Chern-Simons and Other Topological Field Theories

MSRI, November 2021

D. Zack Garza
University of Georgia
dzackgarza@gmail.com

Last updated: 2021-11-18

Table of Contents

Contents

1 | Dan Freed, Introduction to Chern-Simons

See their paper: Some Cohomology classes in Principal Fibers Bundle and Their Applications to Riemannian Geometry.

Remark 1.0.1: Setup: G a Lie group, $\pi_0 G$ finite, a G-invariant p-linear form on

$$\mathfrak{g} : \langle -, -, \cdots, - \rangle : \mathfrak{g}^{\otimes^p} \to \mathbb{R}.$$

These sit inside invariant polynomials on $\mathfrak{g}$, i.e. $(\mathrm{Sym}^p \mathfrak{g}^\vee)^G$.

$P \xrightarrow{\pi} M$ is a principal G-bundle, Θ a connection in $\Omega^1_P(\mathfrak{g})$

Remark 1.0.2: The curvature:

$$\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta].$$

$\mathcal{A}_\pi$ the affine space of all connections, look for a universal connection on the bundle $\mathcal{A}_\pi \times P \to \mathcal{A}_\pi \times M$.

$\Delta^1 \to \mathcal{A}_\pi$ a simplex connecting Θ, Θ_1, define

$$\alpha(\Theta_0, \Theta_1) = \int_{\Delta^1} \omega(\Theta_\pi).$$

Check that by Stokes', $d\alpha = \omega(\Theta_1) - \omega(\Theta_0)$. Basepoint in this affine space: a distinguished flat section (no holonomy) $P \to P^{\times 2}_{M}$ See Chern-Weil form $\alpha(\Theta)$, it satisfies a *transgression* $d\alpha(\theta) = \pi^* d\omega$?

Remark 1.0.3: Descend forms on P to M, need forms evaluated along fiber direction to vanish in order to descent. How to descend the Chern-Simons form? Original paper involves $H^{2p}(\mathbf{B}G; \mathbb{R})$ and reducing to $\mathbb{R}/\mathbb{Z}$ to kill topological obstructions to descent.

Remark 1.0.4: Next steps: Cheeger-Simons define an abelian Lie group of *differential characters* by taking a pullback:

$$\begin{tikzcd}
\mathrm{CS}^q(M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^q_{\mathrm{closed}}(M) \arrow[d] \\
H^q(M; \mathbb{Z}) \arrow[r] & H^q(M; \mathbb{R})
\end{tikzcd}$$

Link to Diagram

The Chern-Weil form of a G-connection is a canonical lift to $\mathrm{CS}^q(M)$. See Maurer-Cartan form.

Remark 1.0.5: Cheeger-Simons groups aren't local, i.e. don't satisfy a sheaf condition. Write

$$S^1 = \underrightarrow{\text{colim}}(I \to I \times I \leftarrow I) \rightsquigarrow \mathbb{R}/\mathbb{Z} = \underrightarrow{\text{colim}}(0 \to 0 \leftarrow 0),$$

since there is no holonomy on an interval. See Hopkins-Singer paper: pull back entire cochain complex, *differential function spaces*. Integrating this over manifold (after cutting into pieces) is packaged as a partition function on an invertible field theory.

Remark 1.0.6: Idea: use local Chern-Simons as a functional to integrate, e.g. along elements of ΩM for M a Riemannian manifold, or e.g. integrating scalar curvature. Can also use alternative cohomology theories, e.g. K-theory, do spin versions etc. Their result: finding obstructions to conformally immersing Riemannian manifolds into $\mathbb{R}^n$.

2 Stephon Alexander, Chern-Simons and Matter-Antimatter symmetry

Remark 2.0.1: Weak interaction: different to other interactions in that it violates parity: experimentally, the mirror version has not been seen. Gravity assumed left-right symmetric. Use resonance and power spectrum analysis to determine that a majority of the energy expenditure of the universe is in the form of dark energy.

Remark 2.0.2: Pillars of cosmology:

- Large scale homogeneous, isotropic,
- Solutions to Einstein field equations predict Hubble expansion law: further away implies moving away from each other faster.
- General relativity

Remark 2.0.3: Gravitational waves: tensorial fluctuations. Gravity waves: satisfies perturbed wave equation. Turn into a quantum thing to get gravitons. Ratio of matter to antimatter is a precise number, on the order of 10^{-10}. In Physics: see Chern-Simons current. Modify Einstein's equations with an scalar (axion?) term. This field is a candidate for what causes inflation.

Remark 2.0.4: Famous result: ABJ/chiral anomaly, related to an index theorem: a standard U_1 current is usually conserved, but can vanish? Chern-Simons form prevents $R\tilde{R}$ from vanishing, which if vanishing would cause equal left and right handedness.

3 Mina Aganagic, Homological Knot Invariants from Mirror Symmetry

Remark 3.0.1: Motivation: knot categorification problem. Recall that the Jones polynomial arises from a skein relation which depends on n, taking $n = 2$. Can take other values for n. Witten: this polynomial comes from Chern-Simons theory, expected value of a collection of Wilson loops in a fundamental representation of a Lie algebra. Alexander polynomial: take a Lie *superalgebra* instead. Categorification starts with quantum groups. Chern-Simons: assign a Hilbert space to $A := \Sigma_{g,n}$. Finite dimensional, spanned by a basis of conformal blocks (solutions to a linear PDE) Allow heavy particles to move in surface, take a path integral in $A \times I$ corresponding to a braid? States in $\mathcal{H}$ are special solutions to the PDE. Can be described by caps and cups? Links to conformal field theory: braiding and fusion of particle trajectories/paths. Khovanov: assign a cohomology theory, take graded Euler characteristic to recover Jones polynomial. Euler characteristic: trace from supersymmetric QM:

$$\mathrm{Tr}(-1)^F e^{-\beta H}.$$

Remark 3.0.2: Action of supercharge on a chain complex is generated by instantons. Khovanov grading: graded by fermion number and another grading. Ben Webster: framework for quantum link invariants for Lie algebras, but very inexplicit. Associate to conformal blocks a bigraded category, braids go to functors, links go to vector spaces of morphisms. Hard to prove they decategorify correctly: new framework makes this automatic and uses mirror symmetry.

Remark 3.0.3: Why Calabi-Yaus: theory of strings on a pair of dual tori. Complex is B-type (algebraic geometry), symplectic is A-type. Counting holomorphic maps from $\mathbb{P}^1$: hard, infinite series of enumerative problems. Easier on mirror? Branes are central objects in mirror symmetry, regard as objects in a category whose morphisms are open branes. Insight: need mirrors to be fibered by dual tori.

Remark 3.0.4: Category of branes: $\mathbb{D}\mathsf{Coh}X$, vs $\mathbb{D}\mathsf{F}$ for F the Fukaya category. Quantum product: count rational curves. See *quantum differential equation*, central to mirror symmetry. Defined on a moduli space of complex/symplectic structures. Solution space finite dimensional, spanned by charges. Solutions are maps from infinite cigar to X.

Remark 3.0.5: New knot theory idea: take an infinite punctured cylinder. Langlands dual group carries magnetic monopoles? Need to consider moduli space of monopoles in $\mathbb{R}^3$, turns out to be Calabi-Yau and in fact hyperKähler. Braids are paths in moduli of Kählers? Braid group is the sigma model on the infinite punctured cigar.

Remark 3.0.6: Fusion: singular monopoles coming together and bubbling. Fusion diagonalizes the action of braiding in conformal field theory. Cups and caps: branes supported on miniscule Grassmannians. Perverse filtrations make certain problems much simpler here. See KLRW algebra, algebraic invariants match up with geometric invariants coming from branes on B-side. Homological mirror symmetry: equivalence of categories of branes on different models. Smooth monopoles indexed by simple roots of a Lie algebra. See Landau-Ginzburg model. Category of A-branes: morphisms come from Floer theory, see More theory approach to QM? Instantons: holomorphic maps from strips!! Can generalize Heegaard-Floer to Lie superalgebras (need simply-laced Lie algebras?). Note that HF produces DGAs. Categorifies the Alexander polynomial. Equivariant homological mirror symmetry: relates moduli space $\mathcal{X}$ to a half-dimensional "core" X. Not an equivalence categories, comes from an adjunction $\mathsf{C} \mathrel{\substack{\xrightarrow{h_*} \\ \perp \\ \xleftarrow[h^*]{}}} \mathsf{C}'$ instead. Can compute $\mathrm{Hom}(A, B) =$

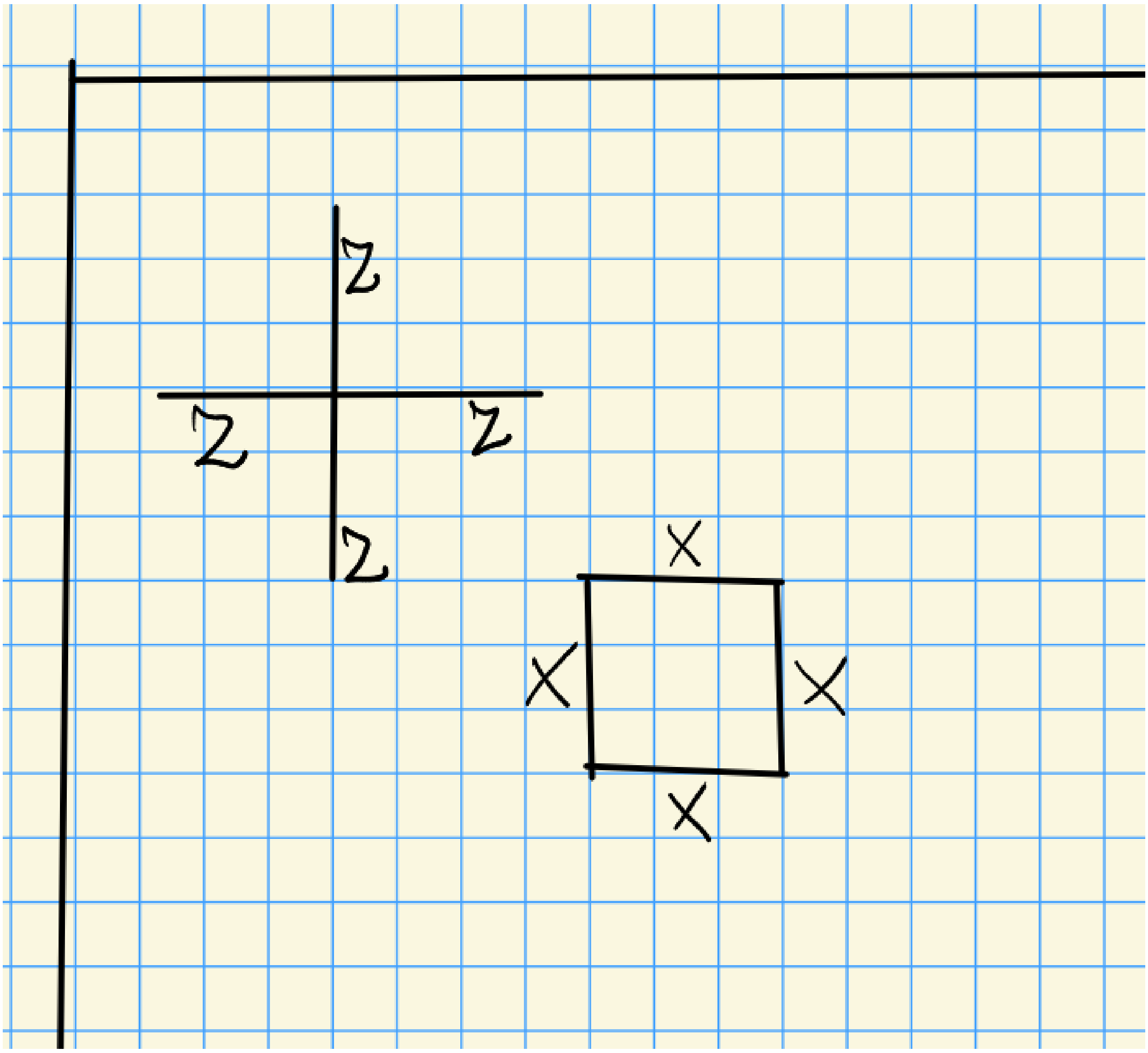
$\mathrm{Hom}(h^* h_* A, B)$. HF reduces curve counting to well-defined problems in 1-dimensional complex analysis, e.g. applications of Riemann mapping. Same story here, yields hard but tractable problems.

Remark 3.0.7: The theory on D-branes is related to 3-dimensional gauge theories of quiver type. Deformation: reply affine Lie algebra with quantum affine Lie algebras. See *defects* of conformal field theories, Koszul dual algebras. Yields some integrable lattice model, solves some analog of Yang-Baxter equations? Braiding matrices are computed by partition functions.

Remark 3.0.8: How computable is this theory, compared to the diagram calculus of Heegaard-Floer diagrams? Probably a few months away from being algorithmic enough for a computer.

4 Xie Chen, Chern-Simons Theory and Fractons

Remark 4.0.1: How to use Chern-Simons in condensed matter physics. Toy model: take a lattice and label edges with DOF given by Pauli matrices $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$:



Write Hamiltonian as

$$H = -\sum A_i - \sum B_i$$

where A_i are the crosses and B_i the boxes, gives an exactly solvable model. Cross with time to get $2+1$ where Chern-Simons terms show up. Generalize to lattices in 3-space. See ground-state degeneracy, minimal/stable if no perturbation can reduce it. Very large here, so doesn't fit into TQFT framework.

Remark 4.0.2: For anyons: quasiparticles that experience excitation, can move around. For fracton models, points can get stuck. By Gauss' law, particles must be created in pairs. Rank 1 tensor gauge theories: conserve charges. For rank 2, additionally need to conserve dipoles, so particles must be created in fours.

Remark 4.0.3: Gauge theories: far simpler when gauge group is abelian, e.g. U_1. An integer symmetric matrix shows up when writing a Lagrangian here, whose properties inform integer/fractional quantum Hall effects. Quasiparticles are restricted to move along submanifolds, e.g. an xy-plane when many 2d models are stacked in the z direction. Growing cubic models: add a new decoupled layer, then do a smooth deformation to get a new model with one more layer. Ground state degeneracy scales like 4^n. Foliated systems have statistics with finite range, vs some systems where statistics span multiple layers with long tails.

Question 4.0.4
Possibly easy to identify matrices that are $\mathrm{SL}_n(\mathbb{Z})$-invariant for finite n, what about as $n \to \infty$? What if you allow adding on $k \times k$ (decoupled) blocks, and locally allow $\mathrm{SL}_k(\mathbb{Z})$ to act on blocks?

Remark 4.0.5: Quasilocal matrices: integer matrices with nonzero entries only on some band about the diagonal. Open question: are there invariants that determine when a system is or is not foliated? What are the equivalence classes of foliated models?

Remark 4.0.6: Why "fracton"? The original models around 2010 had some fractal structure, could also be related to the fractions showing up. Modern interpretation: point particle that doesn't freely move through the entire space.

5 Simon Donaldson, The Chern-Simons Functional and Floer Homology

Remark 5.0.1: Recent work: when a variety admits an Einstein metric, and G_2 manifolds. For an account of the Chern-Simons paper, see Chern's *Complex manifolds without potential theory*, appendix on characteristic classes. 2+1: action functionals

$$\int e^{\mathrm{CS}} \mathrm{d}A.$$

3+1: Floer homology $\mathrm{HF}(M^3)$.

Remark 5.0.2: Consider $P \to Y^3 \in \mathrm{Prin}\,\mathsf{Bun}_G$, define $\mathcal{A}$ the affine space of functions, then $A_1 - A_2 \in \Omega^1(\operatorname{ad} P)$ Curvature $F(A) \in \Omega^2(\operatorname{ad} P)$. Define a form Θ Chern-Simons functional: $\mathrm{CS} : \mathcal{A} \to \mathbb{R}$, descends to $\mathcal{A}/\mathcal{G} \to S^1$ for $\mathcal{G}$ the gauge group. Critical points are flat connections, Hessians at these points are quadratic forms corresponding to a bilinear form from earlier. Alternative definition: pick X^4 such that $\partial X = Y$ and set

$$\mathrm{CS}(A) = \int_X \mathrm{Tr}(F(A)^2),$$

i.e. integrate the Chern-Weil form.

Remark 5.0.3: Examples of functionals: for $\gamma \in \Omega M$,

$$E(\gamma) := \int |\gamma'|^2.$$

Sub-level sets will have compactness properties, Hessians are finite index at critical points. The flow $-\operatorname{grad} E$ is a parabolic PDE, so a nonlinear heat equation. But the Chern-Simons functional doesn't have these properties. Very different!

Remark 5.0.4: For symplectic manifolds:

$$A(\gamma) = \int_{\mathbb{D}} \omega \qquad \text{where } \partial\mathbb{D} = \gamma.$$

For $M = \mathbb{C}^n$, write as a Fourier series $A(\gamma) = \sum c_k |\gamma'|^2$? Fixed points of exact Hamiltonian diffeomorphisms φ correspond to critical points of a deformed functional A_φ on ΩM. Arnold conjecture: $\sharp\mathrm{Fix}\varphi \geq \sum \beta_i$ is bounded below by Betti numbers.

Remark 5.0.5: For a torus, $\Omega T^{2n} \cong T^{2n} \times H^- \times H^+$ where the $H^\pm$ are vector spaces. Arnold conjecture proved for torii by Conley and Zehnder, 1983. Floer's insight: one can still "do Morse theory" with A_φ and CS: while gradient flow isn't defined, flow lines between critical points do make sense.

Remark 5.0.6: Introducing a Riemannian metric yields $\star : \Omega^2 \to \Omega^1$. Get connections A_t over Y^3 as solutions to

$$\frac{\partial A_t}{\partial t} = \star F(A_t).$$

Solutions are Yang-Mills instantons on $Y \times \mathbb{R}$ asymptotic to flat connections at $\pm\infty$. These solve anti-self-dual equations

$$F(A) = - \star F(A),$$

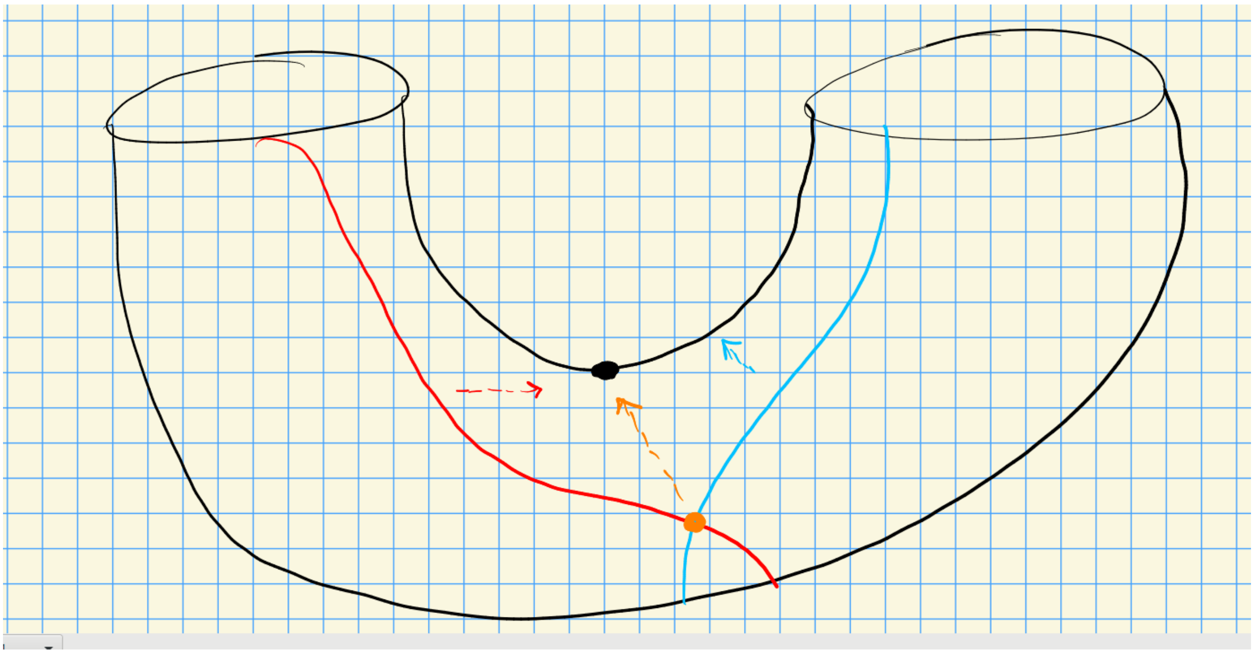
where here the star is a 4-dimensional version. Connected to electromagnetism? In symplectic case, gradient flow lines in ΩM are holomorphic curves in M.

Remark 5.0.7: Usual story: chain complex with generators for each critical point, graded by index, $\mathcal{M}(p,q)$ the space of gradient flow lines $p \to q$, compute $\dim \mathcal{M}(p,q) = i(p) - i(q) - 1$, and define a differential $\partial p = \sum \sharp\mathcal{M}(p,q)$. Problem in infinite dimensions: index isn't well-defined, but the difference is e.g. when $G = \mathrm{SU}_2$. Linearize the instanton equation. Euler characteristic with respect to HF is twice the Casson invariant.

Remark 5.0.8: If $Y \in \mathbb{Z}\mathrm{HS}^3$, then the trivial flat connection is isolated. Otherwise, Floer's construction works when you can avoid "reducible" flat connections, e.g. a nontrivial SO_3 bundle over Y^3. For general $P \to Y$, need an equivariant version of Floer theory – at present, this doesn't seem to exist.

Remark 5.0.9: Toward a 3+1 TFT: solutions to Yang-Mills on a 4-manifold give invariants by counting instantons in a 0-dimensional moduli space. For X a 4-manifold with $\partial X = Y$, these invariants $I(X)$ take values in $\mathrm{HF}(Y)$. Sum over all flat connections C_i, and count number of connections asymptotic to C_i.

Remark 5.0.10: Gluing formula: a type of surgery formula when $X = X_1 \coprod_Y X_2$ to compute $I(X) = I(X_1)I(X_2)$. This uses the pairing $\mathrm{HF}_*(Y) \otimes \mathrm{HF}^*(Y) \to \mathbb{Z}$. Proof: stretching the neck. Analog in Morse theory moves intersections closer to critical points.



Question 5.0.11
Other Floer theories on Y^3: solutions to SW, combinatorial Heegard theory. An outstanding problem: how are all of these Floer theories related?

Remark 5.0.12: Floer's deepest work: work leading to his exact surgery sequence. $K \hookrightarrow Y \in \mathbb{Q}\mathrm{HS}^3$ a knot, take a tubular neighborhood diffeomorphic to a torus with a prefered meridian. Do $+1$ surgery: cut out, glue meridian back along the diagonal. Can also do 0 surgery. Get a LES in HF_* of the surgered pieces, using well-known cobordisms.

Remark 5.0.13: Representation variety: moduli of flat connections! Extending all of this to include surfaces: see bordered Floer theory, Lipshitz-Osvath-Thurston for Seiberg-Witten and Heegard cases. Floer homotopy: see Manolescu, Bauer-Furuta.

Remark 5.0.14: How to complexify this theory? E.g. for $G = \mathrm{SL}_2(\mathbb{C})$ instead of SU_2, or replace Y with a Calabi-Yau threefold?

Remark 5.0.15: For Z a Calabi-Yau, have a nonvanishing holomorphic 3-form ω, so define a $\mathbb{C}$-valued function on $\mathcal{A}$ the space of connections:

$$F(A) = \int_Z \mathrm{CS}(A) \wedge \omega.$$

Critical points are connections with $F^{0,2} = 0$, i.e. $\bar\partial_A^2 = 0$. See "holomorphic Casson invariants" by

R. Thomas, counting holomorphic line bundles over Z. More generally, counting coherent sheaves on Z to generalize DT invariants.

Remark 5.0.16: See Atiyah-Floer conjecture.

6 | Sylvester Gates, SUSY, Topology, Chern-Simons Theory

Remark 6.0.1: Topic: ectoplasm conjecture. Use pure group theory to study supersymmetry and supergravity. M-theory and supergravity: 11 dimensions. Superspace: interior of a sphere, ordinary space is an equatorial plane. Consider the weight lattice for $\mathfrak{su}_3$. Define an action functional using $\star d \star$. See nonlinear sigma model for pions? See super curvature. Covariant derivative allows coupling to matter fields. See potential of a connection? Given degrees of freedom, how are they represented? Scalars? n-forms? What are the representations of $\mathfrak{so}_4$? Which ones are spinor representations? Traceless symmetric tensors correspond to gravitons? Link between Young tableaux and Dynkin labels? There are multiplication rules for both of these. Can replace data of super fields with a poset of Young tableaux, stratified by level, and even just track them in a formal polynomial. Group theory for physicists: representation theory of of compact Lie groups!

7 | Charles Kane, Quantized Nonlinear Conductance in Ballistic Metals

Remark 7.0.1: Applications of Chern-Simons to condensed matter physics. Intro: relation to quantized Hall effect, and a related quantized response in 1D: the Landauer formula. How generalize: the Euler characteristic of the Fermi sea.

Remark 7.0.2: Integer quantized Hall effect: can confine electrons to a 2D plane. Standard experiment: measure conductance as a function of the magnetic field. Remarkably, nearly a sum of step functions! Measures fundamental constant h/e^2 extremely accurately, NIST declared it an *exact* number which in turn defines the Ohm and kilogram. Low energy produces a Chern-Simons TFT.

Remark 7.0.3: Topological band theory: a mean field theory that reduces QM particles to understanding single particles. Band structure: collection of energy eigenvalues/eigenvectors parametrized by momenta (S^1 or more generally a torus due to periodicity). If occupied/empty states are separated by a gap, the N occupied bands forum a U_n bundle over T^d a torus. Problem: classify vector bundles over T^d. In dimension 2, the first Chern class c_1 yields a number: $n = \frac{1}{2\pi} \int_{T^2} \mathrm{Tr}(F), F = da + A \wedge A$ for $A = i\langle u_i,\ du_j \rangle$ the Berry connection.

Remark 7.0.4: In metals, electron states are occupied for $E < E_F$ a constant, so define Fermi sea as the region in momentum space bounded by the associated Fermi surface. Why topology shows up: integrate the connection associated to the above Chern class to get something gauge invariant. This is something that can be measured in a lab! Chern numbers measured how twisted the vector bundle is.

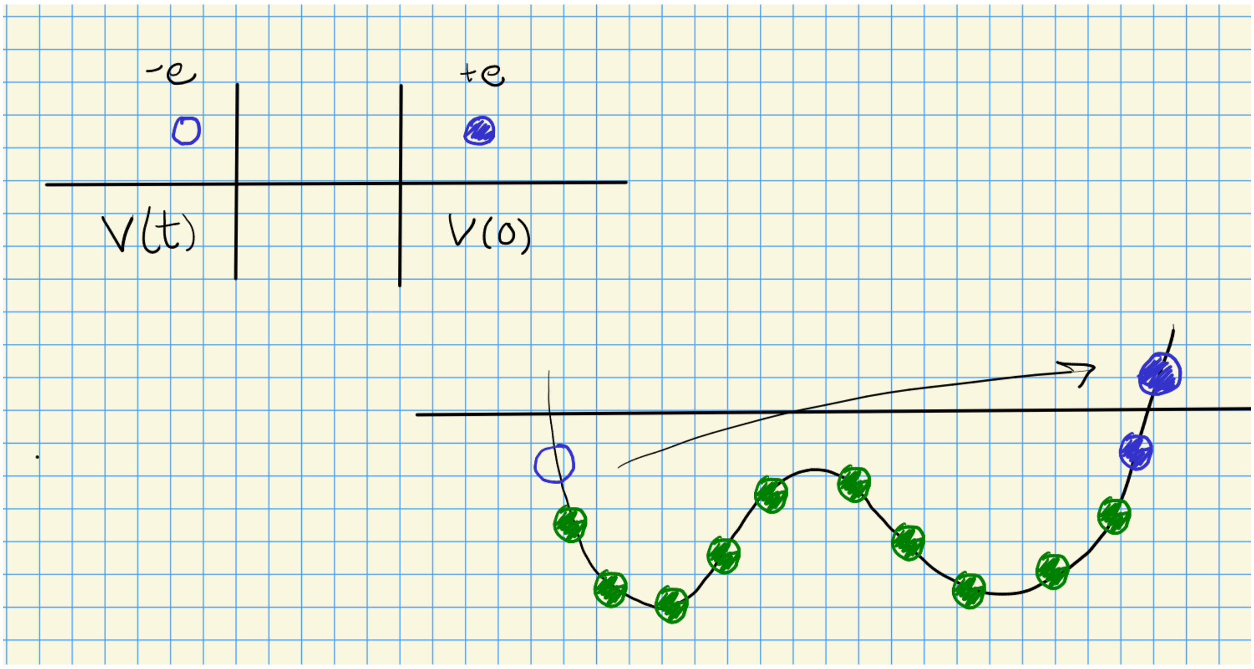
Remark 7.0.5: Seemingly tough problem (to me): put a sphere inside of a regular polyhedron. Puncture "symmetrically" corresponding to opposite faces, add a tube passing through the face, and then glue the resulting faces together. What surface do you get? Shows that Fermi surfaces can have nontrivial genus.

Remark 7.0.6: For 1D: see Landauer conductance. With assumptions (no backscattering of electrons, no reflectance), can get exact model

$$I = \frac{1}{2\pi}\int \mathrm{d}k\, ev_k = \frac{e}{h}\int \mathrm{d}E = \left(\frac{e^2}{h}\right)V.$$

What is the homeomorphism type of the 1D Fermi sea? Use quantized Landauer conductance experiments to probe $\sharp\pi_0$. Idea: χ_F carries more information than $\chi_{\partial F}$ for F the Fermi sea and ∂F its boundary surface. For F, there is a natural Morse functional given by measuring electron energy. Passing through critical points is like a phase transition?

Remark 7.0.7: How to explain the quantization in the Landauer formula: consider dimension $D = 1$. Apply a voltage impulse to move electrons:



See chiral anomaly. Use Morse theory to compute χ.

Remark 7.0.8: See conformal field theories, e.g. 1+1 boson for Fermi liquids? Where else might χ_F show up? Generalizing chiral anomalies? A universal way to characterize quantum entanglement? Generalizations to higher dimensions or non-Fermi liquids?

8 John Lott, Chern-Simons, Differential K-theory, Operator Theory

Remark 8.0.1: Chern character form: M smooth, E a Hermitian vector bundle, ∇ a compatible connection, get a closed form

$$\operatorname{ch}\nabla = \operatorname{Tr} e^{-\nabla^2} \in \Omega^{\text{even}}(M).$$

For two connections ∇_i, take a homotopy $\nabla_s = s\nabla_1 + (1-s)\nabla_0$, the Chern-Simons form is

$$\operatorname{CS}(\nabla_0, \nabla_1) \coloneqq \int_0^1 \operatorname{Tr}\left(\frac{\partial \nabla_s}{\partial s} e^{-\nabla_s^2}\right) ds.$$

Alternative construction of Chern character and Chern-Simons form due to Quillen: instead of interpolating between ∇_0, ∇_1, interpolate between $\begin{bmatrix} \nabla_0 & 0 \\ 0 & \nabla_1 \end{bmatrix}$ and ∞.

Remark 8.0.2: Recall

$$\mathsf{K}^0 = \frac{\mathbb{Z}[\{0 \to A \to B \to C \to 0\}]}{\langle [B] = [A] + [C] \rangle}.$$

Differential $\check{\mathsf{K}}$-theory: for K^0, quadruples $(E, h^E, \nabla^E, \omega)$ with $E \in \mathsf{Bun}\left(\mathrm{GL}_r\right)_{/M}$, h^E a Hermitian metric, ∇ a Hermitian connection, ω an auxiliary form. Quotient by relations $\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$ when the E_i form a SES of vector bundles, $\omega_2 = \omega_1 + \omega_3 + \varepsilon$ where ε is a correction term. Forget everything but the vector bundle to get a map $\check{\mathsf{K}}^0 \to \mathsf{K}^0$ and a Chern character $\check{\mathsf{K}}^0(M) \to \Omega_K^{\text{even}}(M)$. Kernel is K^{-1} and this forms a SES, so this mixes K and Ω.

Remark 8.0.3: Integration over the fiber in K-theory: index maps $\mathsf{K}(M) \to \mathsf{K}(B)$, see Atiyah-Singer index formula. Manifests as an equality of numerical indices. Joint work with Dan Freed establishes a similar result for $\check{K}$ using local index theory. Can compute certain η invariants without using analysis.

Remark 8.0.4: Idea: K-theory of finite dimensional vector bundles with connections. For today, how to generalize to infinite dimensions with super connections. Twisted K-theory: use elements in H^3. Problem: can't define $\operatorname{Tr} e^{-A^2}$ for A an operator, expanding and taking the 0th term already yields $\infty - \infty$! Fix: super connections, a sum $A = \sum A_{[i]}$, and Tr_s a super trace. Idea: $A_{[i]}$ are in $\Omega^i(M; ?)$ with restrictions. See C_2-graded Hilbert bundles over a manifold M.

Remark 8.0.5: What should the structure group G be..? For fibers H, should have a subgroup $G \leq \mathrm{U}(H)$ (the unitary transformations). Needs to be general enough to include $\operatorname{Diff}(Z)$ for Z the fibers. Construct using a type of Dirac operator, pseudo differential operators. Define structure group as even unitary transformations intersected with a certain space of 0th order pseudo differential operators op^0. See Bismut-Cheeger η form, $\eta(A, \infty)$ which "interpolates to ∞" as before.

Remark 8.0.6: Generators for $\check{\mathsf{K}}^0$: triples $(\mathcal{H}, A, \omega)$ with $\mathcal{H} \to M$ a C_2-graded Hilbert bundle, A a super connection, plus conditions. Turns out to be isomorphic to standard $\check{\mathsf{K}}$ from before.

Unclear if the Hopkins-Singer model is isomorphic. Standing assumptions: compact fibers, fiberwise tangent bundle is $\mathrm{Spin}^{\mathbb{C}}$. Need a horizontal distribution. Define an index as the image of a map $\check{\mathsf{K}}^0(M) \to \check{\mathsf{K}}^0(B)$ constructed by pushforwards.

Remark 8.0.7: Twisted K-theory: $H^3(M;\mathbb{Z})$ classifies U_1 gerbes on M. Gerbes: $\mathcal{U} \rightrightarrows M$, line bundles $\mathcal{L}_{ab}$ on double intersections, on triples $\mathcal{L}_{ab} \otimes \mathcal{L}_{bc} \xrightarrow{\sim} \mathcal{L}_{ac}$, and on quadruples a cocycle condition. Can define U_1 connections on a gerbe. Twisted Hilbert bundles: maps $\varphi_{ab} : \mathcal{H}_a \otimes \mathcal{L}_{ab} \to \mathcal{H}_b$. Define super connection on the open cover, plus compatibility on double overlaps to define globally. Closed: $(d + H\wedge)\,\mathrm{ch}(A) = 0$. Take same generators, quotient by new relations. Theorem: this new twisted $\check{\mathsf{K}}$ only depends on the class of the gerbe in H^3, and is independent of choices for connective structures and curving.

Remark 8.0.8: Open question: showing this is isomorphic to other models.

9 | Seiberg, Lattice vs Continuum QFT

Remark 9.0.1: QFT: successful but still not mathematically rigorous! Regularize by discretizing space to a lattice, then the functional integral is well-defined. Then take a continuum limit taking spacing to zero but fixing lengths, compute correlation functions. For condensed matter, find low-energy/long-distance limit. Expect it to be described by an effective continuum field theory.

Remark 9.0.2: Challenges with just regularizing to a lattice and taking a limit:

- Does the limit exist?
- Is the limit independent of details at finite levels?
- Lattice theory doesn't necessarily capture topological features (π_1, characteristic classes, Chern-Simons terms, etc). These are tied to global symmetries, anomalies.
- Some QFTs (e.g. with self-dual forms or fermions) don't allow putting DOF on the lattice, can't write an action.
- Some QFTs may not have a continuum Lagrangian (e.g. $2,0$ theory), so can't do a lattice Lagrangian.

Remark 9.0.3: Opposite problem: given an actual physical lattice, find the low energy continuum model. Often solvable, so can couple with whatever, but continuum limits have divergence issues. E.g. XY-plaquette, fracton models. Some issues:

- Separate symmetric group elements for different subspaces
- Observables vary at lattice scale a, and can become discontinuous in the limit
- Infinite ground state degeneracy in the limit

Remark 9.0.4: Very well understood model: $1+1$ boson. Take a 2d lattice with periodic boundary conditions, action

$$S = -\beta \sum \cos(\Delta_\mu \varphi),$$

where $e^{i\varphi}$ are phases on the lattice points. Sum over "links" (edge..?), Δ_μ is like a discrete derivative. Take limit to get

$$S = \frac{\beta}{2} \int \left(\frac{\partial \varphi}{\partial \mu} \right)^2 d\tau \, dx.$$

A symmetry emerges in the continuum: the winding number of φ, and a certain 't Hooft anomaly. How much is present in the original lattice?

Remark 9.0.5: Try to make those symmetries appear on the lattice. Idea: replace cos by a linear function, add a correction term that sums over plaquettes that is roughly curvature, scale by a constant that forces curvature to be zero. See *Villain form*. This kills the vorticity, which helps make the winding number show up on the lattice. Use Poisson resummation to show self-duality.

Suspected causes of anomalies: infinities, fermions, issues with invariance of measure. But none are present here!

Remark 9.0.6: A slightly more complex model: XY-plaquette in $2+1$ dimensions. Put phases $e^{i\varphi}$ on nodes with action

$$S = -\beta_0 \sum_{\text{edges}} \cos(\delta_\tau \varphi) - \beta \sum_{\text{plaqs}} \cos(\Delta_x \Delta_Y \varphi).$$

Has a global U_1 symmetry, take continuum limit to get

$$S = \int d\tau \, dx \, dy \, \frac{\mu_0}{2} \left(\frac{\partial \varphi}{\partial \tau} \right)^2 + \frac{1}{2\mu} \left(\frac{\partial^2}{\partial_x \partial_y} \varphi \right)^2 .$$

Remark 9.0.7: Other questions to ask to compare lattices to continuum limit:

- What are the (operator) spectra of the Hamiltonians?
- What are the correlation functions?

Importantly: limits don't commute!

Remark 9.0.8: Interesting generalizations:

- Replace U_1 with C_n
- Go to $3+1$ dimensions
- Look at subsystem symmetries

Modify the Villain term, compute the spectra and correlation functions to study. QCD: infinite number of states (vs finite number of states in these kinds of models?) There's a difference between global and gauge symmetries, exotic models land somewhere in between. See UV/IR mixing (short/long distances).

10 Xiao-Gang Wen, Chern-Simons, Nonabelian Topological Order

Remark 10.0.1: Goal: classify all possible phases of matter. Can model *spin liquids*, someone wrote a wave function for it. Quantum Hall states: 2d electron gas. Fractional quantum Hall states have different phases even when there is no symmetry (and thus no symmetry to break). See energy gaps, literally show up as gaps in the spectrum of Hamiltonians. Topological order: having a specific type of ground state degeneracy. "Topological" usually means robust against small perturbations, particularly ones that break symmetries.

Remark 10.0.2: Conjecture: chiral spin states and quantum Hall states are described at low energies by a path integral over an action which includes a Chern-Simons term corresponding to a U_1 gauge. Theme: things locally look the same, but can differ globally.

Remark 10.0.3: Cut electron into partons, write wave function for each and take product (gluing). Degeneracies: locally the same wave functions but globally different wave functions. See nonabelian statistics: modular tensor categories. Monopoles: defects or punctures in spacetime.

Remark 10.0.4: Looking for a mathematical language to describe long-range entanglement. Current directions: category theory, Chern-Simons theory. Framing anomaly: path integral depends on the framing of spacetime. Believe the path integral in Chern-Simons leads to a gravitational Chern-Simons term $\omega_{\mathrm{CS}}^{\mathrm{grav}}$.

11 Simons, Origins of Chern-Simons Theory

Remark 11.0.1: Motivations: a combinatorial formula for the signature of a 4-manifold. Triangulate and integrate the first Pontryagin class, yields a term that doesn't vanish coming from the 3-manifold surrounding a vertex. Setup: M^3 closed oriented Riemannian, then $\mathrm{Frame}(M) \to M \in \mathrm{Prin}\,\mathsf{Bun}_{\mathrm{SO}_3(\mathbb{R})}$. Write θ_{ij} for the connection form and Ω_{ij} its curvature form. Get a 3-form Q which integrates to zero along fibers (since curvature terms are horizontal). 3-manifolds are parallelizable, yielding sections of the frame bundle. Any two sections χ, χ' differ by an integer, so get a well-defined $\Phi(M) := \int_\chi Q \bmod \mathbb{Z}$

Remark 11.0.2: Theorems: $\Phi(M)$ is a conformal invariant, and $\Phi(M) = 0$ is necessary for M to admit a conformal immersion into $\mathbb{R}^4$. Application: $\Phi(\mathbb{RP}^3) = 1/2$, so although immersible and isometrically embeddable into $\mathbb{R}^4$, not conformally.

Interpret Φ as a map from the space of conformal structures to $\mathbb{R}/\mathbb{Z}$, whose critical points are locally conformally flat structures. Well-known fact: a locally conformally flat simply-connected 3-manifold is diffeomorphic to S^3. Could lead to a shorter proof of 3d smooth Poincaré. Chern got this invariant to work in all dimensions, led to Annals paper.

12 Nico Yunes, Astrophysical Observational Signatures for Dynamical Chern-Simons Gravity

Remark 12.0.1: Question: what observable signatures are produced by gravitational waves, black holes, neutron stars? Why are there more baryons than anti-baryons in the universe? General relativity predicts singularities – problematic due to infinities. Unclear if Chern-Simons

a. Has nothing to do with these phenomena, or
b. Explains them entirely.

Remark 12.0.2: New data: gravitational wages at LIGO in 2015, wavelike perturbations of the metric tensor that decay like $1/r$. After traveling $\sim 10^3$ megaparsecs, yields a very weak observable. Around 50 similar events in the years after, high level of confidence in measurements ($> 5\sigma$). Signatures help us figure out what to sift for in experimental data.

Remark 12.0.3: Pontryagin invariant: R^*R? Lagrangian density: perturbed Einstein equations for GR where tensor and scalar field are coupled? Recovers classical GR in a limit:

$$L \sim R - \frac{1}{2}(\nabla_a \nu)(\nabla^a \nu) + \alpha_{\mathrm{dCS}} \nu R^* R,$$

for R the Ricci tensor. Now vary with respect to degrees of freedom. Currents: something whose divergence is the original thing? Third derivatives: not good in physics, causes ghosts, unbounded Hamiltonians. Metric is GR metric plus order α^2 terms. Need to reduce order to get rid of third derivatives, unphysical unstable modes, due to insistence on treating this as an exact theory.

Remark 12.0.4: Chern-Simons form here:

$$\mathrm{Tr}(dA \wedge A + cA \wedge A \wedge A), \qquad c = 2/3.$$

Remark 12.0.5: Spherically symmetric black holes: assuming static and vacuum, the unique solution is the Schwarzschild metric. See box operator, d'Alembertian?

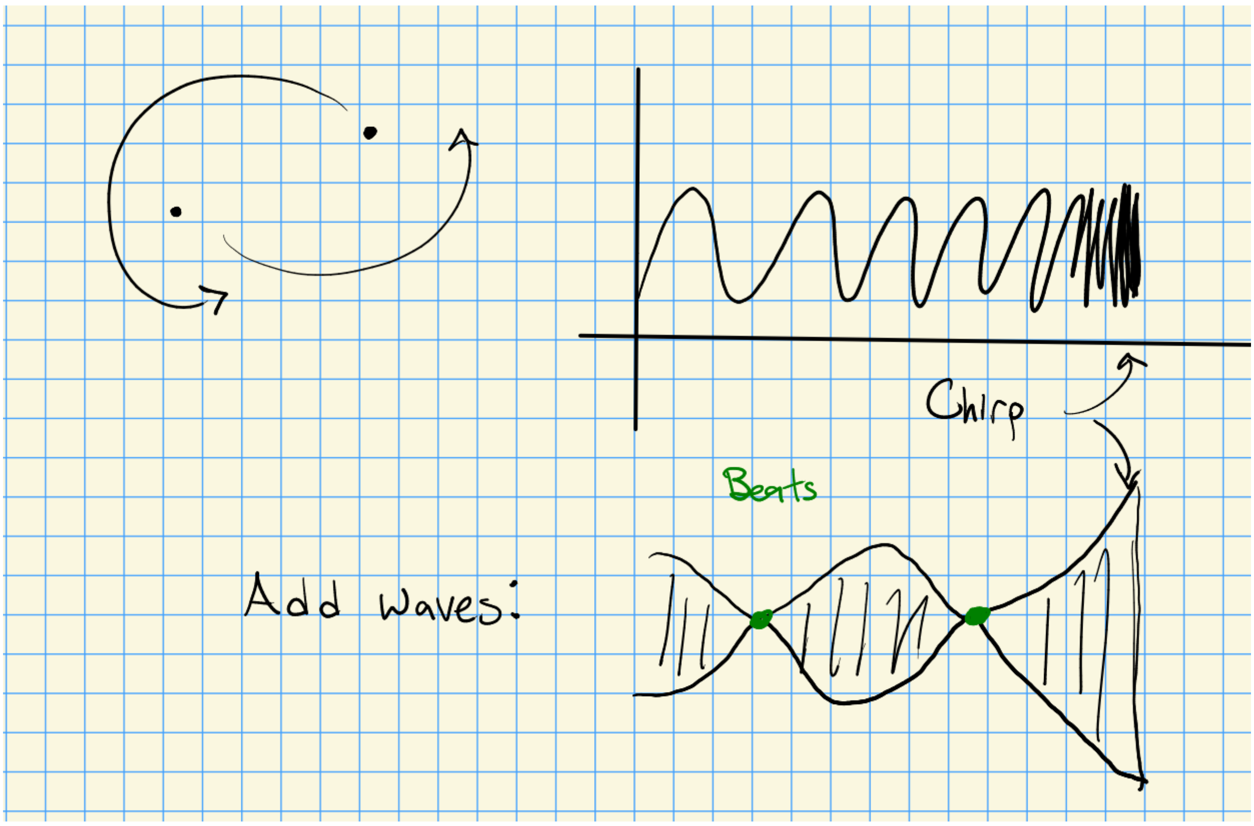
$$\begin{aligned} \square \propto \frac{\partial^2}{\partial t^2} - \Delta &= \mathrm{diag}(1,-1,-1,-1)\nabla^2(t,x,y,z) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \Delta(t,x,y,z) \\ &= \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}. \end{aligned}$$

Effective field theory treatment for axially symmetric black holes, yields 2nd order PDE, no exact solution. Can take first order solution, not bad to do by hand. Second and fifth order, much more complicated but easily handled by a computer, no clear obstructions to higher order expansions. Yields “hairy” black hole solutions: a $1/r^2$ scalar field, a perturbed horizon and ergosphere, no naked singularities, polar “caps”: regions where geodesics need not focus. No killing tensor, so no 4th constant of motion. Are there observational signatures for the caps? Could there be chaos in geodesic motion for test particles in orbit around dynamical Chern-Simons black holes? Chaos in this dynamical system, say around supermassive black holes, should imprint on gravitational waves. Recent results: quasinormal modes carry such a dynamical signature.

Remark 12.0.6: Summary:

- Spinning black holes are not Kerr, bc they excite a scalar field
- Binary black hole space time has two scalar fields anchored with each black hole.
- Dynamical CS induces a $2PN$ correction to the orbital evolution.

dCS is a $v/4$ correction to GR. Plot frequencies of orbit, play as sound – collision causes a chirp as amplitude and frequency spike, correction causes a different chirp. Now add the sinusoids to obtain beats/nodes in the waves, LIGO looks for these!



Remark 12.0.7: Problem with wave detection on Earth: seismic noise, creates a frequency wall. Future projects: labs in space in 2035-2045! Things that haven’t been worked out in dCS yet:

- AdS black holes?
- Exact rotating solutions

- Gravitational collapse
- Singularity theorems
- Area theorems

13 Kevin Costello, Chern-Simons in Dimensions 4,5,6

Remark 13.0.1: Why holomorphic CS is interesting: mirrors counts of curves through mirror symmetry, and shows up in construction of 4d integrable field theories.z

Remark 13.0.2: In higher dimensions: non-topological field theories. Setup: for X a Calabi-Yau 3-fold, $A \in \Omega^{0,1}(X, \mathfrak{g})$ a $\bar{\partial}$ connection, write a Lagrangian

$$\mathrm{hCS}(A) = \int_X \Omega_X \wedge \mathrm{CS}(A).$$

Equations of motion imply flat connection, here $F^{0,2}(A) = 0$, so a holomorphic instead of flat bundle. Translation $\dfrac{\partial}{\partial \bar{z}_i}$ involves a BRST term. For 4-manifolds, take $\Sigma_1 \times \Sigma_2$ a bundle of Riemann surfaces, where $\omega \in \Omega^1(\Sigma_2)$ has no zeros, and $A \in \Omega^1(\Sigma_1 \times \Sigma_2) \operatorname{mod} \Omega^{1,0}(\Sigma_2)$. The Lagrangian is $\displaystyle\int \omega \wedge \mathrm{CS}(A)$ for ω a 1-form, the equations of motion become $\omega F(A) = 0$. Locally looks like $\displaystyle\int zF \wedge F$ if one writes $\omega = dz$.

Remark 13.0.3: 4d CS unifies integrable PDEs and field theories. Basic example of integral PDEs: $G \in \mathrm{Lie}\,\mathsf{Grp}$ compact, $\sigma : \mathbb{R}^2 \to G$, asking σ to be harmonic is an integrable PDE. Importantly *lax*, so can build a connection that is flat iff the harmonic equation holds. An important generalization: include a WZW term. Interpret maps $\mathbb{R} \times S^1 \to G$ as $\mathbb{R} \to \Omega G$, where the phase space is $\mathbf{T}^\vee \Omega G$. Conservation: $\{H,\ m(z)\} = 0$.

Remark 13.0.4: Given a Riemannian manifold with a closed 3-form, when is the harmonic map equation on M integrable? Traditional examples: Riemannian symmetric spaces and their deformations. One example is S^2 with a specific metric, which also satisfies a Ricci flow condition.

Remark 13.0.5: 4d Chern-Simons: equations of motion become maps $\sigma : \mathbb{R}^2 \to \mathsf{Bun}\,(G)_{/\Sigma}$, a moduli of bundles over a Riemann surface trivialized at the poles of ω a 1-form. This has a canonical metric and 3-form. Theorem: the harmonic map equation for these maps is always integrable. Proved painfully without reference to 3D CS.

Remark 13.0.6: Ricci flow is closely tied to integrability. For Riemannian manifolds with closed 3-form, modify the flow $\delta g_{uv} = \mathrm{Ric}_{uv} - ?$. The "ancient" solutions correspond to something unique. Consider the moduli of Ricci flows. There is a natural flow on the moduli space $\mathcal{M}(\Sigma, \omega)$ given by computing periods of the form between its zeros? Turns out to be proportional to a Ricci flow on this space.

Remark 13.0.7: For 4d, mostly classical at this point. Higher dimensions: get a similar Lagrangian

$$\int dz_1\, dz_2\, \mathrm{Tr}(\mathrm{Ad}\, A) + c\, \mathrm{Tr}(A * A * A) \qquad A * B \coloneqq A \wedge B + c \frac{\partial A}{\partial z_1} \wedge \frac{\partial B}{\partial z_2} + \cdots .$$

5d nonabelian CS is a supersymmetric sector of M-theory. Holography: can match supersymmetric things for N $M2$ branes with computations in this 5d CS theory. These are QM particles moving on a moduli space of rank k instantons on $\mathbb{R}^4$ with charge N.

Remark 13.0.8: Most natural variant: holomorphic CS on a Calabi-Yau with $A \in \Omega^{0,1}(X)$:

$$\int \Omega_X \wedge \mathrm{CS}(A).$$

Problem: doesn't exist as a quantum theory due to gauge anomalies. Apply Grothendieck-Hirzebruch-Riemann-Roch to show the canonical doesn't vanish:

$$c_1 \mathsf{Bun}\,(G)(X) = \int_X \mathrm{Td}(\mathbf{T}X)\, \mathrm{ch}(\mathrm{Ad}_{\mathfrak{g}}).$$

Do some anomaly cancellation. Famously not renormalizable. Restricts gauge groups to SO_8 or $G_2 \times G_2$? For $\sigma : \mathbb{R}^4 \to \mathrm{SO}_8$, Lagrangian involves Kähler potential.

14 Minhyong Kim, Arithmetic Field Theories and Invariants

Remark 14.0.1: Problem: classify principal bundles over a point $\mathrm{Spec}\, k \in \mathrm{Spec}(\mathsf{Field})$. Classified by

$$\mathrm{Prin}\, \mathsf{Bun}\,(G)_{/\mathrm{Spec}\, F} \xrightarrow{\sim} H^1(\pi_1\, \mathrm{Spec}\, F; G).$$

Here G could be a p-adic Lie group, e.g. a Tate module $\varprojlim_n A[n]$ for $A \in \mathsf{AbVar}$. If G acts trivially, this becomes a representation variety $\mathrm{Hom}(\pi_1\, \mathrm{Spec}\, F, G)/\, \mathrm{Inn}(G)$, quotienting by conjugation. A complete description is essentially the Langlands reciprocity conjecture!

Remark 14.0.2: Idea: replace F be $\mathcal{O}_F$ its ring of integers, so $X \coloneqq \mathrm{Spec}\, \mathcal{O}_F$ with the étale topology. Behaves like a compact closed 3-manifold. For $v \in \mathrm{mSpec}\, \mathcal{O}_F$, $k_{\widehat{v}} = \mathcal{O}_F/v$ is a finite field, so $\mathrm{Spec}\, k_{\widehat{v}} \hookrightarrow X$ is like an embedding of a knot. Completion at v, $\mathcal{O}_{F,v} \coloneqq (\mathcal{O}_F)^{\widehat{v}}$ is like a formal tubular neighborhood. Completing the original ring at v, so $F^{\widehat{v}}$ is like a tubular neighborhood with the knot deleted X_B. For B a finite set of primes, set $\mathcal{O}_{F,B}$ to be the set of B-integers, i.e. almost algebraic but allowing denominators from B. Then $\mathrm{Spec}\, \mathcal{O}_{F,B}$ is like a 3-manifold with boundary, so

$$\partial X = \coprod_{v \in B} \mathrm{Spec}\, F_v \to X_B \hookrightarrow X.$$

Remark 14.0.3: What are these π_1? Simple structure in nice cases, $\pi_1 \operatorname{Spec} k_{\widehat{v}} = \widehat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$. A finite field extension $K_{/F}$ is **unramified** over $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_F$ if the prime decomposition $\mathfrak{p}\mathcal{O}_K = \prod \mathfrak{q}_i$ has no primes with multiplicity. There is a maximal unramified extension, just take the compositum of all unramified extensions. Can restrict this to just be unramified over primes not in B Note that $\mathsf{Ab}(\pi_1 X) = \operatorname{Pic}\mathcal{O}_F = \operatorname{cl}(F)$ is the ideal class group of F. Old results: $\pi_1 \operatorname{Spec} \mathbb{Z}, \pi_1 \operatorname{Spec} \mathcal{O}_F = 0$ for F imaginary quadratic with $\operatorname{cl}(F) = 1$. Assume GRH to get $\pi_1 \operatorname{Spec} \mathcal{O}_K = A_5$ for $K = \mathbb{Q}\left[\sqrt{653}\right]$, or $\mathrm{PSL}_2(\mathbb{F}_8) \times C_{15}$ for $K = \mathbb{Q}\left[\sqrt{-1567}\right]$.

Question 14.0.4
Central problem: classify arithmetic 3-folds, or more generally understand $H^1(\pi_1 X_B; G)$, isomorphism classes of principal G-bundles over X_B. Write as $\mathcal{M}(X_B, R)$ for R the group, breaks into $\prod_{b \in B} \mathcal{M}(X_b, R)$.

Remark 14.0.5: Assume F is complex, so $F \cong \mathbb{Q}[x]/\langle f \rangle$ where f has no real roots. For gauge theory interpretations, need to write an action

$$S : \mathcal{M}(X_B, R) = H^1(\pi_1 X_B, R) \to K$$

and path integral

$$\int_{\rho \in \mathcal{M}(X_B, R)} \exp(\mathrm{CS}(\rho))\, d\rho.$$

Actions take the form of L-functions:

$$L : \mathcal{M}(X_B, \mathrm{GL}(V)) \to \mathbb{C} \text{ or } \mathbb{C}_p.$$

To a representation $\pi_1(X_B) \to \mathrm{GL}(V)$ assign

$$L(\rho) = \prod_{v \in \operatorname{Spec} \mathcal{O}_F} \frac{1}{\det\left(I - \mathrm{Frob}_v\right) V^{I_v}}.$$

Importantly, this product may not always make sense! It's a big problem to regularize this to get convergence. Hasse-Weil conjecture says one can renormalise to a function of s in $\mathbb{C}$, get convergence for $\Re(s) \gg 0$, and analytically continue to $s = 0$ or $s = 1$ to recover original product. Write $r \coloneqq \mathrm{Ord}_{s=0} L(\rho(s))$ for the order of vanishing. For the trivial representation φ, $r = \operatorname{rank} \mathcal{O}_F^{\times}$. For $\rho = T_p E$ for an elliptic curve, $r = \operatorname{rank} E(\mathbb{Q})$ (the Mordell-Weil group) assuming BSD. Néron-Tate height pairing is a metric, its determinant appears in L function for E.

Remark 14.0.6: Need arithmetic orientations, problematic since dualizing sheaves are often μ_n. Fact:

$$H^3(X; \mu_n) = H^3(\operatorname{Spec} \mathcal{O}_F; \mu_n) = \frac{\frac{1}{n}\mathbb{Z}}{\mathbb{Z}}.$$

More well-known that

$$H^3(X; \mu_n) = H^3(X; \mathbb{G}_m)[n] \cong (\mathbb{Q}/\mathbb{Z})[n].$$

Local CFT yields $H^2(F_v; \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$ by classification of gerbes. Follows from SES in global CFT:

$$0 \longrightarrow H^2(F;\mathbb{G}_m) \xrightarrow{\text{loc}} \bigoplus_v H^2(F_v;\mathbb{G}_m) \xrightarrow{\text{sum}} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Link to Diagram

Assume $\mu_n \subseteq F$, get a map

$$\text{inv} : H^2(\pi_1 X; C_n) \to H^3(X;\mu_n) \cong \frac{\frac{1}{n}\mathbb{Z}}{\mathbb{Z}}.$$

Use this to build a Chern-Simons function $\text{CS} : \mathcal{M}(X;R) \to \frac{\frac{1}{n}\mathbb{Z}}{\mathbb{Z}}$, essentially by pulling back cocycles along the representation. Use Bockstein to define a "path integral" defined as a finite sum over representations. Closed form solution involves Legendre symbol, very nice! Also a determinant of a quadratic form.

Remark 14.0.7: Bockstein is common in arithmetic geometry, $d : H^1(X;C_n) \to H^2(X;C_n)$, used to construct de Rham-Witt complex for crystalline cohomology. On general, having a SES like $0 \to V \to E \to V \to 0$ allows defining such arithmetic functionals. Can define a bilinear pairing

$$\begin{aligned} \text{BF} : H^1(X;C_n) \times H^1(X;\mu_n) &\to 1/n\mathbb{Z}/\mathbb{Z} \\ a \times b &\mapsto \text{inv}(da \smile b). \end{aligned}$$

Nice closed form solutions for the "path integral":

$$\sum_{(a,b)\in H^1(X;C_n)\times H^1(X;\mu_n)} \exp(2\pi i \text{BF}(a,b)) = \sharp \text{Pic}(X)[n] \cdot \sharp(\mathcal{O}_X^\times/(\mathcal{O}_X^\times)^n).$$

Compare to orders of L functions.

Remark 14.0.8: This setup occurs for Néron models of elliptic curves: for $n \gg 0$, there is a SES

$$0 \to \mathcal{E}[n] \to \mathcal{E}[n^2] \to \mathcal{E}[n] \to 0.$$

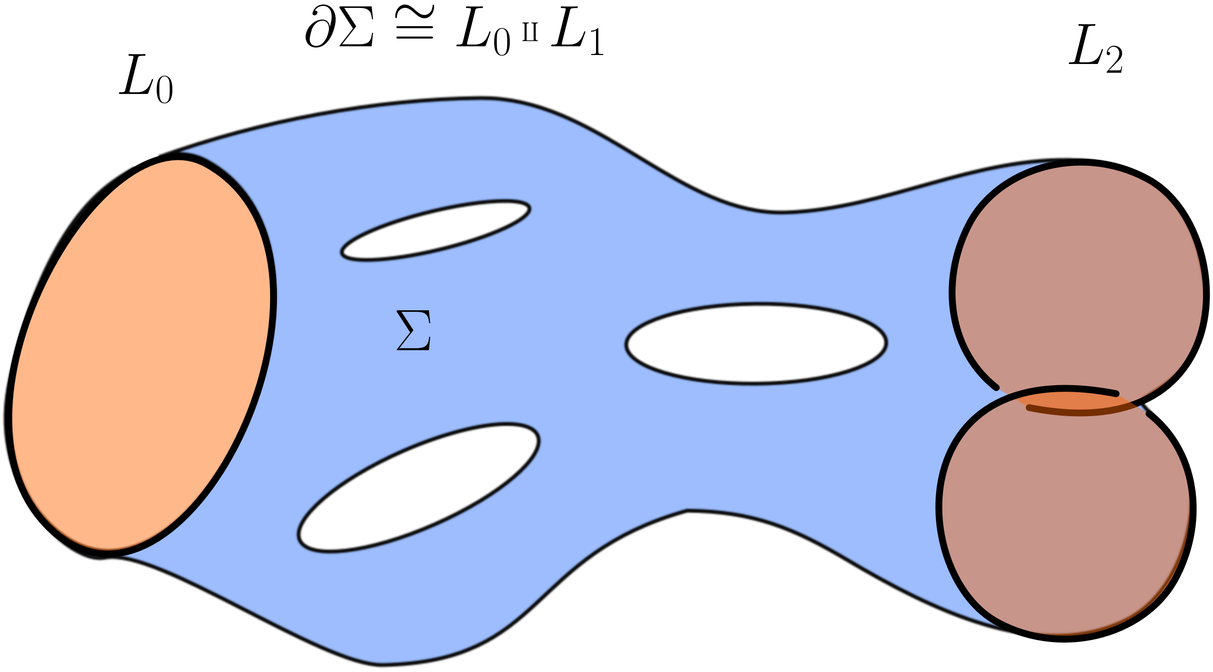
Path integral evaluates to $\sharp Ш(A)[n] \cdot (\sharp E(F)/n)^2$.

Remark 14.0.9: For boundaries: $X_B = \operatorname{Spec} \mathcal{O}_F[\frac{1}{B}]$ for B a finite set of primes. Start with $\mathcal{M}(X_B, R)$, get a local version $\mathcal{M}(\partial X_B, R)$. Here $H^2(\pi_v; C_n) \cong 1/n\mathbb{Z}/\mathbb{Z}$ and vanishes for $i \geq 3$, get global CFT SES. Pulling back cocycles lands in Z^3, but H^3 is trivial, so look at space trivialization (torsors for H^2). Get a bunch of local torsors, then sum to get a single $1/n\mathbb{Z}/\mathbb{Z}$ torsor. Do some local/global and push/pull gymnastics to get a U_1 bundle over $\mathcal{M}(\partial X_B, R)$. Space of sections breaks up as a tensor product of local spaces of sections, can cook up a way to land in a Hilbert space, and this is the "state" you assign to the 3-manifold.

Remark 14.0.10: Recent work: entanglement of primes.

15 | Khovanov, Categorification

Remark 15.0.1: CS functional critical for 3 and 4 dimensional TQFTs. In 3d, path integral in Witten's construction of WRT invariants. In 4d, in instanton Floer homology. Generally a tensor functor from 3-manifolds with 4d cobordisms to some algebraic category like AbGrp. Motivational problem: construct a 4D TQFT that categorifies the WRT invariant for 3-manifolds. Look at tensor functors from the categories of links in $\mathbb{R}^3$ and link cobordisms in $\mathbb{R}^3 \times I$ to some algebraic category.

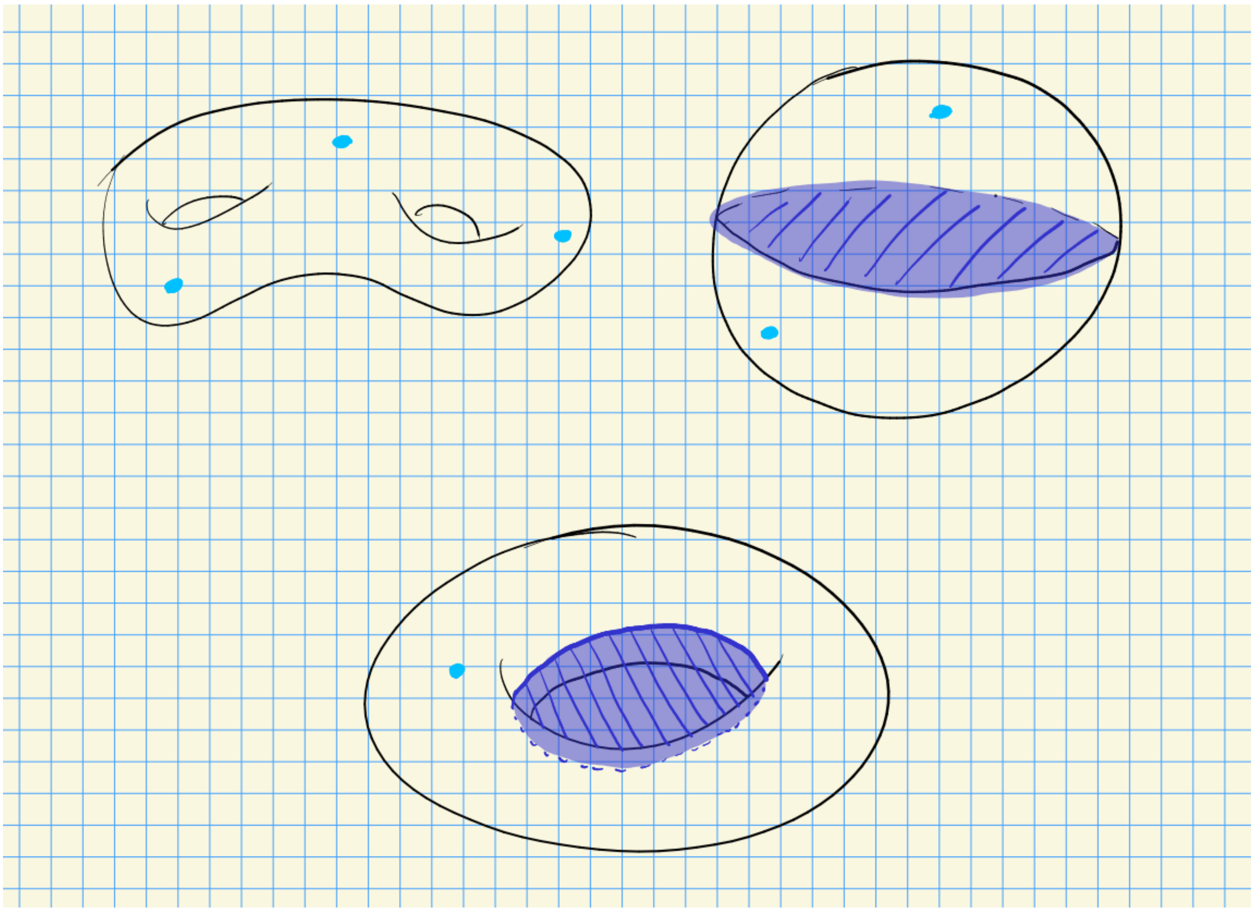


Remark 15.0.2: Components in links colored by irreps of a simple Lie algebra $\mathfrak{g}$. Categorifying: recover polynomials as graded Euler characteristics. In some cases, not clear how to extend link homology to link cobordisms. Link homology is best understood with miniscule representations, e.g. $\mathfrak{g} = \mathfrak{sl}_n$, labeled by $\bigwedge^k V$ for $V \cong \mathbb{C}^N$ the fundamental representation. For various N, recovers Jones, Alexander, HOMFLYPT polynomials. Skein relation: assign a complex to resolutions, find an exact triangle, set the original complex to be the cone of the map between the complexes of the two resolutions. Why this is a good idea: reduces 3d diagram projections to actual 2d planar graphs.

Remark 15.0.3: WRT is an invariant of a planar graph, homology supported in a single degree. Maps between graph homology induced by foams. Look for functors that assign objects to vector spaces and foams to morphisms. A foam: for SL_3, a 2d combinatorial CW complex with generic singularities embedded in $\mathbb{R}^3$. Examples:

- $\Sigma_{g,n}$
- S^2 with a marked point and an equatorial disc attached.

- $S^1 \times S^1$ with an equatorial disc.



Introduce Tait coloring: any two facets sharing an edge must have distinct colors. E.g. the sphere above admits 6 such colorings over $\mathbb{F}_3$, since the 3 facets (north/south hemispheres and equatorial disc) share an edge (equator). Theorem: can produce a closed orientable surface in $\mathbb{R}^3$ from this. The procedure is roughly numbering the facets, then taking F_{ij} to be the surface obtained by leaving $\{i, c\}^c$ facets out. E.g. for index set $\{1, 2, 3\}$, you get F_{12}, F_{13}, F_{23}. Construct an evaluation map by constructing local evaluations with respect to an admissible coloring, then sum over such colorings to get a symmetric function in $k[x_1, \cdots, x_n]^{S_n}$. Initially defined to be rational, i.e. $\langle F \rangle \in k(x_1, x_2, x_3)$, but it turns out that denominators cancel and $\langle F \rangle \in k[x_1, x_2, x_3]^{S_3}$. Can pair foams by flipping and gluing along boundary to get a closed foam, then apply evaluation. Yields a bilinear pairing, consider the kernel to get interesting skein-type relations.

Remark 15.0.4: General construction yields a lax tensor functor: just a map, not an isomorphism.

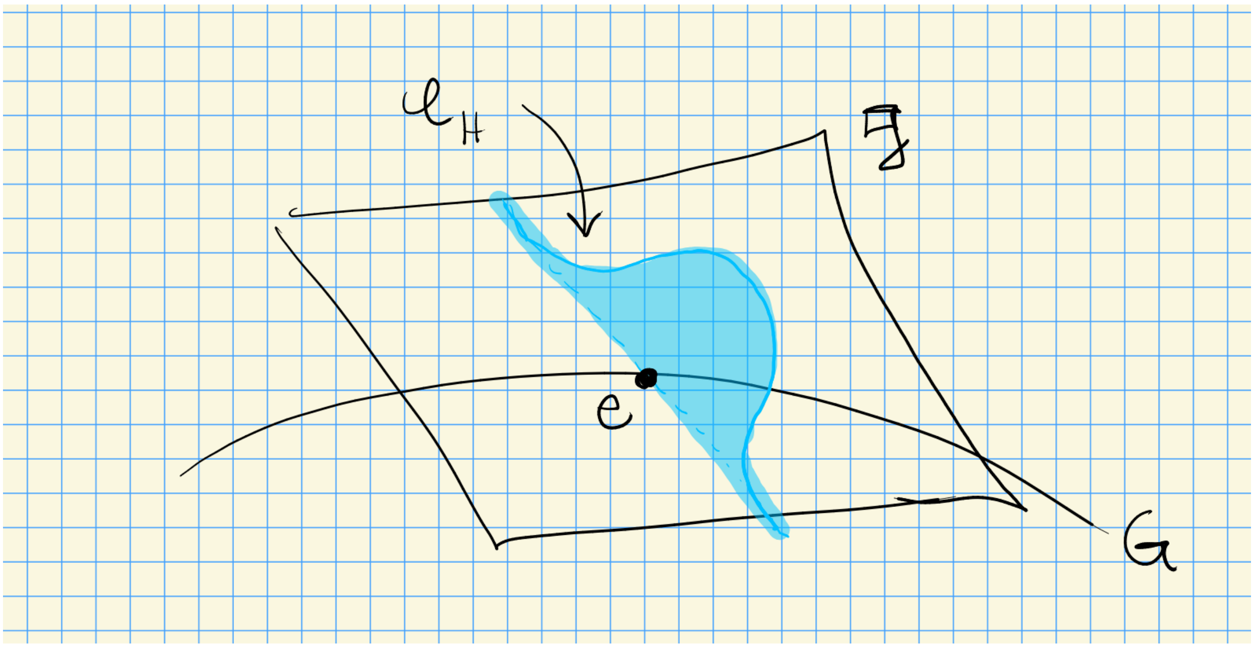
Remark 15.0.5: Conjecture: the state space $\langle \Gamma \rangle$ for Γ a graph is a free R-module of rank r the number of Tait colorings. Known for reducible graphs, i.e. skein relations can eventually break them into empty graphs. Theorem is true after a certain modification to $\langle \Gamma \rangle_\varphi$? Motivations coming from Kronheimer and Mrowka 2019, studying SO_3 instanton Floer homology over $\mathbb{F}_2$ for 3-orbifolds using a CS functional for orbifolds, primarily $\mathbb{R}^3/({C_2}^{\times 2})$ to get a trivalent vertex in a graph. Yields a homology theory for trivalent graphs in $\mathbb{R}^3$. Might give a new way to think about the four color theorem! Reductions: reduce links in $\mathbb{R}^3$ to diagrams in $\mathbb{R}^2$, set up evaluation, then take cones?

Remark 15.0.6: Current work: how to categorify $\zeta^p = 1$? Possibly use cyclotomic rings in characteristic p.

16 Freedman, Universe from a Single Particle, Metric Crystallization

Remark 16.0.1: Fun fact: Freedman did work on topological quantum computing. Interesting fun example: cone on $S^3 \times S^3 \hookrightarrow \mathbb{R}^7$. Idea for today: break metric symmetry on a Lie group.

Remark 16.0.2: Setup toy model for the beginning of the universe: finite dimensional Hilbert space $X \cong \mathbb{C}^n$ with symmetry group $G \coloneqq \mathfrak{su}_n$, and a metric g_{ij} on $\mathfrak{g} \coloneqq \mathfrak{su}_n$. This gives G a left-invariant metric. Now add a probability distribution on $\mathfrak{g}$, which are basically Hermitian matrices, whose draws are random Hamiltonians.



Several layers of randomness: choose metric in a Boltzmann manner, then choose a Hamiltonian using a Gaussian based on the metric.

Remark 16.0.3: Natural metric: Killing form, ad-invariant and a local (global) extremum in the space of metrics. Other metrics yield unit ellipsoids instead of unit spheres. Write an energy functional on this space, essentially scalar curvature (i.e. Ricci scalar curvature). Can extract an explicit formula from a paper of Milnor from the 70s on left-invariant metrics on Lie groups. Need these invariant metrics to sort out how to actually compile a quantum circuit. Nielsen-Brown-

Susskind define a word metric $g_{ij} = \delta_{ij} e^{?w(i)}$ where $w(-)$ counts the number of letters, e.g.

$$w(1 \otimes x \otimes 1 \otimes 1 \otimes y \otimes z \otimes 1 \otimes 1 \otimes 1) = 3.$$

This makes travelling along long words or words with capitals exponentially more difficult, rethinks the combinatorial problem of building a circuit as a differential geometric problem.

Remark 16.0.4: Qubit structure: $J : (C^z)^{\otimes n} \xrightarrow{\sim} C^{z^N}$. Define g_{ij} to be KAQ ("knows about qubits") if it has a basis of principle axes $\{H_a\}$ which admit a tensor product decomposition. Forms about a $\sqrt{d}$ dimension subvariety in a d-dimensional space – codimension 1 is already very thin, so these are exceedingly rare. However, about 30% of critical points for the action functional satisfy this property. Try to do perturb a Gaussian and expand/truncate. Slight issue: get an integral like $\int e^{g_{ij} \cdots + c_{ij} x^i x^j}$, but the x^i, x^j are commuting variables and the structure constants c_{ij} anticommute, so this term vanishes after integrating. Trying to integrate fermionically cancels the first term. See Majorana operators.

Remark 16.0.5: How they studied various metrics: gradient flow with respect to the action functional. I wonder how one actually does this in practice..?

Remark 16.0.6: Conclusions: nature abhors naked Hilbert spaces and attempts to equip them with tensor or Majorana structures. In $D = 2^n$, we see Majorana degree groupings and Brown-Susskind exponential penalty factors. For random initial dimensions, conjecture all but log many dimensions develop such a structure. A few exceptions: leaky universe scenario. Not good! Setup is well-suited to studying pairs of an initial Hamiltonian and initial state to model near-beginning universe scenarios. Consider the configuration space of these pairs and see how entropy evolves. Outstanding problem: where does "space" come from? Finding "space" in a model means finding the Leech lattice, wild! But checking for this by brute force requires 10^{12} qubits.

Remark 16.0.7: Feynman diagrams: if no thickness, kind of old school. Modern treatments involve ribbon diagrams.