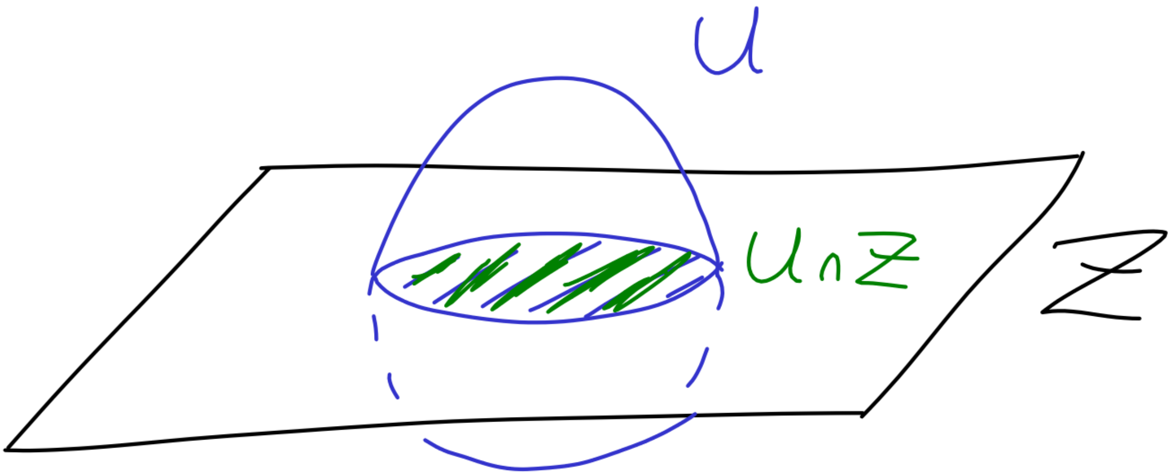
where the submatrix formed by first m columns has a nonzero determinant.

The implicit function theorem: In this situation, there exist a sequence of functions $\{g_k\}_{k=m+1}^n$ such that $g_k(x_1, \cdots x_m) = x_k$ which are smooth.

Compare this to $F(x, y) = 0$ (the two variable case) and $\frac{\partial F}{\partial x} \mid_{x=x_0} \neq 0$, then there exists and f neared near x_0 such that $F(x, f(x)) = 0 \iff y = f(x)$, and now just replace x with $\mathbf{x}$ (add bars everywhere) to get the above theorem.

Say we have $\mathbf{x}^0 = [x_1^0, \cdots, x_n^0]$. such that $\left(\frac{\partial f_i}{\partial x_j^0}\right) \Big|_{\mathbf{x}=\mathbf{x}^0} \neq 0$, then there exists a $U \subset \mathbb{R}^n$ where inside $U \bigcap Z$, all points have the form $(x_1, x_2, \cdots, x_m, g_{m+1}(x_1, \cdots x_m), \cdots g_n(x_1, \cdots x_m))$.

So only the first m variables are free, and the remaining are determined by some functions g_k.



Here U gives a defining region in $\mathbb{R}^m$ for $x_1, \cdots x_m$, and φ_α of this neighborhood satisfies $\varphi_\alpha^{-1}(x_1, \cdots x_m) = (x_1, \cdots, x_m, g_{m+1}, \cdots g_n)$. Now we can look at the transition function $f_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$.

For example,

$$\begin{aligned} \varphi_\alpha(x_1, \cdots x_n) &= (x_1, \cdots x_{m-1}, x_m) \\ \varphi_\beta(x_1, \cdots x_n) &= (x_1, \cdots x_{m-1}, x_{m+1}) \\ \varphi_\alpha^{-1}(x_1, \cdots x_m) &= (x_1, \cdots x_m, g_{m+1}, \cdots g_n) \\ \varphi_\beta^{-1}(x_1, \cdots x_{m-1}, x_{m+1}) &= (x_1, x_{m-1}, h_m, x_{m+1}, h_{m+2}, \cdots h_n) \end{aligned}$$

For $x \in U_\alpha \bigcap U_\beta$, we have $\varphi_\beta \circ \varphi_\alpha^{-1}(x_1, \cdots x_m) = (x_1, \cdots x_{m-1}, g_{m+1})$.

Example: the sphere revisited. Take $F(x_1, \cdots x_n) = 0$, where $F(x_1, \cdots x_m) = -1 + \sum x_i^2$. For any $(x_1, \cdots x_m) \in S^{m-1}$, at least one $x_i \neq 0$, wlog let this be x_1. Then $\left(\frac{\partial F}{\partial x_1}\right)\Big|_{\mathbf{x}} = \partial x_1 \neq 0$.

Example: the torus. We have $\mathbb{T}^n \subset \mathbb{R}^{2n}$, where $\mathbb{T}^n = \prod_{i=1}^n S^1$. Write a point in $\mathbb{R}^{2n}$ as $(x_1, y_1, \cdots x_n, y_n)$, then $\mathbb{T}^n = \{(\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^2)^n \ni x_i^2 + y_i^2 = 1\}$.

Remark (Choice of atlas): If $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas for M, then

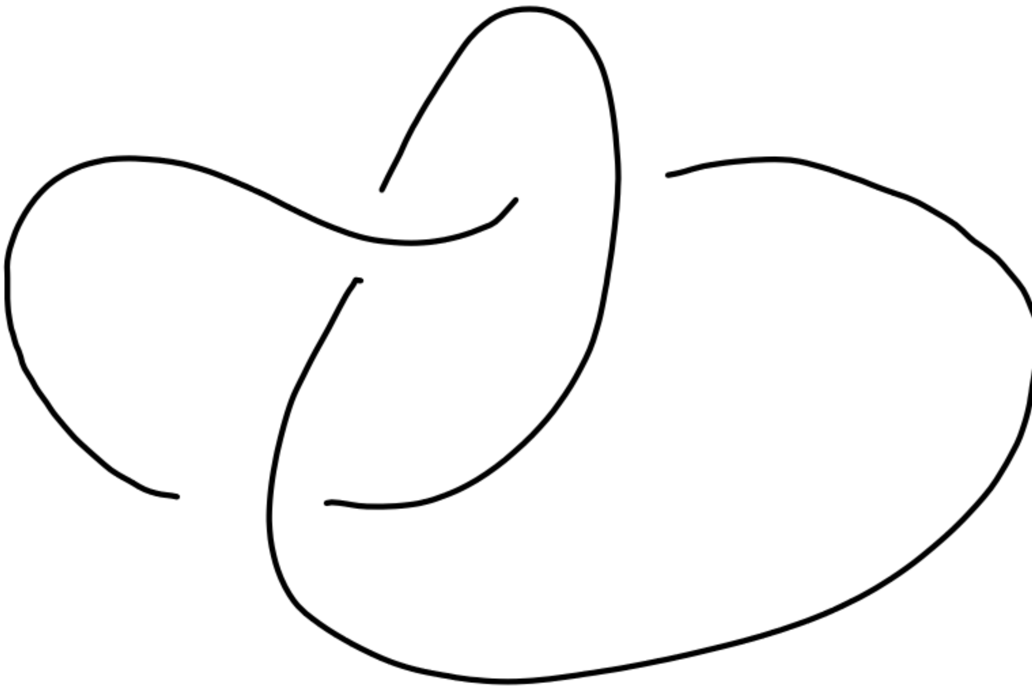
- Changing $\varphi_\alpha \to \varepsilon_\alpha \circ \varphi_\alpha$ where $\varepsilon_\alpha : \varphi_\alpha(U_\alpha) \circlearrowleft$ is a diffeomorphism, then the atlas $\{(U_\alpha, \varepsilon_\alpha \circ \varphi_\alpha)\}$ does not a priori yield the same smooth manifold; we will declare them to be the same though.
- If $\{(U_\beta, \varphi_\beta)\}$ is another atlas of M such that the refinement $\{(U_\alpha, \varphi_\alpha)\} \bigcup \{(U_\beta, \varphi_\beta)\}$ is again an atlas of M, then they define the same smooth manifold M.

2.1 Submanifolds

If $U \subseteq M$ and M is a smooth manifold, then U also has the structure of a smooth manifold. This is obtained by taking an atlas of M and intersecting each U_α with U, and then restricting φ_α to $\varphi_\alpha|_U$.

Examples:

- $\mathrm{GL}(n,\mathbb{R}) \subseteq \mathrm{Mat}(n,\mathbb{R}) = \{X \ni \det X \neq 0\}$. Note that the $\det X = 0$ is a closed subset, so its complement is open.



- Knot complements

Definition: $N^k \subseteq M^n$ is a submanifold if $\forall p \in N, \exists U_\alpha \subseteq M$ with $p \in U_\alpha$ such that $N \bigcap U_\alpha = \{\mathbf{q} \in U \ni x_{k+1}(\mathbf{q}) = \cdots = x_n(\mathbf{q}) = 0\}$ where x_i are the coordinate functions. (Note that we abuse notation here, and we are applying φ_α to everything.)

3 Lecture 3

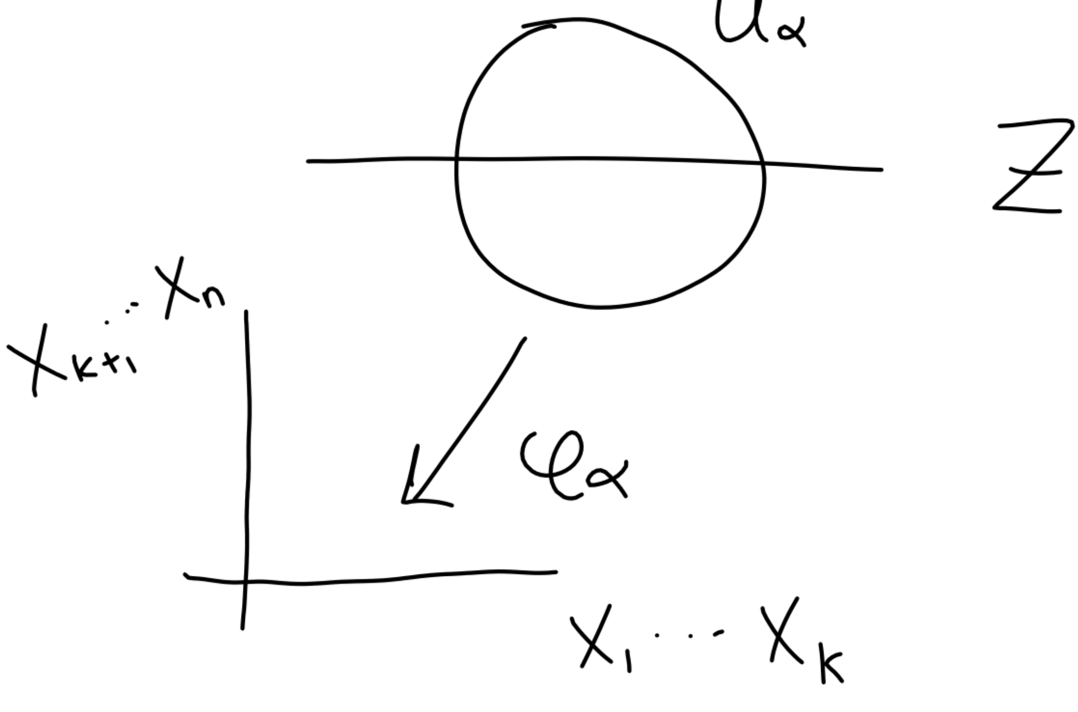


Figure 2: The coordinate chart situation.