1. (a) Consider the Rouquier complex

$$F_{s_i}: \ \ldots \to 0 \to R(1) \to \underline{B_{s_i}} \to 0 \to \ldots \ .$$

Show that the tensor product $F_{s_i} \otimes_R F_{s_i}$ is homotopy equivalent to a complex of the form

$$\ldots \to 0 \to R(2) \to B_{s_i}(1) \to \underline{B_{s_i}(-1)} \to 0 \to \ldots \ .$$

Remember to keep track of (at least some of) the differentials to justify each use of Gaussian elimination.

(b) Part (a) can be viewed as a categorification of the equation

$$H_{s_i}^2 = v^{-1}C_{s_i} - vC_{s_i} + v^2 \tag{1}$$

in the Hecke algebra. Show that (1) is the quadratic relation.

(a) We'll use the following facts:

- $B_{s_i} := R \otimes_{R^{s_i}} R$
- $B_{s_i} \otimes_R B_{s_i} \xrightarrow{\hat{p} := \binom{p^+}{p^-}} B_{s_i}(1) \oplus B_{s_i}(-1)$
- $R(1) \otimes_R B_{s_i} = R(1) \cancel{\otimes_R R} \otimes_{R^{s_i}} R = R(1) \otimes_{R^{s_i}} R(-1)$

$$= R \otimes_{R^{s_i}} R = \underline{B_{s_i}(1)}$$

- $B_{s_i} \otimes_R R(1) = R \otimes_{R^{s_i}} R = \underline{B_{s_i}(1)}.$

Set $F_{s_i} = (\cdots \to 0 \to R(1) \xrightarrow{\partial} \underline{B_{s_i}} \to 0 \to \cdots)$

Then $F_{s_i} \otimes_R F_{s_i} =$

$$\begin{tikzcd}
 & & R(1) \otimes_R B_{s_i} \arrow[dr, "\partial \otimes 1"] & & \\
\cdots \to 0 \arrow[r] & R(1) \otimes_R R(1) \arrow[ur, "-1 \otimes \partial"] \arrow[dr, "\partial \otimes 1"'] & \oplus & \underline{B_{s_i} \otimes_R B_{s_i}} \arrow[r] & 0 \to \cdots \\
 & & B_{s_i} \otimes_R R(1) \arrow[ur, "1 \otimes \partial"'] & &
\end{tikzcd}$$

$$
= \left(\cdots \to 0 \to R(2) \xrightarrow{\begin{pmatrix} -\partial \\ \partial \end{pmatrix}} \begin{matrix} B_{s_i}(1) \\ \oplus \\ B_{s_i}(1) \end{matrix} \xrightarrow{\begin{pmatrix} \partial\otimes 1 & 1\otimes\partial \end{pmatrix}} \underline{B_{s_i} \otimes_R B_{s_i}} \to 0 \to \cdots \right)
$$

$$
\hat{p} := \begin{pmatrix} p^+ \\ p^- \end{pmatrix}
$$

$$
= \left(\cdots \to 0 \to R(2) \xrightarrow{\begin{pmatrix} -\partial \\ \partial \end{pmatrix}} \begin{matrix} B_{s_i}(1) \\ \oplus \\ B_{s_i}\cancel{(1)} \end{matrix} \longrightarrow \cancel{B_{s_i}(1)} \oplus B_{s_i}(-1) \to 0 \to \cdots \right)
$$

$$
\hat{p}\circ(\partial\otimes 1) = \begin{bmatrix} ? \\ ? \end{bmatrix} \qquad \hat{p}\circ(1\otimes\partial) = \begin{bmatrix} ? \\ ? \end{bmatrix}
$$

Want to say this is the identity, but can't trace out why!

Gaussian elim. for $F = \begin{bmatrix} ? & ? \\ ? & id \end{bmatrix}$

$B_{s_i}(1) \oplus \underset{\sim}{B_{s_i}(1)} \xrightarrow{F} B_{s_i}(-1) \oplus \underset{\sim}{B_{s_i}(1)}$

$$
= \left(\cdots \to 0 \to R(2) \xrightarrow{-\partial} B_{s_i}(1) \xrightarrow{\hat{p}\circ(\partial\otimes 1)} B_{s_i}(-1) \to 0 \to \cdots \right). \quad \blacksquare
$$

b) In $\underset{\mathbb{Z}[v,v^{-1}]}{H_n}(v^{-2}, z)$, the quadratic relation is

$$
H_{s_i}^2 = (v^{-1} - v) H_{s_i} + 1 .
$$

Here we have

$$
\begin{aligned}
H_{s_i}^2 &= v^{-1} C_{s_i} - v C_{s_i} + v^2 \cdot 1 \\
&:= v^{-1}(H_{s_i} + v1) - v(H_{s_i} + v1) + v^2 1 \\
&= v^{-1} H_{s_i} + 1 - v H_{s_i} - \cancel{v^2 1} + \cancel{v^2 1} \\
&= (v^{-1} - v) H_{s_i} + 1 . \quad \blacksquare
\end{aligned}
$$