Proof of Leray-Hirsch Theorem

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1 Preliminaries

Definition: Fibre Bundle

Definition: Homology

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Definition: Good Cover

Notation: Let R be an arbitrary ring, and let h denote the contravariant functor

$$\begin{aligned} h(\,\cdot\,; R) &: \textbf{Top} \to \textbf{Ring} \\ X &\mapsto H^*_{\text{sing}}(X; R) \\ (X \xrightarrow{f} Y) &\mapsto (H^*(Y; R) \xrightarrow{f^*} H^*(X; R)) \end{aligned}$$

which corresponds to taking singular cohomology.

> Note that $H^*_{\text{sing}}(X; R)$ is a graded ring with multiplicative structure given by the cup product, and similarly $H^*_{\text{dR}}(X; \mathbb{R})$ is graded ring with multiplication induced by the wedge product of forms.

2 Statement of the Theorem

Let

$$\begin{array}{ccc} F & \xhookrightarrow{i} & E \\ & & \Big\downarrow p \\ & & B \end{array}$$

be a fibre bundle. Taking cohomology induces maps

$$\begin{array}{ccc} h(F;R) & \xleftarrow{i^*} & h(E;R) \\ & & \Big\uparrow p^* \\ & & h(B;R) \end{array}$$

Suppose that

1. $h(F;R)$ is a finitely-generated free R-module in each degree n, and
2. For every fiber F, there exists a collection of chains

   $$C_F := \left\{c_j \;\middle|\; j \in J\right\} \subseteq h(E;R) \quad \text{for some index set} J$$

   such that their restrictions $\left\{i^*(c_j) \;\middle|\; j \in J\right\} \subseteq h(F;R)$ along i^* yield an R-basis for $h(F;R)$, i.e.

   $$h(F;R) = \operatorname{span}_R \left(\left\{i^*(c_j) \;\middle|\; j \in J\right\}\right)$$

   as an R-module.

We can then define the following group action:

$$\begin{aligned} h(B;R) &\curvearrowright h(E;R) \\ b \curvearrowright e &:= p^*(b) \smile e, \end{aligned}$$

and as a result we have

1. Both $h(E;R)$ and $h(F;R) \otimes_R h(B;R)$ are modules over the ring $h(B;R)$,
2. Letting $\left\{b_k \;\middle|\; k \in K\right\} \subseteq h(B;R)$ denote a (not necessarily finite) set of generators,

   $$h(B;R) \otimes_R h(F;R) = \operatorname{span}_R \left\{b_i \otimes i^*(c_j) \;\middle|\; k \in K, j \in J\right\}$$

3. The following map is an isomorphism in the category of $h(B;R)$-modules:

   $$\begin{aligned} \varphi : h(B;R) \otimes_R h(F;R) &\to \quad h(E;R) \\ \sum_{k \in K, j \in J} b_k \otimes_R i^*(c_j) &\mapsto \sum_{k \in K, j \in J} p^*(b_k) \smile c_j \end{aligned}$$

4. As an $h(B;R)$ module,

$$h_E(R) = \mathrm{span}_{h(B;R)}\left(\left\{c_j \;\middle|\; j \in J\right\}\right)$$

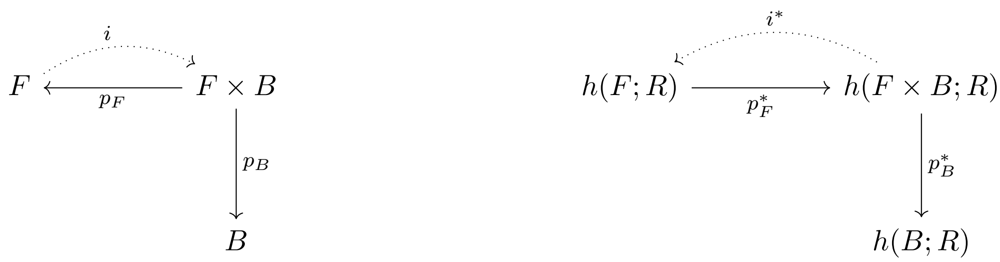
so the cohomology of the total space is given by $h(B;R)$ span of these c_j.

Note: this map is not an isomorphism in the category of rings.

Remark:

The assumption that each C_F exists is necessary, and can be guaranteed when $E \cong B \times F$ is homeomorphic to a product.

Letting $p_B : B \times F \to B$ and $p_F : B \times F \to F$ be the projections supplied by the universal property of the product, since $h(F;R)$ is a free R-module, we can take its R-basis $\left\{f_k \;\middle|\; j \in K\right\} \subseteq h(F;R)$ and pull it back along p_F to obtain $\{p_F^*(f_k)\} \subseteq h(F \times B; R)$.



Then we can note that for a product, $p_F \circ i = \mathrm{id}_{F\times B}$, and so $i^* \circ p_F^* = \mathrm{id}_F$. Thus defining $c_k := p_F^*(f_k)$ satisfies condition (2), i.e. $i^*(c_k) := (i^* p_F^*)(f_k) = f_k$ is an R-basis for $h(F;R)$.

Corollary: Let $h(X) = H^*_{dR}(X;\mathbb{R})$ denote taking deRham cohomology.

If

1. $F \to E \to M$ is a fibre bundle of smooth manifolds where M has a good cover,
2. There exist a set of global forms $\{\omega_j\} \subseteq E$ such that $h(E) = \mathrm{span}_{\mathbb{R}}\left\{\omega_j \;\middle|\; j \in J\right\}$, and
3. For each fiber F, the collection of restrictions $\left\{\omega_j|_F \;\middle|\; j \in J\right\}$ freely generated $h(F)$,

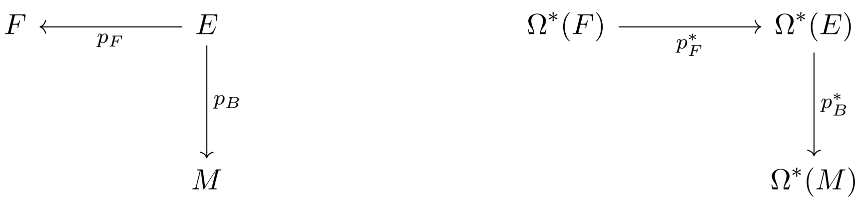
then $h(E)$ is a free $h(M)$-module and

$$h(E) = \mathrm{span}_{\mathbb{R}}\left\{\omega_j \;\middle|\; j \in J\right\} \otimes h(M) \cong h(F) \otimes h(M).$$

3 Proof

We'll prove the special case given in the corollary.

Given a fibre bundle $F \to E \to M$, there are projections p_F and p_M which induce maps in the deRham complex,



This allows us to define a map on forms:

$$\begin{aligned} \phi : \Omega^*(M) \otimes_{\mathbb{R}} \Omega^*(F) &\longrightarrow \Omega^*(E) \\ \omega \otimes_{\mathbb{R}} \phi \quad &\mapsto \quad p_B^*(\omega) \wedge p_F^*(\phi), \end{aligned}$$

which amounts to pulling the forms back and then wedging them.

Claim: ϕ induces a map on the deRham cohomology h^*.

By the claim, we obtain a map

$$\begin{aligned} \tilde{\phi} : h(M) \otimes_{\mathbb{R}} h(F) &\longrightarrow h(E) \\ [\omega] \otimes_{\mathbb{R}} [\phi] \quad &\mapsto \quad [p_B^*(\omega) \wedge p_F^*(\phi)]. \end{aligned}$$

Claim: ϕ is an isomorphism of graded modules (???).

Proof:

We will induct on the cardinality of the good cover $\mathcal{U}$ of M. For the base case, suppose $\#\mathcal{U} = 2$, so $M = U \bigcup V$.

Noting that here $h^*(X) \coloneqq H^*_{\mathrm{dR}}(X; \mathbb{R})$ is a graded ring, we can identify

$$h(M) \otimes_{\mathbb{R}} h(F) = \bigoplus_{i+j=n} h^i(M) \otimes_{\mathbb{R}} h^j(F)$$

and so the kth graded piece is given by

$$(h(M) \otimes_{\mathbb{R}} h(F))^k = \bigoplus_{i+j=k} h^i(M) \otimes_{\mathbb{R}} h^j(F).$$

Similarly, $h(E) = \bigoplus_{i=0}^{n} h^i(E)$, and thus ϕ is an isomorphism iff it is an isomorphism between kth graded pieces for every k. So it suffices to show that the maps

$$\tilde{\phi}_k : \bigoplus_{i+j=k} h^i(M) \otimes_{\mathbb{R}} h^j(F) \to h^k(E)$$

.

are isomorphisms for every $0 \leq k \leq n$.

To this end, fix an arbitrary $0 \leq k \leq n$ and consider the following diagram:

Claim:

$$
\begin{array}{ccccccccc}
0 \leftarrow & \bigoplus_{j=0}^{k+1} h^j(U \cap V) \otimes_R h^{n-k}(F) & \leftarrow & \bigoplus_{j=0}^{k} h^j(U \cup V) \otimes_R h^{n-k}(F) & \leftarrow & \bigoplus_{j=0}^{k} (h^j(U) \otimes h^{n-k}(F)) \oplus (h^j(V) \otimes h^{n-k}(F)) & \leftarrow & \bigoplus_{j=0}^{k} h^j(U \cap V) \otimes_R h^{n-k}(F) & \leftarrow 0 \\
& \downarrow \tilde{\phi}_{k+1} & & \downarrow \tilde{\phi}_k & & \downarrow \tilde{\phi}_k \oplus \tilde{\phi}_k & & \downarrow \tilde{\phi}_k & \\
0 \longleftarrow & h^{k+1}(U \cap V \times F) & \longleftarrow & h^k(E) & \longleftarrow & h^k(U \times F) \bigoplus h^k(V \times F) & \longleftarrow & h^k(U \cap V \times F) & \longleftarrow 0
\end{array}
$$

Figure 1: Main Diagram

- The rows in this diagram are exact,
- The diagram commutes, and
- The 1st, 3rd, and 4th maps are isomorphisms.

If this is the case, the 5 lemma can be applied and this will imply that the 2nd map

$$\tilde{\phi}_k : \bigoplus_{j=0}^{k} h^j(U \cup V) \otimes_R h^{n-k}(F) \longrightarrow h^k(E)$$

is an isomorphism. Since k was arbitrary, $\tilde{\phi}_k$ will be an isomorphism for every k, which is precisely what we want to show.

3.1 The Top Row is Exact

By Mayer-Vietoris, there is a long exact sequence:

$$
\begin{array}{ccccccccc}
0 & \longleftarrow & h^n(U \cup V) & \longleftarrow & h^n(U) \oplus h^n(V) & \longleftarrow & h^n(U \cap V) & & \\
& & & \overset{\delta}{\nearrow} & & & & & \\
& & h^{n-1}(U \cup V) & \longleftarrow & h^{n-1}(U) \oplus h^{n-1}(V) & \longleftarrow & h^{n-1}(U \cap V) & & \\
& & & \overset{\delta}{\nearrow} & & & & & \\
& & h^k(U \cup V) & \longleftarrow & h^k(U) \oplus h^k(V) & \longleftarrow & h^k(U \cap V) & & \\
& & & \overset{\delta}{\nearrow} & & & & & \\
& & h^0(U \cup V) & \longleftarrow & h^0(U) \oplus h^0(V) & \longleftarrow & h^0(U \cap V) & \longleftarrow & 0
\end{array}
$$

where δ is the connecting map supplied by the Snake Lemma.

Since $h^*(F)$ was assumed to be a free R-module, the functor $(\,\cdot\,) \otimes_R h^j(F)$ is exact for any j.

Fixing j for the moment, we note that applying $(\cdot) \otimes h^j(F)$ to the above sequence yields a new long exact sequence:

$$
\begin{array}{c}
0 \longleftarrow h^n(U \cup V) \otimes_R h^j(F) \longleftarrow h^n(U) \otimes_R h^j(F) \oplus h^n(V) \otimes_R h^j(F) \longleftarrow h^n(U \cap V) \otimes_R h^j(F) \\
\nearrow_{\delta \otimes \mathrm{id}_{h^j(F)}} \\
h^{n-1}(U \cup V) \otimes_R h^j(F) \longleftarrow h^{n-1}(U) \otimes_R h^j(F) \oplus h^{n-1}(V) \otimes_R h^j(F) \longleftarrow h^{n-1}(U \cap V) \otimes_R h^j(F) \\
\nearrow_{\delta \otimes \mathrm{id}_{h^j(F)}} \\
h^k(U \cup V) \otimes_R h^j(F) \longleftarrow h^k(U) \otimes_R h^j(F) \oplus h^k(V) \otimes_R h^j(F) \longleftarrow h^k(U \cap V) \otimes_R h^j(F) \\
\nearrow_{\delta \otimes \mathrm{id}_{h^j(F)}} \\
h^0(U \cup V) \otimes_R h^j(F) \longleftarrow h^0(U) \otimes_R h^j(F) \oplus h^0(V) \otimes_R h^j(F) \longleftarrow h^0(U \cap V) \otimes_R h^j(F) \longleftarrow 0
\end{array}
$$

where the tensor product has been distributed across the direct sums in the middle column.

Consider doing this for every j; we then obtain a collection of long exact sequences

$$
\begin{array}{ll}
j = k+1 & 0 \longrightarrow h^0(U \bigcup V) \otimes h^{k+1}(F) \to \cdots \\
j = k & 0 \longrightarrow h^0(U \bigcup V) \otimes h^k(F) \longrightarrow h^0(U) \otimes h^k(F) \oplus h^0(V) \otimes h^k(F) \longrightarrow h^0(U \bigcap V) \otimes h^k(F) \xrightarrow{\delta} h^1(U \bigcup V) \otimes h^k(F) \longrightarrow \cdots \\
j = k-1: & \cdots \xrightarrow{\delta} h^1(U \bigcup V) \otimes h^{k-1}(F) \to h^1(U) \otimes h^{k-1}(F) \oplus h^1(V) \otimes h^{k-1}(F) \to h^1(U \bigcap V) \otimes h^{k-1}(F) \xrightarrow{\delta} h^2(U \bigcup V) \otimes h^{k-1}(F) \to \cdots \\
j = k-2: & \cdots \xrightarrow{\delta} h^2(U \bigcup V) \otimes h^{k-2}(F) \to h^2(U) \otimes h^{k-2}(F) \oplus h^2(V) \otimes h^{k-2}(F) \to h^2(U \bigcap V) \otimes h^{k-2}(F) \xrightarrow{\delta} h^2(U \bigcup V) \otimes h^{k-2}(F) \to \cdots \\
\vdots & \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
j = 0: & \cdots \xrightarrow{\delta} h^k(U \bigcup V) \otimes h^0(F) \longrightarrow h^k(U) \otimes h^0(F) \oplus h^k(V) \otimes h^0(F) \longrightarrow h^k(U \bigcap V) \otimes h^0(F) \xrightarrow{\delta} h^{k+1}(U \bigcup V) \otimes h^0(F) \to \cdots
\end{array}
$$

Then summing along the columns will preserve exactness in each each degree. Moreover, taking the direct sum down the first, second, third, and fourth columns respectively yields

$$\text{Column 1}: C_1 := \bigoplus_{j=0}^{k} h^j(U \bigcup V) \otimes h^{k-j}(F)$$

$$\text{Column 2}: C_2 := \bigoplus_{j=0}^{k} h^j(U) \otimes h^{k-j}(F) \oplus h^j(V) \otimes h^{k-j}(F)$$

$$\text{Column 3}: C_3 := \bigoplus_{j=0}^{k} h^j(U \bigcap V) \otimes h^{k-j}(F)$$

$$\text{Column 4}: C_4 := \bigoplus_{j=0}^{k+1} h^j(U \bigcup V) \otimes h^{k-j}(F)$$

,

and the exactness of the sequence $C_1 \to C_2 \to C_3 \to C_4$ is precisely the exactness of the top row in figure (1). $$ ## The Bottom Row is Exact