1. Let $R = \mathbb{C}[X_1, \ldots, X_n]$. The *Demazure operator* is defined by

$$\partial_{s_i} : R \longrightarrow R^{s_i}(2)\ ,\quad g \mapsto \frac{g - s_i.g}{X_i - X_{i+1}}.$$

Consider the homomorphisms of graded R-bimodules given by

$$\iota_- : R \otimes_{R^{s_i}} R(-2) \longrightarrow R \otimes_{R^{s_i}} R \otimes_{R^{s_i}} R(-2)\ ,\quad f \otimes g \mapsto f \otimes 1 \otimes g$$

$$\iota_+ : R \otimes_{R^{s_i}} R \longrightarrow R \otimes_{R^{s_i}} R \otimes_{R^{s_i}} R(-2)\ ,\quad f \otimes g \mapsto f \otimes \frac{X_i - X_{i+1}}{2} \otimes g$$

$$p_- : R \otimes_{R^{s_i}} R \otimes_{R^{s_i}} R(-2) \longrightarrow R \otimes_{R^{s_i}} R(-2)\ ,\quad f \otimes g \otimes h \mapsto \partial_{s_i}\left(\frac{X_i - X_{i+1}}{2} g\right) f \otimes h$$

$$p_+ : R \otimes_{R^{s_i}} R \otimes_{R^{s_i}} R(-2) \longrightarrow R \otimes_{R^{s_i}} R\ ,\quad f \otimes g \otimes h \mapsto \partial_{s_i}(g)\, f \otimes h$$

(a) Show that

$$p_+ \circ \iota_- = 0\ ,\quad p_- \circ \iota_+ = 0\ ,\quad p_+ \circ \iota_+ = \mathrm{id}\ ,\quad p_- \circ \iota_- = \mathrm{id}\ ,\quad \iota_+ \circ p_+ + \iota_- \circ p_- = \mathrm{id}.$$

(b) Use part (a) to show that

$$B_{s_i} \otimes_R B_{s_i} \cong B_{s_i}(1) \oplus B_{s_i}(-1)$$

by constructing two mutually inverse graded R-bimodule homomorphisms.

(a)

- $$f \otimes g \xmapsto{i_-} f \otimes \mathbb{1} \otimes g \xmapsto{p_+} \left(\frac{\mathbb{1} - s_i \mathbb{1}}{X_i - X_{i+1}}\right) f \otimes g$$

$$= \left(\frac{\mathbb{1} - \mathbb{1}}{X_i - X_{i+1}}\right) f \otimes g = 0 \cdot f \otimes g = 0.$$

- $$f \otimes g \xmapsto{i_+} f \otimes \left(\tfrac{1}{2}(X_i - X_{i+1})\right) \otimes g \xmapsto{p_-} \partial_{s_i}\left(\tfrac{1}{2}(X_i - X_{i+1}) \cdot \tfrac{1}{2}(X_i - X_{i+1})\right) f \otimes g$$

$$= \partial_{s_i}\left(\tfrac{1}{4}(X_i - X_{i+1})^2\right) f \otimes g = \left(\frac{\tfrac{1}{4}(X_i - X_{i+1})^2 - s_i \tfrac{1}{4}(X_i - X_{i+1})^2}{X_i - X_{i+1}}\right) f \otimes g$$

$$= \left(\frac{(X_i - X_{i+1})^2 - (X_{i+1} - X_i)^2}{X_i - X_{i+1}}\right) f \otimes g = \left(\frac{(X_i - X_{i+1})^2 - (X_i - X_{i+1})^2}{X_i - X_{i+1}}\right) f \otimes g$$

$$= 0 \cdot f \otimes g = 0.$$

- $$f \otimes g \xmapsto{i_+} \tfrac{1}{2} f \otimes (X_i - X_{i+1}) \otimes g \xmapsto{P_+} \tfrac{1}{2} \partial_{s_i}(X_i - X_{i+1}) f \otimes g$$

$$= \tfrac{1}{2}\left(\frac{(X_i - X_{i+1}) - (X_{i+1} - X_i)}{X_i - X_{i+1}}\right) f \otimes g = \frac{1}{2}\left(\frac{2X_i - 2X_{i+1}}{X_i - X_{i+1}}\right) f \otimes g = \underline{1} \cdot f \otimes g = f \otimes g.$$

$$\cdot\ f\otimes g \overset{i_-}{\longmapsto} f\otimes \mathbb{1}\otimes g \overset{p_-}{\longmapsto} \partial_{s_i}\left(\frac{x_i - x_{i+1}}{2}\cdot \mathbb{1}\right) f\otimes g$$

$$= \left(\frac{\frac{1}{2}(x_i - x_{i+1}) - \frac{1}{2}(x_{i+1} - x_i)}{x_i - x_{i+1}}\right) f\otimes g = \left(\frac{\frac{1}{2}(2x_i - 2x_{i+1})}{x_i - x_{i+1}}\right) f\otimes g = 1\cdot f\otimes g = f\otimes g.$$

$$\cdot\ f\otimes g\otimes h \overset{p_+}{\longmapsto} \partial_{s_i}(g) f\otimes h \overset{i_+}{\longmapsto} \partial_{s_i}(g) f\otimes \tfrac{1}{2}(x_i - x_{i+1})\otimes$$

$$= (x_i - x_{i+1})^{-1}(g - s_i g) f\otimes \tfrac{1}{2}(x_i - x_{i+1})\otimes h$$

$$= f\otimes (x_i - x_i)^{-1}(g - s_i g)\cdot \tfrac{1}{2}(x_i - x_{i+1})\otimes h \qquad \text{since } \partial_{s_i}(g)\in R^{s_i}$$

$$= f\otimes \tfrac{1}{2}(g - s_i g)\otimes g$$

$$\cdot\ f\otimes g\otimes h \overset{p_-}{\longmapsto} \partial_{s_i}(\tfrac{1}{2}(x_i - x_{i+1})g) f\otimes h \overset{i_-}{\longmapsto} \partial_{s_i}(\tfrac{1}{2}(x_i - x_{i+1})g) f\otimes 1\otimes h$$

$$= f\otimes \partial_{s_i}(\tfrac{1}{2}(x_i - x_{i+1})g)\otimes h = f\otimes \tfrac{1}{2}(g + s_i g)\otimes h$$

$$\Rightarrow (i_+\circ p_+ + i_-\circ p_-)(f\otimes g\otimes h) = (f\otimes \tfrac{1}{2}(g - s_i g)\otimes h) + (f\otimes \tfrac{1}{2}(g + s_i g)\otimes h)$$
$$= f\otimes g\otimes h.$$

(b) WTS $\quad B_{s_i}\otimes_R B_{s_i} \cong B_{s_i}(1)\oplus B_{s_i}(-1)$

Using $B_{s_i} = R\otimes_{R^{s_i}} R(-1)$, wts

$$(R\otimes_{R^{s_i}} R(-1))\otimes_R (R\otimes_{R^{s_i}} R(-1)) \cong R\otimes_{R^{s_i}} R \oplus R\otimes_{R^{s_i}} R(-2)$$

$$R\otimes_{R^{s_i}} R \cancel{\otimes_R} \overset{\shortparallel}{\cancel{R}} \otimes_{R^{s_i}} R(-2), \qquad \text{So take...}$$

$$\left\{ R\otimes_{R^{s_i}} \overset{\shortparallel}{R} \otimes_{R^{s_i}} R(-2) \;\underset{i_+\oplus i_-}{\overset{p_+\oplus p_-}{\rightleftarrows}}\; \underbrace{R\otimes_{R^{s_i}} R}_{A} \oplus \underbrace{R\otimes_{R^{s_i}} R(-2)}_{B} \right\}$$

Thus it suffices to show

$$p_+\circ i_+ \oplus p_-\circ i_- = \mathrm{id}_{A\oplus B} = \underset{p_+\circ i_+ = \mathrm{id}}{\mathrm{id}_A} \oplus \underset{p_-\circ i_- = \mathrm{id}}{\mathrm{id}_B}, \text{ but we know this from (a).}$$

$\blacksquare$