Differentiable Manifolds, Due date: 11/11/2019

1. Let $f : \mathbb{C} \to \mathbb{C}$ be the complex function $f(z) = z + c$ for some constant $c \in \mathbb{C}$. Compactify f to $\widetilde{f} : \mathbb{CP}^1 \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, so that $\widetilde{f}(z) = f(z)$ when $z \in \mathbb{C}$ and $\widetilde{f}(\infty) = \infty$. Is $\widetilde{f}$ a diffeomorphism? Give your reasons.

2. Given two charts $(U; x_1, \ldots x_n)$ and $(V; y_1, \ldots, y_n)$. Show that the change of coordinates induces the following equality for local expressions of tangent vectors.

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i}(p) \frac{\partial}{\partial y_j}.$$

3. Given a smooth map $f : M \to N$. For any $v \in T_pM$ and $h \in C^\infty(N)$, show that the assignment $h \mapsto v(h \circ f)$ defines a tangent vector. Prove this fact using both definitions given in the class as follows.

   (i) Show that this assignment yields a derivation.

   (ii) Given a smooth curve $c(t) \subset M$ such that $c(0) = p$. Show that this tangent vector is the equivalence class given by $f(c(t))$.

   Derive the following local expression sketched in the class. Let $p \in (U; x_1, \ldots, x_m)$ and $f(p) \in (V; y_1, \ldots, y_n)$

$$df(\frac{\partial}{\partial x_i}) = \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i} \cdot \frac{\partial}{\partial y_j}$$

   Given a heuristic explanation why this expression mimics the formula in Problem 2 closely.

4. (i) Let $V = \mathbb{R}^n$. Given a non-degenerate matrix $g = (g_{ij})_{n \times n}$ (meaning $det(g_{ij}) \neq 0$), show that g yields an isomorphism $V \xrightarrow{\sim} V^*$ by sending $v \mapsto \langle v, g(-) \rangle$, where $\langle -, - \rangle$ is the standard inner product.

   (ii) Given a metric $g(-,-)$ on M, show that it induces an isomorphism of vector bundles $g : TM \to T^*M$.

5. (i) Show that the sum of a positive definite matrix and a non-negative definite matrix ($v^T g v \geq 0$) is positive definite.

   (ii) Use partition of unity to show that every smooth manifold M has a metric.

6. (i) Prove that every vector field on S^2 has at least one zero. (Hint: consider the time-1 flow of this vector field, and apply Lefschetz fixed point theorem. Show that the fixed point must come from a zero of the vector field.)

   (ii) Prove that the tangent bundle TS^2 is not isomorphic to a trivial bundle $S^2 \times \mathbb{R}^2$.