MATH 8320 HOMEWORK

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0. Problem Set 0

Exercise 0.1. *There three notions of finite generation in play for a field extension l/k: (i) l is finitely generated as a k-module (equivalently, finite-dimensional as a k-vector space) – we also say that l/k has finite degree – (ii) l is finitely generated as a k-algebra: there are $x_1, \ldots, x_n \in l$ such that $l = k[x_1, \ldots, x_n]$: every element of l can be expressed as a polynomial in $x_1, \ldots, x_n$ with coefficients in k. (iii) l/k is finitely generated as a field extension.*

a) *Show: l/k finitely generated as a module implies l/k finitely generated as a k-algebra implies l/k finitely generated as a field extension.*
b) *Let $k(t)$ be the rational function field over k – the fraction field of the polynomial ring $k[t]$. Show: $k(t)/k$ is finitely generated as a field extension but is not finitely generated as a k-algebra.*
c) *Show: $k[t]/k$ is finitely generated as a k-algebra but not as a k-module. (However $k[t]$ is not a field!)*
d) *Can you exhibit a field extension l/k such that l is finitely generated as a k-algebra but not as a k-vector space?*
(Hint: no, you can't – this is a famous result of commutative algebra!)
e) *Suppose l/k is algebraic and finitely generated as a field extension. Show that l/k has finite degree.*

Exercise 0.2. *Show that every finitely generated field extension $K = k(x_1, \ldots, x_n)$ is the fraction field of a quotient of $k[t_1, \ldots, t_n]$ by a (not necessarily principal) prime ideal.*

Exercise 0.3. *Let k be a field, and let $k(a, b)$ be a field extension of k of transcendence degree 1.*

a) *Let $k[x, y]$ be the polynomial ring in two variables. Let $f : k[x, y] \to k(a, b)$ be the unique k-algebra homomorphism such that $f(x) = a$ and $f(y) = b$. Show that the kernel $\mathfrak{p}$ of f is a prime ideal, and let K be the fraction field of $k[x, y]/\mathfrak{p}$. Show that f induces a k-algebra isomorphism $K \overset{\sim}{\to} k(a, b)$.*
b) *Show: $\mathfrak{p}$ is generated by an irreducible polynomial, and deduce that there is an irreducible polynomial $f \in k[x, y]$, unique up to scaling by an element of $k^\times$, such that $f(a, b) = 0$ and $k(a, b)$ is the fraction field of $k[x, y]/(f)$.*
(Suggestion: by [CA, Cor. 12.17]*, the prime ideal $\mathfrak{p}$ has height 0, 1 or 2. Rule out the possibilities of height 0 and height 2, and then find and use a fact about height one prime ideals in a UFD.)*
c) *Show that if K/k is a separable one variable function field, then $K = k(a, b)$ for some a and b. (Remark: In the third lecture I mention that in this case we can actually take the polynomial f to be geometrically irreducible.)*

Exercise 0.4. *Let k be a field, let G be a finite group of order n, and let $G \hookrightarrow S_n$ be the Cayley embedding. Permutation of variables gives a natural action of S_n and hence also G on $k(t_1, \ldots, t_n)$. Put $l := k(t_1, \ldots, t_n)^G$, so $k(t_1, \ldots, t_n)/l$ is a finite Galois extension with automorphism group G. Notice that this is an instance of the Lüroth problem.*

a) *Let $k = \mathbb{Q}$. Show: if $l/\mathbb{Q}$ is purely transcendental, then G occurs as a Galois group over $\mathbb{Q}$. Thus: an affirmative answer to the Lüroth problem yields an affirmative answer to the Inverse*

Galois Problem over $\mathbb{Q}$.
(Suggestion: This holds whenever k *is a Hilbertian field.)*

b) *Alas,* $l/\mathbb{Q}$ *need not be purely transcendental. Explore the literature on this – the first example was due to Swan, where* G *is cyclic of order* 47.

Exercise 0.5. *Let* R_1 *and* R_2 *be two* k*-algebras that are also domains, with fraction fields* K_1 *and* K_2*. Show that* $R_1 \otimes_k R_2$ *is a domain iff* $K_1 \otimes_k K_2$ *is a domain.*

Exercise 0.6. a) *Let* l/k *be an algebraic field extension. Show:* $l \otimes_k l$ *is a domain iff* $l = k$.

b) *Let* l/k *be any field extension. Show:* $k(t) \otimes_k l$ *is always a domain with fraction field* $l(t)$*. It is already a field iff* l/k *is algebraic.*

Exercise 0.7. *Describe the* $\mathbb{R}$*-algebra* $\mathbb{C}(t) \otimes_{\mathbb{R}} \mathbb{C}$.

Exercise 0.8. a) *Show:* $k(t)/k$ *is regular.*

b) *Show: every purely transcendental extension is regular.*
c) *Show: every extension* K/k *is regular iff* k *is algebraically closed.*
d) *Show:* K/k *is regular iff every finitely generated s subextension is regular.*

Exercise 0.9. *Let* k *be a field, let* $d \geq 2$ *be such that* $4 \nmid d$*, and let* $p(x) \in k[x]$ *be a polynomial of positive degree. In* $\overline{k}[t]$ *we factor* p *as* $(x-a_1)^{e_1} \cdots (x-a_r)^{e_r}$ *with* $a_1, \ldots, a_r$ *distinct elements of* $\overline{k}$ *and* $e_1, \ldots, e_r \in \mathbb{Z}^+$*. Suppose that there is some* $1 \leq i \leq r$ *such that* $d \nmid e_i$*. Show that the*

$$f(x,y) = y^d - p(x) \in k[x,y]$$

is geometrically irreducible and thus the fraction field of $k[x,y]/(y^d - p(x))$ *is a regular one variable function field over* k.
(Suggestion: use [FT, Thm. 9.21]*.)*

Exercise 0.10. *Let* k *be a field of characteristic different from* 2.

a) *Show that the function field* K_f *attached to* $f(x,y) = x^2 - y^2 - 1$ *is rational: i.e., there is* $z \in K$ *such that* $K_f = k(z)$.
b) *Show that the function field* K_f *attached to* $f(x,y) = x^2 + y^2 - 1$ *is rational.*
c) *If* $k = \mathbb{C}$*, show that the function field* K_f *attached to* $f(x,y) = x^2 + y^2 + 1$ *is rational.*
d) *If* $k = \mathbb{R}$*, is the function field attached to* $f(x,y) = x^2 + y^2 + 1$ *rational?*
(Answer: it is not, but at the moment we have precisely no tools to show that a regular function field is not rational, so I don't know how you could prove this. But keep it in mind – as we develop more theory, it will become possible, then easy, then clear.)

Exercise 0.11. *Give a purely algebraic proof of the Lüroth Theorem: for any field* k*, if* K *is a field such that* $k \subsetneq K \subset k(t)$*, then* $K = k(f)$ *for some* $f \in K$.

Exercise 0.12. *Fix* $n \in \mathbb{Z}^+$*. Exhibit a finite degree field extension* l/k *such that needs* $n+1$ *generators: that is,* $l \neq k(x_1, \ldots, x_n)$ *for any* $x_1, \ldots, x_n \in l$.

I do not know how to do the following exercise:

Exercise 0.13. a) *For each* $n \in \mathbb{Z}^+$*, find a one variable function field* K/k *that needs* $n+1$ *generators or show that no such exists.*
(Idea: As in Exercise 0.12, there is a finite degree field extension l/k *that needs* $n+1$ *generators. It seems likely that* $l(t)/k$ *also needs* $n+1$ *generators!)*

b) *Prove or disprove: every one variable function field* K/k *with* $\kappa(K) = k$ *is* 2*-generated.*

1. Problem Set 1

Exercise 1.1. *Let* k *be any field, and let* $\frac{p(t)}{q(t)} \in k(t)$ *be a nonconstant rational function. Show:*

$$\deg[k(t) : k(p/q)] = \max \deg(p), \deg(q).$$

Exercise 1.2. *Let R be a valuation ring with fraction field K. Let F be a subfield of K. Show: $R \cap F$ is a valuation ring with fraction field F. We call it the "restriction of R to F."*

Exercise 1.3. *Let $(G, +, \leq)$ be a totally ordered commutative group, and let*

$$v : K \to G \cup \{\infty\}$$

be a ***Krull valuation*** *on K:*
(VRK0) For all $x \in K$, we have $v(x) = \infty$ iff $x = 0$.
(VRK1) For all $x, y \in K^\times$, we have $v(xy) = v(x) + v(y)$.
(VRK2) For all $x, y \in K^\times$ such that $x + y \neq 0$, we have $v(x + y) \geq \min v(x), v(y)$.
Show that, as expected, if $v(x) \neq v(y)$, then we have $v(x + y) = \min v(x), v(y)$.

Exercise 1.4. *Let k be a field, and let $K = k(t_1, \ldots, t_n)$ be a rational function field in n indeterminates. Let $G := \mathbb{Z}^n$, with the lexicographic ordering. Let $G^{\geq 0} = \mathbb{N}^n$ (it is indeed the submonoid of non-negative elements for the given ordering).*

a) *Observe/recall that the polynomial ring $k[t_1, \ldots, t_n]$ can be viewed as the semigroup algebra $k[G^{\geq 0}]$.*
b) *Define a map $v : k[G^{\geq 0}]^\bullet \to G^{\geq 0}$ by mapping each polynomial to the smallest monomial in it support.*
c) *Extend v to a surjective map $K^\bullet \to G$ that satisfies (VRK1) and (VRK2). Show that $R_v := v^{-1}(G^{\geq 0}) \cup \{0\}$ is a valuation ring with value group G. In particular, if $n \geq 2$ then K carries a valuation of rank $n \geq 2$.*
d) *Suppose now that L/k is any function field in n variables. Show that L carries a valuation of rank n. (It suffices to know that higher rank valuations on a field can be exended to a a finite degree field extension. This is true, although it is not discussed in* [NTII]*.)*

Exercise 1.5. *Define a map $v_\infty : k(t)^\times \to \mathbb{Z}, x = \frac{p(t)}{q(t)} \mapsto \deg q - \deg p$.*

a) *Show that v_∞ is a $k[1/t]$-regular discrete valuation on $k(t)$.*
b) *Deduce from the above discussion that the valuations v_∞ and $v_{1/t}$ are equivalent: i.e., have the same valuation ring.*
c) *Show that $v_\infty = v_{1/t}$.*

Exercise 1.6. *Let K/k be a one-variable function field.*

a) *Show that $\Sigma(K/k)$ is infinite.*
b) *More precisely, show that the cardinality of $\Sigma(K/k)$ is equal to the number of monic irreducible polynomials $p \in k[t]$, which is $\# \max(\#k, \aleph_0)$.*

Exercise 1.7. *Let K/k be a one variable function field, let $v \in \Sigma(K/k)$, let R_v be the valuation of v, $\mathfrak{m}_v$ its maximal ideal, and $k(v) = R_v/m_v$ its residue field. We showed in the lecture that $[k(v) : k]$ is finite using "afine grounding" and Zariski's Lemma. In* [St]*, Stichtenoth gives a different proof. He chooses $f \in K$ such that $v(f) = 1$ and shows that $[k(v) : k] \leq [K : k(f)]$. Show this by showing first that if $\overline{v}$ is the restriction of v to $k(f)$ then $k(\overline{v}) = k$ and then applying the Degree Equality (**??**).*

Exercise 1.8. *Let K/k be a one variable function field with constant field $\kappa(K)$. Show that for all $v \in \Sigma(K/k)$, we have*

$$[\kappa(K) : k] \mid \deg v.$$

In particular, if $\kappa(K) \supsetneq k$, then K has no degree one points.

Exercise 1.9. *Let K/k be a one variable function field. Show that the following are equivalent:*

(i) *Every $v \in \Sigma(K/k)$ has degree 1.*
(ii) *The ground field k is algebraically closed.*

Exercise 1.10. *For any field k, let $\mathbb{P}^1(k)$ denote the set $k \cup \{\infty\}$. (You can certainly go ahead and think of this as the set of lines through the origin in k^2. However it is not necessary, or even immediately helpful, to think in terms of algebraic varieties.) Show that there is a natural bijection*

$$\Sigma_1(k(t)/k) = \mathbb{P}^1(k).$$

Combining with Exercise 1.9 we get: $\Sigma(k(t)/k) = \mathbb{P}^1(k)$ *iff* k *is algebraically closed.*

Exercise 1.11. *Show: If* A *is an affine Dedekind domain with fraction field* K*, then we have* $A = R^{\Sigma(K/k)\setminus \operatorname{MaxSpec} A}$.

Exercise 1.12. *Let* $Z \subset \Sigma(K/k)$ *be infinite and proper. Show:* R^Z *is a Dedekind domain with fraction field* K *that is* not *finitely generated as a* k*-algebra.*

Exercise 1.13. *Let* K/k *be a one-variable function field. Show: there are affine Dedekind domains* A_1, A_2 *over* k *with fraction field* K *such that* $\Sigma(K/k) = \operatorname{MaxSpec} A_1 \cup \operatorname{MaxSpec} A_2$ *(the union is very far from being disjoint).*

2. Problem Set 2

Exercise 2.1. *Let* K/k *be a one variable function field.*

a) *Show: If* $\Sigma_1(K/k) \neq \varnothing$*, then* K *has index* 1.
b) *We will see later that if* k *is finite,* K *always has index* 1 *but* $\Sigma(K/k)$ *may be empty. You can try to prove this now if you like!*
c) *Deduce: if* k *is algebraically closed, then* K *has index* 1.
d) *Show: The index of* K *is divisible by* $[\kappa(K) : k]$.

Exercise 2.2. *Let* K/k *be a one variable function field, and let* $f \in K^\times$*. Show that the divisor of* f *is* 0 *iff* f *lies in the constant subfield of* K.

Exercise 2.3. *Let* $f, g \in K^\times$.

a) *Show:* $(\frac{1}{f}) = -(f)$.
b) *Show:* $(fg) = (f) + (g)$.
c) *Deduce the principal divisors form a subgroup of* $\operatorname{Div}^0 K$*, denoted* $\operatorname{Prin} K$.

Exercise 2.4. a) *Show that every degree zero divisor on* $k(t)$ *is the divisor of a rational function.*
b) *Deduce that the degree map induces an isomorphism* $\operatorname{Cl} k(t) \xrightarrow{\sim} \mathbb{Z}$ *and that* $\operatorname{Cl}^0 k(t) = (0)$.

Exercise 2.5. *This exercise takes place in the setting of Rosen's Theorem* [NTII, Thm. 3.28].

a) *Show:* $D^0(S) \cong \mathbb{Z}^{\#S-1}$.
b) *Suppose that* $S = \{P\}$ *consists of a single place, of degree* $d \in \mathbb{Z}^+$*. Show that (**??**) simplifies to*

$$0 \to \operatorname{Cl}^0(K) \xrightarrow{\alpha} \operatorname{Cl} R^S \xrightarrow{\beta} C(d/I(K)) \to 0.$$

Deduce that in this case α *is an isomorphism iff* $I(K) = d$.
c) *Deduce that if* S *consists of a single degree* 1 *place, then* $\alpha : \operatorname{Cl}^0 K \xrightarrow{\sim} \operatorname{Cl} R^S$.

Exercise 2.6. *This exercise takes place in the setting of Rosen's Theorem* [NTII, Thm. 3.28].

a) *Suppose that* $\operatorname{Cl}^0 K$ *is finite. Show that every affine Dedekind domain* R^S *in* K *has finite ideal class group.*[1]
b) *Suppose* $\operatorname{Cl}^0 K$ *is infinite and finitely generated. Show that for any nonempty finite subset* $S \subset \Sigma(K/k)$*, there is a nonempty finite subset* $S' \supset S$ *such that* $\operatorname{Cl} R^{S'}$ *is finite.*

Exercise 2.7. a) *Show:* $\mathbb{R}[\cos\theta, \sin\theta] \cong \mathbb{R}[x, y]/(x^2 + y^2 - 1)$*. Show that the latter is an affine Dedekind domain. By Exercise 0.10, its fraction field* K *is isomorphic to* $\mathbb{R}(t)$.
b) *Use Rosen's Theorem to show that* $\operatorname{Cl} \mathbb{R}[\cos\theta, \sin\theta] \cong \mathbb{Z}/2\mathbb{Z}$.
c) *Show:* $\mathbb{C}[\cos\theta, \sin\theta] = \mathbb{C}[e^{i\theta}, e^{-i\theta}]$ *and deduce that* $\mathbb{C}[\cos\theta, \sin\theta]$ *is a PID.*
d) *Use Rosen's Theorem to show that* $\operatorname{Cl} \mathbb{C}[\cos\theta, \sin\theta]$ *is trivial.*

[1] Later we will show that $\operatorname{Cl}^0 K$ is always finite when k is a finite field. Thus this exercise shows the finiteness of all the class groups $\operatorname{Cl} R^S$, which is the function field analogue of the finiteness of the class group of the ring of integers (or better, of the rings of S-integers; but the latter follows easily from the former) of a number field.

References

[CA] P.L. Clark, *Commutative Algebra.* `http://math.uga.edu/~pete/integral2015.pdf`

[FT] P.L. Clark, *Field Theory.* `http://math.uga.edu/~pete/FieldTheory.pdf`

[NTII] P.L. Clark, *Algebraic Number Theory II: Valuations, Local Fields and Adeles.* `http://math.uga.edu/~pete/8410FULL.pdf`

[St] H. Stichtenoth, *Algebraic function fields and codes.* Second edition. Graduate Texts in Mathematics, 254. Springer-Verlag, Berlin, 2009.