Problem Set 1

D. Zack Garza

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1 Problem 1

We'll use the following definition of a smooth map between manifolds:

> Definition: Let M, N be smooth manifolds of dimensions m, n respectively and $f : M \to N$ a continuous map. Then f is *smooth* iff for every $p \in M$, there exists a chart (U, ϕ) with $p \in U$ and a chart (V, ψ) with $f(p) \in V$ such that $f(U) \subseteq V$, and the induced map
>
> $$\begin{aligned}\overline{f} &: \phi(U) \to \psi(V) \\ \overline{f} &= \psi \circ f \circ \phi^{-1}\end{aligned}$$
>
> is smooth as a map from $\mathbb{R}^m \to \mathbb{R}^n$.

We will thus show that both $\tilde{f} : \mathbb{CP}^1 \to \mathbb{CP}^1$ and $\tilde{f}^{-1} : \mathbb{CP}^1 \to \mathbb{CP}^1$ are both smooth bijections, from which we can conclude that f is a diffeomorphism.

So identify $0 = [0,1]$ and $\infty = [1,0]$ in $\mathbb{CP}^1$ and choose the following charts on $\mathbb{CP}^1$ in terms of homogeneous coordinates:

$$\begin{aligned}(U,\phi) &:= \\ U = \mathbb{CP}^1 \setminus \{\infty\} = \{[x,y] \mid x,y \in \mathbb{C},\ y \neq 0\} \\ \phi : \mathbb{CP}^1 &\to \mathbb{C} \\ [x,y] &\mapsto x/y. \\ \\ \phi^{-1} : \mathbb{C} &\to \mathbb{CP}^1 \\ z &\mapsto [z,1].\end{aligned}$$

and

$$\begin{aligned}(V,\psi) &:= \\ V = \mathbb{CP}^1 \setminus \{0\} = \{[x,y] \mid x,y \in \mathbb{C},\ x \neq 0\} \\ \psi : \mathbb{CP}^1 &\to \mathbb{C} \\ [x,y] &\mapsto y/x. \\ \\ \psi^{-1} : \mathbb{C} &\to \mathbb{CP}^1 \\ z &\mapsto [1,z].\end{aligned}$$

Now define

$$\begin{aligned}\tilde{f} : \mathbb{CP}^1 &\to \mathbb{CP}^1 \\ p &\mapsto \begin{cases} p, & p = \infty \\ p + c & \text{otherwise} \end{cases}\end{aligned}$$

We then need to determine a formula for $\tilde{f}$ in homogeneous coordinates. We compute

$$\begin{aligned}p \in U \implies p = [a,b],\ a,b, \in \mathbb{C},\ b \neq 0 \\ \implies \tilde{f}([a,b])\Big|_U &= (\phi^{-1} \circ f \circ \phi)([a,b]) \\ &= (\phi^{-1} \circ f)(\frac{a}{b}) \\ &= \phi^{-1}(\frac{a}{b} + c) \\ &= [\frac{a}{b} + c, 1] \\ &= [a + bc, b]\end{aligned}$$

and

$$
\begin{aligned}
p \in V \implies p &= [a,b],\ a,b, \in \mathbb{C},\ a \neq 0 \\
\implies \tilde{f}([a,b])\Big|_V &= (\psi^{-1} \circ f \circ \psi)([a,b]) \\
&= (\psi^{-1} \circ f)(\frac{b}{a}) \\
&= (\psi^{-1})(\frac{b}{a} + c) \\
&= [1, \frac{b}{a} + c] \\
&= [a, b + ac]
\end{aligned}
$$

Since $\mathbb{CP}^1 = U \bigcup V$, we can note that if $p \in M$ then either $p \in U$ or $p \in V$. Moreover, $p \in U \implies \tilde{f}(p) \in \tilde{f}(U) = U$, since $p = [a,b]$ with $b \neq 0 \implies f(p) = [a + bc, b]$ where $b \neq 0$ as well, so $\tilde{f}(p) \in U$ and $\tilde{f}(U) \subseteq U$. Similarly, $\tilde{f}(V) \subseteq V$. So it only remains to check that the following two compositions are smooth:

- $f_U : \mathbb{C} \to \mathbb{C}, f_U \coloneqq \phi \circ \tilde{f} \circ \phi^{-1}$, and
- $f_V : \mathbb{C} \to \mathbb{C}, f_V \coloneqq \psi \circ \tilde{f} \circ \psi^{-1}$.

We can compute

$$
\begin{aligned}
f_U(z) &\coloneqq (\phi \circ \tilde{f} \circ \phi^{-1})(z) \\
&= (\phi \circ \tilde{f})([z,1]) \\
&= \phi([z+c,1]) \\
&= z + c
\end{aligned}
$$

$$
\begin{aligned}
f_V(z) &\coloneqq (\phi \circ \tilde{f} \circ \phi^{-1})(z) \\
&= (\phi \circ \tilde{f})([1,z]) \\
&= \phi([1,z+c]) \\
&= z + c
\end{aligned}
$$

And $\frac{\partial}{\partial z} f_U(z) = \frac{\partial}{\partial z} f_V(z) = 1$, so these are smooth maps on their domains.

To Summarize: Let $p \in M$ be arbitrary. The map $\tilde{f}$ will be smooth iff there are charts $(U_\alpha, \varphi_\alpha : U_\alpha \to \mathbb{C}), (U_\beta, \varphi_\beta : U_\beta \to \mathbb{C})$ with $U_\alpha, \beta \subseteq \mathbb{CP}^1$ such that

- $p \in U_\alpha$,
- $\tilde{f}(p) \in U_\beta$,
- $\tilde{f}(U_\alpha) \subseteq U_\beta$
- $\varphi_\beta \circ \tilde{f} \circ \varphi_\alpha^{-1}$ is smooth.

By cases,

- If $p \neq \infty$, then choose $U_\alpha = U_\beta = U$ and $\varphi_\alpha = \varphi_\beta = \phi$. Then $\tilde{f}(p) \neq \infty$, so $\tilde{f}(U) \subseteq U$, and the composition $\phi \circ \tilde{f} \circ \phi^{-1}(z) = z + c$ is smooth.

- If $p = \infty$, then choose $U_\alpha = U_\beta = V$ and $\varphi_\alpha = \varphi_\beta = \psi$. Then $\tilde{f}(p) \neq 0$, so $\tilde{f}(V) \subseteq V$, and the composition $\psi \circ \tilde{f} \circ \psi^{-1}(z) = z + c$ is again smooth.

Note: I'm almost certain this argument is *not* correct, but I do not know why.

2 Problem 2

Following the example in Lee's Smooth Manifolds (pp. 63), we want to show the following:

$$d\left(\psi \circ \varphi^{-1}\right)_{\varphi(p)} \left(\frac{\partial}{\partial x^i} \bigg|_{\varphi(p)} \right) = \frac{\partial y^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial y^j}\bigg|_{\psi(p)}$$

where

- (U, ϕ) and (V, ψ) are charts containing p,
- $\psi \circ \phi^{-1} : \phi(U) \to \psi(V)$ is the corresponding change of coordinates,
- $\left\{ \frac{\partial}{\partial x_i} \right\}_{i=1}^n$ are the vectors spanning T_pM,
- $\left\{ \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right\}_{i=1}^n$ are the basis vectors spanning $T\varphi(U \bigcap V) \cong T\mathbb{R}^n$ at the point $\varphi(p) \in \varphi(U \bigcap V)$, and
- $\left\{ \frac{\partial}{\partial y^j} \Big|_{\psi(p)} \right\}_{i=1}^n$ are the basis vector spanning $T\psi(U \bigcap V)$ at the point $\psi(p) \in \psi(U \bigcap V)$.

Define

$$\begin{aligned} F &: \varphi(U) \to \psi(V) \\ F(v) &= (\psi \circ \phi^{-1})(v) \end{aligned}$$

which, at the point $\varphi(p)$, induces a map

$$\begin{aligned} dF_p &: T_{\varphi(p)}\mathbb{R}^n \to T_{\psi(p)}\mathbb{R}^n \\ dF(v) &= d(\psi \circ \varphi^{-1})(v) \end{aligned}$$

since $F(\varphi(p)) = \psi(p)$.

Identifying elements in the tangent space as derivations, we first note that given any $F : M_1 \to M_2$, at a point $p \in M_1$ we define

$$\begin{aligned} dF_p &: T_pM_1 \to T_{F(p)}M_2 \\ dF_p(v) \curvearrowright (f : M_2 \to \mathbb{R}) &\coloneqq v(f \circ F : M_1 \to \mathbb{R}) \end{aligned} \tag{1}$$

which is well-defined because $v \in T_pM_1$ means that $v : C^\infty(M_1) \to \mathbb{R}$ is a derivation, and $f \circ F \in C^\infty(M_1)$, so it makes sense to evaluate v on this composition.

We can then compute a formula for F in coordinates by computing its action on smooth functions $f : M \to \mathbb{R}$ where $f \in C^\infty(M)$:

$$\begin{aligned}
dF_p\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\right) \curvearrowright f &:= \frac{\partial}{\partial x_i}(f \circ F) && \text{by equation (1)} \\
&= \sum_k \frac{\partial f}{\partial y_k}(F(p)) \frac{\partial F_k}{\partial x_i}(p) && \text{by the chain rule} \\
&= \sum_k \frac{\partial F_k}{\partial x_i}(p) \frac{\partial f}{\partial y_k}(F(p)) \\
&:= \left(\sum_k \frac{\partial F_k}{\partial x_i}(p) \frac{\partial}{\partial y_k}\Big|_{F(p)}\right) \curvearrowright f
\end{aligned}$$

But then we can write

$$\begin{aligned}
\frac{\partial}{\partial x_i}\Big|_p &:= d(\varphi^{-1})\Big|_{\varphi(p)} \left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\right) \\
&= d(\mathrm{id} \circ \varphi^{-1})\Big|_{\varphi(p)} \left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\right) \\
&= d((\psi^{-1} \circ \psi) \circ \varphi^{-1})\Big|_{\varphi(p)} \left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\right) \\
&= d(\psi^{-1})\Big|_{\psi(p)} \circ d(\psi \circ \varphi^{-1})\Big|_{\varphi(p)} \left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\right) && \text{by Lee Proposition 3.6b} \\
&= d(\psi^{-1}) \left(\sum_k \frac{\partial F_k}{\partial x_i}(p) \frac{\partial}{\partial y_k}\Big|_{F(p)}\right) && \text{by previous computation} \\
&:= \sum_k \frac{\partial F_k}{\partial x_i}(p) \frac{\partial}{\partial y_k}\Big|_p
\end{aligned}$$

which is what we wanted to show. $\square$

3 Problem 3

> Note: Throughout this question, we will identify $\{f : C^\infty(M) \to \mathbb{R}\} \cong C^\infty(M)^\vee$ as vector spaces.

Let M, N be smooth manifolds and $f : M \to N$ be a fixed smooth map, and define a map

$$\begin{aligned}
\phi : C^\infty(N) \times TM &\to \mathbb{R} \\
(h, v) &\mapsto v(h \circ f)
\end{aligned}$$

3.1 Part 1

Using the derivation definition, we can identify this assignment as a map

$$\begin{aligned}
\phi : C^\infty(N) \times C^\infty(M)^\vee &\to \mathbb{R} \\
(h, v) &\mapsto v(h \circ f)
\end{aligned}$$

We'd like to show that this yields a well-defined element of $T_pM = C^\infty(M)$. So for some fixed $v \in T_pM$, define a map

$$\begin{aligned}\phi_v : C^\infty(N) &\to \mathbb{R}\\ h &\mapsto v(h \circ f),\end{aligned}$$

which will be an element of TM if it is a derivation. For $x \in N$, we have

$$\begin{aligned}\phi_v(h_1 \cdot h_2)(x) &\coloneqq v((h_1 h_2) \circ f)(x)\\ &= v((h_1 \circ f)(h_2 \circ f))(x)\\ &= v(h_1 \circ f)(x) \cdot h_2(x) + h_1(x) \cdot v(h_2 \circ f)(x) \quad \text{since } v \text{ is a derivation}\\ &= \phi_v(h_1)(x) \cdot h_2(x) + h_1(x) \cdot \phi_v(h_2)(x),\end{aligned}$$

so this is indeed a derivation.

3.2 Part 2

Given $c(t) : I \to M$, we define the map

$$\begin{aligned}\psi : TM &\to TN\\ v \coloneqq [c(t)] &\mapsto v_c \coloneqq [(f \circ c)(t)]\end{aligned}$$

where $c_1 \sim c_2 \iff \frac{\partial}{\partial t} c_1(t) \Big|_{t=0} = \frac{\partial}{\partial t} c_2(t) \Big|_{t=0}$. We can then associate $[c(t)]$ with the derivation

$$\begin{aligned}D_c : C^\infty(M) &\to \mathbb{R}\\ g &\mapsto \frac{\partial}{\partial t}(g \circ c)(t) \Big|_{t=0}\end{aligned}$$

and similarly we can define

$$\begin{aligned}D_{f\circ c} : C^\infty(N) &\to \mathbb{R}\\ h &\mapsto \frac{\partial}{\partial t}(h \circ (f \circ c))(t) \Big|_{t=0}\end{aligned}$$

and the question now is whether $v_c(h \circ f) = \frac{\partial}{\partial t}(h \circ (f \circ c))(t) \Big|_{t=0}$, where $v_c \in TN$ is the tangent vector obtained by applying ψ.

Thus the preimage of v_c under ψ is a class $[c(t)]$, and by definition we have

$$\begin{aligned}v(h \circ f) &= \frac{\partial}{\partial t}((h \circ f) \circ c)(t) \Big|_{t=0}\\ &= \frac{\partial}{\partial t}(h \circ (f \circ c))(t) \Big|_{t=0}\\ &= D_{f\circ c}(h),\end{aligned}$$

which is what we wanted to show.

3.3 Part 3

Not sure how to proceed.

4 Problem 4

4.1 Part 1

Let $V = \mathbb{R}^n$ as a vector space, let g be a nonsingular matrix, and define a map

$$\begin{aligned} \phi : V &\to V^\vee \\ v &\mapsto (\phi_v : w \mapsto \langle v,\ gw \rangle) \end{aligned}$$

The claim is that ϕ is a natural isomorphism. It is clearly linear (following from the linearity of the inner product and matrix multiplication), so it remains to check that it is a bijection.

To see that $\ker \phi = 0$, so that only the zero gets sent to the zero map, we can suppose that $x \in \ker \phi$. Then $\phi_x : w \to \langle x,\ gw \rangle$ is the zero map. But the inner product is nondegenerate by definition, i.e. $\langle x,\ y \rangle = 0\ \forall y \implies x = 0$. So x could only have been the zero vector to begin with.

But $\dim V = \dim V^\vee$, so any injective linear map will necessarily be surjective as well.

4.2 Part 2

Let $g : TM \otimes TM \to \mathbb{R}$ be a metric, and consider the tangent space TM. By definition, the cotangent space $T_p^* M = (T_p M)^\vee$

5 Problem 5

5.1 Part 1

Let $A \in \mathrm{Mat}(n, n)$ be a positive definite $n \times n$ matrix, so

$$\langle v,\ Av \rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

and $B \in \mathrm{Math}(n, n)$ be positive semi-definite, so

$$\langle v,\ Bv \rangle \geq 0 \quad \forall v \in \mathbb{R}^n.$$

We'd like to show

$$\langle v,\ (A+B)v \rangle \geq 0 \quad \forall v \in \mathbb{R}^n,$$

which follows directly from

$$\begin{aligned} \langle v,\ (A+B)v \rangle &= \langle v,\ Av \rangle + \langle v,\ Bv \rangle \\ &> \langle v,\ Av \rangle + 0 \\ &\geq 0 + 0 \\ &= 0. \end{aligned}$$

5.2 Part 2

Let M be a smooth manifold with tangent bundle TM and a maximal smooth atlas $\mathcal{A}$. Choose a covering of M by charts $\mathcal{C} = \{(U_i, \phi_i) \mid i \in I\} \subseteq \mathcal{A}$ such that $M \subseteq \bigcup_{i\in I} U_i$.Then choose a partition of unity $\{f_i\}_{i\in I}$ subordinate to $\mathcal{C}$, so for each i we have

$$\begin{aligned} f_i : M \to I \\ \forall p \in M, \quad \sum_{i\in I} f_i(p) = 1 \end{aligned}$$

In each copy of $\phi_i(U_i) \cong \mathbb{R}^n$, let g^i be the Euclidean metric given by the identity matrix, i.e. $g^i_{jk} \coloneqq \delta_{jk}$. We then have

$$\begin{aligned} g^i : T\phi_i(U_i) \otimes T\phi_i(U_i) &\to \mathbb{R} \\ (\partial x_i, \partial x_j) &\mapsto \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

which is defined for pairs of vectors in $T\phi_i(U_i) \cong T\mathbb{R}^n = \operatorname{span}_{\mathbb{R}} \{\partial x_i\}_{i=1}^n$ on basis vectors as the Kronecker delta and extended linearly.

Note that each coordinate function $\phi_i : U_i \to \mathbb{R}^n$ induces a map $\tilde{\phi}_i : TU_i \to T\mathbb{R}^n$.

Let G^i be the pullback of g^i along these induced maps $\tilde{\phi}_i$, so

$$\begin{aligned} G^i : TU_i \otimes TU_i &\to \mathbb{R} \\ G^i(x, y) \coloneqq \left(\left(\tilde{\phi}_i \right)^* g^i \right)(x, y) &\coloneqq g^i(\tilde{\phi}_i(x), \tilde{\phi}_i(y)) \end{aligned}$$

Then, for a point $p \in M$, define the following map:

$$\begin{aligned} g_p : T_pM \otimes T_pM &\to \mathbb{R} \\ (x, y) &\mapsto \sum_{i\in I} f_i(p) G^i(x, y). \end{aligned}$$

The claim is that g_p defines a metric on M, and thus the family $\{g_p \mid p \in M\}$ yields a tensor field and thus a Riemannian metric on M. If we define the map

$$\begin{aligned} g : M &\to (TM \otimes TM)^\vee \\ p &\mapsto g_p \end{aligned}$$

then g can be expressed as

$$g = \sum_{i\in I} f_i G^i.$$

We can check that this is positive definite by considering $x \in T_pM$ and computing

$$\begin{aligned} g(x, x) &\coloneqq g_p(x, x) \\ &= \sum_{i\in I} f_i(p)\ G^i(v, v) \\ &= \sum_{i\in I} f_i(p)\ g^i(\tilde{\phi}_i(x), \tilde{\phi}_i(y)), \end{aligned}$$

where each term is positive semi-definite, and *at least one term* is positive definite because $\sum_i f_i(p)$ must equal 1. By part 1, this means that the entire expression is positive definite, so g is a metric. □

6 Problem 6

6.1 Part 1

Let $M = S^2$ as a smooth manifold, and consider a vector field on M,

$$X : M \to TM$$

We want to show that there is a point $p \in M$ such that $X(p) = 0$.

Every vector field on a compact manifold without boundary is complete, and since S^2 is compact with $\partial S^2 = \emptyset$, X is necessarily a complete vector field.

Thus every integral curve of X exists for all time, yielding a well-defined flow

$$\phi : M \times \mathbb{R} \to M$$

given by solving the initial value problems

$$\begin{aligned} \frac{\partial}{\partial s}\phi_s(p)\Big|_{s=t} &= X(\phi_t(p)), \\ \phi_0(p) &= p \end{aligned}$$

at every point $p \in M$.

This yields a one-parameter family

$$\phi_t : M \to M \in \text{Diff}(M, M).$$

In particular, $\phi_0 = \text{id}_M$, and $\phi_1 \in \text{Diff}(M, M)$. Moreover ϕ_0 is homotopic to ϕ_1 via the homotopy

$$\begin{aligned} H : M \times I &\to M \\ (p, t) &\mapsto \phi_t(p). \end{aligned}$$

We can now apply the Lefschetz fixed-point theorem to ϕ_0 and ϕ_1. For an arbitrary map $f : M \to M$, we have

$$\Lambda(f) = \sum_k \text{Tr}\left(f_* \Big|_{H_k(X;\mathbb{Q})} \right).$$

where $f_* : H_*(X;\mathbb{Q}) \to H_*(X;\mathbb{Q})$ is the induced map on homology, and

$$\Lambda(f) \neq 0 \iff f \text{ has at least one fixed point.}$$

In particular, we have

$$\begin{aligned}\Lambda(\mathrm{id}_M) &= \sum_k \mathrm{Tr}(\mathrm{id}_{H_k(X;\mathbb{Q})}) \\ &= \sum_k \dim H_k(X;\mathbb{Q}) \\ &= \chi(M),\end{aligned}$$

the Euler characteristic of M.

Since homotopic maps induce equal maps on homology, we also have $\Lambda(\phi_1) = \chi(M)$.

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

we have $\chi(S^2) = 2 \neq 0$, and thus ϕ_1 has a fixed point p_0, thus

$\frac{\partial}{\partial t}\phi_t(p_0)\Big|_{t=1}$ so

$$\begin{aligned} \phi_t(p) &= p \\ \implies \frac{\partial}{\partial t}\phi_t(p) &= \frac{\partial}{\partial t} p = 0 && \text{by differentiating wrt } t \\ \implies \frac{\partial}{\partial t}\phi_t(p)\Big|_{t=1} &= 0\Big|_{t=0} = 0 && \text{by evaluating at } t = 0 \\ \implies X(\phi_1(p_0)) \coloneqq \frac{\partial}{\partial t}\phi_t(p)\Big|_{t=1} &= 0 && \text{by definition of } \phi_1 \end{aligned}$$

so $X(\phi_1(p_0)) = 0$, which shows that p_0 is a zero of X. So X has at least one zero, as desired. $\square$

6.2 Part 2

The trivial bundle

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & S^2 \times \mathbb{R}^2 \\ & & \downarrow \uparrow s \\ & & S^2 \end{array}$$

has a nowhere vanishing section, namely

$$\begin{aligned} s : S^2 &\to S^2 \times \mathbb{R}^2 \\ \mathbf{x} &\to (\mathbf{x}, [1, 1]) \end{aligned}$$

which is the identity on the S^2 component and assigns the constant vector $[1, 1]$ to every point.

However, as part 1 shows, the bundle

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & TS^2 \\ & & \big\downarrow \; \big\uparrow s \\ & & S^2 \end{array}$$

can *not* have a nowhere vanishing section. $\square$