① $tr_4(T_2T_1^{-1}T_2T_3) = tr_4(T_2[q^{-1}(T_1 - z1)]T_2T_3$

$= q^{-1}[tr_4(T_2T_1T_2T_3) - z\,tr_4(T_2^2T_3)]$

Aside: $tr_4(T_2T_1T_2T_3) = tr_3(T_2T_1T_2)$

$= tr_3(T_1T_2^2)$

$= tr_3(T_1(zT_2 + q1))$

$= z\,tr_3(T_1T_2) + q\,tr_3(1)$

$= z\,tr_2(T_1) + q\left(\frac{1-q}{z}\right)^3$

$= z\,tr_1(1) + q\left(\frac{1-q}{z}\right)^3$

$\boxed{= z\left(\frac{1-q}{z}\right) + q\left(\frac{1-q}{z}\right)^3}.$

$tr_4(T_2^2T_3) = tr_4((zT_2 + q1)T_3)$

$= z\,tr_4(T_2T_3) + q\,tr_4(T_3)$

$= z\,tr_3(T_2) + q\,tr_3(1)$

$= z\,tr_2(1) + q\left(\frac{1-q}{z}\right)^3$

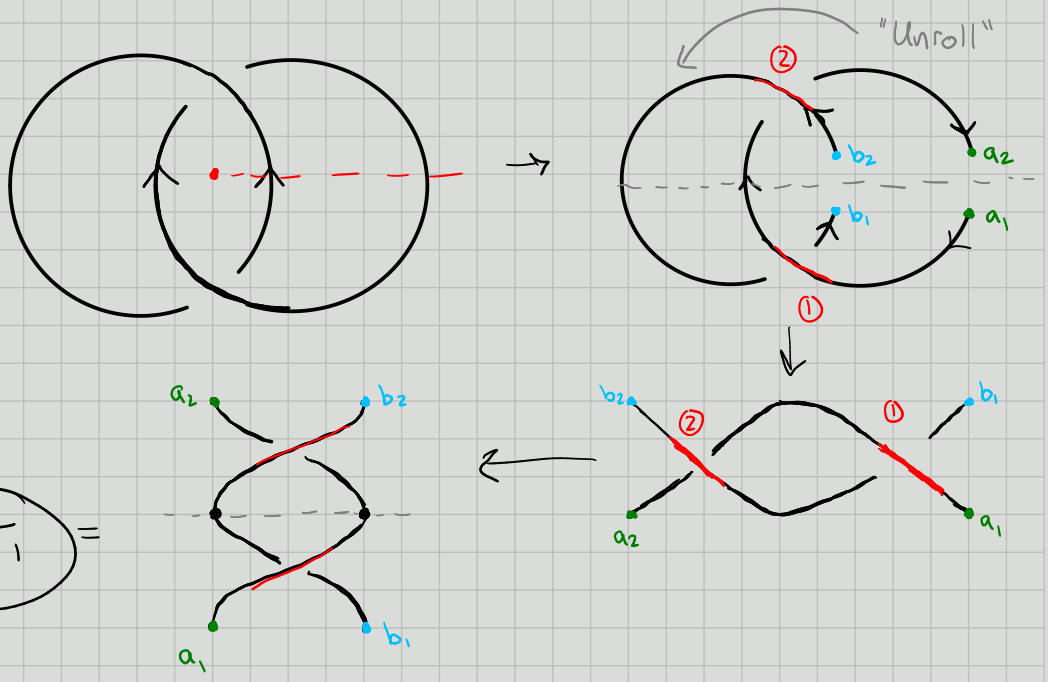
$\boxed{= z\left(\frac{1-q}{z}\right)^2 + q\left(\frac{1-q}{z}\right)^3}.$

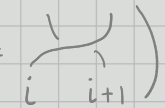
$= q^{-1}\left[z\left(\frac{1-q}{z}\right) + q\left(\frac{1-q}{z}\right)^3 - z\left(z\left(\frac{1-q}{z}\right)^2 + q\left(\frac{1-q}{z}\right)^3\right)\right]$

$= q^{-1}(1-q) + \left(\frac{1-q}{z}\right)^3 - q^{-1}(1-q)^2 - z\left(\frac{1-q}{z}\right)^3$

$\boxed{= 1 - q + (q^3 - 3q^2 + 3q - 1)(z^{-2} - z^{-3}).}$

(2)



$(\sigma_i =$  $i \quad i+1)$

$\sigma_1\sigma_1 =$

$B_2 = \langle \sigma_1 \mid \cdots \rangle$

$\omega_2 : B_2 \to H_2^R(q,z)^\times = \langle T_1 \mid T_1^2 = zT_1 + q1 \rangle$

$\sigma_1 \mapsto T_1$

$\Rightarrow (tr_2 \circ \omega_2)(\sigma_1^2) = tr_2(T_1^2) = tr_2(zT_1 + q1)$

$= z\,tr_2(T_1) + q\,tr_2(1)$

$= z\,tr_1(1) + q\left(\frac{1-q}{z}\right)^2$

$= z\left(\frac{1-q}{z}\right) + q\left(\frac{1-q}{z}\right)^2$

$= 1 - q + q\left(\frac{1-q}{z}\right)^2$

$q = a^{-2},\ z = a^{-1}(v - v^{-1})$ ⟶

$= 1 - a^{-2} + \left(\frac{a^{-2}-1}{v-v^{-1}}\right)^2$

$= a^{-1}(a - a^{-1}) + \left(\frac{-a^{-1}(a-a^{-1})}{v-v^{-1}}\right)^2$

$= a^{-1}(a-a^{-1}) + a^{-2}\left(\frac{a-a^{-1}}{v-v^{-1}}\right)^2$ ∎

Relation

$= -a(a - a^{-1}) + a^2\left(\frac{a-a^{-1}}{v-v^{-1}}\right)^2$

Almost