- $C_e \longmapsto H_e = 1$

- <u>Claim</u>: $C_{s_i} \mapsto H_{s_i} + v$

Note only $e < s_i$, so $\quad C_{s_i} := H_{s_i} + \sum_{x < s_i} h_x H_x$

$$= H_{s_i} + h_1 H_e$$
$$= H_{s_i} + h_i \cdot 1$$

What is h_i? Self-duality forces

$$C_{s_i} = \overline{C_{s_i}} = \overline{H_{s_i} + h_i \cdot 1} = \overline{H_{s_i}} + \overline{h_i} \cdot 1 = H_{s_i}^{-1} + \overline{h_i} \cdot 1$$
$$= H_{s_i} + (v - v^{-1}) \cdot 1 + \overline{h_i} \cdot 1$$
$$= H_{s_i} + (v - v^{-1} + \overline{h_i}) \cdot 1$$
$$= H_{s_i} + h_i \cdot 1 \quad \Rightarrow \quad v - v^{-1} - (h_i - \overline{h_i}) = 0$$
$$\Rightarrow h_i = v \quad .$$

$$\boxed{\therefore C_{s_i} = H_{s_i} + v = \overline{C_{s_i}}.}$$

- $C_{s_2 s_3 s_2} \longmapsto ?$

$$C_{s_2 s_3} = C_{s_2} C_{s_3} = (H_{s_2} + v)(H_{s_3} + v) = v^2 + v(H_{s_2} + H_{s_3}) + H_{s_2} H_{s_3}$$

$- (H_{s_3} + v)(H_{s_2} + v) = \overline{C_{s_2}}\, \overline{C_{s_3}} = \overline{C_{s_2} C_{s_3}}$, so self-dual.

$$C_{s_2 s_3 s_2} = C_{s_2 s_3} \cdot C_{s_2} = (v^2 + v(H_{s_2} + H_{s_3}) + H_{s_2 s_3})(H_{s_2} + v)$$

$$= v^2 H_{s_2} + v^3 + v(H_{s_2} + H_{s_3}) H_{s_2} + v^2 (H_{s_2} + H_{s_3}) + H_{s_2 s_3 s_2} + v H_{s_2 s_3}$$
$$= v^2 H_{s_2} + v H_{s_2}^2 + v H_{s_3 s_2} + v^2 H_{s_2} + v^2 H_{s_3} + H_{s_3 s_2 s_3} + v H_{s_2 s_3} + v^3$$
$$= H_{s_2 s_3 s_2} + H_{s_3 s_2}.(v) + H_{s_2 s_3}(v) + H_{s_2}(v^2 + v^2) + H_{s_3}^2(v) + H_{s_3}(v^2) + H_e(v^3)$$

$$= H_{s_2 s_3 s_2} + H_{s_3 s_2}(v) + H_{s_2 s_3}(v) + H_{s_2}(2v^2 + 1 - v^2) + H_{s_3}(v^2) + H_e(v^3 + v)$$

$$\boxed{= H_{s_2s_3s_2} + H_{s_3s_2}(v) + H_{s_2s_3}(v) + H_{s_2}(v^2+1) + H_{s_3}(v^2) + H_e(v(v^2+1))}$$

$$\therefore\ C_{s_2s_3s_2} - C_{s_2} = \cdots - H_{s_2} - v$$

$$= H_{s_2s_3s_2} + H_{s_3s_2}(v) + H_{s_2s_3}(v) + H_{s_2}(v^2) + H_{s_3}(v^2) + H_e(v(v^2+1) - v)$$

$$\boxed{= 1\cdot H_{s_2s_3s_2} + vH_{s_3s_2} + vH_{s_2s_3} + v^2H_{s_2} + v^2H_{s_3} + v^3H_e}$$

• $C_{s_2s_1s_3s_2} = C_{s_2s_1}\cdot C_{s_3s_2} = (v^2+(H_{s_2}+H_{s_1})v + H_{s_2s_1})(v^2+(H_{s_3}+H_{s_2})v+H_{s_3s_2})$

$$= v^4 + v^3((H_{s_2}+H_{s_1})+(H_{s_3}+H_{s_2})) + v^2(H_{s_2s_1}+H_{s_3s_2}+(H_{s_2}+H_{s_1})(H_{s_3}+H_{s_2}))$$

$$+ v((H_{s_2}+H_{s_1})H_{s_3s_2} + H_{s_2s_1}(H_{s_3}+H_{s_2}))$$

$$= v^4+v^3(H_{s_1}+2H_{s_2}+H_{s_3}) + v^2(H_{s_2s_1}+H_{s_3s_2}+H_{s_2s_3}+H_{s_2}^2+H_{s_1s_3}+H_{s_1s_2})$$

$$+v(H_{s_2s_3s_2}+H_{s_1s_3s_2}+H_{s_2s_1s_3}+H_{s_2s_1s_2})$$

$$= v^4+v^3(H_{s_1}+2H_{s_2}+H_{s_3}) + v^2(H_{s_2s_1}+H_{s_3s_2}+H_{s_2s_3}+(v-v^{-1})H_{s_2}+1+H_{s_1s_3}+H_{s_1s_2})$$

$$+v(H_{s_2s_3s_2}+H_{s_1s_3s_2}+H_{s_2s_1s_3}+H_{s_2s_1s_2})$$

$$\boxed{\begin{aligned}&= vH_{s_1s_3s_2}+vH_{s_2s_1s_2}+vH_{s_2s_1s_3}+vH_{s_2s_3s_2}\\&+v^2H_{s_1s_2}+v^2H_{s_1s_3}+v^2H_{s_2s_1}+v^2H_{s_2s_3}+v^2H_{s_3s_2}\\&+v^3H_{s_1}+(v-v^{-1}+2v^3)H_{s_2}+v^3H_{s_3}\\&+(v^2+v^4)H_e.\end{aligned}}$$