1. Consider the graded $\mathbb{C}$-algebra $R = \mathbb{C}[X_1, X_2]$ with $|X_1| = |X_2| = 2$.

   (a) Explicitly write down a free resolution of R as a graded $R \otimes_{\mathbb{C}} R$-module. (Hint: Tensor the resolution for $\mathbb{C}[X]$ discussed in class with itself (over $\mathbb{C}$) and use the isomorphism of graded $\mathbb{C}$-algebras $\mathbb{C}[X] \otimes_{\mathbb{C}} \mathbb{C}[X] \cong \mathbb{C}[X_1, X_2]$.)

   (b) Use part (a) to compute the Hochschild homology $\mathrm{HH}_*(R, R)$. Do not forget to keep track of the (internal) grading on each $\mathrm{HH}_i(R, R)$.

① Labeling the original free resolution: (Set $A := \mathbb{C}[x]$, $A^{\otimes n} := \underbrace{A \otimes_{\mathbb{C}} A \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} A}_{n\text{ terms}}$)

$$P_\bullet = \overset{P_3}{0} \xrightarrow{\partial_3} \overset{P_2}{A^{\otimes 2}} \xrightarrow{\partial_2} \overset{P_1}{A^{\otimes 2}} \xrightarrow{\partial_1} \overset{P_0}{A} \xrightarrow{\partial_0} \overset{P_{-1}}{0}$$

We then take $(P_\bullet \otimes_{\mathbb{C}} P_\bullet)_n := \bigoplus_{i+j=n} P_i \otimes_{\mathbb{C}} P_j$, which yields

$$0 \to A^{\otimes 4} \xrightarrow{\hat\partial_4} A^{\otimes 4} \oplus A^{\otimes 4} \xrightarrow{\hat\partial_3} A^{\otimes 3} \oplus A^{\otimes 4} \oplus A^{\otimes 3} \xrightarrow{\hat\partial_2} A^{\otimes 3} \oplus A^{\otimes 3} \xrightarrow{\hat\partial_1} A^{\otimes 2} \xrightarrow{\hat\partial_0} 0$$

$\hookrightarrow = \mathbb{C}[x]^{\otimes 2} \cong \mathbb{C}[x, Y]$

where $\hat\partial_n := \sum_{i+j=n} \partial_i \otimes_{\mathbb{C}} 1 + (-1)^i \, 1 \otimes \partial_j$.

$$\Rightarrow \left.\begin{aligned}
\hat\partial_4 &= \partial_2 \otimes 1 + 1 \otimes \partial_2 \\
\hat\partial_3 &= (\partial_1 \otimes 1 - 1 \otimes \partial_2) + (\partial_2 \otimes 1 + 1 \otimes \partial_1) \\
\hat\partial_2 &= (1 \otimes \partial_2) + (\partial_1 \otimes 1 + 1 \otimes \partial_1) + (\partial_2 \otimes 1) \\
\hat\partial_1 &= (1 \otimes \partial_1) + (\partial_1 \otimes 1)
\end{aligned}\right\} \text{ using } \partial_0 = \partial_4 = 0$$

Define $X \hat\otimes Y := X \otimes_{\mathbb{C}[x]^e} Y$ where $X^e := X \otimes_{\mathbb{C}} X^{op}$, and

and $\mathbb{1} := \mathrm{id}_{\mathbb{C}[x]} \in \mathbb{C}[x]^e\text{-mod}$

zero map

From the example in lecture, we know $\mathbb{1} \hat\otimes \partial_2 = 0$, and note that the functor $\mathbb{C}[x] \hat\otimes \cdot$ satisfies $\mathbb{C}[x] \hat\otimes \mathbb{C}[x]^{\otimes n} = \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[x]^{n-2}$

since $\mathbb{C}[x]^{op} = \mathbb{C}[x] \Rightarrow \mathbb{C}[x] \hat\otimes \mathbb{C}[x]^{\otimes n} = \mathbb{C}[x] \otimes_{\mathbb{C}[x]^{\otimes 2}} \mathbb{C}[x]^{\otimes n} = \mathbb{C}[x] \otimes_{\cancel{\mathbb{C}[x]^{\otimes 2}}} (\cancel{\mathbb{C}[x]^{\otimes 2}} \otimes_{\mathbb{C}} \mathbb{C}[x]^{\otimes n-2})$

② Applying the functor yields

$$0 \longrightarrow \underbrace{A^{\otimes 2}}_{HH_3} \xrightarrow{\tilde{\partial}_4} \underbrace{A^{\otimes 2} \oplus A^{\otimes 2}}_{HH_2} \xrightarrow{\tilde{\partial}_3} \underbrace{A^{\otimes 1} \oplus A^{\otimes 2} \oplus A^{\otimes 1}}_{HH_1} \xrightarrow{\tilde{\partial}_2} \underbrace{A^{\otimes 1} \oplus A^{\otimes 1}}_{HH_0} \longrightarrow 0$$

Where

$$\tilde{\partial}_4 = \mathbb{1} \hat{\otimes} \left[\partial_2 \cancel{\otimes} 1 + 1 \cancel{\otimes} \partial_2\right] = 0$$

$$\tilde{\partial}_3 = \mathbb{1} \hat{\otimes} \left[(\partial_1 \otimes 1 - 1 \cancel{\otimes} \partial_2) + (\partial_2 \cancel{\otimes} 1 + 1 \otimes \partial_1)\right] = \mathbb{1} \hat{\otimes} \partial_1 \otimes 1 + \mathbb{1} \hat{\otimes} 1 \otimes \partial_1$$

$$\tilde{\partial}_2 = \mathbb{1} \hat{\otimes} \left[(1 \cancel{\otimes} \partial_2) + (\partial_1 \otimes 1 + 1 \otimes \partial_1) + (\partial_2 \cancel{\otimes} 1)\right] = \mathbb{1} \hat{\otimes} \partial_1 \otimes 1 + \mathbb{1} \hat{\otimes} 1 \otimes \partial_1$$

Should be this, but maybe a calculation mistake somewhere earlier. :(

Thus $HH_{n \geq 3} = 0$ and

Claim: $\tilde{\partial}_2 = \tilde{\partial}_3 = 0$, so

$$\boxed{\begin{aligned} &HH_0 \overset{?}{\cong} A \otimes_{\mathbb{C}} A = \mathbb{C}[X] \otimes_{\mathbb{C}} \mathbb{C}[X] = \underline{\mathbb{C}[X,Y]} \\ &HH_1 \cong \mathbb{C}[X] \oplus \mathbb{C}[X,Y] \oplus \mathbb{C}[X] \\ &HH_2 \cong \mathbb{C}[X,Y] \oplus \mathbb{C}[X,Y] \end{aligned}}$$

Recall $\partial_1 : \mathbb{C}[X] \otimes \mathbb{C}[X] \longrightarrow \mathbb{C}[X]$

$$f \otimes g \longmapsto fg$$

Identity on remaining elts

$$(\mathbb{1} \hat{\otimes} \partial_1 \otimes 1 + \mathbb{1} \hat{\otimes} 1 \otimes \partial_1)(f \hat{\otimes} \cdots \otimes g \otimes \cdots \otimes h) = (f \hat{\otimes} \cdots \otimes gh \otimes \cdots) \pm (f \hat{\otimes} \cdots\cdots \otimes gh)$$

$$= (f(gh) \hat{\otimes} \cdots) \pm (f(gh) \hat{\otimes} \cdots\cdots)$$

$$= (f(gh) \pm f(gh)) \hat{\otimes} \cdots$$

$$\overset{?}{=} 0$$

(Sign error somewhere?)