THE h-COBORDISM THEOREM

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Abstract. This paper begins with an introduction to some of the concepts of algebraic topology. The main result that will be proven is the h-cobordism theorem, a powerful tool for manifolds of dimension 5 or greater. We will only address dimensions 6 and greater, as dimension 5 limits the theorem to topological, rather than smooth, manifolds and involves significant complications in the proof. The h-cobordism theorem is then used to prove the Poincaré conjecture for high dimensions. Some familiarity with algebraic topology is assumed, though a brief review of basic topics is provided. Necessary concepts in homology are only briefly addressed, and further reading is suggested to the interested reader.

Contents

1. Basic Concepts

The h-cobordism theorem in dimension 6 or greater is concerned with diffeomorphisms between smooth manifolds. We won't address the definitions rigorously here, but [4] has a good introduction to the topic. The standard definitions for homotopy, isotopy, diffeotopy, homotopy equivalence, homeomorphism, embeddings, and diffeomorphisms will be assumed.

Notation 1.1. A homotopy, isotopy, or diffeotopy is a function $F : X \times [0, 1] \to Y$. For $0 \leq t \leq 1$, we let $F_t(x) := F(x, t)$.

Notation 1.2. Two diffeomorphic spaces X and Y will be denoted $X \cong Y$.

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1.1. **Algebra.** Algebraic topology involves the application of group structures to topological objects. One way of doing this is with homotopy groups, which classify the maps of spheres into a space.

Definition 1.3. The *n-th homotopy group* of a space X with basepoint x_0, denoted $\pi_n(X, x_0)$, or often just $\pi_n(X)$, is the group whose elements are equivalence classes of maps $f : S^n \to X$ under (based) homotopy: that is, each map f must send some element $y \in S^n$ to x_0, and the homotopies F between the maps f must be based at x_0: $F_t(y) = x_0$ for all $0 \leq t \leq 1$. The *fundamental group* is another name for the first homotopy group π_1.

Definition 1.4. A space X is *connected* if $\pi_0(X)$ is the trivial group. It is *simply connected* if $\pi_1(X)$ and $\pi_0(X)$ are both trivial. It is *n-connected* if $\pi_i(X)$ is trivial for all $i \leq n$.

There is also a notion of relative homotopy groups:

Definition 1.5. The *relative n-th homotopy group* of a space X with subspace X' and basepoint $x_0 \in X'$ is denoted $\pi_n(X, X')$, or $\pi_n(X, X', x_0)$. Its elements are equivalence classes of maps $f : D^n \to X$ under (based) homotopy, with the restrictions that $f(\partial D^n) = f(S^{n-1}) \subset X'$ and $x_0 = f(y)$ for some fixed $y \in S^{n-1}$. The homotopies must be similarly based at x_0. Note that the relative homotopy group $\pi_n(X, x_0, x_0)$ is the same as the absolute homotopy group given in definition 1.3, since D^n with its boundary mapped to a single point is equivalent to a map from S^n.

The notion of n-connectedness is different for maps than it is for spaces. First, note that a map between topological spaces $f : X \to Y$ induces a map $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ by simply sending the image of S^n in X to its composition with f in Y. This is well defined: if s_1, s_2 are images of S^n in X that are homotopic by a homotopy $H : S^n \times [0, 1] \to X$, then $f(s_1)$ and $f(s_2)$ will be homotopic in Y by $f \circ H : S^n \times [0, 1] \to X \to Y$. This notion of induced maps allows us to define an n-connected map.

Definition 1.6. A map $f : X \to Y$ is *n-connected* if the induced maps $\pi_i(f) : \pi_i(X) \to \pi_i(Y)$ are isomorphisms for $i < n$ and $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ is a surjection.

Example 1.7. For $X \subset Y$, the inclusion $f : X \to Y$ is 1-connected if the addition of $Y - X$ to X does not change the path components of X ($\pi_0(f)$ is an isomorphism) and if there are no new elements of the fundamental group added by the rest of Y ($\pi_1(f)$ is a surjection). The inclusion of the red subset into the space in figure 1 is 1-connected: both the space and the subspace are connected, and $\pi_1(f)$ maps $\mathbb{Z} = \pi_1(X)$ to the trivial group $\pi_1(Y)$.

In addition to homotopy groups, the proof of the h-cobordism theorem employs homology groups. The overview of homology here is sketchy and informal. For a more complete approach, see [1, Chapter 2].

Any topological space can be seen as a CW complex, which is a combination of D^n 'cells' joined along their boundaries to D^k 'cells', where $k < n$. For instance, figure 2 shows the decomposition of the sphere S^2 into a single 0-cell, a single 1-cell, and two 2-cells.

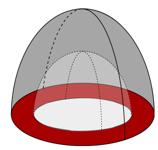

FIGURE 1. The inclusion map of the red subset into the entire space is 1-connected. (See example 1.7.)

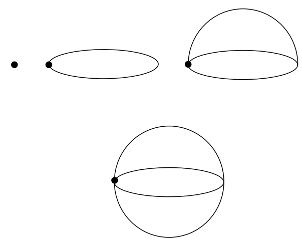

FIGURE 2. The construction of a sphere S^2 from a single D^0, a single D^1 with its boundary attached to the D^0, and two D^2 with their boundaries attached to the D^1.

First, we'll consider an object that we'll call the *group of cellular n-chains* of a space X, denoted by $C_n(X)$. It is the free abelian group generated by the n-cells in a given cellular decomposition of a space. That is, if the n-cells are denoted $\{\alpha_1, \alpha_2, \cdots, \alpha_{k_n}\}$, then the elements of $C_n(X)$ (called *cellular n-chains*) are formal sums $z_1\alpha_1 + z_2\alpha_2 + \cdots + z_{k_n}\alpha_{k_n}$ for $z_j \in \mathbb{Z}$.

We'll also define a *boundary operator* $d_n : C_n(X) \to C_{n-1}(X)$. The boundary operator acts on the basis of $C_n(X)$ by sending each α_j to the $(n-1)$-cells that its boundary is mapped to, with either a positive or negative sign indicating orientation. The image of other elements in the group follows naturally from this definition on the basis.

Example 1.8. The boundary operator would send the top 2-cell in figure 2 to the 1-cell with positive orientation, while it would send the bottom 2-cell to the 1-cell with negative orientation.

Remark 1.9. The composition $d_n \circ d_{n+1}$ is the zero map. This is because any collection of n-cells that form the boundary of an $(n+1)$-cell must itself have an empty boundary.

This trivial composition motivates the definition of a homology group, which we'll think of as follows: the *n-th homology group* of a space X is defined as the kernel of d_n with the image of d_{n+1} identified. That is, $H_n(X) = \ker(d_n)/\operatorname{im}(d_{n+1})$.

Just as with homotopy groups, we also have relative homology groups. See [1, Chapter 2] for a discussion of these relative groups.

2. Introduction to the h-Cobordism Theorem

The h-cobordism theorem is a powerful result in algebraic topology that allows us to prove that two spaces are diffeomorphic. It was first proven in 1962 by Stephen Smale, then an instructor at the University of Chicago (now a professor at the Toyota Technological Institute at Chicago).

The basic objects that we will be working with in the proof of the theorem are called cobordisms.

Definition 2.1. An n-dimensional *cobordism* $(W; M_0, f_0, M_1, f_1)$ consists of a compact n-dimensional manifold W, closed $(n-1)$-dimensional manifolds M_0, M_1, a decomposition of the boundary of W as $\partial W = \partial_0 W \amalg \partial_1 W$, and diffeomorphisms $f_i : M_i \to \partial_i W$ for $i = 0, 1$.

Usually f_0 and f_1 are clear from the definition of the cobordism, and so we can just write cobordisms as $(W; \partial_0 W, \partial_1 W)$. We also want a notion of equivalent cobordisms:

Definition 2.2. If $(W; M_0, f_0, M_1, f_1)$ and $(W'; M_0, f_0', M_1', f_1')$ are two cobordisms with the same dimension, they are *diffeomorphic relative to* M_0 if there exists a diffeomorphism $F : W \to W'$ such that $F \circ f_0 = f_0'$.

As the name of the theorem suggests, we won't actually be working with general cobordisms, but instead with a refinement known as h-cobordisms.

Definition 2.3. A cobordism $(W; \partial_0 W, \partial_1 W)$ is an *h-cobordism* if the inclusions $\partial_0 W \to W$ and $\partial_1 W \to W$ are homotopy equivalences.

Our goal with h-cobordisms is to prove that certain spaces are diffeomorphic, which motivates the following definition.

Definition 2.4. An h-cobordism $(W; \partial_0 W, \partial_1 W)$ is *trivial* if it is diffeomorphic relative $\partial_0 W$ to $(\partial_0 W \times [0,1]; \partial_0 W \times \{0\}, \partial_0 W \times \{1\})$.

With these definitions, we can state the main result of this paper and a corollary.

Theorem 2.5 (h-Cobordism Theorem (4.27))**.** *Any h-cobordism $(W; M_0, f_0, M_1, f_1)$ for M_0 simply connected and closed and $\dim(W) \geq 6$ is trivial.*

Theorem 2.6 (Poincaré Conjecture (5.3))**.** *For $n \geq 6$, any simply connected, closed n-manifold M whose homology groups $H_p(M)$ are isomorphic to $H_p(S^n)$ for all $p \in \mathbb{Z}$ is homeomorphic to S^n.*

3. Handles and Handlebody Decomposition of Manifolds

Handles are crucial to the proof of the h-cobordism theorem. They are geometric structures that can be attached to or detached from generic manifolds in ways that preserve diffeomorphism classes. These structures are called handles because a relatively basic version of them looks like a typical handle attached to an object. Consider a solid ball, D^3. Clearly, D^3 is a 3-manifold with boundary, where $\partial D^3 = S^2$, the unit sphere in three dimensions. We can take a bent 'full' cylinder $D^2 \times D^1$ and embed the two ends into the surface of the sphere, creating a sphere with an attached handle. See figure 3.

The following definition of a handle is a generalization of this intuitive understanding.

Definition 3.1. An n-dimensional handle of index q is a structure diffeomorphic to $D^q \times D^{n-q}$. We will refer to this as an (n, q)-handle or, if the dimension is clear, simply a q-handle. Handles must also be embedded into a topological space in a specific way: see definition 3.4.

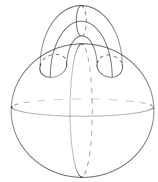

FIGURE 3. A solid ball with an attached handle.

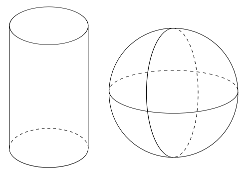

FIGURE 4. A $(3, 2)$ handle and a $(3, 3)$ handle.

The first thing to note is that (n, q)-handles will always be n-manifolds with boundary, as the product of q- and $(n - q)$-manifolds with boundary. Figure 4 shows a $(3, 2)$-handle and a $(3, 3)$-handle.

The relationship between the handles, their boundaries, and the way they will be embedded into other manifolds is easier to understand by first defining two notions:

Definition 3.2. The *core* of an (n, q)-handle is $D^q \times \{0\}$. The *cocore* of an (n, q)-handle is $\{0\} \times D^{n-q}$. Note that the boundary of the core is $S^{q-1} \times \{0\}$ and the boundary of the cocore is $\{0\} \times S^{n-q-1}$.

Definition 3.3. The *transverse sphere* of a handle (ϕ^q) is the boundary of the cocore, $\{0\} \times S^{n-q-1}$.

Figure 5 shows the core and cocore of a $(3, 1)$-handle, while figure 6 shows the core and cocore of a $(3, 2)$ handle. (In each diagram, the core is in red and the cocore is in blue.) The notions of core and cocore demonstrate the product structure of the handle, and help differentiate between (n, q) and $(n, n-q)$ handles, even though they appear identical as geometric structures in n-space.

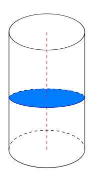

FIGURE 5. Core and cocore of a $(3, 1)$ handle.

Handles are useful because they can be embedded into manifolds with boundary in order to form 'new' manifolds. This embedding must be done in a specific way. The handle was attached to the ball above by embedding the disks at each end of the cylinder. Defining that cylinder as a $(3, 1)$-handle makes it clear that the embedding is being done on $S^0 \times D^2$. In other words, the embedding sends the product of the boundary of the core with the cocore into the boundary of the initial manifold. This is generalized as follows:

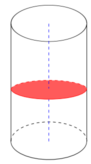

FIGURE 6. Core and cocore of a $(3, 2)$ handle.

Definition 3.4. Given an n-dimensional manifold M with boundary ∂M and a smooth embedding $\phi^q : S^{q-1} \times D^{n-q} \to \partial M$, we can attach a q-handle to M. This operation generates a new manifold $M + (\phi^q) = M \cup_{\phi^q} D^q \times D^{n-q}$.

Remark 3.5. It is important to check that $M + (\phi^q)$ is actually a manifold. Topologically, this follows easily: since M and $D^q \times D^{n-q}$ are both n-manifolds, any point not in $\phi^q \left(S^{q-1} \times D^{n-q}\right)$ clearly has a neighborhood homeomorphic to either $\mathbb{R}^n$ or $\{x \in \mathbb{R}^n \,|\, x_0 \geq 0\}$. What remains is to examine the image of ϕ^q. Recall that ϕ^q sends a subset of the boundary of the q-handle to the boundary of M. Therefore, any point in the image has a neighborhood homeomorphic to one copy of $\{x \in \mathbb{R}^n \,|\, x_0 \geq 0\}$ in the handle and another copy of it in M. These two copies will be glued along a piece of the embedded part of the handle, which will correspond precisely to $x_0 = 0$ in each half Euclidean space. The space $\{x \in \mathbb{R}^n \,|\, x_0 \geq 0\} \cup_{x_0=0} \{x \in \mathbb{R}^n \,|\, x_0 \geq 0\}$ is homeomorphic to $\mathbb{R}^n$, so $M + (\phi^q)$ must be a topological n-manifold. Guaranteeing that $M + \phi^q$ is smooth requires rounding out the corner where the manifold is joined to the handle. This can clearly be done, so I will omit the details.

Remark 3.6. From remark 3.5 it clearly follows that the boundary of $M + (\phi^q)$ can be found by taking the boundary of M, removing the interior of the image of ϕ^q, and adding in those parts of the boundary of the q-handle that are not embedded into M. Since the q-handle is $D^q \times D^{n-q}$, its boundary must be $S^{q-1} \times D^{n-q} \cup D^q \times S^{n-q-1}$, which means that $D^q \times S^{n-q-1}$ will be a part of the boundary of $M + (\phi^q)$.

Remark 3.7. Note that this 'addition' between manifolds and handles is not necessarily commutative. The manifold $(W + (\phi^q)) + (\psi^p)$ is not necessarily diffeomorphic to $(W + (\psi^p)) + (\phi^q)$. That is because each added handle changes the boundary, and thus affects the ways the next handle can be attached: ψ^p's codomain is $\partial(W + (\phi^q))$ in the first manifold, but only ∂W in the second. The conditions under which we have quasi-commutativity will be discussed in lemma 4.4.

One important concept later on will be that of trivial handle embeddings.

Definition 3.8. If W is an $(n-1)$-manifold, an embedding $\phi : S^{q-1} \times D^{n-q} \to W$ is *trivial* if it can be written as the composition of two embeddings $\phi = f \circ g$ where $f : D^{n-1} \to W$ and $g : S^{q-1} \times D^{n-q} \to D^{n-1}$. In other words, a trivial embedding is one that sends $S^{q-1} \times D^{n-q}$ injectively to a contractible subspace of W. Figure 7 shows an example of a trivial and a non-trivial embedding.

Remark 3.9. Since all D^k are homotopic to points, note that saying that $\phi : S^{q-1} \times D^{n-q} \to W$ is trivial doesn't depend at all on the D^{n-q} portion of the map. Because of this, we will often talk about the restriction $\phi|_{S^{q-1}}$ being trivial, which is an equivalent notion.

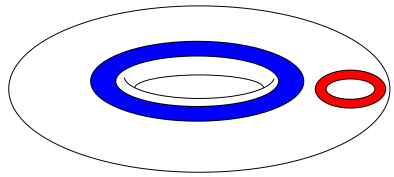

FIGURE 7. A 'solid' torus $W = S^1 \times D^2$ with the images of two embeddings $\phi_1^2, \phi_2^2 : S^1 \times D^1 \to \partial W$. The red embedding is clearly trivial (see definition 3.8) while the blue embedding is not.

Now that embedding a handle into a manifold is defined, it can be applied to the manifolds at issue in the h-cobordism theorem. Recall that we are interested in $(n+1)$-dimensional manifolds W with $\partial W = \partial_0 W \amalg \partial_1 W$, where $\partial_0 W$ and $\partial_1 W$ are both n-dimensional manifolds. The goal is to trivialize W: that is, show that it is diffeomorphic relative $\partial_0 W$ to $\partial_0 W \times [0,1]$. The method for doing this is to construct a 'handlebody decomposition' of W.

Definition 3.10. A *handlebody decomposition* of a manifold W with $\partial W = \partial_0 W \amalg \partial_1 W$ (relative to $\partial_0 W$) is another manifold W' diffeomorphic to W with

$$W' = \partial_0 W \times [0,1] + (\phi_1^{q_1}) + (\phi_2^{q_2}) + \cdots + (\phi_r^{q_r}).$$

In order to not change $\partial_0 W \times \{0\}$, we require that the image of $\phi_i^{q_i}$ be contained in $\partial_1(\partial_0 W \times [0,1] + (\phi_1^{q_1} + \cdots + (\phi_{i-1}^{q_{i-1}})))$. Note that each q_j need not be distinct: we can have multiple 5-handles, for instance. Also note that since the addition is not commutative, the sequence of q_j will not necessarily be increasing, though we will order it later.

A handlebody decomposition of W relative to $\partial_0 W$ thus gives the trivial $(n+1)$-manifold for $\partial_0 W$ with arbitrary handles attached to $\partial_0 W \times \{1\}$, so trivializing it will involve diffeomorphically removing all of the handles. The following lemma is derived from Morse theory. (See [2, Chapter 6] for an introduction to Morse theory.)

Lemma 3.11. *If W is a compact manifold of dimension $n \geq 6$ with $\partial W = \partial_0 W \amalg \partial_1 W$, then there exists a handlebody decomposition of W rel $\partial_0 W$.*

This lemma allows us to decompose any h-cobordism into the trivial h-cobordism of $\partial_0 W$ together with arbitrary handles. In order to show that the h-cobordism is diffeomorphic to the trivial h-cobordism, then, we need only find a way to smoothly remove the handles.

Remark 3.12. To connect handlebody decompositions with the homology theory mentioned in section 1.1, we will assert that q-handles $D^q \times D^{n-q}$ can be thought of as q-cells D^q in a cellular decomposition, with the homology notions understood accordingly. For a discussion of this that is beyond the scope of this paper, see [3, Section 1.2]. Note that while Lück discusses homology groups over the universal covers of spaces, we will be working with simply connected spaces. Since the universal cover of a simply connected space is just the space itself, we can dispense with that hassle.

4. Simplifying Handlebody Decompositions

This section closely follows the proof of the s-Cobordism Theorem in [3, Chapter 1].

Once an h-cobordism is decomposed into a handlebody, there are various tools that allow us to diffeomorphically alter the decomposition into a new one that brings us closer to the trivial h-cobordism. The lemmas presented in this section are all important ways of changing between handlebody decompositions.

The isotopy lemma will enable us to change between two handles of the same index if the embeddings of those handles are sufficiently similar. First, though, we need the following isotopy extension lemma from [2, Chapter 8, Theorem 1.3]. The proof is beyond the scope of this paper, but can be seen there.

Lemma 4.1. *Let $V \subset W$ be a compact submanifold and $F : V \times [0,1] \to W$ be an isotopy of V. If either $F(V \times [0,1]) \subset \partial W$ or $F(V \times [0,1]) \subset W - \partial W$ then F extends to a diffeotopy G of W that has a compact support. That is, the closure of the set $\{x \in W \mid G(x,t) \neq G(x,0)$ for some $t\}$ is compact.*

Lemma 4.2 (Isotopy Lemma)**.** *Let ϕ^q and ψ^q be embeddings attaching two q handles to an n dimensional manifold W with $\partial W = \partial_0 W \amalg \partial_1 W$. If ϕ^q and ψ^q are isotopic, then there is a diffeomorphism $W + (\phi^q) \to W + (\psi^q)$ relative to $\partial_0 W$.*

Proof. Recall that ϕ^q and ψ^q both send $S^{q-1} \times D^{n-q} \to \partial_1 W$. Thus an isotopy between them is $i : (S^{q-1} \times D^{n-q}) \times [0,1] \to \partial_1 W$ with the relevant restrictions to ϕ^q and ψ^q. By lemma 4.1, taking $V = S^{q-1} \times D^{n-q}$ compact, we can extend i to a diffeotopy $H : W \times [0,1] \to W$ with $H_0 = \mathrm{id}_W$. Since H is an extension of i, it follows that $H_1 \circ \phi^q = \psi^q$. We can further guarantee that H is stationary on $\partial_0 W$ by considering only a noncompact subset of W that contains $\partial_1 W$ but not $\partial_0 W$. Since H is stationary on all but a compact subset of this noncompact set, it can be extended to a diffeotopy that is stationary on $\partial_0 W$.

This tells us that $H_1 = H|_{W \times \{1\}} : W \to W$ is a diffeomorphism relative to $\partial_0 W$ with $H_1 \circ \phi^q = i|_{S^{q-1} \times D^{n-q} \times \{1\}} = \psi^q$. Since H_1 is a diffeomorphism of W that sends the embedded part of the handle (ϕ^q) to the embedded part of the handle (ψ^q), we need only join it with the identity map on the remainder of the handle to find a diffeomorphism from $W + (\phi^q) \to W + (\psi^q)$. $\square$

The following lemma allows us to consider the base manifold only up to diffeomorphism when attaching handles. In essence, it says that embedding a handle into two diffeomorphic manifolds is exactly the same.

Lemma 4.3 (Diffeomorphism Lemma)**.** *Take two compact manifolds W and W' with boundaries $\partial W = \partial_0 W \amalg \partial_1 W$ and $\partial W' = \partial_0 W' \amalg \partial_1 W'$. Let $F : W \to W'$ be a diffeomorphism that induces a diffeomorphism $f_0 : \partial_0 W \to \partial_0 W'$.*

Take a handle (ϕ^q) embedded by $\phi^q : S^{q-1} \times D^{n-q} \to \partial_1 W$. We can find an embedding $\overline{\phi}^q : S^{q-1} \times D^{n-q} \to \partial_1 W'$ and a diffeomorphism $F' : W + (\phi^q) \to W' + (\overline{\phi}^q)$ that induces f_0 on $\partial_0 W$. That is, attaching the handles does not change the diffeomorphism on the base part of the boundary.

Proof. We prove this explicitly by taking $\overline{\phi}^q = F \circ \phi^q$. This is clearly an embedding with the right domain and target. To complete the proof we need only find F' and verify the properties.

Let $F' : W \cup_{\phi^q} D^q \times D^{n-q} \to W' \cup_{\overline{\phi}^q} D^q \times D^{n-q}$ be defined by $F \cup_{\phi^q} \mathrm{id}$. Since $F \circ \phi^q = \overline{\phi^q}$, this definition yields a diffeomorphism. And since $\phi^q(S^{q-1} \times D^{n-q}) \subset \partial_1 W$, F' induces the same f_0 as F on $\partial_0 W$. □

The first step in simplifying a handlebody decomposition is to put the handles in order of increasing degree. As mentioned in remark 3.7, addition of handles to a manifold is not obviously commutative, since each added handle changes the boundary to which the subsequent handle can be embedded. Thus it is not immediately clear that we can list the handles in order of ascending degree. The following lemma, however, will allow us to do just that.

Lemma 4.4 (Ordering Lemma)**.** *Let W be a compact n-manifold with boundary $\partial W = \partial_0 W \amalg \partial_1 W$. Let $V = W + (\psi^r) + (\phi^q)$ for two handle embeddings with $q \leq r$. Then V is diffeomorphic relative $\partial_0 W$ to $V' = W + (\overline{\phi}^q) + (\psi^r)$ for an appropriate $\overline{\phi}^q$.*

Proof. The first thing we want to do is separate the embedding ϕ^q from (ψ^r). To do this, first consider the restriction $\phi^q|_{S^{q-1}\times\{0\}} : S^{q-1} \times \{0\} \to \partial_1 W + (\psi^r)$. The dimension of this restriction of the embedding is $q-1$, and the dimension of the transverse sphere of (ψ^r) is $n-r-1$. Since $q \leq r$, we have that $(q-1)+(n-r-1) < n-1$, where $n-1$ is the dimension of the boundary of W.

Since the restriction of ϕ^q together with the transverse sphere of (ψ^r) do not make up the full dimension of the boundary, we can perturb the former slightly into an unoccupied dimension at any intersection of the two. This perturbation can be written as an isotopy, so by lemma 4.1, ϕ^q is isotopic to a map $\hat{\phi}^q$ that does not intersect the transverse sphere of (ψ^r) anywhere along $\hat{\phi}^q|_{S^{q-1}\times\{0\}}$.

Because any embedding of D^k can be isotoped arbitrarily close to the image of any point in the disk, we can isotope $\hat{\phi}^q$ to a map $\overline{\phi}^q$ that sends $S^{q-1} \times D^{n-q}$ to an arbitrarily small neighborhood $U \supset \hat{\phi}^q(S^{q-1} \times \{0\})$. We can take this neighborhood to be disjoint from a closed neighorhood V of the transverse sphere of (ψ^r).

Then there is a clear diffeotopy of $\partial_1(W + (\psi^r))$ that sends V to the boundary of (ψ^r) (that is, $D^r \times S^{n-1-r}$) and the rest of $\partial_1(W + (\psi^r))$ to parts of the boundary completely disjoint from the r-handle. Under this diffeotopy, the q-handle's embedding is completely separate from the r-handle, and so we can switch their order. Therefore, $W + (\psi^r) + (\phi^q) \cong W + (\overline{\phi}^q) + (\psi^r)$ relative to $\partial_0 W$. □

The ordering lemma allows us to sort the handlebody decomposition according to the degree of the handles and to reorder the handles of the same degree however we want. Therefore, combining the ordering lemma with lemma 3.11 we can write any compact manifold W of dimension $n \geq 6$ with disjoint boundary components as

$$W \cong W' = \partial_0 W \times [0,1] + \sum_{i=1}^{p_0} (\phi_i^0) + \sum_{i=1}^{p_1} (\phi_i^1) + \cdots + \sum_{i=1}^{p_n} (\phi_i^n). \tag{4.5}$$

Sorting the decomposition allows us to consider the relationship between handles of subsequent degree. Since our goal is to gradually eliminate the handles, it makes sense to examine situations in which subsequent handles trivialize each other. For example, $(\phi^q) + (\phi^{q+1})$ attached to an n-manifold is trivial if it is diffeomorphic to D^n embedded along a D^{n-1}. Such an attachment can be deformed into the interior, with its boundary simply replacing the part of the boundary that the handles were embedded into.

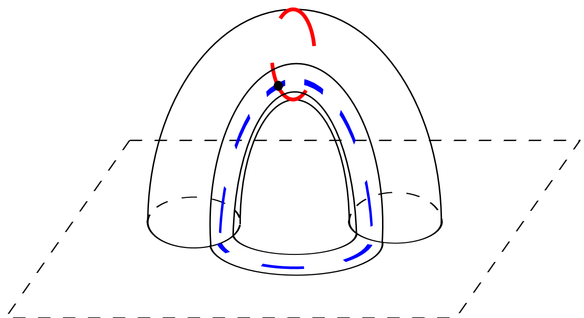

FIGURE 8. Trivial combination of (3,1) and (3,2) handles.

One example is found in figure 8. The rectangle represents a portion of the boundary of a 3-manifold. Attached is a 1-handle, as in figure 3, and a 2-handle. The 2-handle is also a cylinder, but it is embedded along $S^1 \times D^1$. Half of the S^1 lies along $\partial_1 W$, while the other half lies on the boundary of the 1-handle. The D^1 part of the embedding provides substance to the intersection, so that the 2-handle shares a strip with the original boundary and a strip with the 1-handle. The 2-handle itself fills the space between the 1-handle and the original boundary. Once the two handles are joined together, they can be squished into the original manifold and become trivialized.

For our purposes, what is interesting is the intersection of the transverse sphere $\{0\} \times S^{n-q-1}$ of the lower index handle and a single sphere $S^q \times \{0\}$ of the embedded portion of the higher index handle, which is the blue dashed circle in figure 8. Their intersection is indicated with a black dot, and it is this point of intersection that we look at in the cancellation lemma.

The term *transversal* comes up in what follows. Rather than defining it rigorously, we'll say that two objects are transversal if they're not in a special relationship to each other: that is, if they don't happen to coincide in some unexpected way, such as two curves which happen to be tangent rather than intersecting, or two submanifolds that happen to coincide over a certain number of dimensions.

Lemma 4.6 (Cancellation Lemma). *Let W be a compact n-manifold with boundary $\partial W = \partial_0 W \amalg \partial_1 W$. Take two handle embeddings, ϕ^q into $\partial_1 W$ and ψ^{q+1} into $\partial_1(W + (\phi^q))$. If $\psi^{q+1}(S^q \times \{0\})$ is transversal to the transverse sphere of (ϕ^q) and meets the transverse sphere in a single point, then there is a diffeomorphism rel $\partial_0 W$ such that $W \cong W + (\phi^q) + (\psi^{q+1})$.*

Proof. Since $\psi^{q+1}(S^q \times \{0\})$ has dimension q and the transverse sphere of (ϕ^q) has dimension $n - q - 1$ and both lie within the $n - 1$ dimensional $\partial_1(W + (\phi^q))$, transversality simply means that together they 'fill up' the full dimension.

Given a neighborhood $U \subset \partial_1(W + (\phi^q))$ of the transverse sphere $\{0\} \times S^{n-q-1}$, we can find a diffeotopy $F : \partial_1(W + (\phi^q)) \times [0, 1] \to \partial_1(W + (\phi^q))$ such that $F_0 = \mathrm{id}$ and F_1 maps the complement of U to $\partial_1(W + (\phi^q)) - \phi^q(S^q \times \mathrm{int}(D^{n-1-q}))$, while $F(\{0\} \times S^{n-q-1} \times \{t\}) = \{0\} \times S^{n-q-1}$ for all t. That is, F diffeotopes U to cover the whole boundary of the handle (ϕ^q) while pushing the rest of the handle's boundary onto the rest of the manifold's boundary and holding the transverse sphere fixed.

Assume without loss of generality that the intersection between the transverse sphere and $\psi^{q+1}(S^q \times \{0\})$ occurs at the image of the 'north pole' of the q-sphere. Then we can take U to include the image of the northern hemisphere but not include any of the southern hemisphere.

The new embedding obtained by composing $F_1 \circ \psi^{q+1}$ is diffeotopic to the original embedding, so by the isotopy lemma 4.2 we may treat it as though it were the same. Thus ψ^{q+1} sends the lower hemisphere of $S^q \times \{0\}$ outside of the handle (ϕ^q). We can further isotope it so that the upper hemisphere is mapped into the disk $D^q \times \{x\}$ for some $x \in S^{n-q-1}$. This is guaranteed by transversality and single intersection.

This new embedding ψ^{q+1} gives us a picture analogous to that in figure 8. The first step in proving that $W + (\phi^q) + (\psi^{q+1}) \cong W$ is to collapse (ϕ^q) into (ψ^{q+1}).

Consider each non-embedded disk of (ϕ^q): $\{y\} \times D^{n-q}$ for all $y \in D^q$. One of these, $\{0\} \times D^{n-q}$, is the disk whose boundary corresponds to the transverse sphere. Clearly each of these intersect the embedded upper hemisphere $D^q \times \{x\}$ in a point (y, x). Thus they intersect the image of ψ^{q+1} along $\{y\} \times D^{n-1-q}$. Taking the full handle $D^q \times D^{n-q}$, we see that this is an n-manifold embedded in another n-manifold along $D^q \times D^{n-1-q}$. Therefore, we can collapse (ϕ^q) into (ψ^{q+1}).

Now we are left with W and an object $D^{q+1} \times D^{n-1-q}$ that is embedded in W along $S^q_- \times D^{n-1-q}$, where $S^q_- \cong D^q$ denotes the lower hemisphere of the q sphere. The manifold $D^{q+1} \times D^{n-1-q}$ embedded in W along $D^q \times D^{n-1-q}$ is trivial, and so we have a diffeomorphism $W \cong W + (\phi^q) + (\psi^{q+1})$. □

At this point, some notation will be useful moving forward. Recall our current statement of the handlebody decomposition in equation (4.5). The following tools are derived from that statement.

Notations 4.7. Let $-1 \leq q \leq n$. Define

$$\begin{aligned} W_q &:= \partial_0 W \times [0,1] + \sum_{i=1}^{p_0} (\phi_i^0) + \sum_{i=1}^{p_1} (\phi_i^1) + \cdots + \sum_{i=1}^{p_q} (\phi_i^q), \\ \partial_1 W_q &:= \partial W_q - \partial_0 W \times \{0\}, \text{ and} \\ \partial_1^\circ W_q &:= \partial_1 W_q - \coprod_{i=1}^{p_{q+1}} \phi_i^{q+1}(S^q \times \operatorname{int}(D^{n-1-q})). \end{aligned}$$

The submanifold W_q consists of the trivial product structure together with all of the handles up to index q. The 'upper' component of the boundary of W_q is $\partial_1 W_q$, and $\partial_1^\circ W_q$ gives us the the same upper boundary minus the interiors of the regions into which $(q+1)$-handles will be embedded in W_{q+1}. It can be thought of as the 'safe' or 'unaffected' part of the boundary.

The following lemma characterizes handles attached by trivial embeddings (recall definition 3.8) as particularly easy to cancel.

Lemma 4.8 (Trivial Cancellation)**.** *If $\phi^q : S^{q-1} \times D^{n-q} \to \partial_1 W$ is a trivial embedding then there is an embedding $\phi^{q+1} : S^q \times D^{n-1-q} \to \partial_1(W + (\phi^q))$ such that $W \cong W + (\phi^q) + (\phi^{q+1})$.*

Proof. Since ϕ^q is a trivial embedding, we can write it as the composition of embeddings $\phi^q = f \circ g$ with $f : D^{n-1} \to \partial_1 W$ and $g : S^{q-1} \times D^{n-q} \to D^{n-1}$. By the cancellation lemma 4.6, we want to find an embedding ϕ^{q+1} such that $\phi^{q+1}(S^q \times \{0\})$ is transversal to the transverse sphere of (ϕ^q) and meets it in a single point.

The transverse sphere of (ϕ^q) is $\{0\} \times S^{n-1-q}$ and it is a piece of the boundary of the handle $D^q \times S^{n-1-q}$. Let $p_1 : D^q \to D^q \times S^{n-1-q}$ be defined by some fixed point $x \in S^{n-1-q}$ and the identity on D^q. The image of p_1 intersects the transverse sphere transversally and exactly once, and the boundary of the D^q lies in $\partial_1 W$ through the embedding ϕ^q, along the boundary of the embedded $S^{q-1} \times D^{n-q}$. Then, since that $S^{q-1} \times D^{n-q}$ lies in a D^{n-1} (through f^{-1}), we can define $p_2 : D^q \to D^{n-1}$ to avoid the image $g(S^{q-1} \times D^{n-q}) \subset D^{n-1}$. By letting p_1 and p_2 define the embeddings of the upper and lower hemispheres of an S^q and extending the embedding to $S^q \times D^{n-1-q}$, we have a ϕ^{q+1} that satisfies the conditions of the cancellation lemma and thus $W \cong W + (\phi^q) + (\phi^{q+1})$. □

The elimination lemma will provide a way of replacing handles with handles of higher index, but before we approach its proof we need another lemma from [2, Chapter 8].

Lemma 4.9. *Let V be a compact submanifold of a manifold N, and let $f_0, f_1 : V \to M - \partial M$ be isotopic embeddings. If f_0 extends to an embedding $N \to M$ then so does f_1. Further, the extension of f_1 is isotopic to the extension of f_0.*

Proof. Recall that by the definition of isotopy we have a function $H : V \times [0,1] \to M - \partial M$ such that $H_0 = f_0$, $H_1 = f_1$, and each H_t is a homeomorphism. We can find a new isotopy $H' : f_0(V) \times [0,1] \to M - \partial M$ by simply taking $H' = H \circ (f_0^{-1} \times \mathrm{id})$. Note that $H'_1 = H_1 \circ f_0^{-1} = f_1 f_0^{-1}$.

By lemma 4.1, H' extends to a diffeotopy $G : M \to M$. Since it is an extension, we know that $G_1|_{f_0(V)} = H'_1 = f_1 f_0^{-1}$, or $G_1 f_0 = f_1 f_0^{-1} f_0 = f_1$.

Let $j : N \to M$ be the given extension of f_0. Then we can set $k = G_1 j : N \to M$ and it will be an extension of f_1. And since G_1 is a diffeomorphism (since G is a diffeotopy), we have that j and k are isotopic. □

Lemma 4.10 (Elimination Lemma)**.** *Fix q such that $1 \leq q \leq n-3$, and suppose that the handlebody decomposition has no handles of index $j < q$. Fix i_0 with $1 \leq i_0 \leq p_q$ to denote a specific q-handle. Further suppose that there is an embedding $\psi^{q+1} : S^q \times D^{n-1-q} \to \partial_1^\circ W_q$ that satisfies these conditions:*

(1) If we restrict $\psi^{q+1}|_{S^q \times \{0\}}$, it is isotopic in $\partial_1 W_q$ to another embedding $\chi : S^q \times \{0\} \to \partial_1 W_q$, where χ meets the transverse sphere of $(\phi^q_{i_0})$ transversally and in exactly one point, and where χ is also disjoint from the transverse spheres of all other q-handles.

(2) The same restriction of ψ^{q+1} to $S^q \times \{0\}$ is isotopic in $\partial_1 W_{q+1}$ to a trivial embedding $\chi' : S^q \times \{0\} \to \partial_1 W_{q+1}$.

Then, we can replace $(\phi^q_{i_0})$ with a $(q+2)$-handle, meaning that the handlebody decomposition can be written as

$$\partial_0 W \times [0,1] + \sum_{i=1, i \neq i_0}^{p_q} (\phi^q_i) + \sum_{i=1}^{p_{q+1}} (\phi^{q+1}_i) + (\psi^{q+2}) + \sum_{i=1}^{p_{q+2}} (\phi^{q+2}_i) + \cdots + \sum_{i=1}^{p_n} (\phi^n_i).$$

Remark 4.11. Note that the handles of index $q+1$ and greater will not have the same embedding functions as they did before the switch. Since we only really care about the number of handles of each index, this isn't a concern.

Proof. The first thing to do is to extend χ and χ' to legitimate $(q+1)$-handle embedding functions $\psi_1^{q+1}, \psi_2^{q+1}$ using lemma 4.9 with $V = S^q \times \{0\}$, $N = S^q \times D^{n-1-q}$, and $M = \partial_1 W_q$ or $\partial_1 W_{q+1}$.

By the lemma, ψ^{q+1} is isotopic to ψ_1^{q+1}, which is isotopic to ψ_2^{q+1}. We obviously have that $\psi_1^{q+1}|_{S^q\times\{0\}}$ meets the transverse sphere of $(\phi_{i_0}^q)$ transversally and in exactly one point, since that is a property of χ in (1). And further, since χ is disjoint from the transverse spheres of all the other q-handles, we can make ψ_1^{q+1} similarly disjoint from those transverse spheres. Also, since χ' is trivial its extension ψ_2^{q+1} must also be trivial.

Since we have stayed away from higher index handles, we can safely ignore them and add new ones back in at the end. That is, if we show a diffeomorphsim

$$\partial_0 W\times[0,1]+\sum_{i=1}^{p_q}(\phi_i^q)+\sum_{i=1}^{p_{q+1}}(\phi_i^{q+1}) \cong \partial_0 W\times[0,1]+\sum_{i=1,i\neq i_0}^{p_q}(\phi_i^q)+\sum_{i=1}^{p_{q+1}}(\phi_i^{q+1})+(\psi^{q+2})$$

then we can show a diffeomorphism between the two with the handles of index $q+2$ or greater added back in by the diffeomorphism lemma 4.3.

Since ψ_2^{q+1} is trivial, by the trivial cancellation lemma 4.8 we know there exists some ψ^{q+2} such that

$$\begin{aligned}&\partial_0 W\times[0,1]+\sum_{i=1}^{p_q}(\phi_i^q)+\sum_{i=1}^{p_{q+1}}(\phi_i^{q+1})\\ \cong\ &\partial_0 W\times[0,1]+\sum_{i=1}^{p_q}(\phi_i^q)+\sum_{i=1}^{p_{q+1}}(\phi_i^{q+1})+(\psi_2^{q+1})+(\psi^{q+2}),\end{aligned}$$

which, by the isotopy lemma 4.2,[1] is diffeomorphic to

$$\partial_0 W\times[0,1]+\sum_{i=1}^{p_q}(\phi_i^q)+\sum_{i=1}^{p_{q+1}}(\phi_i^{q+1})+(\psi_1^{q+1})+(\psi^{q+2}).$$

By a simple application of the ordering lemma 4.4 we can move $(\phi_{i_0}^q)$ and (ψ_1^{q+1}) and get that the above is diffeomorphic to

$$\partial_0 W\times[0,1]+\sum_{i=1,i\neq i_0}^{p_q}(\phi_i^q)+(\phi_{i_0}^q)+(\psi_1^{q+1})+\sum_{i=1}^{p_{q+1}}(\phi_i^{q+1})+(\psi^{q+2}).$$

Since $\psi_1^{q+1}(S^q\times D^{n-1-q})$ meets the transverse sphere of $(\phi_{i_0}^q)$ transversally and in a single point by assumption, by the cancellation lemma 4.6[2] we get that

$$\partial_0 W\times[0,1]+\sum_{i=1}^{p_q}(\phi_i^q)+\sum_{i=1}^{p_{q+1}}(\phi_i^{q+1}) \cong \partial_0 W\times[0,1]+\sum_{i=1,i\neq i_0}^{p_q}(\phi_i^q)+\sum_{i=1}^{p_{q+1}}(\phi_i^{q+1})+(\psi^{q+2}).$$

As stated above, we can at this point use the diffeomorphism lemma to add in the $(q+2)$-and higher handles to both sides of the diffeomorphism, and obtain the diffeomorphism called for in the lemma. □

[1]For this step we also technically need the diffeomorphism lemma and a different $(q+2)$-handle embedding, since ψ_1^{q+1} has a different boundary available for embedding than ψ_2^{q+1}. This is a minor complication, though, since it all ends up the same.

[2]Again, we technically need to use the diffeomorphism lemma and change the $(q+1)$- and $(q+2)$-handle embeddings. But the numbers of the handles remains the same, so changing the notation is unnecessary.

While the elimination lemma is useful, it is not the final solution. In order for it to apply we must have eliminated at least the 0-handles by another method (since $1 \leq q$) and it will not change the total number of handles. It will merely shift the handles to ones of higher index, up to shifting handles from index $n-3$ to index $n-1$. More tools are necessary in order to simplify the handlebody decomposition.

The next lemma allows us to eliminate handles of index 0 and 1, and be left with only higher index handles.

Lemma 4.12. *Take W to be a compact manifold of dimension at least 6, with $\partial W = \partial_0 W \amalg \partial_1 W$. If the inclusion $\partial_0 W \to W$ is 1-connected, then we can eliminate the 0- and 1- handles. That is,*

$$W \cong W' = \partial_0 W \times [0,1] + \sum_{i=1}^{p_2} (\phi_i^2) + \cdots + \sum_{i=1}^{p_n} (\phi_i^n).$$

Proof. First, we want to eliminate the 0-handles. Consider what they look like: $D^0 \times D^n \cong D^n$, embedded along $\partial D^0 \times D^n$, which is empty. So 0-handles are simply D^ns that are disjoint from $\partial_0 W \times [0,1]$. However, since the inclusion $\partial_0 W \to W$ is 1-connected, we know there is an isomorphism between the path components of $\partial_0 W$ and those of W. Since $\pi_0(\partial_0 W \times [0,1])$ is isomorphic to $\pi_0(\partial_0 W)$, that means that $\pi_0(\partial_0 W \times [0,1])$ is isomorphic to $\pi_0(W)$.

That means that for each (ϕ_i^0) there must be a higher degree handle that connects it to a path component of $\partial_0 W \times [0,1]$. Since S^0 is the only disconnected S^k, that connecting handle must be a 1-handle. Therefore, for each (ϕ_i^0) there exists (ϕ_j^1) such that $\phi_j^1(\{-1\} \times D^{n-1}) \subset \partial_0 W \times \{1\}$ and $\phi_j^1(\{1\} \times D^{n-1}) \subset S^{n-1} = \partial(\phi_i^0)$, if we let $S^0 = \{-1, 1\}$.

The transverse sphere of a 0-handle is simply its boundary S^{n-1}. Looking at the requirements of the cancellation lemma 4.6, we see that $\phi_j^1(S^0 \times \{0\})$ is transversal to the transverse sphere of (ϕ_i^0) and meets it in exactly one point, $(1,0)$. Therefore, for each i, j we can cancel (ϕ_i^0) and (ϕ_j^1) by the cancellation lemma.

Of course, there may be more 1-handles left that we also need to eliminate. First, reindex the handlebody decomposition so that it is

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_1} (\phi_i^1) + \cdots + \sum_{i=1}^{p_n} (\phi_i^n). \tag{4.13}$$

We will get rid of the remaining 1-handles using the elimination lemma 4.10. To do this, we need to construct an embedding $\psi_i^2 : S^1 \times D^{n-2} \to \partial_1^\circ W_1$ for each ϕ_i^1 that, when restricted to $S^1 \times \{0\}$, is isotopic in $\partial_1 W_1$ to two other embeddings. One must intersect the transverse sphere of (ϕ_i^1) transversally and in exactly one point and be disjoint from the other 1-handles' transverse spheres. The other must be isotopic in $\partial_1 W_2$ to a trivial embedding. Using this 2-handle we will replace (ϕ_i^1) with a 3-handle (ψ_i^3).

We will construct ψ_i^2 first on $S^1 \times \{0\}$, which we will do by considering the two halves of S^1: S^1_-, S^1_+. First, let $\psi_{i_+}^2 : S^1_+ \to \partial_1^\circ W_1$ be the embedding of the upper half of the circle, and define it by a standard embedding of $S^1_+ \to D^1$, with $D^1 = D^1 \times \{x_0\}$ representing a line along the boundary of $(\phi_i^1) = D^1 \times D^{n-1}$. Clearly, this intersects the transverse sphere $\{0\} \times S^{n-2}$ transversally and in exactly one point $(0, x_0)$.

Now we want to embed the lower half of the circle S^1_- in such a way that the overall ψ_i^2 is isotopic to a trivial embedding. First, though, a detour into algebra is necessary.

The unaffected boundary $\partial_1^\circ W_0$ is formed from $\partial_1 W_0 = \partial_0 W \times \{1\}$ (since we have no 0-handles left) by removing a finite number of disks D^{n-1}, the images of $S^0 \times D^{n-1}$ for each 1-handle. (There will be $2p_1$ such disks removed.) In any manifold of dimension $n \geq 3$, the removal of a D^n does not change the fundamental group because each $\pi_1(S^n)$ is isomorphic to the trivial group for $n \geq 2$. Therefore, the fundamental group of $\partial_1^\circ W_0$ is isomorphic to the fundamental group of $\partial_1 W_0$, which is the same as the fundamental group of $\partial_0 W$.

Recall that the 1-connectedness of the inclusion $\partial_0 W \to W$ means that we have an induced surjection between the fundamental group of $\partial_0 W$ (equivalently the fundamental group of $\partial_1^\circ W_0$) and the fundamental group of W.

Now, to find an appropriate embedding $\psi_{i_-}^2$, start by picking an arbitrary embedding $p : S^1_- \to \partial_1^\circ W_0$ that joins up with $\psi_{i_+}^2$ at the endpoints. This gives us a loop $l \in W$, which corresponds to an element $\alpha \in \pi_1(W)$. Look at α^{-1}, and take some preimage l' of α^{-1} in $\partial_1^\circ W_0$, which exists (though not necessarily uniquely) by the surjection between the fundamental groups. By combining p and l' we get a single embedding $\psi_{i_-}^2 : S^1_- \to \partial_1^\circ W_0$.[3]

Since the isotoping does not change the homotopy class, our new loop $\psi_i^2 = \psi_{i_+}^2 \cup \psi_{i_-}^2$ corresponds to the element $\alpha\alpha^{-1} = 0$ in the fundamental group, and thus ψ_i^2 is nullhomotopic in W.

We want ψ_i^2 to be trivial in $\partial_1 W_2$, so we first want nullhomotopy in W to imply nullhomotopy in $\partial_1 W_2$.

Note that any 3-or-greater-handle is embedded along $S^{q-1} \times D^{n-q}$, where $q \geq 3$. This embedded boundary will always be simply connected, since S^k is simply connected for $k \geq 2$. Thus any homotopy that passes through a 3-or-greater-handle can be homotoped around that handle. So any two functions in W_2 that are homotopic in W are also homotopic in W_2.

Now we want to show that functions in $\partial_1 W_2$ that are homotopic in W_2 are also homotopic in $\partial_1 W_2$. If the homotopy passes through $\partial_0 W \times [0, 1)$, we can obviously homotope it so that it only passes through $\partial_0 W \times \{1\}$ and the 1 and 2 handles. If it passes through a 1-handle $D^1 \times D^{n-1}$ we can homotope it so that passes only through one of the embedded D^{n-1}s and the boundary. And, as mentioned above, since we are in an $(n-1)$-manifold with $n - 1 \geq 3$, the removal of a D^{n-1} does not change the fundamental group, so if the homotopy passes through the embedded D^{n-1}s we can homotope it so that it passes only through the boundary.

The only problem that remains is if the homotopy passes through a 2-handle. However, since we know that $n \geq 6$, the boundary of any 2-handle will be of the form $D^2 \times S^{n-3}$, with $n - 3 \geq 3$. Since disks are simply connected and S^k is simply connected for $k \geq 2$, this means that the boundaries of the 2-handles are simply connected, and thus any homotopy that passes through a 2-handle can be homotoped to only lie on its boundary. Therefore, nullhomotopy in W implies nullhomotopy in $\partial_1 W_2$.

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[3]We will have to isotope it slightly first, so that the joining of p and l' occurs away from the endpoint of the embedding and to eliminate all the other self-intersections of the embedding, but this is not a problem.

Our nullhomotopy in $\partial_1 W_2$ is a continuous map $H : S^1 \times [0,1] \to \partial_1 W_2$ where $H(S^1 \times \{1\})$ is constant, so we can equivalently think of it as a continuous map $H : D^2 \to \partial_1 W_2$. This continuous map is not necessarily an embedding, however, so we don't yet have triviality. For that, we need the "easy Whitney embedding theorem" (see [2, Chapter 1, Theorem 3.5] and the notes following it for the proof) which tells us that any continuous map from an n-manifold to $\mathbb{R}^{2n+1}$ (or greater dimensional Euclidean space) can be approximated by an embedding into the same Euclidean space. Since, by assumption, $\partial_1 W_2$ has dimension at least 5, D^2 has dimension only 2, and $\partial_1 W_2$ is a manifold (we can treat it locally as though it were a subset of $\mathbb{R}^{\geq 5}$), we can approximate our nullhomotopy by an embedding of D^2 into $\partial_1 W_2$. Since $\partial_1 W_2$ has no boundary, this approximation is also guaranteed to be contained in $\partial_1 W_2$.

So we now have our embedding ψ_i^2 defined on $S^1 \times \{0\}$ in a way that it intersects the transverse sphere of (ϕ_i^1) transversally and in exactly one point, and such that it is trivial in $\partial_1 W_2$. We can easily extend it to the full $S^1 \times D^{n-2}$, at which point it satisfies the conditions of the elimination lemma 4.10, and we can replace our handle (ϕ_i^1) by a handle (ψ_i^3). By repeating this process for each 1-handle, we eliminate all of the 1-handles, and we can rewrite the handlebody decomposition (4.13) as

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_2} (\phi_i^2) + \cdots + \sum_{i=1}^{p_n} (\phi_i^n). \tag{4.14}$$

□

Lemma 4.15. *Given an h-cobordism $(W; \partial_0 W, \partial_1 W)$ where $\partial_0 W$ is simply connected and a handlebody decomposition as in equation (4.14), each W_q and $\partial_1 W_q$ is also simply connected.*

Proof. Since W is an h-cobordism, the inclusion $\partial_0 W \to W$ is a homotopy equivalence, which gives us an isomorphism on the homotopy groups. Thus W is simply connected. Since all handles of index ≥ 2 (index 1 and 0 have been removed) are simply connected manifolds embedded along simply connected manifolds, removing them only simplifies the fundamental group: any homotopy of loops in the full space that passes through a handle can be changed to a new homotopy that passes along the embeddings of those handles. Thus there is a surjection from $\pi_1(W) \to \pi_1(W_q)$ for all q. Since W is simply connected, this means that each W_q is simply connected.

For the boundary components $\partial_1 W_q$, note that $\partial_1 W$ is simply connected since it is homotopy equivalent to W. Each $\partial_1 W_q$ is constructed from $\partial_1 W_{q+1}$ by removing and adding simply connected manifolds, so again we have a surjection $\pi_1(\partial_1 W) \to \pi_1(\partial_1 W_q)$ for all q. Thus each $\partial_1 W_q$ is simply connected. □

We now have to step back into homology. Recall from remark 3.12 that we can treat the q-handles of a decomposition as q-cells for purposes of relative homology groups.

Lemma 4.16 (Homology Lemma)**.** *Let $(W; \partial_0 W, \partial_1 W)$ be an $n \geq 6$ dimensional h-cobordism with $\partial_0 W$ simply connected. Fix a handle of index $2 \leq q \leq n-3$ such that there are no handles of index less than q, and select a specific handle $(\phi_{i_0}^q)$. Let $f : S^q \to \partial_1 W_q$ be an embedding. Let $[f]$ represent the image of the class of* *maps homotopic to f under the following composition of functions:*

$$\pi_q(W_q) \to \pi_q(W_q, \partial_0 W \times [0,1]) \to H_q(W_q, \partial_0 W \times [0,1]) = C_q(W)$$

If $[f] = \pm[\phi^q_{i_0}] \in C_q(W)$, then f is isotopic to an embedding $g : S^q \to \partial_1 W_q$ where g meets the transverse sphere of $(\phi^q_{i_0})$ transversally and in exactly one point, and is disjoint from the transverse spheres of the handles (ϕ^q_i) for $i \neq i_0$.

Proof. First, note that the equality shown above, $H_q(W_q, \partial_0 W \times [0,1]) = C_q(W)$, holds. The q-th relative homology group here consists of $\mathbb{Z}$-combinations of q-handles that embed in $\partial_0 W \times [0,1]$ (which is all of them, since there are no $< q$ handles), modded out by q-handles which are the images of $q+1$ handles under the boundary operator, which is none of them, because W_q has no $q+1$ handles by definition. This combination, then, is exactly $C_q(W)$.

Now, examine what the composition does. First, it associates the embedding f to its element in the q-th homotopy group, i.e. it equates it with all other homotopic embeddings of S^q into W_q. The next step equates it to all homotopic embeddings of S^{q-1} into $\partial_0 W \times [0,1]$ that are the boundary of embeddings of D^q into W_q. Of course, since f is an embedding of S^q into $\partial_1 W_q$, this will always have S^{q-1} identified to a point. However, we can stretch out the point of the S^q that lies in $\partial_0 W \times [0,1]$. The part added in the middle of this 'stretching' will form the lower hemisphere of a new, isotopic embedding f, and the upper hemisphere will be a D^q such that its boundary S^{q-1} lies in $\partial_0 W \times [0,1]$.

Recall that q-handles are $D^q \times D^{n-q}$ embedded along $S^{q-1} \times D^{n-q}$, so the equivalence classes in $\pi_q(W_q, \partial_0 W \times [0,1])$ can be equated to q-handles in $H_q(W_q, \partial_0 W \times [0,1]) = C_q(W)$. Therefore, $[f] \in C_q(W)$ denotes the handle (ψ^q) such that the embedding of the upper hemisphere of S^q under f is isotopic to the core of (ψ^q).

If $[f] = \pm[\phi^q_{i_0}]$ then the upper hemisphere of f is isotopic to the core of the handle $(\phi^q_{i_0})$. We can then isotope it to the boundary of that handle, guaranteeing that it intersects the transverse sphere of that handle transversally and in a single point. To show that it is disjoint from the other transverse spheres, only note that we can define the lower hemisphere of this new f such that it lies entirely within the original product structure minus the interior of the embeddings of the q-handles. The upper hemisphere lies entirely within the target q-handle, so we have disjointness. □

Lemma 4.17 (Modification Lemma)**.** *Let $(W; \partial_0 W, \partial_1 W)$ be an h-cobordism with $\partial_0 W$ simply connected, and let $f : S^q \to \partial_1^\circ W_q$ be an embedding. Take $x_j \in \mathbb{Z}$ for $1 \leq j \leq p_{q+1}$. Then there is another embedding $g : S^q \to \partial_1^\circ W_q$ that is isotopic to f in $\partial_1 W_{q+1}$ and, using the notation developed in the homology lemma 4.16, $[g] = [f] + \sum_{j=1}^{p_{q+1}} x_j \cdot d_{q+1}[\phi^{q+1}_j] \in C_q(W)$. Note that d_{q+1} is the boundary operator from $C_{q+1}(W, \partial_0 W) \to C_q(W, \partial_0 W)$: that is, d_{q+1} maps $(q+1)$-handles to q-handles.*

Proof. First, to clarify the notation, note that $[g] = [f] + \sum_{j=1}^{p_{q+1}} x_j \cdot d_{q+1}[\phi^{q+1}_j]$ means that g is homologous to f joined to the core of each handle (ϕ^{q+1}_j) x_j times.

It suffices to prove that we can find g such that $[g] = [f] \pm d_{q+1}[\phi^{q+1}_j]$ for some arbitrary j: from that point we can proceed by induction.

For a given (ϕ^{q+1}_j), let t_j be an embedding $S^q \to S^q \times S^{n-2-q} \subset \partial(\phi^{q+1}_j) \subset \partial_1 W_q$ given by fixing a point $z \in S^{n-2-q}$ and sending S^q to $S^q \times \{z\}$. Note that t_j is trivial in $\partial_1 W_{q+1}$ because it is contained in $D^{q+1} \times \{z\} \subset \partial(\phi^{q+1}_j) \subset \partial_1 W_{q+1}$.

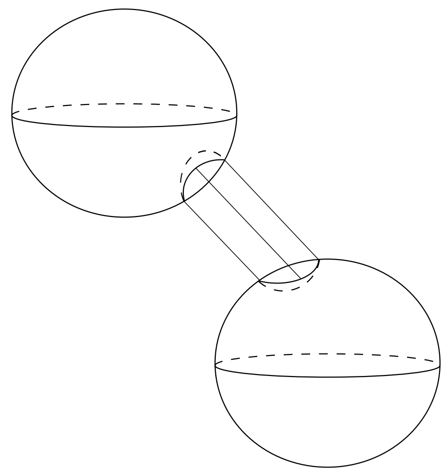

FIGURE 9. The connect sum of two S^2 along a $D^2 \times [0,1]$ with each end embedded in one of the spheres.

Since $\partial_1 W_q$ is path connected, we can define a path w_j as an embedding $[0,1] \to \partial_1 W_q$ connecting the images of f and t_j. That is, $w_j(0) \in f(S^q)$, $w_j(1) \in t_j(S^q)$. Since $\partial_1 W_q$ is simply connected, w_j is unique up to homotopy.

By thickening w_j to an embedding of $[0,1] \times D^q$, we can obtain a new embedding $v : S^q \to \partial_1 W_q$ as the connect sum of f and t_j, where the boundary of w_j defines the connection between the two spheres (see figure 9).

Since t_j is trivial, the embedded S^q minus a D^q can be isotoped back into that removed D^q, and then w_j can be isotoped back to the D^q removed from the image of f, so v is isotopic to f. That means that we can also isotope it just slightly so that we get a new embedding g isotopic to f that lies in $\partial_1^\circ W_q$.

When we pass g through the composition of functions we get that $[g] = [f] \pm d_{q+1}[\phi_j^{q+1}]$, where the sign may be altered by changing the orientation of t_j. □

The next step in the proof eliminates almost all of the remaining handles, leaving only handles of two consecutive indices $q, q+1$. This mostly reduced form is called the 'normal form' of the handlebody decomposition. The proof of the normal form lemma involves two steps. First, we eliminate all the handles of index $k < q$. Then we use the dual handlebody lemma to turn the handles of index $k > q+1$ into handles of index $k < q$ and repeat the first step.

Lemma 4.18 (Dual Handlebody Lemma)**.** *Let $(W; \partial_0 W, \partial_1 W)$ be an n-dimensional cobordism and suppose that*

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_0} (\phi_i^0) + \sum_{i=1}^{p_1} (\phi_i^1) + \cdots + \sum_{i=0}^{p_n} (\phi_i^n).$$

Then we can build W up from the other boundary component by finding a handle of index $(n-k)$ for each k-handle in the original decomposition. That is,

$$W \cong \partial_1 W \times [0,1] + \sum_{i=1}^{p_n} (\psi_i^0) + \sum_{i=1}^{p_{n-1}} (\psi_i^1) + \cdots + \sum_{i=0}^{p_0} (\psi_i^n).$$

The proof of the dual handlebody lemma is rather technical, so we'll omit the details here and relegate them to Appendix A.

Lemma 4.19 (Normal Form Lemma)**.** *Let $(W, \partial_0 W, \partial_1 W)$ be an oriented, compact h-cobordism of dimension at least 6 with $\partial_0 W$ simply connected. Then for any q with $2 \le q \le n-3$, we have handles such that*

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}).$$

Proof. The basic approach of this proof is inductive: we will eliminate k-handles with the assumption that there are no $(k-1)$-handles. Because of lemma 4.12 the base cases $k=0$ and $k=1$ are already solved. In order to remove each k-handle we will add a $(k+1)$- and a $(k+2)$- handle, such that each pair $(\phi_i^k), (\psi_i^{k+1})$ and $(\psi_i^{k+1}), (\psi_i^{k+2})$ cancel each other. The net result will be a diffeomorphic handlebody decomposition where the k-handle is replaced by a $(k+2)$-handle.

Note that since the inclusion $\partial_0 W \to W$ is a homotopy equivalence, all of the relative homology groups $H_p(W, \partial_0 W)$ are trivial. Recall that $H_p(W, \partial_0 W)$ is the p-handles that attach directly to the product structure, modded out by the embedding targets of $(p+1)$-handles. Since we have that $H_k(W, \partial_0 W) = 0$, and there are no $< k$ handles, this means that all of the k-handles are targets of the embeddings of $(k+1)$-handles, so the boundary operator d_{k+1} is surjective.

Now we can select a k-handle (ϕ_i^k) and write

$$[\phi_i^k] = \sum_{j=1}^{p_{k+1}} x_j \cdot d_{k+1}[\phi_j^{k+1}]$$

by the surjectivity of d_{k+1}, for $x_j \in \mathbb{Z}$.

Now fix an arbitrary trivial embedding $\overline{\psi}_i^{k+1} : S^k \times D^{n-1-k} \to \partial_1^\circ W_k$. Since $\overline{\psi}_i^{k+1}|_{S^k \times \{0\}}$ is nullhomotopic ($\overline{\psi}_i^{k+1}$ is trivial), $[\overline{\psi}_i^{k+1}] = 0$. By the modification lemma 4.17 we can find an embedding ψ_i^{k+1} that is isotopic to $\overline{\psi}_i^{k+1}$ with $[\psi_i^{k+1}] = [\phi_i^k]$.

Now, by letting $f_i = \psi_i^{k+1}|_{S^k \times \{0\}}$ and applying the homology lemma 4.16 we can find an isotopic embedding that meets the transverse sphere of (ϕ_i^k) transversally and in a single point. Since ψ_i^{k+1} is still isotopic to the trivial embedding $\overline{\psi}_i^{k+1}$, we can apply the elimination lemma 4.10 and replace the handle (ϕ_i^k) with another handle (ψ_i^{k+2}). By repeating this process for each k-handle, and then proceeding inductively over k up to $q-1$, we can eliminate all of the handles of index less than q at the cost of increasing the number of handles of index q and $q+1$.

At this point, we have a handlebody decomposition of the form

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \cdots + \sum_{i=1}^{p_n} .(\phi_i^n). \tag{4.20}$$

An application of the dual handlebody lemma 4.18 yields a handlebody decomposition of the form

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_0} (\phi_i^0) + \cdots + \sum_{i=1}^{p_{n-q}} (\phi_i^{n-q}). \tag{4.21}$$

We can apply lemma 4.12 to eliminate the 0 and 1 handles, and then repeat the argument above in this proof with $q' = n - q - 1$ (note that since $2 \le q \le n-3$, we have that $2 \le n-q-1 \le n-3$). Thus (after another application of the dual handlebody lemma) we obtain a handlebody decomposition of the form

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}). \tag{4.22}$$

□

Now that we have this easily manageable handlebody, we want to describe it in terms of a matrix. To do this, we look at the relative homology groups $H_{q+1}(W_{q+1}, W_q)$ and $H_q(W_q, W_{q-1})$. Clearly $\{[\phi_i^{q+1}]\}_{1\le i\le p_{q+1}}$ is a $\mathbb{Z}$-basis for $H_{q+1}(W_{q+1}, W_q)$ and $\{[\phi_i^q]\}_{1\le i\le p_q}$ is a $\mathbb{Z}$-basis for $H_q(W_q, W_{q-1})$. It turns out (because these two relative homology groups are the only nonzero entries in a long exact sequence) that the boundary operator is a bijection between these two groups, so their bases are the same size, i.e. $p_q = p_{q+1}$.

Definition 4.23. The *representative matrix* of an h-cobordism in normal form (as in equation (4.22)) is a p_q by p_q matrix describing the action of the boundary operator on the basis $\{[\phi_i^{q+1}]\}_{1\le i\le p_{q+1}}$ in terms of the basis $\{[\phi_i^q]\}_{1\le i\le p_q}$.

Remark 4.24. The representative matrix will be invertible. This follows from the fact that the map it describes is a bijection, and thus invertible.

Lemma 4.25. *Take an h-cobordism $(W; \partial_0 W, \partial_1 W)$ with $\dim(W) \ge 6$ and $\partial_0 W$ simply connected, and its representative matrix $A \in M_{p_q}(\mathbb{Z})$. Let $B \in M_j(\mathbb{Z})$ be any matrix formed from A using any of the following operations:*

(1) B is obtained from A by adding a multiple of the k-th row to the l-th row, for $k \ne l$;
(2) B is obtained from A by multiplying the k-th row by -1;
(3) B is obtained from A by interchanging two rows or two columns;
(4) B is of the form $A \oplus I_1$, i.e. $B = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$; or
(5) A is of the form $B \oplus I_1$, i.e. $A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$.

Then there is another handlebody decomposition of W that has B as a representative matrix.

Remark 4.26. For general h-cobordisms in normal form, the group of $\mathbb{Z}\pi$ matrices under the equivalence class of the operations above is called the Whitehead group. This general theory is used to prove the s-cobordism theorem.

Proof. We will approach this proof separately for each operation. Operation 4 is included for completeness, even though it it not necessary in order to prove the h-cobordism theorem.

(1) We want to change the l-th row of the matrix, which represents the image $d_{q+1}[\phi_l^{q+1}]$ as a $\mathbb{Z}$-vector relative to the basis of q-handles. Consider

$$W' = \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1, i\ne l}^{p_q} (\phi_i^{q+1}).$$

By the modification lemma 4.17 we can find an embedding $\overline{\phi}_l^{q+1}$ that is isotopic to ϕ_l^{q+1} with $[\overline{\phi}_l^{q+1}] = [\phi_l^{q+1}] + x \cdot d_{q+1}[\phi_k^{q+1}]$ $(x \in \mathbb{Z})$. By the isotopy lemma 4.2, $W = W' + (\phi_l^{q+1}) \cong W' + (\overline{\phi}_l^{q+1})$. And if we take the representative matrix of the new handlebody decomposition of W, the l-th row will have changed by x times the k-th row, since the k-th row is the representation of $d_{q+1}[\phi_k^{q+1}]$ in the matrix.

(2) We want to change the k-th row, which represents (ϕ_k^{q+1}). Multiplying this row by -1 is represented by changing the orientation of the embedding ϕ_k^{q+1}, which can be achieved by composing the embedding with a diffeomorphism $S^q \to S^q$ of degree -1.

(3) This is simply an application of the ordering lemma 4.4: we can reorder the handles however we wish, which corresponds to either changing rows or columns depending on which index handles we rearrange.

(4) First, we attach a q-handle (ψ^q) that does not touch any of the existing handles, giving us the matrix $\begin{pmatrix} A & 0 \end{pmatrix}$. Then we can find an embedding f of S^q such that $[f] = \pm[\psi^q]$. Then we can extend f to a $(q+1)$-handle embedding. Our new handlebody has the desired matrix, and by the homology lemma 4.16 and the cancellation lemma 4.6 we can isotope the handles so that they cancel each other, and our new handlebody is diffeomorphic to our original one.

(5) This is the exact same process as (4), but in reverse. Looking at our last handles $(\phi_{p_q}^q), (\phi_{p_q}^{q+1})$, they satisfy the homology lemma and thus we can cancel them with the cancellation lemma, giving us the smaller matrix. □

Theorem 4.27 (h-Cobordism Theorem)**.** *Any h-cobordism $(W; \partial_0 W, \partial_1 W)$ with $\dim(W) \geq 6$ and $\partial_0 W$ simply connected is diffeomorphic relative to $\partial_0 W$ to the trivial h-cobordism $(\partial_0 W \times [0,1]; \partial_0 W \times \{0\}, \partial_0 W \times \{1\})$.*

Proof. By the normal form lemma 4.19 we can find a handlebody decomposition diffeomorphic to W relative to $\partial_0 W$ in the form of equation (4.22). Then we can take its representative matrix, an invertible p_q by p_q matrix over $\mathbb{Z}$. Since every invertible matrix in $M_n(\mathbb{Z})$ can be changed to I_n by operations 1-3 in lemma 4.25, we can find a handlebody decomposition diffeomorphic to W relative to $\partial_0 W$ whose representative matrix is I_n.

Then, by n iterations of operation 5 in lemma 4.25, we can find a handlebody diffeomorphic to W relative to $\partial_0 W$ whose representative matrix is a 0 by 0 matrix. By definition, $p_q = 0$, so this handlebody is of the form $W \cong \partial_0 W \times [0,1]$. □

5. Proof of the Poincaré Conjecture

The idea of the Poincaré conjecture is that a less precise form of equality suffices to show homeomorphism for spheres S^n. That is, any simply connected n-manifold that has the same homology groups as S^n is homeomorphic to S^n.

The proof requires a few lemmas, which we present first.

Lemma 5.1. *For $n \geq 6$, let M be a simply connected n-manifold with $H_j(M)$ isomorphic to $H_j(S^n)$ for all $j \in \mathbb{N}$. Take two disjoint disks $D_i^n \subset M$ for $i = 0, 1$. Let $N = M - \text{int}(D_0^n) - \text{int}(D_1^n)$. Then the inclusion of the boundary spheres $S_i^{n-1} \to N$ is a homotopy equivalence for $i = 0, 1$.*

Proof. The first thing we want to do is show that the relative homology group $H_j(M - \text{int}(D_0^n) - \text{int}(D_1^n), S_0^{n-1}) = 0$ for all j. By excision [1, Theorem 2.20], we have that this relative homology group is isomorphic to $H_j(M - \text{int}(D_1^n), D_0^n)$, so we will instead show that this equivalent group is 0 for all j.

Now, we will use long exact sequence of a pair (see [1, page 117]) by considering the pair $(M - \text{int}(D_1^n), D_0^n)$. This sequence is

$$\cdots \to H_j(D_0^n) \to H_j(M - \text{int}(D_1^n)) \to H_j(M - \text{int}(D_1^n), D_0^n) \to \\ H_{j-1}(D_0^n) \to H_{j-1}(M - \text{int}(D_1^n)) \to H_{j-1}(M - \text{int}(D_1^n), D_0^n) \to \cdots .$$

Since all disks are homotopic to points, $H_j(D_0^n) = 0$ for all j, so this sequence can be rewritten as

$$\cdots \to 0 \to H_j(M - \text{int}(D_1^n)) \to H_j(M - \text{int}(D_1^n), D_0^n) \to \\ 0 \to H_{j-1}(M - \text{int}(D_1^n)) \to H_{j-1}(M - \text{int}(D_1^n), D_0^n) \to 0 \to \cdots .$$

Any time two terms in a long exact sequence are bracketed by trivial groups, they must be isomorphic. So, instead of looking at the relative homology group $H_n(M - \text{int}(D_1^n), D_0^n)$, we can equivalently look at $H_j(M - \text{int}(D_1^n))$. To do this, we can consider another pair $(M, M - \text{int}(D_1^n))$, which gives us a new long exact sequence,

$$\cdots \to H_j(M - \text{int}(D_1^n)) \to H_j(M) \to H_j(M, M - \text{int}(D_1^n)) \to \\ H_{j-1}(M - \text{int}(D_1^n)) \to H_{j-1}(M) \to H_{j-1}(M, M - \text{int}(D_1^n)) \to \cdots .$$

By excising $M - D_1^n$ from $H_j(M, M - \text{int}(D_1^n))$, we can change this sequence to

$$\cdots \to H_j(M - \text{int}(D_1^n)) \to H_j(M) \to H_j(D_1^n, S_1^{n-1}) \to \\ H_{j-1}(M - \text{int}(D_1^n)) \to H_{j-1}(M) \to H_{j-1}(D_1^n, S_1^{n-1}) \to \cdots .$$

The homology $H_j(D_1^n, S_1^{n-1})$ is isomorphic to the j-th homology of an n-sphere, which is 0 for $j \neq n$ and $\mathbb{Z}$ for $j = n$. By hypothesis, $H_j(M)$ is the same. So, for all $j \neq n, n-1$ we have a short exact sequence

$$\cdots \to 0 \to H_j(M - \text{int}(D_1^n)) \to 0 \to \cdots ,$$

which means that $H_j(M - \text{int}(D_1^n)) = 0$ for $j \neq n, n-1$. For the other cases, we have

$$\cdots \to 0 \to H_n(M - \text{int}(D_1^n)) \to \mathbb{Z} \to \mathbb{Z} \to H_{n-1}(M - \text{int}(D_1^n)) \to 0 \to \cdots .$$

The map $\mathbb{Z} \to \mathbb{Z}$ is the map $H_n(M) \to H_n(M, M - \text{int}(D_1^n))$, which is an isomorphism since M is an n-manifold. Therefore, $H_j(M - \text{int}(D_1^n)) = 0$ for $j = n, n-1$, as well.

Thus we have that $H_j(M - \text{int}(D_1^n)) = H_j(N, S_0^n) = 0$ for all j. By an analogous argument, $H_j(N, S_1^n) = 0$ for all j. Therefore, since everything is simply connected, the inclusions of each of these spheres into the full space are homotopy equivalences. □

Lemma 5.2. *Any homeomorphism $h : S^k \to S^k$ can be extended to a homeomorphism $H : D^{k+1} \to D^{k+1}$.*

Proof. First note that we can think of D^{k+1} as the product $S^k \times [0, 1]$ with $S^k \times \{0\}$ identified to a single point. Define H by letting $H(x, t) = (t \cdot h(x), t)$. The fact that H is a homeomorphism follows directly from h being a homeomorphism. Note that we cannot extend this lemma to diffeomorphisms, because problems will arise near $t = 0$. Thus diffeomorphisms $h' : S^k \to S^k$ only extend to homeomorphisms $H' : D^{k+1} \to D^{k+1}$. □

Theorem 5.3 (Poincaré Conjecture)**.** *For $n \geq 6$, let M be a simply connected n-manifold with $H_j(M) = H_j(S^n)$ for all $j \in \mathbb{N}$. Then M is homeomorphic to S^n.*

Proof. Take two disjoint disks $D_i^n \subset M$ for $i = 0, 1$, and let $N = M - \text{int}(D_0^n) - \text{int}(D_1^n)$. Clearly $(N; S_0^{n-1}, S_1^{n-1})$ is a cobordism, and it is an h-cobordism by lemma 5.1. Since S^{n-1} is simply connected, we can apply the h-cobordism theorem 4.27 and get that $(N; S_0^{n-1}, S_1^{n-1}) \cong (S_0^{n-1} \times [0,1]; S_0^{n-1} \times \{0\}, S_0^{n-1} \times \{1\})$. This diffeomorphism induces a diffeomorphism $f : S_0^{n-1} \to S_1^{n-1}$, which we can extend by lemma 5.2 to a homeomorphism $F : D_0^n \to D_1^n$.

By filling the interiors of the disks back in according to this homeomorphism F, we get a homeomorphism $M = N \cup D_0^n \cup D_1^n \to S_0^{n-1} \times [0,1] \cup D_0^{n-1} \cup D_0^{n-1}$. The target of this is clearly S^n. □

Appendix A. The Dual Handlebody Lemma

This is a more technical version of the proof of the dual handlebody lemma.

Lemma A.1 (Dual Handlebody Lemma 4.18)**.** *Let $(W; \partial_0 W, \partial_1 W)$ be an n-dimensional cobordism and suppose that*

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_0} (\phi_i^0) + \sum_{i=1}^{p_1} (\phi_i^1) + \cdots + \sum_{i=0}^{p_n} (\phi_i^n).$$

Then we can build W up from the other boundary component by finding a handle of index $(n-k)$ for each k-handle in the original decomposition. That is,

$$W \cong \partial_1 W \times [0,1] + \sum_{i=1}^{p_n} (\psi_i^0) + \sum_{i=1}^{p_{n-1}} (\psi_i^1) + \cdots + \sum_{i=0}^{p_0} (\psi_i^n).$$

Proof. First, consider a simpler case. Let $W \cong \partial_0 W \times [0,1] + (\phi^q)$. Let M denote $\partial_0 W \times \{1\}$ minus the interior of the q-handle embeddings. That is, $M = \partial_0 W \times \{1\} - \text{int}\{\phi^q(S^{q-1} \times D^{n-q})\}$. It is clear that M is an $(n-1)$-manifold with $\partial M = \phi^q(S^{q-1} \times S^{n-1-q})$.

Figure 10 shows an example of this: $W = S^1 \times [0,1] + (\phi^1)$, where ϕ^1 sends $S^0 \times D^1$ to two arcs of $\partial_1 W$. In this case, $M = S^1 - D_0^1 - D_1^1$.

Rather than building W up from the interval product of $\partial_0 W$, we can construct it from a product of this manifold M. First, take $M \times [0,1]$ (an example is in figure 10). Then we want to attach a new piece along the boundary of the embedding, i.e. the boundary of the piece that was removed, all the way through the product. As noted above, $\partial M = \phi^q(S^{q-1} \times S^{n-1-q})$, so we want to attach our new piece to $\phi(S^{q-1} \times S^{n-1-q}) \times [0,1]$.

Let N be the new piece we are attaching. In order to restore W, we need to both fill in the removed bits along the entire product and also restore the original q-handle. So $N = S^{q-1} \times D^{n-q} \times [0,1] \cup_{S^{q-1} \times D^{n-q} \times \{1\}} D^q \times D^{n-q}$.

Therefore, we have that

$$\begin{aligned} W \cong\ & (M \times [0,1]) \cup_{\phi(S^{q-1} \times S^{n-1-q}) \times [0,1]} \\ & (S^{q-1} \times D^{n-q} \times [0,1] \cup_{S^{q-1} \times D^{n-q} \times \{1\}} D^q \times D^{n-q}). \end{aligned}$$

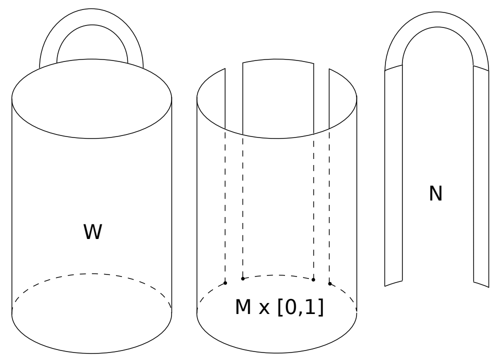

Figure 10. A 2-manifold $W = S^1 \times [0,1]$ with an attached 1 handle, together with the associated manifolds $M \times [0,1]$ and N mentioned in the proof of lemma A.1.

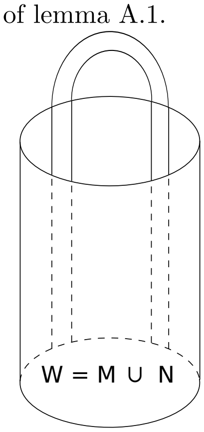

Figure 11. The manifold W from figure 10 written as the union of $M \times [0,1]$ and N from figure 10.

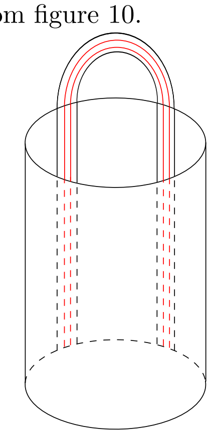

Figure 12. The manifold W in the proof of lemma A.1 in its decomposition into $M \times [0,1]$ and N, with Y (See equation (A.3)) shown in red.

The goal is to write W as the product of $\partial_1 W$ with an attached $n-q$ handle. Recall that an $n-q$ handle is $D^{n-q} \times D^q$, embedding along $S^{n-1-q} \times D^q$. This handle can be found inside of N. Let

$$N \supset X = S^{q-1} \times D^{n-q}_{\frac{1}{2}} \times [0,1] \cup_{S^{q-1} \times D^{n-q}_{\frac{1}{2}} \times \{1\}} D^q \times D^{n-q}_{\frac{1}{2}}, \tag{A.2}$$

$$X \supset Y = S^{q-1} \times S^{n-1-q}_{\frac{1}{2}} \times [0,1] \cup_{S^{q-1} \times S^{n-1-q}_{\frac{1}{2}} \times \{1\}} D^q \times S^{n-1-q}_{\frac{1}{2}}, \tag{A.3}$$

where $D^k_{\frac{1}{2}} = \{x \in \mathbb{R}^k \mid \|x\| \leq \frac{1}{2}\}$ and $S^{k-1}_{\frac{1}{2}} = \partial D^k_{\frac{1}{2}}$. See figure 12.

Since $S^{q-1} \times [0,1] \cup_{S^{q-1} \times \{1\}} D^q$ is diffeomorphic to D^q, we have that $X \cong D^q \times D^{n-q} \cong D^{n-q} \times D^q$. Similarly, $Y \cong D^q \times S^{n-1-q} \cong S^{n-1-q} \times D^q$.

Thus X embeded along Y (as it is inside N) is an $n-q$ handle. Now, consider $Z = W - \operatorname{int}(X)$. Clearly $W = Z + (\psi^{n-q})$. All that remains is to show that $Z = \partial_1 W \times [0,1]$.

Recall that

$$\partial_1 W = M \cup_{\phi^q(S^{q-1} \times S^{n-1-q})} D^q \times S^{n-1-q}.$$

So

$$\partial_1 W \times [0,1] = M \times [0,1] \cup_{\phi^q(S^{q-1} \times S^{n-1-q}) \times [0,1]} D^q \times S^{n-1-q} \times [0,1].$$

Meanwhile,

$$\begin{aligned} Z &= W - \operatorname{int}(X) \\ &\cong M \times [0,1] \cup_{\phi(S^{q-1} \times S^{n-1-q}) \times [0,1]} S^{q-1} \times (D^{n-q} - \operatorname{int}(D^{n-q}_{\frac{1}{2}})) \times [0,1] \\ &\quad \cup_{S^{q-1} \times (D^{n-q} - \operatorname{int}(D^{n-q}_{\frac{1}{2}})) \times \{1\}} D^q \times (D^{n-q} - \operatorname{int}(D^{n-q}_{\frac{1}{2}})). \end{aligned}$$

Since $D^k - D^k_{\frac{1}{2}} = S^{k-1} \times [0,1]$ and, as before, $S^{q-1} \times [0,1] \cup_{S^{q-1} \times \{1\}} D^q = D^q$, this means that $Z = \partial_1 W \times [0,1]$.

This process can be iterated for handlebodies with more than a single handle. By combining this with the ordering lemma 4.4, we can find a dual handlebody decomposition

$$\partial_0 W \times [0,1] + \sum_{i=1}^{p_0} (\phi_i^0) + \cdots + \sum_{i=0}^{p_n} (\phi_i^n) \cong \partial_1 W \times [0,1] + \sum_{i=1}^{p_n} (\psi_i^0) + \cdots + \sum_{i=0}^{p_0} (\psi_i^n).$$

□

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