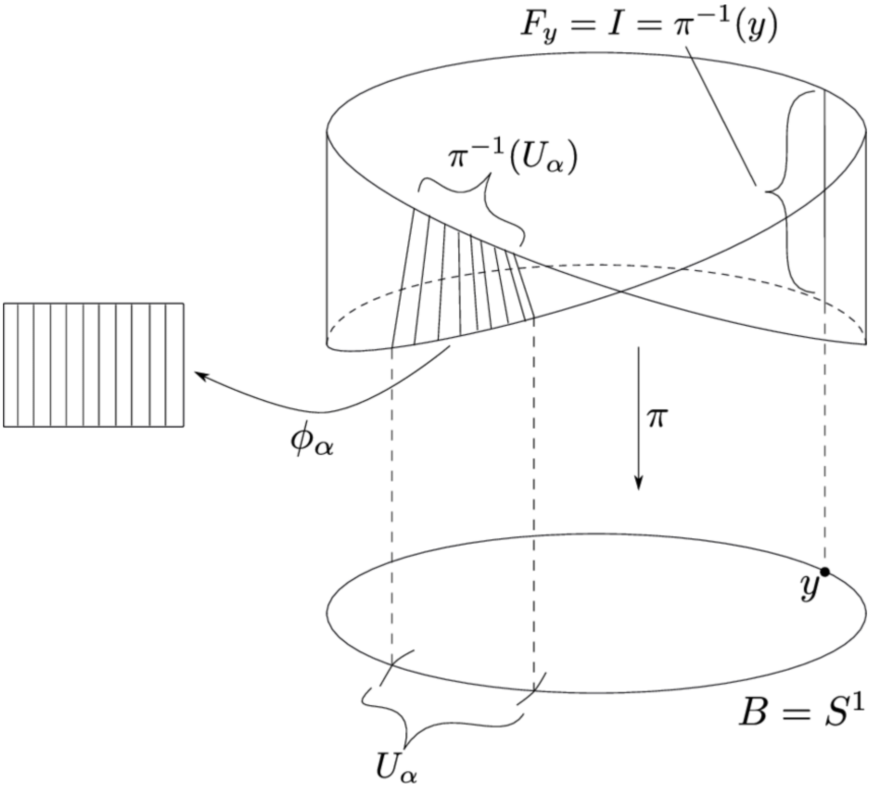

Notes: These are notes live-tex'd from a graduate course in characteristic classes taught by Akram Alishahi at the University of Georgia in Fall 2021. As such, any errors or inaccuracies are almost certainly my own.

Characteristic Classes

Lectures by Akram Alishahi. University of Georgia, Fall 2021

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Table of Contents

Contents

1 | Thursday, August 19

1.1 Intro and Overview

Remark 1.1.1: Course website: https://akramalishahi.github.io/CharClass.html

Description from Akram's syllabus:

> *This course is about characteristic classes, which are cohomology classes naturally associated to vector bundles or, more generally, principal bundles. They are a key tool in modern {algebraic, differential}×{topology, geometry}. The course starts with an introduction to vector bundles and principal bundles. It then discusses their main characteristic classes—the Euler class, Stiefel-Whitney classes, Chern classes, and Pontryagin classes. The last part of the class discusses some applications of characteristic classes to bordisms. In the process, we will see some nice applications (e.g., to immersions) and review some important parts of algebraic topology (e.g., obstruction theory).*

References:

- [**Hu**] Husemoller, Fiber bundles.
- [**MS**] Milnor and Stasheff, Characteristic classes.
- [**S**] Steenrod, The topology of fibre bundles.
- [**Ha**] Hatcher, Vector bundles and K-theory .
- [**BottTu**] Bott and Tu, Differential forms in algebraic topology.

Prerequisites:

- Smooth manifolds: smooth maps and derivatives, differential forms.
- Algebraic topology: homology and cohomology.

Remark 1.1.2: An overview of what we'll cover:

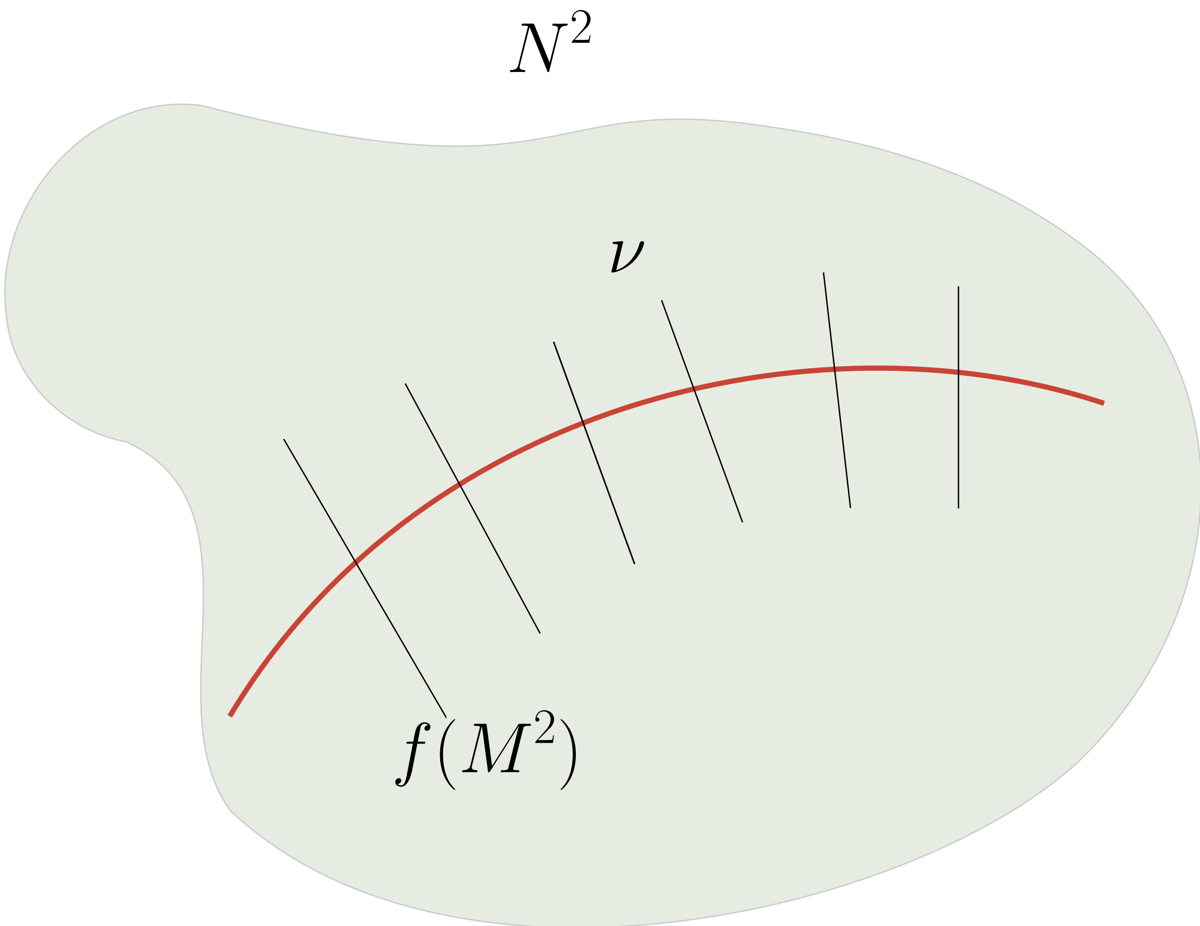
- **General definitions and constructions related to *vector bundles* and *fiber bundles.***

  Why bundles? For a bundle $E \xrightarrow{\pi} B$, characteristic classes will be cohomology classes in $H^*(B)$. Natural examples include

  - The tangent bundle $TX \to X$, and vector fields will be sections.

- Exterior products $\bigwedge^n TX$, where differential forms live
- Normal bundles ν, giving directions an embedded submanifold can be deformed.

Also note that manifolds locally look like vectors spaces ($\mathbb{R}^n$!) and so embedded manifolds locally look like vector bundles. In particular, if $f : M^n \hookrightarrow N^k$ is an embedding, locally ν is locally a $k - n$ dimensional vector bundle over $\mathbb{R}^n$ (and globally a bundle of the form $\nu : E \to f(M_n)$)



- **Characteristic Classes: Euler, Stiefel-Whitney, Pontryagin, etc.**

  These package geometric information into algebraic invariants that are often computable. Some examples:

  - Stiefel-Whitney classes can detect if $M^n = \partial M^{n+1}$ is a boundary (for smooth closed manifolds).
  - Euler classes can prove the Hairy Ball theorem, i.e. S^2 admits no nonvanishing continuous vector fields, which can be generalized to S^{2n} and to splitting the tangent bundle.
  - Pontryagin classes: Milnor used these to produce exotic S^7s! These are manifolds M^7 which are homeomorphic but not diffeomorphic to S^7.

– Chern classes.

1.2 Fiber Bundles

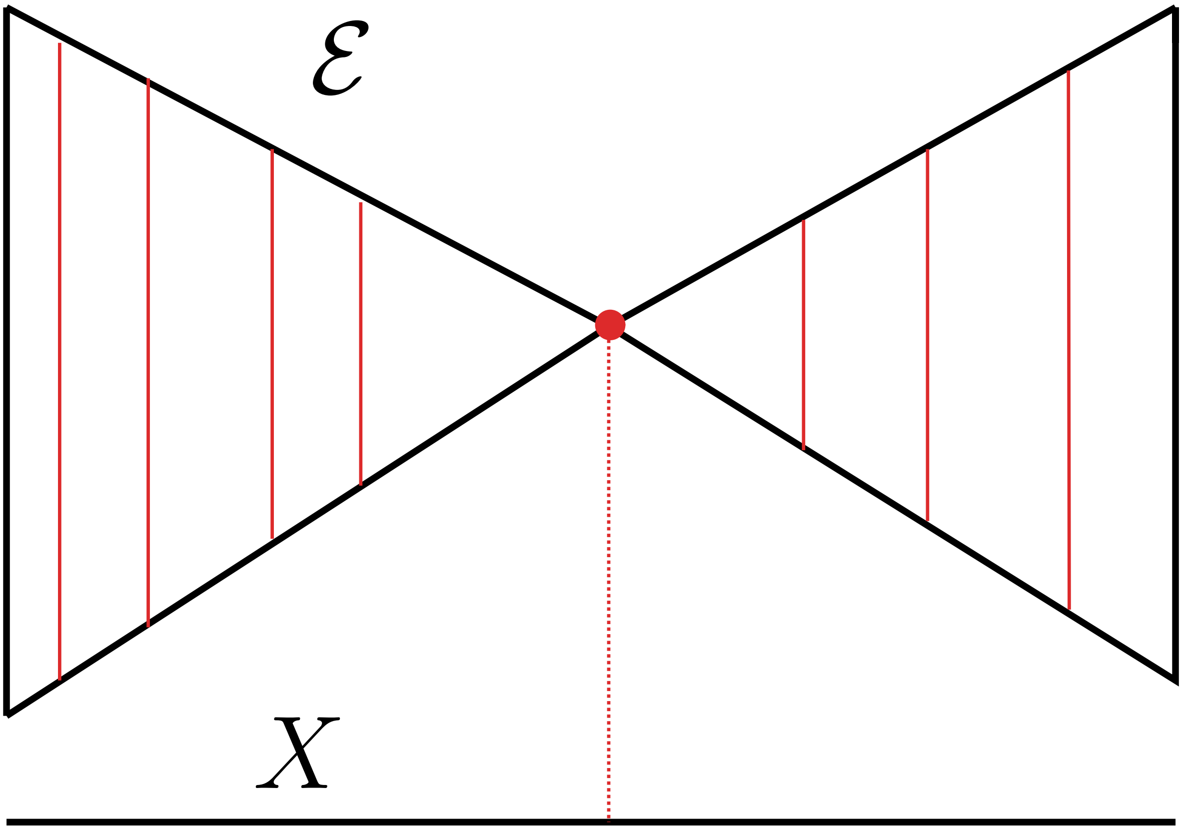
Definition 1.2.1 (Fiber bundle)
A **fiber bundle** over B with fiber F is a continuous map $\pi : E \to B$ where each $b \in B$ admits an open neighborhood $U \subseteq B$ and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes in Top:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\varphi\quad} & U \times F \\ & {\scriptstyle\pi}\searrow \qquad \swarrow{\scriptstyle p_2} & \\ & U & \end{array}$$

Here the square is $[0,1]^{\times 2}$.

Link to Diagram

Remark 1.2.2: Note that this necessarily implies that all fibers are homeomorphic, noting that $F_b \coloneqq \pi^{-1}(b) \xrightarrow{\varphi} \{b\} \times F$. We have inclusions: vector bundles $\implies$ fiber bundles $\implies$ fibrations. For a fibration that's not a fiber bundle, one can collapse a fiber in a trivial bundle, e.g.



Example 1.2.3 ***(?):*** An **atlas bundle** for $\pi : E \to B$ is a collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ such that $\{U_\alpha\} \rightrightarrows B$.

Example 1.2.4 ***(?):***

- $E \coloneqq B \times F \xrightarrow{p_2} F$ the trivial/product bundle.

- $\widehat{X} \to X$ any covering space. Note that the fibers are discrete.

- The Möbius band:

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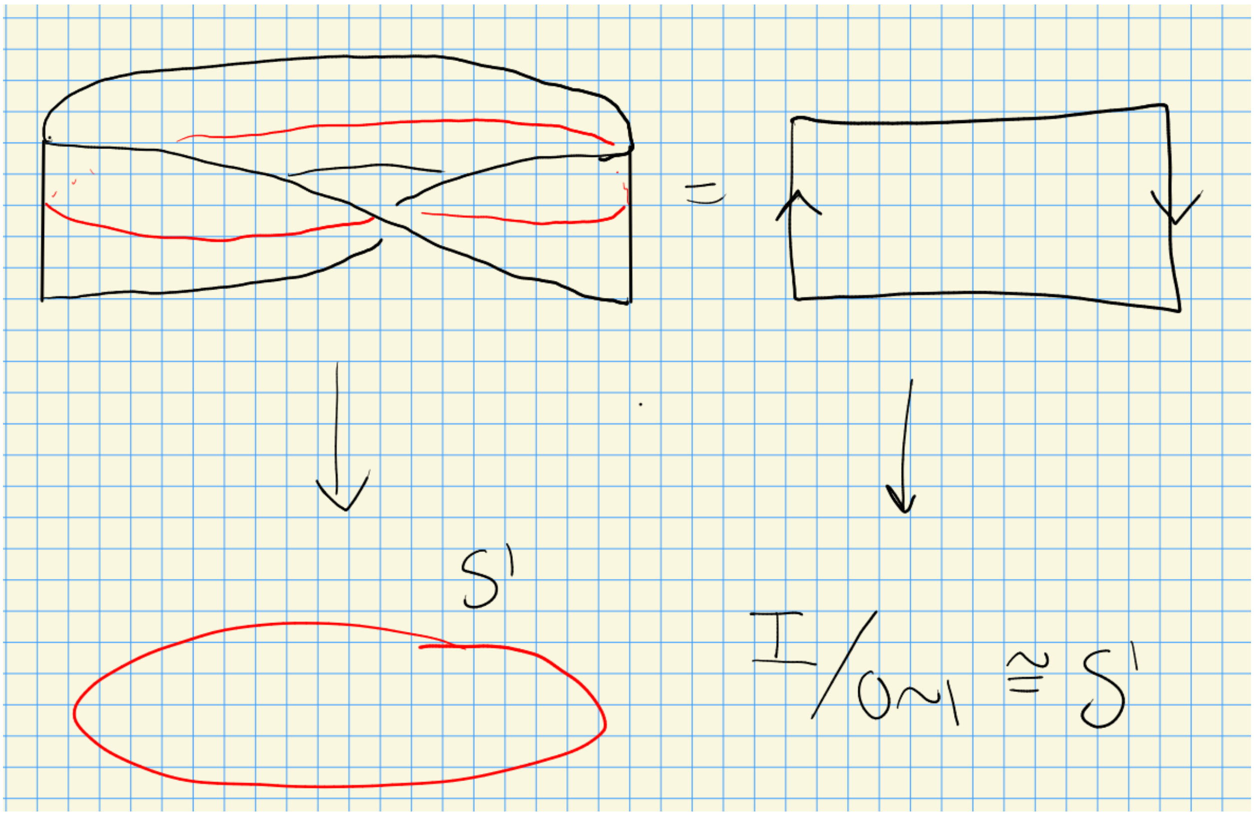
08-19_13-41.xoj –>



This is a fiber bundle with fibers $[0, 1]$. For a fiber bundle, include the boundary, but to make this a vector bundle do not include it!

Remark 1.2.5: Consider the following setup:

- $B \in \mathsf{Top}$
- $\pi : E \to B$ is a map of underlying sets
- There is a bundle atlas $\{\varphi_\alpha\}$, each φ_α being a bijection.

Then there exists *at most* one topology on E such that $\pi : E \to B$ is a fiber bundle with the given atlas.

Exercise 1.2.6 (?)
Find necessary conditions for at least one topology to exist!

1.3 Vector Bundles

Definition 1.3.1 (Vector bundle)
An n-dimensional real (resp. complex) **vector bundle** over B is a fiber bundle $\pi : E \to B$ along with a real vector space structure on each fiber F_b such that for each $b \in B$ there exists a neighborhood $U \ni b$ and a chart $(U, \varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n)$ (resp. $\mathbb{C}^n$) where $\varphi|_{F_b} : F_b \xrightarrow{\sim} \mathbb{R}^n$ (resp. $\mathbb{C}^n$) is an isomorphism of vector spaces.

Example 1.3.2 *(?):*

- The trivial (product) bundle $B \times \mathbb{R}^n \xrightarrow{p_1} B$.
- The tangent bundle TX.
- Identifying the Möbius band as $[0,1] \times (0,1)/\sim$ as $I \times \mathbb{R}/(0,t) \sim (1,-t)$ yields a 1-dimensional bundle $M \to S^1$.

Remark 1.3.3: We have some natural operations:

1. Direct sums.

For $E_1, E_2 \in \mathsf{Bun}\,(\mathrm{GL}_r)_{/B}$, so $E_1 \xrightarrow{\pi_1} B$ and $E_2 \xrightarrow{\pi_2} B$, we can form $E_1 \oplus E_2 \xrightarrow{\pi} B$. As a set, take

$$E_1 \oplus E_2 \coloneqq \bigcup_{b \in B} F_{1,b} \oplus F_{2,b}$$

as a union of direct sums of vector spaces. For the bundle map, take $\pi(F_{1,b} \oplus F_{2,b}) \coloneqq \{b\}$. For charts, for any $b \in B$ pick individual charts about b, say (U_1, φ_1) for E_1 and (U_1, φ_2) for E_2, form charts

$$\left\{ (U_1 \cap U_2, \varphi : \pi^{-1}(U_1 \cap U_2) \to \mathbb{R}^{n_1 + n_2}) \right\}$$

where $n_1 \coloneqq \dim_{\mathbb{R}} F_{1,b}$ and $n_2 \coloneqq \dim_{\mathbb{R}} F_{2,b}$ and define $(b, (v_1, v_2)) \xrightarrow{\varphi} (\varphi_1(v_1), \varphi_2(v_2))$.

2 Fiber Bundles with Structure and Principal G- Bundles (Tuesday, August 24)

Remark 2.0.1: Setup:

- $B \in \mathsf{Top}$ is a space.
- $\pi : E \to B$ is a map of sets with fibers/preimages $F \coloneqq F_b \coloneqq \pi^{-1}(\{b\})$.
- A *bundle atlas* for π is φ where $\varphi_U : \pi^{-1}(U) \to U \times F$ is a bijection of sets

Then there is at most one topology on E making $\pi : E \to B$ into a fiber bundle with the specified atlas.

Definition 2.0.2 (Dual of a vector bundle)
Given a vector bundle $\pi : E \to B$, form the **dual bundle** $\pi^\vee : E^\vee \to B$ by setting

- $E^\vee := \coprod_{b \in B} F_b{}^\vee$
- Set $\pi^\vee(F_b{}^\vee) = \{b\} \in B$.
- Given $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$, set

$$\varphi^\vee : (\pi^\vee)^{-1}(U) = \coprod_{b \in U} F_b{}^\vee \longrightarrow U \times (\mathbb{R}^n)^\vee \cong U \times \mathbb{R}^n.$$

Here $A \subseteq \pi^{-1}(U)$ is open iff $\varphi_U(A)$ is open in B.

Remark 2.0.3: Consider what happens on overlapping charts – looking at maps fiberwise yields:

$$\begin{array}{ccccc} \pi^{-1}(U) & \hookleftarrow & \pi^{-1}(U \cap V) & \hookrightarrow & \pi^{-1}(V) \\ \downarrow{\scriptstyle \varphi_U} & & & & \downarrow{\scriptstyle \varphi_V} \\ U \times F & \hookleftarrow & (U \cap V) \times F & \hookrightarrow & V \times F \end{array}$$

Link to Diagram

Starting at $(U \cap V) \times F$ and running the diagram clockwise yields a map

$$\begin{aligned} \varphi_V \circ \varphi_U^{-1} : (U \cap V) \times F &\to (U \cap V) \times F \\ (b, f) &\mapsto (b, \varphi_{VU}(f)) \end{aligned}$$

where φ_{VU} is the following continuous map, defining a **transition function**:

$$\varphi_{VU} : U \cap V \to \mathrm{Homeo}(F),$$

where $\mathrm{Homeo}(F) := \underset{\mathsf{Top}}{\mathrm{Hom}}(F, F)$ with the compact-open topology.

Definition 2.0.4 (The compact-open topology)
Let $\mathrm{Maps}(X, Y) := \underset{\mathsf{Top}}{\mathrm{Hom}}(X, Y)$ be the set of continuous maps $X \to Y$, then a map $Z \to \mathrm{Maps}(X, Y)$ is continuous iff the following map is continuous:

$$\begin{aligned} Z \times X &\to Y \\ (z, x) &\mapsto f(z)(x). \end{aligned}$$

If X is Hausdorff and locally compact then $\mathrm{Maps}(X, Y)$ will have this property for all Y. A subbasis for this topology will be given by taking $C \subseteq X$ compact, $U \subseteq Y$ open and taking the basic opens to be

$$S(C, U) := \left\{ f \in \mathrm{Maps}(X, Y) \;\middle|\; f(X) \subseteq U \right\}.$$

If Y has a metric, then this will coincide with the *compact convergence topology*, which has a basis

$$\left\{ S(f, C, E) \;\middle|\; C \subseteq X \text{ compact}, \forall \varepsilon > 0, \, \forall f \in \operatorname{Maps}(X, Y) \right\},$$
$$S(f, C, E) := \left\{ g \in \operatorname{Maps}(X, Y) \;\middle|\; d(f(x), g(x)) < \varepsilon \; \forall x \in C \right\}.$$

Definition 2.0.5 (Structure Groups)
Let $G \subseteq \operatorname{Homeo}(F)$, then a **fiber bundle with structure group** G is a fiber bundle $F \to E \xrightarrow{\pi} B$ together with a bundle atlas such that $G \subseteq \operatorname{Homeo}(F)$.

Example 2.0.6 ***(?):*** An $\mathbb{R}^n$-bundle is just a bundle where $F = \mathbb{R}^n$ for all fibers, where we ignore the vector space structure and only take transition functions to be homeomorphisms. An $\mathbb{R}^n$-bundle with a $G := \mathrm{GL}_n(\mathbb{R})$ is exactly a vector bundle, where we can use the structure group to put a vector space structure on the fibers. We have charts $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^n$, so for all $b \in U$, writing $F_b := \pi^{-1}(\{b\})$ and get $\varphi_U(F_b) = \Big| \, b \times \mathbb{R}^n$. We can then define addition and multiplication for $w_1, w_2 \in F_b$ as

$$cw_1 + w_2 := \varphi_U^{-1} \left(c\varphi_U(w_1) + \varphi_U(w_2) \right).$$

This is well-defined because for any other chart containing $V \ni b$, we have $\varphi_{VU} \in \mathrm{GL}_n(\mathbb{R})$. This follows by just setting $A := \varphi_V \circ \varphi_U^{-1}$ and writing

$$\begin{aligned}
\varphi_V(w_1 + w_2) &= A\varphi_U(w_1 + w_2) \\
&:= A\left(\varphi_U(w_1) + \varphi_U(w_2)\right) \\
&= A\varphi_U(w_1) + A\varphi_U(w_2) \\
&= \varphi_V(w_1) + \varphi_V(w_2) \\
&:= \varphi_V(w_1 + w_2).
\end{aligned}$$

Example 2.0.7 ***(Bundles with structure):*** An $\mathbb{R}^n$-bundle with a $\mathrm{GL}_n^+(\mathbb{R})$ structure is an orientable vector bundle, where

$$\mathrm{GL}_n^+(\mathbb{R}) = \left\{ A \in \mathrm{GL}_n(\mathbb{R}) \;\middle|\; \det(A) > 0 \right\}.$$

A $G := O_n(\mathbb{R})$ structure yields vector bundles with Riemannian metrics on fibers, where $O_n(\mathbb{R}) := \left\{ A \in \mathrm{GL}_n(\mathbb{R}) \;\middle|\; AA^t = \mathrm{id} \right\}$. Here we use the fact that there is an equivalence between metrics (symmetric bilinear pairings) and choices of an orthonormal basis, e.g. using that if $\{e_1, \cdots, e_n\}$, one can specify an inner product completely by writing

$$v := \sum v_i e_i, \quad w := \sum w_i e_i \qquad\qquad \implies \langle v, \; w \rangle = \sum v_i w_i.$$

Definition 2.0.8 (Principal G-bundles)
A **principal** G**-bundle** is a fiber bundle $\pi : P \to B$ with a right G-action $\psi : P \times G \to P$ such that

1. $\psi(F_b) = F_b$, so the action preserves each fiber, and
2. ψ is free and transitive.

3 | Principal G-bundles (Thursday, August 26)

Remark 3.0.1: Today: relating $\text{Prin}\,\mathsf{Bun}_{/G}$ to fiber bundles with a G-structure. Recall that a principal G-bundle is a fiber bundle $\pi : P \to B$ with a fiberwise G-action $P \times G \to P$ which induces a free and transitive action on each fiber. Note that we assume $G \in \mathsf{TopGrp}$. Any bundle in $\text{Prin}\,\mathsf{Bun}_{/G}$ is a fiber bundle with fibers F homeomorphic to G and admits a G-structure:

$$\begin{aligned} G &\hookrightarrow \text{Homeo}(G) \\ g &\mapsto (h \mapsto gh). \end{aligned}$$

Using that $F \cong G$, taking charts $(U, \varphi), (V, \psi)$ for $\pi : P \to B$, we can identify

$$\begin{array}{ccccccc} \pi^{-1}(U \cap V) & \xrightarrow{\varphi_U, \cong} & (U \cap V) \times G & \xrightarrow{\varphi_V \circ \varphi_U^{-1}} & (U \cap V) \times G & \xleftarrow{\varphi_V, \cong} & \pi^{-1}(U \cap V) \\ & & (b, 1) & \longmapsto & (b, g) & & \\ & & (b, h) & \longmapsto & (b, gh) & & \end{array}$$

Link to Diagram

So every transition function is given by left-multiplication by some element in G, as opposed to arbitrary homeomorphisms of G as a topological group.

Example 3.0.2 ***(of principal bundles):***

- Trivial actions: $B \times G \xrightarrow{p_1} B$.

- Regular covering spaces $\pi : \tilde{X} \to X$, then $G = \text{Deck}(\tilde{X}/X)$ with the discrete topology.

- Given an n-dimensional vector bundle $\pi : E \to B$, take

$$\text{Frame}(F_b) := \{(e_1, \cdots, e_n) \in F_b\} \subseteq F_b^{\times n},$$

the collection of all ordered bases of F_b. Then set

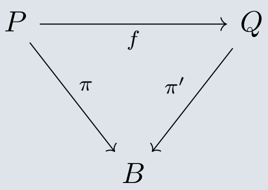
$$\underset{n}{\text{Frame}} := \textstyle\coprod_{b \in B} \text{Frame}(F_b) \to B$$

to get a principal G-bundle for $G = \mathrm{GL}_n(F_b)$ under the following action: picking a framing $(e_1, \cdots, e_n)$ in F_b, then for $A \in \mathrm{GL}_n(F_b)$ regarded as a linear map, define

$$(\mathbf{e}_1, \cdots, \mathbf{e}_n) \cdot A := \left(\sum_i a_{i,1} \mathbf{e}_i, \sum_i a_{i,2} \mathbf{e}_i, \cdots, \sum_i a_{i,n} \mathbf{e}_i \right).$$

- Given an *oriented* n-dimensional vector bundle $\pi : E \to B$, one gets a $G := \mathrm{GL}_n^+(\mathbb{F}_b)$ by taking positively oriented frames.
- Given a vector bundle with a Riemannian metric, we get a principal $\mathcal{O}_n(\mathbb{R})$-bundle by taking orthonormal frames.

Definition 3.0.3 (?)
Given two principal G-bundles $\pi : P \to B$ and $\pi' : Q \to B$, an **isomorphism of principal bundles** is a G-equivariant map $P \xrightarrow{f} Q$ commuting over B:



Link to Diagram

Here *equivariant* means commuting with the G-action, in the following precise sense: let (U, φ) and (V, ψ) be charts for π, π', then consider the composition

$$F : \left((U \cap V) \times F \xrightarrow{\varphi^{-1}} \pi^{-1}(U \cap V) \xrightarrow{f} (\pi')^{-1}(U \cap V) \xrightarrow{\psi} (U \cap V) \times F \right).$$

Note that this fixes every point $b \in U \cap V$, so we can regard $F : U \cap V \to \mathrm{Homeo}(F)$, using that f commutes with the projection maps:

$$(b, ?) \mapsto \pi^{-1}(b) \mapsto (f \circ \pi^{-1})(b) = (\pi')^{-1} b \mapsto b.$$

We say f is a G-isomorphism iff F sends everything to G.

3.1 Sending Fiber Bundles to Principal G-bundles

Remark 3.1.1: Given a principal G-bundle $\pi : P \to B$ and a $F \in \mathsf{Top}$ with a left G-action. Then define

$$P \underset{G}{\times} F / (pg, f) \sim (p, gf)$$

as a fiber bundle over B using π as the projection. Note that this looks like a tensor product, and this works in general for any space P with a right G-action and F with a left G-action. This will be a fiber bundle with fiber F and structure group $G \leq \text{Homeo}(F)$.

Locally there is a homeomorphism:

$$(U \times G) \overset{G}{\times} F \xrightarrow{\sim} U \times F$$
$$(p, g, f) \mapsto (p, gf).$$

This is well defined since (p, gh, f) and (p, g, hf) map to (p, ghf). The inverse is $(p, f) \mapsto (p, 1, gf)$.

> **Exercise 3.1.2** (?)
> Check that this is a fiber bundle with G-structure.

4 | Tuesday, August 31

Remark 4.0.1: We want to show the equivalence between (isomorphism classes) of fiber bundles with G structures with fiber F and principal G-bundles. Recall that $\text{Prin}\,\mathsf{Bun}_{/G}$ are fiber bundles $P \xrightarrow{\pi} B$ with a right fiberwise G-action which is free and transitive on each fiber.

To send fiber bundles to principal bundles, we used a *mixing* construction. Since $G \curvearrowright F$, we get an identification $G \subseteq \text{Homeo}(F, F)$. We constructed

$$P \underset{G}{\times} F := (P \times F)/(pg, f) \sim (p, gf).$$

A lemma was that $P \underset{G}{\times} F \to B$ is a fiber bundle with fiber F and projection $\pi(p, f) := \pi(p)$.

Today we'll talk about the reverse direction. Note the composition of sending E to $\text{Prin}\,\mathsf{Bun}_{/G}$ and then mixing recovers E when E is a vector bundle, but not generally.

$$\left\{ \substack{\text{Fiber bundles with } G\text{-structures} \\ \text{and fiber } F} \right\} \underset{\text{Mixing}}{\overset{\text{Clutching}}{\rightleftarrows}} \left\{ \text{Principal } G\text{-bundles} \right\}$$

Example 4.0.2 ***(?)*****:** For $E \xrightarrow{\pi} B$ a real vector bundle, we sent it to $\text{Frame}(E)$, which is a principal $\text{GL}_n(\mathbb{R})$-action Using a left action $\text{GL}_n \curvearrowright \mathbb{R}^n$, we can form $\text{Frame}(E) \underset{\text{GL}_n}{\times} \mathbb{R}^n$, a fiber bundle with a $G := \text{GL}_n$ structure, i.e. exactly a vector bundle.

> **Exercise 4.0.3** ***(?)***
> Show that there is a homeomorphism
>
> $$\text{Frame}(E) \underset{\text{GL}_n}{\times} \mathbb{R}^n \xrightarrow{\sim} E.$$

For the reverse map, take a map f defined by $(\mathbf{e}_1, \cdots, \mathbf{e}_n) \in \pi^{-1}(b) \subset \text{Frame}(E)$ and $[b_1, b_2, \cdots, b_n]^t \in \mathbb{R}^n$ to $\sum_{i=1}^n b_i \mathbf{e}_i$. For this to be well-defined, one needs to show the following:

$$f((\mathbf{e}_1, \cdots, \mathbf{e}_n)A, \mathbf{b}) = f((\mathbf{e}_1, \cdots, \mathbf{e}_n), A\mathbf{b}) \qquad \forall A \in \text{GL}_n(\mathbb{R}).$$

The left hand side is

$$b_1(a_{1,1}\mathbf{e}_1 + \cdots + a_{n,1}\mathbf{e}_n) + \cdots + b_n(a_{1,n}\mathbf{e}_1 + \cdots + a_{n,n}\mathbf{e}_n) = \sum_{i=1}^n b_i \left(\sum_{j=1}^n a_{j,i}\mathbf{e}_j \right).$$

The right-hand side is

$$(a_{1,1}b_1 + \cdots + a_{1,n}b_n)\mathbf{e}_1 + \cdots + (a_{n,1}b_1 + \cdots + a_{n,n}b_n)\mathbf{e}_n = \sum_{i=1}^n \left(\sum_{j=1}^n a_{i,j}b_i \right) \mathbf{e}_i,$$

and one can check that these sums match term by term.

Remark 4.0.4: Note that if we choose a basis for the fibers, we can set $A' := [\mathbf{e}_1, \cdots, \mathbf{e}_n]^t$ to be the matrix with columns $\mathbf{e}_i$, the map f is given by $f(A', \mathbf{b}) := A'\mathbf{b}$, and we're showing that $(A'A)\mathbf{b} = A'(A\mathbf{b})$. However, this involves choosing an isomorphism between the abstract fibers and $\mathbb{R}^n$.

Remark 4.0.5: What are local charts for a principal bundle? For $P \underset{G}{\times} F$, pick charts (U, φ) for $P \xrightarrow{\pi} B$:

$$\begin{aligned} \varphi : \pi^{-1}(U) &\to U \times G \\ x &\mapsto (\pi(x), \gamma(x)). \end{aligned}$$

Then a local chart for the principal bundle is of the form

$$\begin{aligned} \pi^{-1}(U) \underset{G}{\times} F &\xrightarrow{\tilde{\varphi}} U \times F \\ (x, f) &\mapsto (\pi(x), \gamma(x)f). \end{aligned}$$

We also have

$$\begin{aligned} (U \times G) \underset{G}{\times} F &\to U \times F \\ ((x, g), f) &\mapsto (x, gf). \end{aligned}$$

One can invert $\tilde{\varphi}$ using $(a, f) \mapsto (\varphi^{-1}(a, 1), f)$. This yields transition functions: writing

$$\begin{aligned} \varphi_V : \pi^{-1}(V) &\to V \times G \\ x &\mapsto (\pi(x), \psi(x)), \end{aligned}$$

then

$$\begin{aligned}\varphi_{VU} = \varphi_V \circ \varphi_U^{-1} : (a,f) & \\ &\xrightarrow{\varphi_U^{-1}} (\varphi_U^{-1}(a,1),\ f) \\ &\xrightarrow{\varphi_V} (\pi\varphi_U^{-1}(a,1),\ \psi(\varphi_U^{-1}(a,1))f) \\ &= (a,\ \psi(\varphi_U^{-1}(a,1))f).\end{aligned}$$

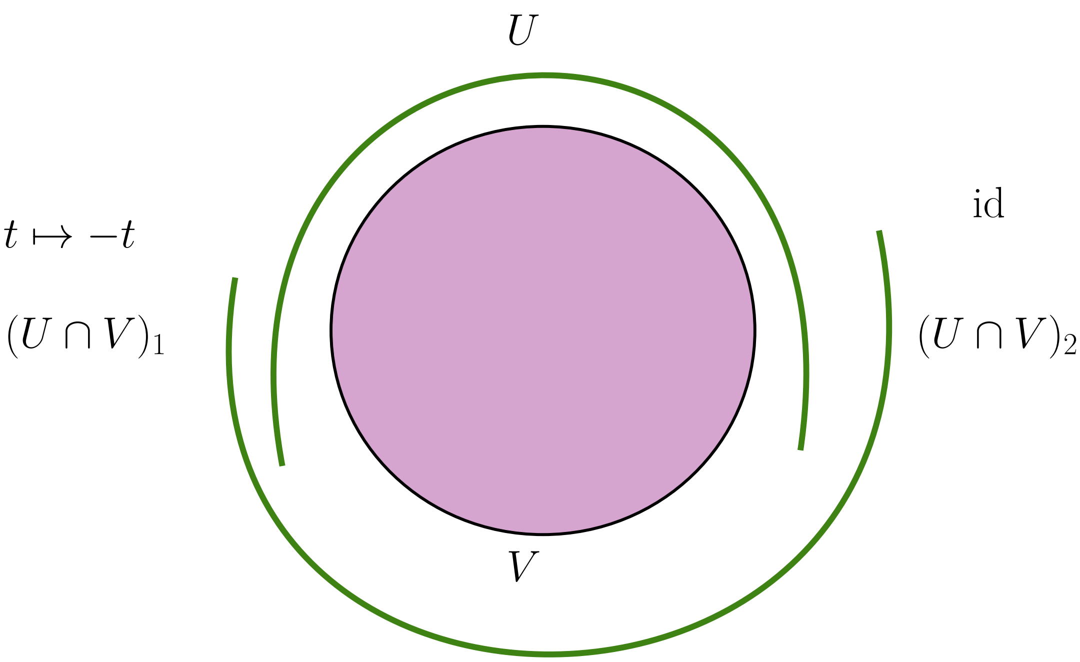
This says that $(a,1) \mapsto \psi(\varphi_U^{-1}(a,1))$.

Remark 4.0.6: In general, for a bundle $E \xrightarrow{\pi} B$, taking local trivializations φ_U, φ_V, we get $\varphi_{VU} : (U \cap V) \times F \circlearrowleft$, or currying an argument, $\varphi_{VU} : U \cap V \to \text{Homeo}(F, F)$. If the bundle satisfies the cocycle condition $\varphi_{UW} = \varphi_{VW} \circ \varphi_{UV}$. Given a covering $\{U_i\}_{i \in I} \rightrightarrows B$, we get $\varphi_{ij} : U_i \cap U_j \to G$ and a topological space F with $G \subseteq \text{Homeo}(F, F)$ satisfying the cocycle condition $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$, then we can build a fiber bundle with fiber F and structure group G by setting $E = \coprod_{i \in I}(U_i \times F)/\sim$ We then set for $b \in U_i \cap U_j$ the equivalence

$$(U_i \times F) \ni (b, f) \sim (b, \varphi_{ij}(b)f) \in (U_j \times F).$$

This is an equivalence relation precisely when the cocycle condition holds. This is referred to as **clutching data**.

Example 4.0.7 ***(The Mobius band is clutch):*** Let $\mathbb{Z}/2 \curvearrowright \mathbb{R}$ by $t \mapsto -t$ with U, V defined as follows:



Labeling the intersections as 1, 2, we set

$$\begin{aligned}\varphi_{VU} : (U \cap V) = (U \cap V)_1 \coprod (U \cap V)_2 &\to \mathbb{Z}/2 \subseteq \text{Homeo}(\mathbb{R}) \\ x \coprod y &\mapsto x \coprod -y.\end{aligned}$$

This yields the open Mobius band.

Question 4.0.8
Actually, several questions. Assume F is a fixed fiber common to all of the following constructions, since bundles with non-homeomorphic fibers can't be isomorphic.

1. Given clutching data $\{\varphi_{ij}\}$, when is the resulting fiber bundle trivial?
2. Given two sets of clutching data $\{\varphi_{ij}\}$ and $\{\psi_{ij}\}$ with the same open cover $\{U_i\} \rightrightarrows X$, when are the corresponding bundles G-isomorphic?
3. Given two sets of clutching data $\{\varphi_{ij}\}$ and $\{\psi_{ij}\}$ with the *different* open cover $\{U_i\} \rightrightarrows X$ and $\{V_i\} \rightrightarrows X$, when are the corresponding bundles G-isomorphic?

Lemma 4.0.9 ***(?)*.**
The fiber bundle obtained from φ_{ij} is trivial iff there exists a map $\gamma_i : U_i \to G$ such that $\varphi_{ij} = \gamma_i \gamma_j^{-1}$.

Proof (?).
The trivial bundle is $B \times F \to B$, so if we have $E \to B$, we can take a map

$$\begin{aligned} U_i \times F &\to U_i \times F \\ (b, f) &\mapsto (b, \gamma_i(b) f). \end{aligned}$$

Use that $B \times F$ is a trivial bundle, so it is its own trivialization.

To be continued next time.

■

5 | Thursday, September 02

Remark 5.0.1: Recall that we have a correspondence

$$\left\{\text{Vector bundles } E\right\} \underset{\text{mixing}}{\overset{\text{clutching}}{\rightleftarrows}} \left\{\text{Principal } \mathrm{GL}_n\text{-bundles Frame}(E)\right\}$$

We saw that $E \cong \mathrm{Frame}(E) \underset{\mathrm{GL}_n(\mathbb{R})}{\times} \mathbb{R}^n$. If we take $\mathrm{Frame}(E)$, mix, and apply the clutching construction, is the result bundle-isomorphic to the frame bundle?

Remark 5.0.2: Recall the clutching construction: we take a cover $\{U_i\}_{i \in I}$ and $\varphi_{ij} : U_i \cap U_j \to G$ satisfying the cocycle condition $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$, then $G \subseteq \mathrm{Homeo}(F, F)$ and we construct a fiber bundle $\displaystyle\bigcup_{i \in I} U_i \times F / \sim$ where for $b \in (U_i \cap U_j)$ and

$$(b, f) \in (U_i \cap U_j) \times F \subseteq U_i \times F,$$

we send this to

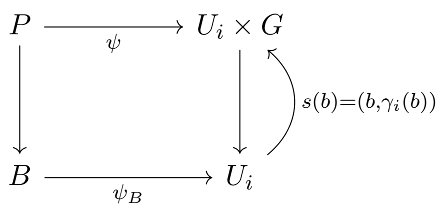
$$(b, \varphi_{ji}(b)f) \in (U_i \cap U_j) \times F \subseteq U_j \times F.$$

This will be a fiber bundle with fiber F and structure group G. Moreover, if $F = G$, this will be a principal G-bundle using right-multiplication.

Question 5.0.3
How can we tell when two fiber bundles constructed via clutching are isomorphic?

Lemma 5.0.4 ***(when clutched bundles are trivial).***
The bundle formed by the clutching data $\{\varphi_{ij}\}$ is trivial (so isomorphic to the trivial bundle) iff there exist $\gamma_i : U_i \to G$ such that $\varphi_{ji} = \gamma_j \circ \gamma_i^{-1}$.

Remark 5.0.5: For principal bundles, these γ_i will give sections assembling to a global section obtained from clutching data:



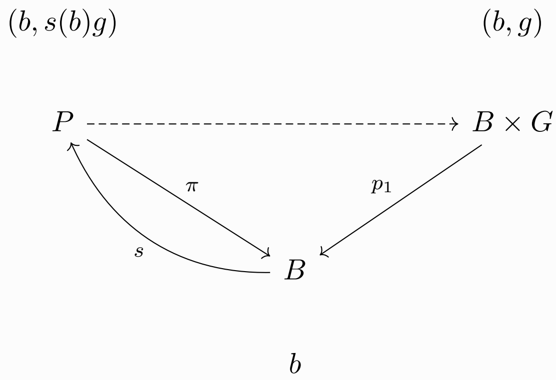
Link to Diagram

The map on $U_i \to \bigcup_i U_i \times F$ will be $(b, \gamma_i(b))$, and we can use that

$$(b, \gamma_i(b)) \sim (b, \varphi_{ji}(b)\gamma_i(b)) \sim (b, \gamma_j(b)),$$

so these agree on overlaps.

Lemma 5.0.6 ***(?).***
If a principal bundle $P \to B$ has a global section, then P is trivial, so $P \cong B \times G$ as bundles.
The idea:



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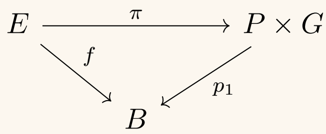
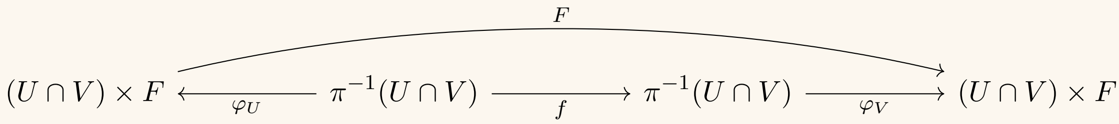
Proof (of lemma about when clutched bundles are trivial).
$\implies$:
If E is trivial, we have an isomorphism



We have a G-isomorphism $E_1 \xrightarrow{f} E_2$, and so a composition



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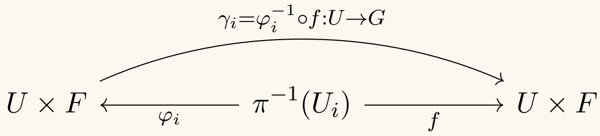
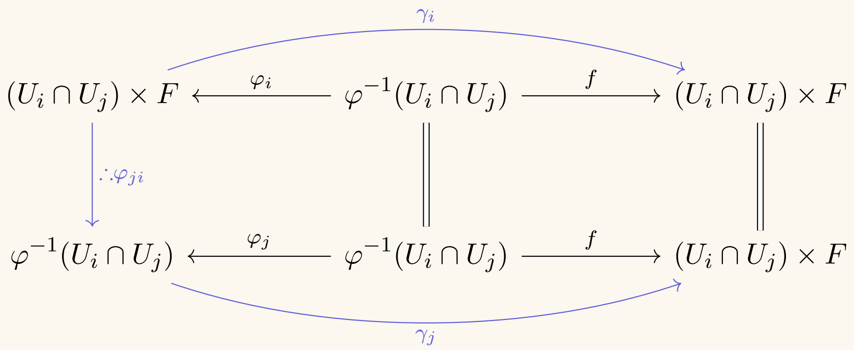
Here we've used that f commutes with the projection maps. We want to show $\mathrm{im}(F) \subseteq G$. We have a composite



Link to Diagram

We can fill this in to a commutative diagram:



Link to Diagram

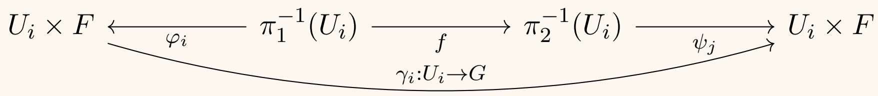
The converse direction proceeds similarly!

■

Lemma 5.0.7 *(?).*
A G-isomorphism between the bundles E_1, E_2 obtained from clutching data $\{\varphi_{ij}\}$ and $\{\psi_{ij}\}$ respectively with the same cover $\{U_i\}_{i\in I}$ give maps $\gamma_i : U_i \to G$ such that

$$\gamma_j \varphi_{ji} \gamma_i^{-1} = \psi_{ji}.$$

Proof (?).
We can form the composite



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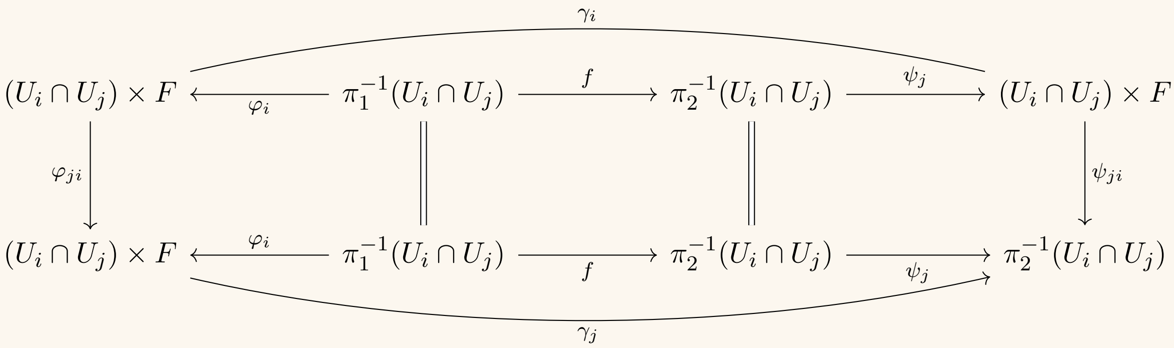
And then assemble a commuting diagram:



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■

5.1 Nonabelian Čech Cohomology

Definition 5.1.1 (Čech complex)
Let $\mathcal{U} := \{U_i\}_{i\in I} \rightrightarrows B$ an open cover, and define

$$\check{C}^0(\mathcal{U}; G) := \{\{\gamma_i : U_i \to G\}_{i\in I}\},$$

which is a group under pointwise multiplication. Define

$$\check{C}^2(\mathcal{U}; G) := \left\{\{\varphi_{ij} : U_i \cap U_j \to G\}_{i,j\in I}\right\}$$
$$\check{C}^3(\mathcal{U}; G) := \left\{\{\varphi_{ijk} : U_i \cap U_j \cap U_k \to G\}_{i,j,k\in I}\right\},$$

and boundary maps

$$\delta^0 : \check{C}^0(\mathcal{U}; G) \to \check{C}^1(\mathcal{U}; G)$$
$$\{\gamma_i : U_i \to G\} \mapsto \left\{\varphi_{ji} := \gamma_j\gamma_i^{-1} : U_i \cap U_j \to G\right\},$$

$$\delta^1 : \check{C}^1(\mathcal{U}; G) \to \check{C}^2(\mathcal{U}; G)$$
$$\{\varphi_{ij} : U_i \cap U_j \to G\} \mapsto \left\{\eta ijk := \varphi_{ij}\varphi_{jk}\varphi_{ik}^{-1} : U_i \cap U_j \cap U_k \to G\right\}.$$

Remark 5.1.2: One can check that $\delta^1 \circ \delta^0 = 0$ is trivial. And 1-cocycle will yield a fiber bundle.

Lemma 5.1.3 *(1)*.
A bundle is trivial iff it is a 1-coboundary, where we take $Z^1(\mathcal{U}; G) := \ker \delta^1$, $B^1(\mathcal{U}; G) := \operatorname{im} \delta^0$.

Warning 5.1.4
We'd like to define homology as Z/B, but since these aren't abelian groups, the coboundaries B may not be normal in Z and the quotient may not yield a group.

Definition 5.1.5 (First Čech cohomology)
There is an action of $\check{C}^0(\mathcal{U}; G) \curvearrowright \check{C}^1(\mathcal{U}; G)$ given by taking $\gamma := \{\gamma_i\}_{i\in I}$ and setting $(\gamma\varphi)_{ij} = \gamma_i\varphi_{ij}\gamma_j^{-1}$, which descends to an action on Z^1. We can take the quotient by this action to define

$$\check{H}^1(\mathcal{U}; G) := Z^1(\mathcal{U}; G)/\sim.$$

Lemma 5.1.6 *(2)*.
Two bundles are isomorphic iff they yield the same element in $\check{H}^1(\mathcal{U}; G)$.

Remark 5.1.7: This works when bundles have the same open cover, and if not, we can take a common refinement.

6 | Tuesday, September 07

Remark 6.0.1: Recall that given a $B \in \mathsf{Top}$ and $\mathcal{U} \rightrightarrows B$, we defined $\check{H}_1(\mathcal{U};G)$ which classified isomorphism classes of fiber bundles $E \xrightarrow{\pi} B$ with fiber F, $G \subseteq \mathrm{Homeo}(F)$, and structure group G, given by clutching data using $\mathcal{U}$. The cochains were given by the following:

$$\begin{aligned}
\check{C}^0(\mathcal{U};G) &= \{\{\gamma_i : U_i \to G\}_{i \in I}\} \\
\check{C}^1(\mathcal{U};G) &= \left\{\{\varphi_{ij} : U_i \cap U_j \to G\}_{i,j \in I}\right\} \\
\check{C}^2(\mathcal{U};G) &= \left\{\{\eta_{ijk} : U_i \cap U_j \cap U_k \to G\}_{i,j,k \in I}\right\}
\end{aligned}$$

with boundary maps $\delta_i : \check{C}^{i-1} \to \check{C}^i$:

$$\begin{aligned}
(\delta_1 \gamma)_{ij} &= \gamma_i \gamma_j^{-1} \\
(\delta_2 \varphi)_{ijk} &= \varphi_{ij} \varphi_{jk} \varphi_{ik}^{-1}.
\end{aligned}$$

Note that

- $\delta_2 \circ \delta_1 = 0$
- $\ker \delta_2 = Z^1(\mathcal{U};G)$ yields clutching data, i.e. a fiber bundle with fiber F,
- $\operatorname{im} \delta_1$ yields trivial bundles,
- $\check{H}^1(\mathcal{U};G) := Z^1(\mathcal{U};G) / \operatorname{im}(\check{C}^0(\mathcal{U};G) \to Z^1(\mathcal{U};G))$.

We'll see that $(\gamma\varphi)_{ij} = \gamma_i \varphi_{ij} \gamma_j^{-1}$, and by a lemma this will prove the above claim about classifying isomorphism classes.

> **Definition 6.0.2** (Refinement of covers)
> We say a cover $\mathcal{V} := \{V_j\}_{j \in J}$ is a **refinement** of $\mathcal{U} := \{U_i\}_{i \in I}$ iff there exists a function $f : J \to I$ between the index sets where $V_j \subseteq U_{f(j)}$ for all j.
>
> *DZG: I'll write $\mathcal{V} \leq \mathcal{U}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$.*

Remark 6.0.3: Since any two covers have a common refinement, we'll assume $\mathcal{V} \leq \mathcal{U}$ is always a refinement. We can then restrict clutching data from $\mathcal{U}$ to $\mathcal{V}$: given $\{\varphi_{ij}\}_{i,j \in I}$, we can set $\psi_{ij} := \varphi_{f(i),f(j)}\Big|_{V_i \cap V_j}$, noting that if $V_j \subseteq U_{f(j)}$ and $V_i \subseteq U_{f(i)}$ then $V_i \cap V_j \subseteq U_{f(i)} \cap U_{f(j)}$. These yield maps $\psi_{ij} : V_i \cap V_j \to G$ satisfying the cocycle condition, so $\psi_{ij} \in Z^1(\mathcal{V};G)$. This means that we have map $Z^1(\mathcal{U};G) \to Z^1(\mathcal{V};G)$ which respects the actions of $\check{C}^0(\mathcal{U};G), \check{C}^0(\mathcal{V};G)$ respectively. Since the category of covers with morphisms given by refinements come from a preorder, we can take a colimit to define

$$\check{H}^1(B;G) \coloneqq \underset{\overrightarrow{\mathcal{U} \rightrightarrows B}}{\operatorname{colim}} \check{H}^1(\mathcal{U};G).$$

Lemma 6.0.4 ***(?).***
There is a bijection

$$\left\{ \substack{\text{Fiber bundles with fiber } F \\ \text{and structure group } G} \right\}_{/\sim} \rightleftharpoons \check{H}^1(B;G)$$

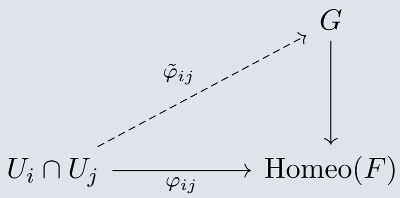
In particular, these classes are independent of F.

Corollary 6.0.5 ***(?).***
There is an equivalence of categories

$$\left\{ \substack{\text{Fiber bundles with fiber } F \\ \text{and structure group } G \\ \text{over } B} \right\}_{/\sim} \rightleftharpoons \text{Prin}\,\mathsf{Bun}(G)_{/B},$$

where the right-hand side are principal G-bundles.

Definition 6.0.6 (G-structures)
Given a map $G \to \text{Homeo}(F)$, a **G-structure** on an F-bundle $E \xrightarrow{\pi} B$ is the following data: given clutching data φ_{ij}, lifts of the following form that again satisfy the cocycle condition:



Link to Diagram

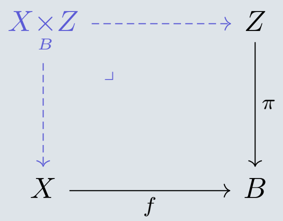
Remark 6.0.7: Note that we need to impose the cocycle condition, since lifts may not be unique and some choices may not glue correctly!

Example 6.0.8 ***(***Spin_n***-structures):*** Using the known Spin double covers, we can form the composition

$$\text{Spin}_n(\mathbb{R}) \xrightarrow{\times 2} \text{SO}_n(\mathbb{R}) \hookrightarrow \text{Homeo}(\mathbb{R}^n).$$

Then a Spin_n-structure on any $\mathbb{R}^n$-bundle is a lift of transition functions from $\text{Homeo}(\mathbb{R}^n)$ to Spin_n satisfying the cocycle condition.

Definition 6.0.9 (Fiber products)
We can fill in a commutative square in the following way:



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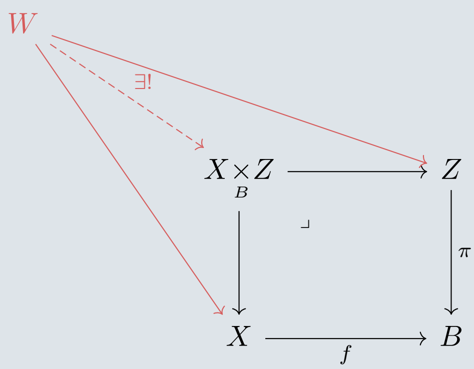
Here we can construct the fiber product as

$$X \underset{B}{\times} Z = \left\{ (x, e) \;\middle|\; \pi(e) = f(x) \right\}.$$

It satisfies the following universal property:



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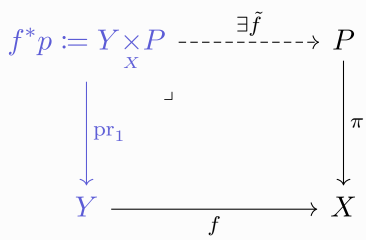
Lemma 6.0.10 ***(?).***
If $\pi : P \to X$ is a principal G-bundle and $f : Y \to X$ is a continuous map, then the following highlighted portion of the pullback is again a principal G-bundle:



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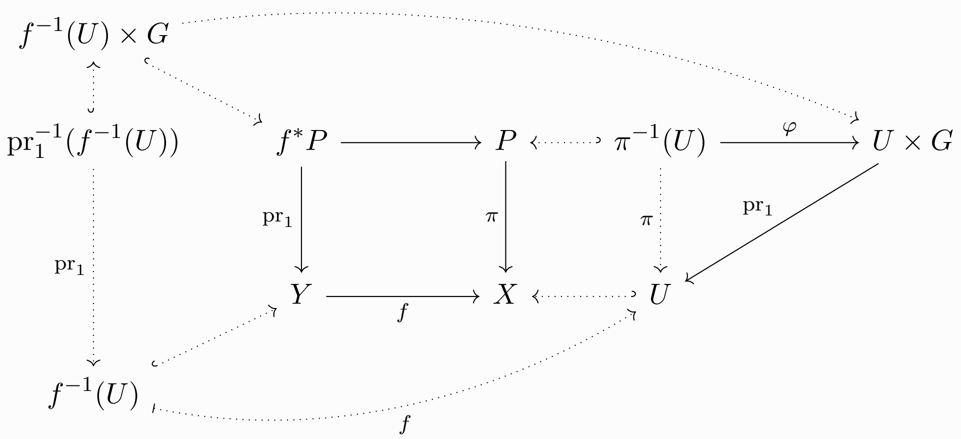
We in fact obtain $\mathrm{pr}_1^{-1}(y) = \pi^{-1}(f(y)) \cong G$, and there will be a right G-action on each fiber. Behold this gnarly diagram:

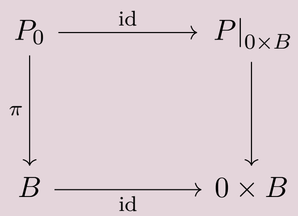


[Link to Diagram]

If $P \to X$ is trivial, this says the pullback will be trivial and $U \times G \mapsto f^{-1}(U) \times G$ will be a homeomorphism.

Remark 6.0.11: So the functor $X \mapsto \operatorname{Prin}_G(X)$ is contravariant functor.

Theorem 6.0.12 ***(Bundle homotopy lemma).***
Suppose B is paracompact and Hausdorff, then there is a principal G-bundle $P \xrightarrow{\pi} I \times B$. Consider the fiber bundle $P_0 := P|_{\{0\} \times B} \to B$, then there is a diagram:



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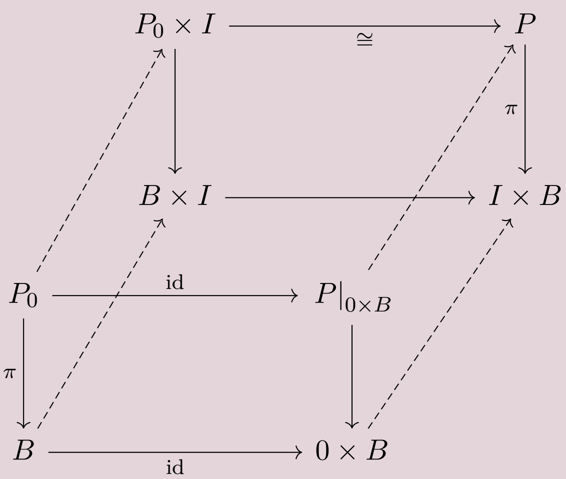
This extends to an isomorphism $I \times P_0 \to I \times B$ and $P \to I \times B$:



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Corollary 6.0.13 *(?).*
$P_1 = P|_{1\times B} \cong P_0$.

Corollary 6.0.14 *(?).*
If $f_0 \sim f_1 : Y \to X$ are homotopic and $P \to X$, then $f_0^* P \cong f_1^* P$.

Proof (?).
Use the homotopy lifting property to get a map h:

$$\begin{array}{ccc} h^*P & \longrightarrow & P \\ \downarrow & \lrcorner & \downarrow \pi \\ I \times Y & \xrightarrow{h} & Y \end{array}$$

Link to Diagram

Then $h^*P|_{0\times Y} \simeq h^*P|_{1\times Y} \cong f_1^* P$. ∎

7 | Thursday, September 09

7.1 Corollaries of the homotopy bundle lemma

Remark 7.1.1: Last time: the bundle homotopy lemma. If $P \to I \times X \in \operatorname{Prin}\mathsf{Bun}(G)$, then there is a bundle isomorphism

$$\begin{array}{ccc} I \times P_0 & & P \\ \downarrow & \xRightarrow{f\cong} & \downarrow \\ I \times X & & I \times X \end{array}$$

Link to Diagram

where $f|_{0\times P_0}$ is the identity.

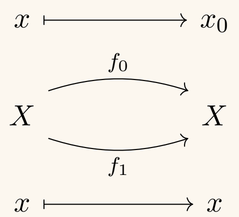
7 Thursday, September 09

Corollary 7.1.2 *(?).*
If $P \to I \times X \in \operatorname{Prin}\mathsf{Bun}(G)$ then $P_0 \cong P_1$ where $P_i := P|_{i \times X}$.

Corollary 7.1.3 *(?).*
If $f_0, f_1 : Y \to X$ with $P \xrightarrow{\pi} X$, then $f_0^* P \cong f_1^* P$ are isomorphic bundles.

Corollary 7.1.4 *(?).*
If X is contractible, then any $P \in \operatorname{Prin}\mathsf{Bun}(G)_{/X}$ is trivial.

Proof (?).
Consider the two maps



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Then $f_0 \simeq f_1$, and conclude by noting that

$$f_0^* P = X \underset{x_0}{\times} P = X \times \pi^{-1}(x_0) = X \times G$$

and $f_1^* P = P$.

■

7.2 Existence/Uniqueness of Metrics

Definition 7.2.1 (Riemannian metrics)
A **Riemannian metric** on a vector bundle $E \xrightarrow{\pi} X$ is a continuous map $E \underset{X}{\times} E \to \mathbb{R}$ which restricts to an inner product on each fiber.

Proposition 7.2.2 *(?).*
A Riemannian metric on E corresponds to a restriction of the structure group from $\mathrm{GL}_n(\mathbb{R})$ to $\mathrm{O}_n(\mathbb{R})$.

Proposition 7.2.3 *(?).*
Every vector bundle over a paracompact X has a unique Riemannian metric.

Proof (?).
Existence: Cover X by charts and choose a locally finite[a] refinement $\mathcal{U} = \{U_i\}_{i\in I}$ and pick a partition of unity $\{\chi_i\}_{i\in I}$ subordinate to $\mathcal{U}$.
Define an inner product g_i on $\pi^{-1}(U_i)$ where $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ by pulling back the inner product on $\mathbb{R}^n$, i.e. taking $e_1 \xrightarrow{\varphi_i} (p_1, \mathbf{v}_1)$ and $e_2 \xrightarrow{\varphi} (p_2, \mathbf{v}_2)$ and setting

$$g_i(e_1, e_2) \coloneqq \langle \mathbf{v}_1,\ \mathbf{v}_2\rangle_{\mathbb{R}^n}.$$

Then define

$$g_p(-,-) \coloneqq \sum_i \chi_i(p) g_i(-,-).$$

Uniqueness: Consider two inner products $g_0(-,-), g_1(-,-)$ on the bundle $E \xrightarrow{\pi} X$, then define

$$g_t(-,-) = t g_0(-,-) + (1-t) g_1(-,-).$$

Then $I \times E \xrightarrow{\mathrm{id},\pi} I \times X$ is a bundle, and g_t is a Riemannian metric on $I \times E$. Consider its corresponding principal bundle

$$P \to I \times X \in \operatorname{Prin}\mathsf{Bun}(\mathrm{O}_n(\mathbb{R}))$$

These correspond to restricting $I \times E$ to $0, 1$, yielding P_0, P_1 with Riemannian metrics g_0, g_1. But $P_0 \cong P_1$ are isomorphic principal bundles, and using the correspondence between bundles with metric and bundles with structure group O_n, this shows the two bundles with metric are isomorphic.

■

[a] Here *locally finite* means every point is covered by finitely many opens in the cover.

Definition 7.2.4 (Universal G-bundles)
A **universal G-bundle** is a principal G-bundle $\pi : EG \to \mathbf{B}G$ such that $\pi_i EG = 0$ for all i (so EG is *weakly contractible*).

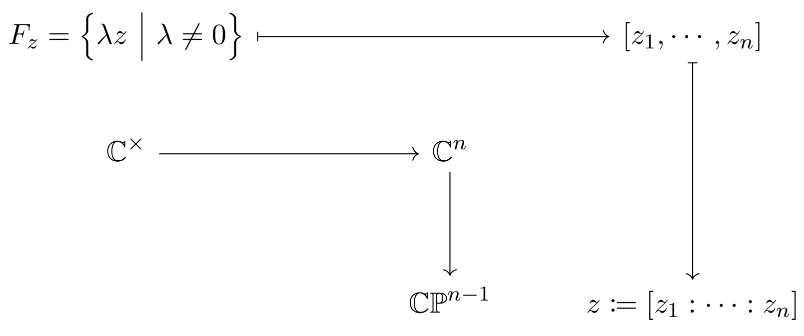
Example 7.2.5 *(?):*

- $\left(\mathbb{R} \to S^1\right) \in \operatorname{Prin}\mathsf{Bun}(\mathbb{Z})_{/S^1}$ since all of the regular covers are principal bundles. Since $\mathbb{R}$ is contractible, this is the universal $\mathbb{Z}$-bundle, so $S^1 \simeq \mathbf{B}\mathbb{Z}$.

- $(S^\infty \to \mathbb{RP}^\infty) \in \operatorname{Prin}\mathsf{Bun}(C_2)$ is a universal C_2-bundle, so $\mathbb{RP}^\infty \simeq \mathbf{B}C_2$

- $(S^\infty \to \mathbb{CP}^\infty)$ is a universal $S^1 = U_1$ bundle, so $\mathbb{CP}^\infty \simeq \mathbf{B}U_1 \simeq \mathbf{B}S^1 \simeq \mathbf{B}\mathbb{C}^\times$:

7 Thursday, September 09



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Theorem 7.2.6 *(?).*
If $X \in \mathsf{CW} \subseteq \mathsf{Top}$ and $EG \xrightarrow{\pi} \mathbf{B}G \in \mathsf{Bun}_G$ is universal, then there is a bijection

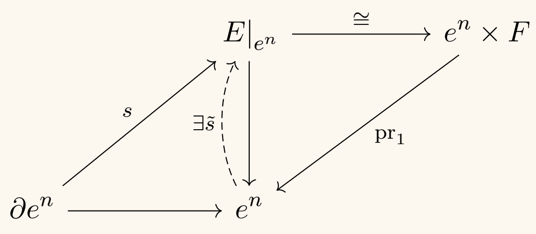
$$\operatorname{Prin}\mathsf{Bun}(G)_{/X} \rightleftharpoons [X, \mathbf{B}G]$$
$$f * EG \leftarrow\!\shortmid f.$$

Lemma 7.2.7 *(?).*
If $E \xrightarrow{\pi} X$ is a fiber bundle with fiber F and X is weakly contractible then

1. π admits a section, and
2. Any two sections are homotopic (through other sections).

Proof (of lemma, part 1).
Step 1: build a section inductively.

- Define a section over the 0-skeleton arbitrarily.
- Inductively, suppose the section is defined on the $n-1$ skeleton, so it's defined over every n-cell boundary ∂e^n.
- Write $E|_{e_n} = e^n \times F$, which is contractible since e_n is contractible.
- Then $s : \partial e^n = S^{n-1} \to F$ with $\pi_n(F) = 0$, so the section extends:



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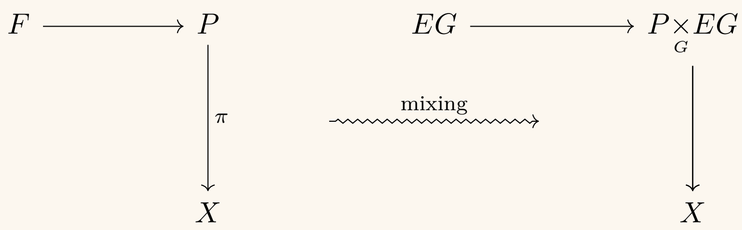
Step 2: Build the homotopy between sections inductively cell-by-cell as in part (1). ■

Proof (of theorem).
We want to show that the assignment $f \mapsto f^*EG$ is bijective.
Surjectivity: Note that EG has a left G-action defined by $g \cdot e := eg^{-1}$. Recall that we can use the mixing construction:



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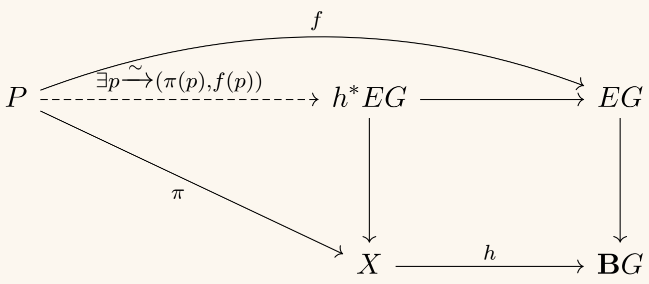
Sections of the mixed bundle biject with G-equivariant maps $P \to EG$. Writing $s(x) = [P, e] \sim [Pg, g \cdot e] := [Pg, g^{-1}e]$, so given $p \in \pi^{-1}(x)$ we can send $p \mapsto e \in EG$ such that $[p, e] \in s(x)$. This is essentially currying an argument. Conversely, given a G-equivariant map

$$\begin{aligned} P &\to EG \\ p &\mapsto e, \end{aligned}$$

we can define $s(x) := [p, e]$ where $x = \pi(p)$. This is well-defined: if $x = \pi(pg)$, then $s(x) = [pg, eg] = [p, e]$. Now note that EG is weakly contractible, so $EG \to P \underset{G}{\times} EG \to X$ has a section $s : X \to P \underset{G}{\times} EG$ and this we get a G-equivariant map $P \to EG$ which induces a map $P/G \xrightarrow{h} EG/G$, where $P/G = X$ and $EG/G = \mathbf{B}G$.



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Exercise *(?)*
Show that this map is an isomorphism.

■

8 Universal Bundles (Thursday, September 16)

Definition 8.0.1 (Universal G-bundles)
A **universal G-bundle** is a principal G-bundle $EG \xrightarrow{\pi} \mathbf{B}G$ such that EG is weakly contractible, i.e. $\pi_*(EG) = 0$.

Remark 8.0.2: We looked at a theorem stating the correspondence

$$\text{Prin}\,\mathsf{Bun}\,(G)_{/X} \rightleftharpoons [X, \mathbf{B}G].$$

Proof (of surjectivity in theorem, continued).
We showed surjectivity of the following map:

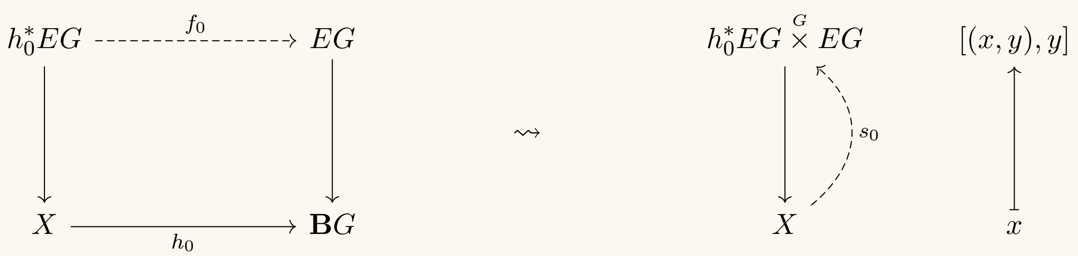
$$\begin{aligned}[X, \mathbf{B}G] &\twoheadrightarrow \text{Prin}\,\mathsf{Bun}\,(G)_{/X} \\ (f \in [X, \mathbf{B}G]) &\mapsto f^* EG.\end{aligned}$$

Given a principal G-bundle $P \xrightarrow{\pi} B$, the mixing construction used an action $G \curvearrowright F$ to construct the fiber bundle $P \overset{G}{\times} F \xrightarrow{\pi} B$. Then the data of an equivariant map $f : P \to F$, so $f(pg) = f(p) \cdot g := g^{-1} f(p)$ is equivalent to a section $s : B \to P \overset{G}{\times} F$. Note that fixing the first coordinate p in $[p, y]$ also fixes the second coordinate. For $b \in B$, we can set $s(b) = [p, y] \sim [pg, g^{-1}y]$ (noting that these are equivalent in the mixed space), and we can define $f(p) = y$ to get an equivariant map since $f(pg) = g^{-1}y = g^{-1}f(p)$
So send $P \xrightarrow{\pi} X \in \text{Prin}\,\mathsf{Bun}\,(G)_{/X}$ to $P \overset{G}{\times} EG \to X$. We proved that this has a section $s : X \to P \overset{G}{\times} EG$ and any two sections are homotopic, so from this we extract a G-equivariant map $f : P \to EG$. Modding out the G action yields $h : P/G \to EG/G$. But $P/G \cong X$ and $EG/G \cong \mathbf{B}G$, so $h : X \to \mathbf{B}G$, and moreover $h^* EG \cong P$.

■

Proof (of injectivity in theorem).
Suppose $h_0, h_1 \in [X, \mathbf{B}G]$, then $h_0^* EG \xrightarrow{\varphi, \cong} h_1^* EG$. We can construct the following section s_0:

$$(x, y) \longmapsto y := f_0([x, y])$$



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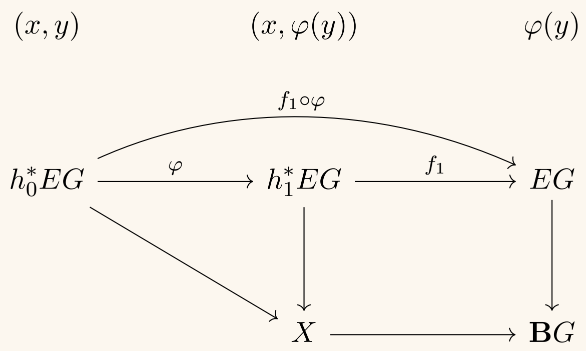
We can build another section s_1 in the following way: use the isomorphism $\varphi : h_0^* EG \to h_1^* EG$ to construct the composite



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So we have

$$\begin{aligned} s_0(x) &\coloneqq [x, y, y] \\ s_1(x) &\coloneqq [x, y, \varphi(y)]. \end{aligned}$$

By the lemma, $s_0 \simeq s_1$ through sections, so there is a homotopy

$$\begin{aligned} s : I \times X &\to I \times h_0^* EG \overset{G}{\times} EG \\ (t, x) &\mapsto (t, x, y, z). \end{aligned}$$

But this is a section of a principle EG-bundle over a CW complex, which yields a G-equivariant map

$$\begin{aligned} f : I \times h_0^* EG &\to EG \\ (t, x, y) &\mapsto z. \end{aligned}$$

Then

- At $t = 0$, we have $(0, x) \mapsto (0, x, y, y)$, so $f(0, x, y) = y$,
- At $t = 1$, we have $(1, x) \mapsto (1, x, y, \varphi(y))$, so $f(1, x, y) = \varphi(y)$.

Since f is G-equivariant, we can quotient both sides by G to get a map

$$\begin{aligned} h : I \times X &\to \mathbf{B}G \\ (0, x) &\mapsto h_0(x) \\ (1, x) &\mapsto h_1(x). \end{aligned}$$

■

Exercise 8.0.3 (?)
Try this proof yourself!

Remark 8.0.4: The same proof shows the following:

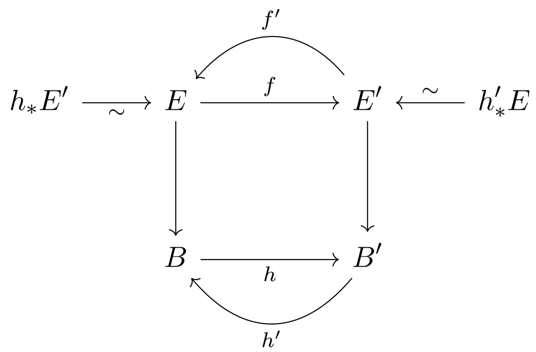
> **Lemma 8.0.5** *(?).*
> If $F \to E \xrightarrow{\pi} B$ is a fiber bundle and $B \in \mathsf{CW}$, if $\pi_{0 \leq i \leq n}(F) = 0$ then we can inductively build sections over skeleta $B_{(k)}$ fir $k \leq n$ to construct a section over $B_{(n+1)}$. Moreover, any two sections over the n-skeleton are homotopic.

> **Proposition 8.0.6** *(?).*
> If $P \xrightarrow{\pi} B \in \text{Prin}\,\mathsf{Bun}\,(G)_{/X}$ and $\pi_{0 \leq i \leq n} P = 0$ (so B is a "weak universal bundle") then $[X, B] \twoheadrightarrow \text{Prin}\,\mathsf{Bun}\,(G)_{/X}$ for any $X \in \mathsf{CW}$ with $\dim(X) \leq n+1$, and it is bijective if $\dim X \leq n$. Here the map is again $h \mapsto h^* P$.

8.1 Existence of Universal Bundles

> **Theorem 8.1.1** *(Milnor, 1966).*
> For any group $G \in \mathsf{TopGrp}$, there is a universal bundle $EG \to \mathbf{B}G$.

Remark 8.1.2: Uniqueness up to homotopy:



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Then since $(h'h^{-1})^* E \cong E$, $h'h^{-1} \simeq \text{id}$ and $h(h')^{-1} \simeq \text{id}$, so we get a homotopy equivalence $B \simeq B'$.

> **Exercise 8.1.3** (?)
> Show $E \simeq E'$.

Remark 8.1.4: We'll prove this theorem using Segal's construction. For discrete groups G, the construction is covered in Hatcher 1B. Hatcher constructs $K(G, 1)$, a space with $\pi_1 = G$ and contractible universal cover. Then the universal cover $\widehat{X} \to X$ is a principle G-bundle, and since $\widehat{X}$ is contractible, it is universal:

$$\begin{array}{ccc} & & \tilde{X} \simeq \mathrm{pt} \\ & & \downarrow{\scriptstyle \pi} \\ K(G,1) & \longrightarrow & X \end{array} \qquad \in \mathrm{Prin}\,\mathsf{Bun}\,(G)_{/X} \text{ universal}$$

Link to Diagram

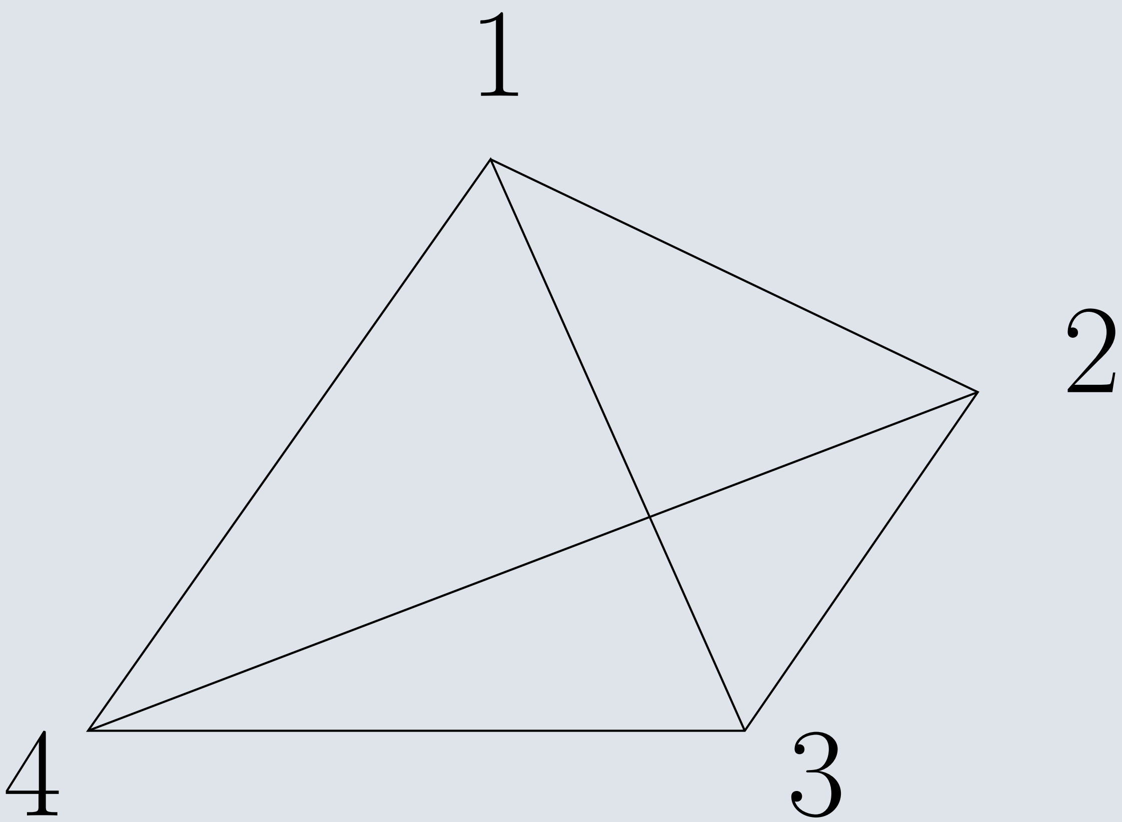
Definition 8.1.5 (Nerve of a category)
Given a category C, the **nerve** $\mathcal{N}(\mathsf{C})$ is the following Δ-complex:

- 0-simplices are objects of C
- n-simplices for $n \geq 1$ are sequences of composable morphisms

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} x_n.$$

- Gluing data for 1-simplices: for $x_0 \xrightarrow{f} x_1$, set $\partial_1(f) = x_1, \partial_0(f) = x_0$.
- Gluing data for n-simplices: the ith boundary maps are given by dropping vertex i:

$$\partial_i(f_1, \cdots, f_n) = \begin{cases} (f_2, f_3, \cdots, f_n) & i = 0 \\ (f_1, f_2, \cdots, f_{i+1} \circ f_i, \cdots, f_n) & i = 0 \\ (f_1, f_2, \cdots, f_{n-1}) & i = n. \end{cases}$$



9 | Thursday, September 16

Remark 9.0.1: Let $G \in \mathsf{Grp}$, and consider the following two categories. $\mathbf{B}G$ will be the category:

- $\mathrm{Ob}(\mathbf{B}G) = \{\mathrm{pt}\}$.
- $\mathrm{Mor}_{\mathbf{B}G}(\mathrm{pt}, \mathrm{pt}) = \{g \in G\}$, i.e. there is one morphism for every group element, with composition $g_1 \circ g_2 \coloneqq g_1 g_2$ given by group multiplication.

EG will be the category:

- $\mathrm{Ob}(EG) = \{g \in G\}$, one object for each element of G,
- $\mathrm{Mor}(g, h) = \left\{g^{-1}h\right\}$, a single (conveniently labeled!) morphism for each ordered pair (g, h).

Note that $G \curvearrowright EG$:

$$\begin{array}{ccc} g_0 & \xrightarrow{g_0^{-1}g_1} & g_1 \\ & \Downarrow & \\ gg_0 & \xrightarrow{g \cdot g_0^{-1}g_1} & gg_1 \end{array}$$

Link to Diagram

This induces an action on $\mathcal{N}(EG) \in \mathsf{Top}$, where the 0-simplices correspond to elements of G. and n-simplices are chains

$$g_0 \xrightarrow{g_0^{-1}g_1} g_1 \xrightarrow{g_1^{-1}g_2} g_2 \to \cdots \to g_n.$$

Acting on this by G yields

$$gg_0 \xrightarrow{g_0^{-1}g_1} gg_1 \xrightarrow{g_1^{-1}g_2} gg_2 \to \cdots \to gg_n,$$

noting we leave the morphism labeling unchanged, and that uniqueness of morphisms makes the simplicial boundary map behave nicely.

Exercise 9.0.2 (?)
Show that

$$\mathcal{N}(EG)/G = \mathcal{N}(\mathbf{B}G).$$

Remark 9.0.3: Note that

$$\begin{aligned} \mathcal{N}(\mathbf{B}G) &= \Delta^0 \coprod \Delta^1 \times G \coprod \Delta^2 \times G^{\times^2} \coprod \Delta^3 \times G^{\times^3} \cdots \\ \mathcal{N}(EG) &= \Delta^0 \times G \coprod \Delta^1 \times G^{\times^2} \coprod \Delta^2 \times G^{\times^3} \cdots, \end{aligned}$$

where the gluing data for $\mathcal{N}(\mathbf{B}G)$ is given by

$$\begin{aligned} \partial_n : \Delta^n \times G^{\times^n} &\to \Delta^{n-1} \times G^{\times^{n-1}} \\ (\mathbf{x}, \mathbf{g}) &\mapsto (\mathbf{x} \setminus \{x_n\}, \mathbf{g} \setminus \{g_n\}) \end{aligned}$$

and for $\mathcal{N}(EG)$ is

$$\begin{aligned} \partial_n : \Delta^n \times G^{\times^{n+1}} &\to \Delta \times G^{\times^n} \\ (\mathbf{x}, \mathbf{g}) &\mapsto (\mathbf{x} \setminus \{x_n\}, \mathbf{g} \setminus \{g_n\}). \end{aligned}$$

The action $G \curvearrowright EG$ is the following:

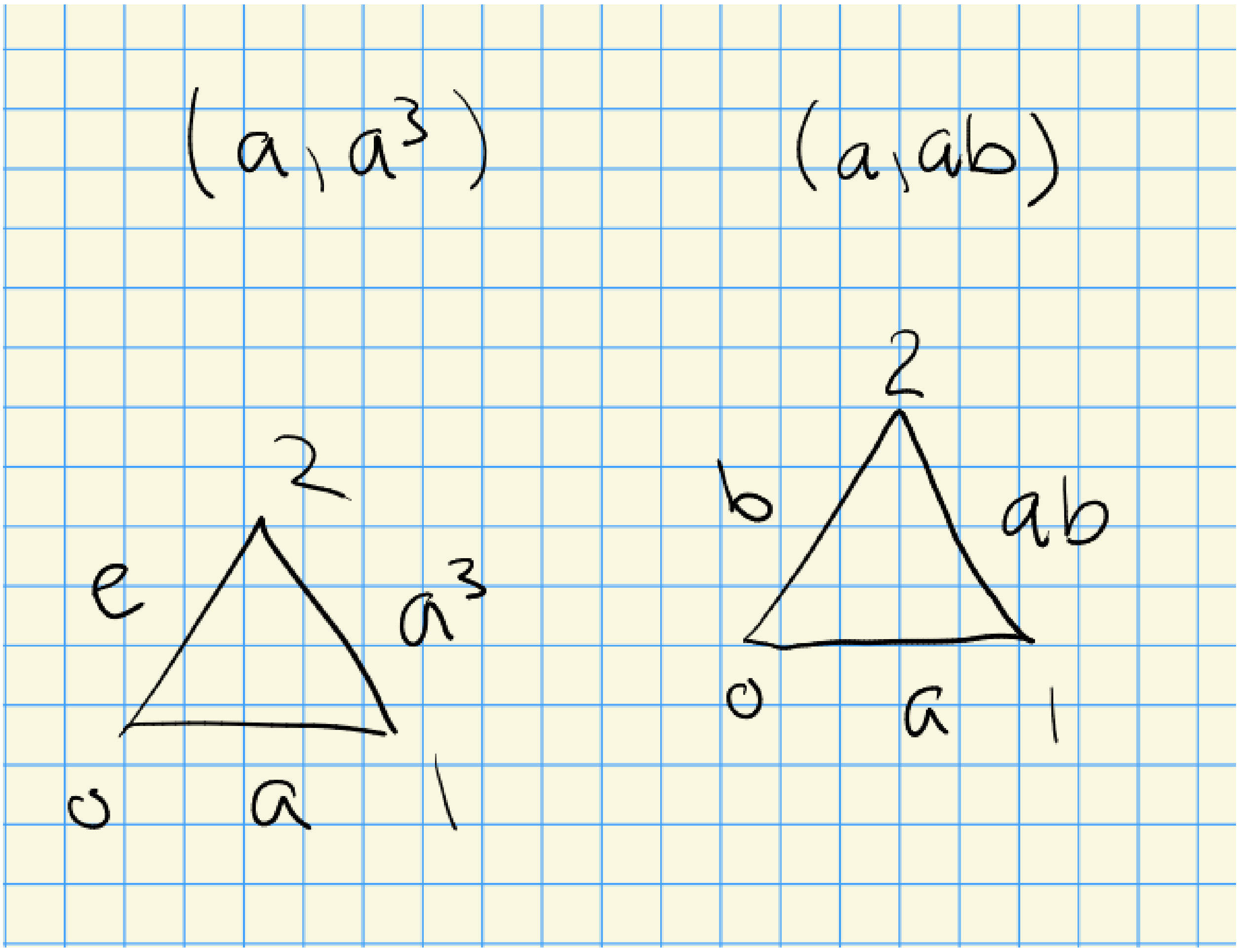
$$g \cdot (\mathbf{x}, \mathbf{g}) \mapsto (\mathbf{x}, [gg_0, gg_1, \cdots, gg_n]).$$

Example 9.0.4 *(?)*: Take $G = C_4, G' = {C_2}^{\times 2}$, and $[(x_0, x_1, x_2), (a, a^2)] \in \Delta^2 \times G^{\times 2}$. Then its faces are

$$
\begin{aligned}
[(0, x_1, x_2), (a, a^2)] &\sim [(x_1, x_2), (a^2)] \\
[(x_0, 0, x_2), (a, a^2)] &\sim [(x_0, x_2), (a)] \\
[(x_0, x_1, 0), (a, a^2)] &\sim [(x_0, x_1), (a)]
\end{aligned}
$$

.

These describe a 2-simplex mapping into $\mathbf{B}C_4$ by $a \to a^2 \to a^3$, yielding the relation $a \cdot a^2 = a^3$. One can check that in $\mathbf{B}G$, these groups yield distinct higher simplices:



> **Lemma 9.0.5** *(?)*.
> If C has an initial or terminal object, then $\mathcal{N}(\mathsf{C})$ is contractible.

Remark 9.0.6: This clearly holds for EG, since every object is initial and terminal.

> *Proof (?).*
> Suppose $y \in \mathsf{C}$ is terminal and any other object $x \in \mathsf{C}$, denote $f_x : x \to y$ the unique morphism. Then for any sequence ending in y, deformation retract to y: $x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \cdots \xrightarrow{f_x} y \rightsquigarrow y$. If

a sequence doesn't end in y, add it on: $x_0 \xrightarrow{f_0} x_1 \cdots \to x_n \xrightarrow{f_{x_n}} y \rightsquigarrow y$.

■

Corollary 9.0.7 *(?).*
$\mathcal{N}(EG)$ is contractible, and the quotient $\mathcal{N}(EG) \to \mathcal{N}(\mathbf{B}G)$ is a universal G-bundle.

Exercise 9.0.8 (?)
Construct EG and $\mathbf{B}G$ for $G = C_4, {C_2}^{\times^2}$ and explicitly compare their 3-skeleta.

10 | Tuesday, September 21

Remark 10.0.1: Today: a short discussion on generalizations of $\mathbf{B}G$ to topological groups.

Definition 10.0.2 (Topological categories)
A **topological category** is a category where the objects are topological spaces and morphisms form topological spaces in a coherent way, i.e. the following maps should be continuous:

- $\text{source}, \text{target} : \underset{\mathsf{C}}{\text{Mor}} \to \text{Ob}(\mathsf{C})$,
- $\text{id} : \text{Ob}(\mathsf{C}) \to \underset{\mathsf{C}}{\text{Mor}}$,
- Composition: $\mathsf{C}(x, y) \times \mathsf{C}(y, z) \to \mathsf{C}(x, z)$.

I.e. it is a category enriched over topological spaces (plus conditions).

Example 10.0.3 *(?):* If $G \in \mathsf{TopGrp}$, then $\mathcal{B}G$ is a topological category since the morphism space $\text{Mor}(\text{pt}, \text{pt}) = G$ has a topology. Similarly $\mathcal{E}G$ is a topological category.

Remark 10.0.4: We can take nerves of topological categories; this just requires tracking the additional enrichment (i.e. the various topologies). The same proof will yield a principal G-bundle $\mathcal{N}(\mathcal{E}G) \xrightarrow{\pi} \mathcal{N}(\mathcal{B}G)$, noting that G again acts on $\mathcal{N}(\mathcal{E}G)$.

Definition 10.0.5 (Absolute Neighborhood Retract)
A space is called an **absolute neighborhood retract** (ANR) if for any $X \hookrightarrow Y$ (as a closed subspace) into a metric space, X is a retract of a neighborhood in Y.

Example 10.0.6 *(?):* Every CW complex is an ANR. This is also true if every point of X has a contractible neighborhood.

Lemma 10.0.7 *(?).*
If G is ANR, then $EG = \mathcal{N}(\mathcal{E}G) \to \mathcal{N}(\mathcal{B}G) = \mathbf{B}G \in \text{Prin}\,\mathsf{Bun}\,(G)$.

Proof (?).
Note that $\mathbf{B}G$ is a Δ-complex, so we'll try to build bundle charts by inducting over the skeleta. Each graded piece of the complex is of the form $\Delta^i \times G^{\times^i}$, so pick an interior point $((x_0, \cdots, x_i), (g_1, \cdots, g_i))$ so $x_i \neq 0$ for every i. Define a map

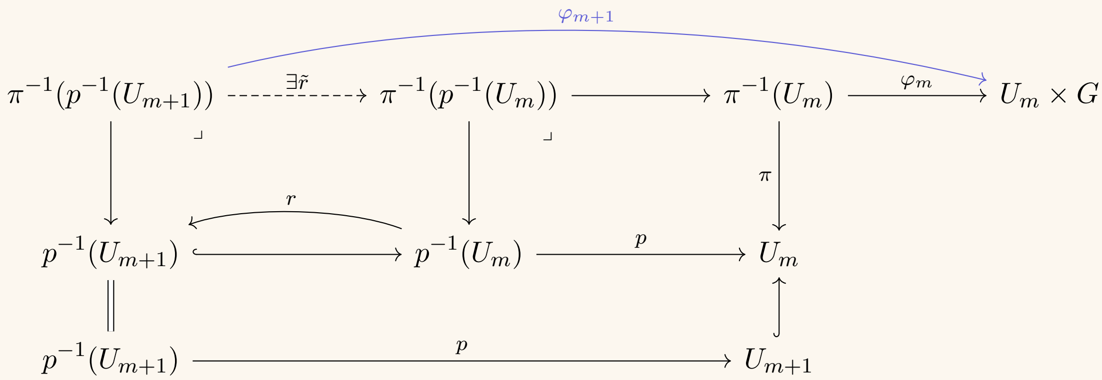
$$\begin{gathered}\Delta^i \times G^{\times^i} \times G \to EG \\ ((x_0, \cdots, x_i), (g_1, \cdots, g_i), g) \mapsto (\mathrm{id}(\cdots), (g, gg_1, gg_1g_2, \cdots, gg_1 \cdots g_i)),\end{gathered}$$

which corresponds to the sequence of composable morphisms

$$(g \xrightarrow{g_1} gg_1 \xrightarrow{g_2} gg_1g_2 \to \cdots \to gg_1 \cdots g_i).$$

> **Exercise** *(?)*
> Show that this is not compatible with the gluing!

Write $p : \Delta^i \times G^{\times^i} \to \mathbf{B}G$ for the quotient attaching map, so we can write the m-skeleton as $\mathbf{B}G^{(m)} = \bigcup_{i \leq m} p(\Delta^i \times G^{\times^i})$. Now suppose (U_m, φ_m) is a chart for $\mathsf{E}G|_{\mathbf{B}G^{(m)}} \to \mathbf{B}G^{(m)}$, we want to extend this to a chart or $\mathbf{B}G^{(m+1)}$. We have a retraction $r : U_{m+1} \to U_m$ where $U_{m+1} \subseteq \mathbf{B}G^{(m+1)}$ is an open inclusion. We construct a trivialization of $\pi^{-1}(U_{m+1}) \to U_{m+1}$:



Link to Diagram

This extends the chart to $\mathbf{B}G^{(m+1)}$, noting that p is a quotient map and thus preserves open sets.

■

Remark 10.0.9: We can't necessarily extend over the entire $m+1$ skeleton! But here extending it over a retractable neighborhood was enough, so we needed G to be an ANR in order for $\mathbf{B}G$ to be an ANR. Why: consider

$$p^{-1}(U_m) \subseteq \bigcup_{i \leq m} \Delta^i \times G^{\times^i} \subseteq \bigcup_{i \leq m-1} \Delta^i \times G^{\times^i}.$$

If G is an ANR, use that Δ^i is an ANR and so their product will be, then pick a neighborhood and apply p to get the required open.

10.1 Building $\mathbf{BO}_n$ and EO_n

Remark 10.1.1: We'll assume all spaces paracompact from this point forward! We have a correspondence

$$\left\{ n\text{-dimensional CW complexes} \right\} \rightleftharpoons \left\{ \substack{n\text{-dimensional vector bundles} \\ \text{with an } \mathrm{O}_n\text{-structure}} \right\} \rightleftharpoons \mathsf{Prin}\,\mathsf{Bun}(\mathrm{O}_n)_{/X} \rightleftharpoons [X, \mathbf{BO}_n]$$

Our next goal is to construct $\mathbf{BO}_n$ and EO_n as spaces. Let $V_n(\mathbb{R}^k) \coloneqq \{(\mathbf{v}_1, \cdots, \mathbf{v}_n) \text{orthonormal}\}$. Note that $\mathrm{O}_n \curvearrowright V_n(\mathbb{R}^k)$ by

$$(\mathbf{v}_1, \cdots, \mathbf{v}_n) \cdot A = \left(\sum_i a_{i,1}\mathbf{v}_i, \sum_i a_{i,2}\mathbf{v}_i, \cdots, \sum_i a_{i,n}\mathbf{v}_i \right).$$

There is a projection

$$\begin{array}{ccccc}
F_{\mathbf{v}_1} = V_{n-1}(\mathbb{R}^{k-1}) & \longrightarrow & V_n(\mathbb{R}^k) & & (\mathbf{v}_1, \cdots, \mathbf{v}_1) \\
 & & \downarrow & & \downarrow \\
 & & V_1(\mathbb{R}^k) & & \mathbf{v}_1
\end{array}$$

Link to Diagram

We'll use the fact that $V_1(\mathbb{R}^k)$ is $(k-2)$-connected, since it's homotopy equivalent to S^{k-1}.

Lemma 10.1.2 *(?)*.
$V_n(\mathbb{R}^k)$ is $(k-n-1)$-connected.

Proof (?).
Induct on n using the homotopy LES for the fiber bundle:

$$\begin{array}{ccccc}
 & & \cdots & \longrightarrow & \pi_{i+1}V_{n-1}\mathbb{R}^{k-1} \cong \pi_{i+1}(S^{k-1}) \\
\pi_i V_{n-1}\mathbb{R}^{k-1} = 0 & \longrightarrow & \pi_i V_n \mathbb{R}^k & \longrightarrow & \pi_i V_1 \mathbb{R}^k \cong \pi_i S^{k-2} = 0 \\
\substack{(k-n-1)\text{-connected} \\ i \leq k-n-1} & & \therefore \text{zero} & & i \leq k-n-1 \implies i \leq k-3
\end{array}$$

Link to Diagram

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Remark 10.1.3: Using the inclusions $V_n(\mathbb{R}^k) \hookrightarrow V_n(\mathbb{R}^{k+1})$, we can define $V_n(\mathbb{R}^\infty) = \operatorname{colim}_{\overrightarrow{k}} V_n(\mathbb{R}^k) = \bigcup_{k\geq 0} V_n(\mathbb{R}^k)$. We equip it with the **weak topology**, i.e. $U \subseteq V_n(\mathbb{R}^\infty)$ is open iff $U \cap V_n(\mathbb{R}^k)$ is open for all k.

Lemma 10.1.4 *(?).*

$$\pi_* V_n(\mathbb{R}^\infty) = 0.$$

Proof (?).
By compactness, any sphere S^m maps to $V_n(\mathbb{R}^k)$ for some large k, and using $V_n(\mathbb{R}^k) \hookrightarrow V_n(\mathbb{R}^\ell)$ with $\ell - n - 1 > m$ where $\pi_n V_n(\mathbb{R}^\ell) = 0$ to make the map nullhomotopic.

■

Definition 10.1.5 (?)

$$V_n(\mathbb{R}^\infty)/\mathrm{O}_n = \mathrm{Gr}_n(\mathbb{R}^\infty).$$

Remark 10.1.6: It will turn out that $\mathsf{EO}_n = V_n(\mathbb{R}^\infty)$, sometimes referred to as the *Stiefel manifold* of n-frames in $\mathbb{R}^\infty$.

11 | Thursday, September 23

Remark 11.0.1: Last time: we were trying to construct EO_n and $\mathbf{BO}_n$, and we defined $V_n(\mathbb{R}^\infty) = \varinjlim_k V_n(\mathbb{R}^k)$, where $V_n(\mathbb{R}^k)$ was the space of n orthonormal vectors in $\mathbb{R}^k$. There is a map $V_n(\mathbb{R}^\infty) \to \mathrm{Gr}_n(\mathbb{R}^\infty)$, which will be our candidate for $\mathsf{EO}_n \to \mathbf{BO}_n$.

Lemma 11.0.2 *(?).*
$V_n(\mathbb{R}^\infty) \xrightarrow{\pi} \mathrm{Gr}_n(\mathbb{R}^\infty) \in \mathrm{Prin}\,\mathsf{Bun}(\mathrm{O}_n)$.

Proof (?).
We'll show this directly in charts. Let $W \in \mathrm{Gr}_n(\mathbb{R}^\infty)$ be an n-dimensional plane, the consider an open neighborhood

$$U_W := \left\{ W' \in \mathrm{Gr}_n(\mathbb{R}^\infty) \,\middle|\, W^\perp \cap W' = 0 \right\}.$$

For any such W', we have a map $W' \to W$ given by orthogonal projection, which is an isomorphism since $W^\perp \cap W' = 0$.

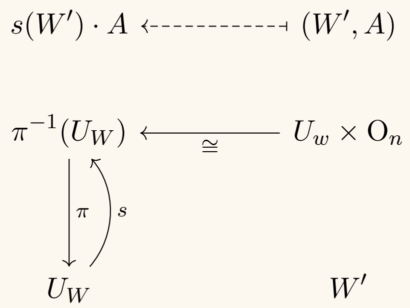
Claim:

$$\pi^{-1}(U_W) \cong U_W \times \mathrm{O}_n.$$

Fix some $\alpha \in \pi^{-1}(U_W)$ (an orthonormal basis for W), apply f^{-1} to get $f^{-1}(\alpha)$, then apply Gram-Schmidt to get $\tilde{a}$, an orthonormal basis for W'. Define $F_{W'}$ to be this composition; this yields a bijection $\pi^{-1}(W) \xrightarrow{\sim} \pi^{-1}(W')$ for all $W' \in U_W$, namely

$$\begin{aligned} U_w \times \mathrm{O}_n &\to \pi^{-1}(W) \\ (W', A) &\mapsto F_{W'}(\alpha) \cdot A. \end{aligned}$$

The claim is that this trivializes the bundle, since this constructs a local section using O_n translations:



Link to Diagram

Summary: pick an orthonormal basis for W, say α, then $s(W) = \alpha$ and we define $s(W')$ by sending α to a basis for W' by P^{-1} and the applying Gram-Schmidt to get an orthonormal basis for W'.

■

Remark 11.0.3:

- Replace O_n with U_n and $\mathbb{R}$ with $\mathbb{C}$ to get $\mathrm{Gr}_n(\mathbb{C}^\infty) = \mathbf{B}U_n$.
- $V_n(\mathbb{R}^\infty)/\mathrm{SO}_n = \mathbf{B}\mathrm{SO}_n$ yields the Grassmannian of *oriented* planes.
- For $H \leq G$, we have $\mathbf{E}H = \mathbf{E}G$ and $\mathbf{B}H = \mathbf{E}G/H$.

Question: can you get $\mathbf{B}\mathrm{Spin}_n$ *this way?*

Remark 11.0.4: An alternative description of $\mathsf{E}\mathrm{O}_n$ and $\mathbf{B}\mathrm{O}_n$, due to Milnor-Stasheff: write $\mathbf{B}\mathrm{O}_n = \mathrm{Gr}_n(\mathbb{R}^\infty)$ and define the **canonical bundle** γ. Recall that every principal O_n bundle is a pullback of the following form:

$$\begin{array}{ccc} P = f^* \mathsf{EO}_n & \dashrightarrow & \mathsf{EO}_n = V_n(\mathbb{R}^\infty) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & \mathbf{B}\mathrm{O}_n \end{array}$$

Link to Diagram

Moreover, $\mathrm{Prin}(\mathrm{O}_n)_{/X} = [X, \mathbf{B}\mathrm{O}_n] = [X, \mathrm{Gr}_n(\mathbb{R}^\infty)]$. Then $\gamma_n \xrightarrow{\pi} \mathrm{Gr}_n(\mathbb{R}^\infty)$ is the $\mathbb{R}^n$-bundle where $\pi^{-1}(v) = v = V$, regarded as a plane in $\mathbb{R}^\infty$. Another description comes from the mixing construction: $\gamma_n = V_n(\mathbb{R}^\infty) \overset{\mathrm{O}_n}{\times} \mathbb{R}^n \to \mathrm{Gr}_n(\mathbb{R}^\infty)$.

$$[\mathbf{v}_1, \cdots, \mathbf{v}_n] \times [t_1, \cdots, t_n] \longmapsto \sum t_i \mathbf{v}_i$$

$$\begin{array}{ccc} V_n(\mathbb{R}^\infty) \times \mathbb{R}^n & \xrightarrow[\exists\cong]{} & \gamma_n \\ \downarrow \pi' & \swarrow \pi & \\ \mathrm{Gr}_n(\mathbb{R}^\infty) & & \end{array}$$

Link to Diagram

Definition 11.0.5 (Subbundles)
$E' \leq E$ is a **subbundle** iff there is an embedding $E' \hookrightarrow E$ over X:

$$\begin{array}{ccc} E' & \overset{f}{\hookrightarrow} & E \\ & {\scriptstyle\pi'}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & X & \end{array}$$

Link to Diagram

We also require that restrictions to fibers $f_x : E'|_x \to E|_x \in \mathrm{Mat}(m \times n; \mathbb{R})$ is a linear map to an n-dimensional subspace $E|_x$, where $\dim_{\mathbb{R}} E'|_x = n$ and $\dim_{\mathbb{R}} E|_x = m$.

Lemma 11.0.6 *(?).*
Every vector bundle $E \xrightarrow{\pi} X$ with $X \in \mathsf{CW}$ compact is a subbundle of a trivial bundle.

Proof (?).
Take $\{(U_i, \varphi_i)\}_{i=1}^m \rightrightarrows X$ a finite cover by charts, and choose a subordinate partition of unity

$\{\chi_i\}_{i=1}^m$ such that $\operatorname{supp}\chi_i \subseteq U_i$. Then define

$$\begin{aligned}\psi : E &\to \mathbb{R}^{nm} = \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \\ v &\mapsto [\chi_1(v)\varphi_1(v), \chi_2(v)\varphi_2(v), \cdots, \chi_m(v)\varphi_m(v)].\end{aligned}$$

This exhibits $(E \to X) \leq (X \times \mathbb{R}^{nm} \to X)$ as a subbundle.

■

Lemma 11.0.7 ***(?).***
Every $(E \to X) \in \mathsf{Bun}\,(\mathrm{GL}_n)_{/X}$ for $X \in \mathsf{CW}$ compact is a pullback of the canonical bundle along some map $f : X \to \mathbf{BO}_n$.

Example 11.0.8 ***(?):*** For $E \xrightarrow{\pi} X$ and $f : X \to \mathbf{BO}_n$, $\psi : E \to \mathbb{R}^{nm} \subseteq \mathbb{R}^\infty$ and we can take $f(x) \coloneqq \psi(\pi^{-1}(x)) \in \mathrm{Gr}_n(\mathbb{R}^\infty)$ to get $f^*\gamma_n \cong E$.

Lemma 11.0.9 ***(?).***
If $f^*\gamma_n \cong E$ and $g^*\gamma_n \cong E$, then $f \simeq g$.

Proof (?).
Corresponding to $f^*\gamma_n \cong E$ we get a map $\tilde{f} : E \to \mathbb{R}^\infty$ which restricts to an embedding on all fibers, and similarly $g^*\gamma_n \cong E$ yields $\tilde{g} : E \to \mathbb{R}^\infty$. So take

$$\begin{aligned}L_t : \mathbb{R}^\infty &\to \mathbb{R}^\infty \\ \mathbf{x} &\mapsto (t-1)[x_1, x_2, \cdots] + t[x_1, 0, x_2, 0, x_3, 0, \cdots],\end{aligned}$$

which is a homotopy between identity and the self-embedding that maps into only odd coordinates. Composing $L_t \circ \tilde{f}$ yields a homotopy between $\tilde{f}$ and a map F' whose image has only odd coordinates. Similarly, we can construct a G_t for $\tilde{g}$ to get a homotopy between $\tilde{g}$ and a map G' whose images has only even coordinates. Now take a linear homotopy $F' \to G'$, this yields a homotopy through embeddings (where we've first made them "transverse").

■

12 Vector Bundle Classification Theorem (Tuesday, September 28)

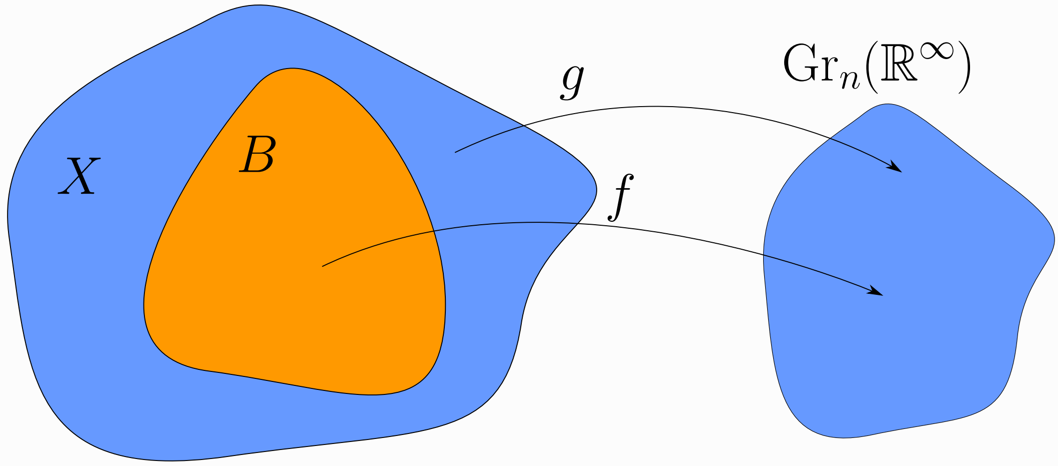
See homework posted on the website! Turn in 2 total problem sets, one by mid-October.

Theorem 12.0.1 ***(?).***

$$\begin{aligned}[X, \mathbf{BO}_n] \cong [X, \mathrm{Gr}_n(\mathbb{R}^\infty)] &\xrightarrow{\sim} \mathsf{Bun}\,(\mathrm{GL}_n)(\mathbb{R}, X) \\ f &\mapsto f^*\gamma_n.\end{aligned}$$

Remark 12.0.2: We proved surjectivity last time for $X \in \mathsf{CW}$ compact, using compactness to embed any bundle into a trivial bundle. We proved injectivity in the form of $f^*\gamma_n \cong g^*\gamma_n \implies f \simeq g$, again for $X \in \mathsf{CW}$ compact. So we need to handle the case of X not compact.

Lemma 12.0.3 *(?).*
Let $\pi : E \to X$ be a vector bundle over X. Suppose for $B \in \mathsf{CW}$ compact with $B \subseteq X$, we have $f : B \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ such that $f^*\gamma_n \cong E|_B$. Suppose also that there exists a $g : X \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ with $g^*\gamma_n \cong E$. Then there exists an $h : X \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ such that $h|_B = f$ and $h^*\gamma_n \cong E$.



Remark 12.0.4: Idea: write $X = \operatorname{colim}_{\overrightarrow{n}} X^{(n)}$ as a limit of compact finite skeleta, define maps $f_n : X^{(n)} \to \mathbf{B}\mathrm{O}_n$ and $f_{n+1} : X^{(n+1)} \to \mathbf{B}\mathrm{O}_n$, then modify $\tilde{f}_{n+1} \simeq f_{n+1}$ to extend f_n in such a way that $f_n^*\gamma_i = E|_{B_n}$.

Proof (?).
For $g^*\gamma_n \cong E$ with $(g|_B)^*\gamma_n \cong E|_B$ and $f^*\gamma_n \cong E|_B$, then $f \simeq g|_B$ by injectivity for compact B. We can then extend the homotopy $H : I \times X \to \mathbf{B}\mathrm{O}_n$ where $H_0 = g$ and $h := H_1$ with $h|_B = f$. ■

12.1 Characteristic Classes

Definition 12.1.1 (Characteristic classes and representability)
Let F, G be two contravariant functors with source C. A **characteristic class of** F **valued in** G is a natural transformation $c : F \to G$. F is **representable** if there exists an object $\mathbf{B}F$ such that $F(X) = [X, \mathbf{B}F]$ for every $X \in \mathrm{Ob}(\mathsf{C})$.

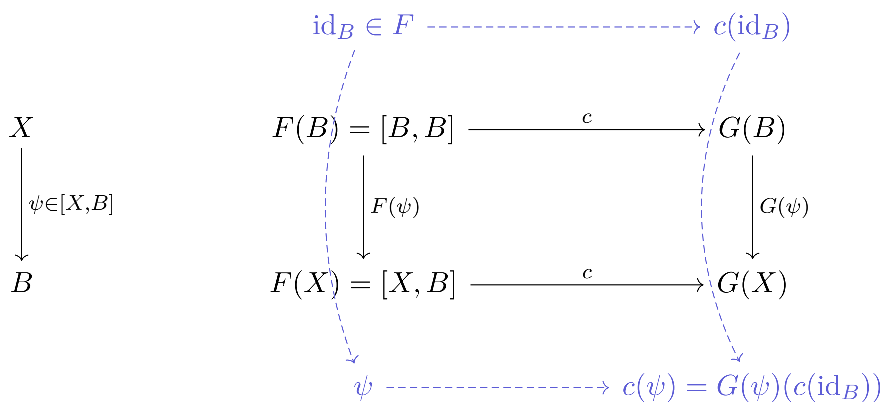
Note: we aren't requiring the target categories to coincide!

Example 12.1.2 ***(?):***

- $F(X) := \text{Prin}\,\mathsf{Bun}(\mathrm{O}_n)_{/X} = \mathsf{Bun}\,(\mathrm{GL}_n)_{/X} = [X\mathbf{B}\mathrm{O}_n]$ is a contravariant functor $\mathsf{hoCW}^{\mathrm{op}} \to \mathsf{Set}$, where contravariance is due to pullbacks.
- $G(X) := H^j_{\text{sing}}(X)$, which is representable: for any $X \in \mathsf{CW}$ and any $G \in \mathsf{AbGrp}$, we have $H^j(X; M) = [X, K(G, j)]$. This comes from taking the sphere that generates $\pi_j K(G, j) = \langle \alpha \rangle$ and pulling any $f : X \to K(G, j)$ back to $f^*\alpha$.

Lemma 12.1.3 ***(?).***
If $F = [-, \mathbf{B}F]$ is representable, then characteristic classes of F valued in a functor G biject with $G(B)$

Remark 12.1.4: We can write $F(B) = [B, B] \ni \mathrm{id}_B$, and it turns out that the characteristic class is determined by where id_B is sent:



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For us, taking $B := \mathrm{Gr}_n(\mathbb{R}^\infty)$ and $G(B) = H^j(\mathrm{Gr}_n(\mathbb{R}^\infty)) \ni \alpha = c(\mathrm{id}_B)$, so we can pullback to define $c(f) = f^*\alpha \in H^j(X)$.

Example 12.1.5 ***(?):*** Take $F(X) = \text{Prin}\,\mathsf{Bun}(U_n)_{/X} = [X\mathrm{BU}_n]$ and $G(X) = H^j(X)$, then $\alpha \in H^j(\mathrm{BU}_n)$ maps to to $c_\alpha(E) = f^*(\alpha)$ for any $f \in [X, \mathrm{BU}_n]$.

Example 12.1.6 ***(?):*** Take $F(X) = H^n(X; M)$ and $G(X) = H^m(X; N)$ with $M, N \in \mathsf{AbGrp}$. Then $F(X) = [X, K(M, n)]$, and taking $G(K(M, n)) = H^m(K(M, n), N) \ni \alpha$ yields a map

$$\begin{aligned} H^n(X; M) &\to H^m(X; N) \\ (f : X \to K(M, n)) &\mapsto f^*\alpha, \end{aligned}$$

i.e. a cohomology operation. If for example

$$\varphi \in \operatorname*{Hom}_{\mathsf{Grp}}(M, N) = \operatorname*{Hom}_{\mathsf{Grp}}(H_n(K(M, n); \mathbb{Z}), N) = H^n(K(M, n); N),$$

using that $H_n(K(M,n);\mathbb{Z}) \cong M$. This yields a change of coefficient morphism

$$H^n(X;M) \to H^n(X;N),$$

which turns out to be the same map as above. So any element in $H^m(K(M,n),N)$ yields a map $H^n(X,M) \to H^m(X,N)$ by sending $f : X \to K(M,n)$ to $f^*\alpha$. Taking $n = m$ yields $H^n(X;M) \to H^n(X;N)$.

13 | Thursday, September 30

13.1 Line Bundles: Chern and Stiefel-Whitney Classes

Remark 13.1.1: Last time: defining characteristic classes. Recall that given $F, G \in \mathsf{Fun}(\mathsf{Top}, -)$, a characteristic class with values in G is a natural transformation $c : F \xrightarrow{\sim} G$. If F is representable, then characteristic classes $c : F \to G$ is of the form $c(\mathrm{id}_B) \in G(B)$ for $\mathrm{id}_B \in [B,B] \cong F(B)$, since c is determined by where it sends id_B. Recall that Eilenberg-MacLane spaces $K(G,n)$ represent $H^n(-;G)$ for $G \in \mathsf{AbGrp}$, and are characterized by $\pi_i(K(G,n)) = G$ only in degree $i = n$.

Example 13.1.2 ***(?)*****:**

- $\mathsf{Bun}\,(\mathrm{GL}_n)(\mathbb{R}, X) \xrightarrow{\sim} [X, \mathbf{BO}_n]$ and we realized $\mathbf{BO}_n \simeq \mathrm{Gr}_n(\mathbb{R}^\infty)$. For $\alpha \in H^j(\mathbf{BO}_n;\mathbb{Z})$, we can take a homotopy class $f : X \to \mathbf{BO}_n$ and pullback to get $f^*(\alpha) \in H^j(X;\mathbb{Z})$.

- $\mathsf{Bun}\,(\mathrm{GL}_n)(\mathbb{C}, X) \xrightarrow{\sim} [X, \mathrm{BU}_n]$, where $\mathrm{BU}_n \simeq \mathrm{Gr}_n(\mathbb{C}^\infty)$ and we can again pullback cohomology classes.

Example 13.1.3 ***(?)*****:** For line bundles, we can identify $\mathrm{BU}_1 \simeq \mathrm{Gr}_1(\mathbb{C}^\infty) \simeq \mathbb{CP}^\infty$, so $\mathsf{Bun}\,(\mathrm{GL}_1)(\mathbb{C}, X) \xrightarrow{\sim} [X, \mathbb{CP}^\infty$ The claim is that line bundles are uniquely characterized by their first Chern classes. Using that $H^2(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z} = \langle \alpha \rangle$, where we've chosen a positive generator, we obtain the **first Chern class** $c_1 := f^*(\alpha) \in H^2(X;\mathbb{Z})$. Note that $\mathbb{CP}^\infty \simeq K(\mathbb{Z},2)$, and $[X, K(G,n)] \xrightarrow{\sim} H^n(X;M)$ where $f \mapsto f^*(\alpha)$ for $H^n(K(G,n);\mathbb{Z}) = \langle \alpha \rangle$, so there is an isomorphism

$$\begin{aligned} [X, \mathbb{CP}^\infty] &\xrightarrow{\sim} H^2(X;\mathbb{Z}) \\ f &\mapsto f^*(\alpha), \quad \langle \alpha \rangle = H^2(\mathbb{CP}^\infty). \end{aligned}$$

So the set of bundles is an affine space over $H^2(X)$.

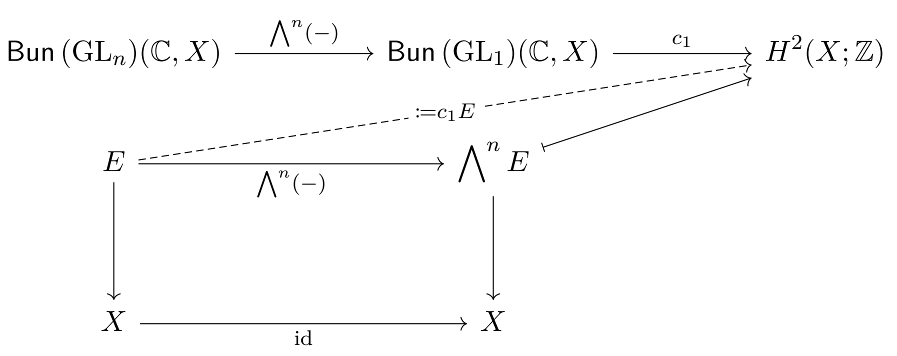
Corollary 13.1.4 ***(?)*****.**
There is a bijection

$$c_1 : \mathsf{Bun}\,(\mathrm{GL}_1)(\mathbb{R}, X) \xrightarrow{\sim} H^2(X),$$

Example 13.1.5 *(?):* For $\mathsf{Bun}\,(\mathrm{GL}_1)(\mathbb{R}, X)$ we can identify $\mathbf{BO}_1 \simeq \mathrm{Gr}_1(\mathbb{R}^\infty) = \mathbb{RP}^\infty$, so for $\langle \alpha \rangle = H^2(\mathbb{RP}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2$, we obtain the **first Stiefel-Whitney class** w_1 and a bijection

$$\begin{aligned} w_1 : [X, \mathbb{RP}^\infty] &\xrightarrow{\sim} H^1(X; \mathbb{Z}/2) \\ f &\mapsto f^*(\alpha). \end{aligned}$$

Remark 13.1.6: We can define c_1 for vector bundles of any dimension by taking a top exterior power to get a line bundle:



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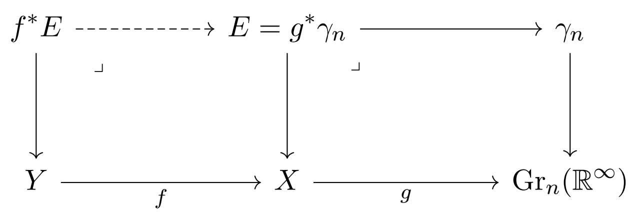
So

$$c_1(E) := c_1(\textstyle\bigwedge^n E).$$

Remark 13.1.7: There is a natural isomorphism $c(f^*(E)) \cong f^*(c(E))$, since we can take iterated pullbacks:



So we can identify $c(f^*(E)) = (g \circ f)^*\alpha$ and $f^*(c(E)) = f^*(g^*\alpha) = (g \circ f)^*\alpha$.

13.2 Euler Classes and the Thom Isomorphism

Remark 13.2.1: Note that any vector bundle with a Riemannian metric admits a unit disc bundle.

Definition 13.2.2 (Oriented disc bundles)
A unit disc bundle $D \xrightarrow{\pi} B$ is **oriented** if there is a locally coherent choice of a generator of $H^n(D_b := \pi^{-1}(b), S^b := \partial D_b; \mathbb{Z})$.

Example 13.2.3 *(?)*: The unit disc bundle for an oriented $E \in \mathsf{Bun}\,(\mathrm{GL}_n)(\mathbb{R}, X)$ with a Riemannian metric will be oriented as a disc bundle.

Remark 13.2.4: Given a bundle $E \to X$ and taking its disc bundle $\mathbb{D}E \to X$, taking boundaries on fibers yields a sphere bundle $\mathbb{S}E \to X$, so $\mathbb{S}E_b := \partial \mathbb{D}E_b$ on fibers. Note the $\mathbb{D}E \simeq X$ by a deformation retraction.

Theorem 13.2.5 ***(Thom Isomorphism Theorem).***
Let $\mathbb{D}E \to X \in \mathsf{Bun}\,(\mathrm{GL}_n)(\mathbb{R}, X)$ be an oriented disc bundle and $\mathbb{S}E \to X$ its corresponding sphere bundle. Then

1. $H^{i<n}(\mathbb{D}E, \mathbb{S}E; \mathbb{Z}) = 0$
2. There exists a generator, the **Thom class** $u_E \in H^n(\mathbb{D}E, \mathbb{S}E; \mathbb{Z})$ mapping to a distinguished generator of $H^n(\mathbb{D}E_x, \mathbb{S}E_x)$, inducing isomorphisms

$$\begin{aligned}(-) \smile u_E : H^j(\mathbb{D}E) &\xrightarrow{\sim} H^{j+n}(\mathbb{D}E, \mathbb{S}E) \qquad \forall j \geq 0 \\ \eta &\mapsto \eta \smile u_E,\end{aligned}$$

Proof (of theorem).

- **Step 1**: Look locally to see why we might expect this result! If $\mathbb{D}E \to X$ is trivial, then the claims hold.
- **Step 2**: If the claims hold for $\mathbb{D}E|_U, \mathbb{D}E|_V, \mathbb{D}E|_{U\cap V}$, then it holds for $\mathbb{D}E|_{U\cup V}$. As a corollary, the claims hold for compact X.
- **Step 3**: Prove claims for $H^*(-; k)$ for $k \in \mathsf{Field}$
- **Step 4**: Prove claims for $H^*(-; \mathbb{Z})$.

■

Proof (step 1).
Trivial bundles are products, and we have formulas for cohomology of products. Write $\mathbb{D}E = X \times \mathbb{D}^n$, and note that $H^*(\mathbb{D}^n, S^{n-1}; \mathbb{Z}) = \mathbb{Z}[n]$, which is always torsionfree. Thus

$$\begin{aligned}H^k(\mathbb{D}E, \mathbb{S}E; \mathbb{Z}) &= H^k(\mathbb{D}^n \times X, S^{n-1} \times X; \mathbb{Z}) \\ &\cong \bigoplus_{0\leq i\leq k} H^i(X; \mathbb{Z}) \otimes_{\mathbb{Z}} H^{k-i}(\mathbb{D}^n, S^{n-1}; \mathbb{Z}) \\ &\cong \bigoplus_{0\leq i\leq k} H^i(X; \mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}[i+n]) \\ &= H^{k-n}(X; \mathbb{Z}) \qquad \text{when } k > n, \text{ else } 0.\end{aligned}$$

So pick $u \in H^n(\mathbb{D}^n, S^{n-1})$ be the generator specified by the orientation, and take $u_E \in$

$H^n(\mathbb{D}E, \mathbb{S}E; \mathbb{Z})$ to be the corresponding generator pulled back along the Kunneth isomorphism (which recall was induced by a cup product).

■

Proof (step 2).
Use Mayer-Vietoris:

$$\cdots \to H^{k-1}(\mathbb{D}E\big|_{U\cap V}, \mathbb{S}E\big|_{U\cap V}) \to H^k(\mathbb{D}E\big|_{U\cup V}, \mathbb{S}E\big|_{U\cup V}) \longrightarrow H^k(\mathbb{D}E\big|_{U}, \mathbb{S}E\big|_{U}) \oplus H^k(\mathbb{D}E\big|_{V}, \mathbb{S}E\big|_{V}) \longrightarrow H^k(\mathbb{D}E\big|_{U\cap V}, \mathbb{S}E\big|_{U\cap V}) \to \cdots$$

Link to Diagram

- If $k < n$, the union terms vanish in degree k, since they're surrounded by zeros.
- If $k = n$, the kernel of $\oplus \to \cap$ is isomorphic to $\mathbb{Z}$, so pick a generator $u_{U\cup V} = u_E|_{U\cup V}$ that lifts $u_E|_U$ and $u_E|_V$.

Next time: we'll show that $u_{U\cup V} \smile (-)$ yields the isomorphism in part 2 of the theorem statement.

■

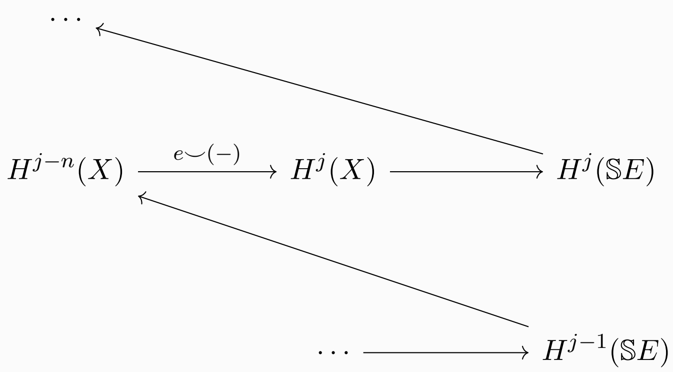
14 | Tuesday, October 05

Remark 14.0.1: Recall that we were proving the Thom isomorphism theorem. Some motivation:

Corollary 14.0.2 ***(The Gysin Sequence).***
Consider an oriented sphere bundle:

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & \mathbb{S}E \\ & & \downarrow \\ & & X \end{array}$$

Then there is a LES induced by the Euler class $e \in H^n(X)$



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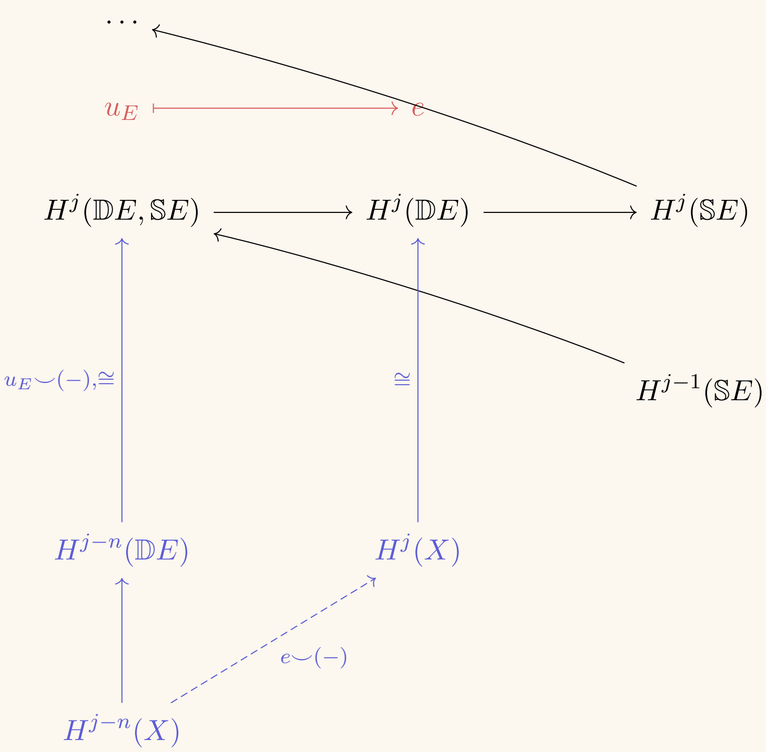
Proof (of corollary).
Corresponding to $\mathbb{S}E \xrightarrow{\pi} X$, the mapping cone of π is $\mathbb{D}E$. So consider the LES in relative homology:



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After identifying terms, we see that $u_E \in H^n(\mathbb{D}E, \mathbb{S}E)$ maps to the Euler class $e \in H^n(X)$, which carries interesting geometric information.

■

Proof (of theorem, for fields).

Suppose the claim holds for compact X, and write $X = \bigcup_i C_i$ for C_i compact CW skeleta. Then $H_i(X) = \varinjlim_j H_i(C_j)$ since simplices are also compact. Note that

$$
\begin{aligned}
H^i(X;k) &\cong H_i(X;k)^\vee \\
&:= \operatorname{Hom}(H_i(X;k),k) \\
&= \operatorname{Hom}(\varinjlim_j H_i(C_j;k);k) \\
&= \varprojlim_j \operatorname{Hom}(H_i(C_j;k),k) \\
&= \varprojlim_j H^i(C_j;k).
\end{aligned}
$$

Similarly,

$$
H^i(\mathbb{D}E,\mathbb{S}E;k) \xrightarrow{\sim} \varprojlim_j H^i\left(\mathbb{D}E|_{C_j}, \mathbb{S}E|_{C_j};k\right) = \begin{cases} 0 & i < n \\ ? & i \geq n. \end{cases}.
$$

Assembling the maps and isomorphisms we have:

$$
\begin{array}{ccc}
H^i(\mathbb{D}E;k) & \xrightarrow{u_E \smile (-)} & H^m(\mathbb{D}E,\mathbb{S}E;k) \\
\downarrow & & \downarrow \\
H^i(\mathbb{D}E|_{C_j};k) & \xrightarrow{u_E|_{\mathbb{D}E} \smile (-), \cong} & H^i(\mathbb{D}E|_{C_j}, \mathbb{S}E|_{C_j};k)
\end{array}
$$

Link to Diagram

Now use that the vertical maps become isomorphisms after a colimit, so the top map must become an isomorphism as well.

■

Proof (of theorem, for arbitrary rings).
Consider $H_i(\mathbb{D}E,\mathbb{S}E;\mathbb{Z}) \to \varprojlim_j H_i(\mathbb{D}E|_{C_j}, \mathbb{S}E|_{C_j};\mathbb{Z})$ Using universal coefficients, we have

$$
H^i(\mathbb{D}E,\mathbb{S}E;\mathbb{Z}) \cong \operatorname{Hom}(H_i(\mathbb{D}E,\mathbb{S}E;\mathbb{Z}),\mathbb{Z}) \oplus \operatorname{Ext}(H_{i-1}(\mathbb{D}E,\mathbb{S}E;\mathbb{Z}),\mathbb{Z}) = 0 \qquad i < n,
$$

since each summand will be zero. For $i = n$, the Ext term vanishes, and we have

$$
\begin{aligned}
H^n(\mathbb{D}E,\mathbb{S}E;\mathbb{Z}) &\cong \operatorname{Hom}(H_n(\mathbb{D}E,\mathbb{S}E;\mathbb{Z}),\mathbb{Z}) \\
&\cong \operatorname{Hom}(\varinjlim_j H_n(\mathbb{D}E|_{C_j}, \mathbb{S}E|_{C_j};\mathbb{Z}),\mathbb{Z}) \\
&\cong \varprojlim_j \operatorname{Hom}(H_n(\mathbb{D}E|_{C_j}, \mathbb{S}E|_{C_j};\mathbb{Z}),\mathbb{Z}) \\
&\cong \varprojlim_j H_n(\mathbb{D}E|_{C_j}, \mathbb{S}E|_{C_j};\mathbb{Z}) \\
&\cong \langle u_E \rangle \cong \mathbb{Z},
\end{aligned}
$$

using the distinguished generator $u_E \in H^n(\mathbb{D}E, \mathbb{S}E)$. So we can define a chain map

$$u_E \frown (-) : C_{j+n}(\mathbb{D}E, \mathbb{S}E; \mathbb{Z}) \to C_j(\mathbb{D}E),$$

which shifts degree by $-n$ by capping against u_E. This induces the cup product $u_E \smile (-) : H^j(\mathbb{D}E; \mathbb{Z}) \to H^{j+n}(\mathbb{D}E, \mathbb{S}E; \mathbb{Z})$.

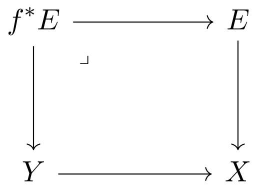
Lemma 14.0.3 ***(Milnor-Stasheff 10.6).***
Given chain complexes of $\mathbb{Z}$-modules C_* and D_* and a chain map $f : C_* \to D_*$, if $f^* : H^*(D_*; k) \to H_*(C_*; k)$ are isomorphisms for every $k \in \mathsf{Field}$, then f_*, f^* are isomorphisms over any $R \in \mathsf{Ring}$.

We showed that $u_E \smile (-)$ was an isomorphism for all k, so now we get an isomorphism over every R and this completes the proof.

■

Remark 14.0.4: Without the oriented assumption, this theorem still holds with C_2 coefficients.

Note also that we have naturality for characteristic classes: given a pullback



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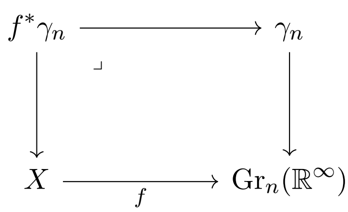
Then

$$c(f^*E) = f^*c(E).$$

Note that we can always pull back the canonical:



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If the Euler class e is natural, then $e(f^*\gamma_n) = f^*e(\gamma_n)$ where $e(\gamma_n) \in H^n(\mathrm{Gr}_n(\mathbb{R}^\infty))$, so e is the characteristic class defined by $e(\gamma_n)$.

Lemma 14.0.5 *(?)*.
The Euler class e is natural with respect to pullback and thus a characteristic class.

Proof (?).
We'll check naturality for the Thom class u_E. Consider pulling back a disc bundle:

$$\begin{array}{ccc} D' := f^*\mathbb{D}E & \longrightarrow & D := \mathbb{D}E \\ \big\downarrow & \lrcorner & \big\downarrow \\ Y & \xrightarrow{\ f\ } & X \end{array}$$

Link to Diagram

Recall that $u_E \in H^n(\mathbb{D}E, \mathbb{S}E; \mathbb{Z})$ such that $u_E|_x : H^n(\mathbb{D}E|_x, \mathbb{S}E|_x; \mathbb{Z})$ is the generator. We get an element in the fibers of the pullback in the following way:

$$\begin{array}{ccc} u_E & & u' \\ H^n(\mathbb{D}E, \mathbb{S}E; \mathbb{Z}) & \longrightarrow & H^n(f^*\mathbb{D}E, f^*\mathbb{S}E; \mathbb{Z}) \\ \big\downarrow & & \big\downarrow \\ H^n(\mathbb{D}E|_{f(x)}, \mathbb{S}E|_{f(x)}; \mathbb{Z}) & \xrightarrow{\ \cong\ } & H^n(f^*\mathbb{D}E|_{f(x)}, f^*\mathbb{S}E|_{f(x)}; \mathbb{Z}) \\ u_E|_x & & u'|_x \end{array}$$

Link to Diagram

We then get naturality of the Euler class from the following:

$$\begin{array}{ccc} u_E & \longrightarrow & e \\ H^n(\mathbb{D}E, \mathbb{S}E; \mathbb{Z}) & \longrightarrow & H^n(\mathbb{D}E; \mathbb{Z}) \cong H^n(X) \\ \big\downarrow & & \big\downarrow \\ H^n(f^*\mathbb{D}E, f^*\mathbb{S}E; \mathbb{Z}) & \longrightarrow & H^n(Y) \\ u' & \longrightarrow & e' \end{array}$$

Link to Diagram

■

Example 14.0.6 *(?)***:** $e(\mathbb{CP}^1)$ is a generator of $H^2(\mathbb{CP}^1)$ Apply the Gysin sequence, taking the canonical line bundle $\mathbb{S}E$ over $\mathbb{CP}^1$:

$$H^{j-1}(\mathbb{S}E) \longrightarrow H^{j-2}(\mathbb{CP}^1) \longrightarrow H^j(\mathbb{CP}^1) \longrightarrow H^j(\mathbb{CP}^1) \longrightarrow H^j(\mathbb{S}E)$$

Link to Diagram

The claim is that the total space here is the Hopf fibration:

$$\begin{array}{ccc} S_1 \to \mathbb{S}E \cong S^3 & \xrightarrow{\subseteq} & E \\ & \searrow & \downarrow \\ & & \mathbb{CP}^1 \cong S^2 \end{array}$$

What is the Hopf fibration? Write $\mathbb{CP}^1 = \left\{ [z_0 : z_1] \,\middle|\, z_0^2 + z_1^2 = 1 \right\} / \sim$, then realize $S^3 = \left\{ z_0^2 + z_1^2 = 1 \right\} \subseteq \mathbb{C}^2$. Then take a map $S^3 \to \mathbb{CP}^1$, whose fibers are $\left\{ \lambda \in \mathbb{C} \,\middle|\, |\lambda| = 1 \right\} = \mathbb{C}^\times \cong S^1$. Then identifying elements and maps in the Gysin sequence yields the following:

$$\begin{array}{llcl} & H^j(\mathbb{S}E) & \longrightarrow & \cdots \\ j = 3: & H^j(\mathbb{S}^3) & \nwarrow & \\ & H^{j-1}(\mathbb{S}E) \longrightarrow & H^{j-2}(\mathbb{CP}^1) \longrightarrow & H^j(\mathbb{CP}^1) \\ j = 2: & H^{j-1}(S^3) = 0 & H^0(\mathbb{CP}^1) \xrightarrow{e \smile (-)} & H^2(\mathbb{CP}^1) \end{array}$$

Link to Diagram

So e is a generator of $H^2(\mathbb{CP}^1)$

Exercise 14.0.7 (?)
Check that $\mathbb{S}E \to \mathbb{CP}^1$ with $\mathbb{S}E$ the canonical is the Hopf fibration.

Exercise 14.0.8 (?)
Try to compute $e(\mathbf{T}S^2)$! You may need to add on a bundle to trivialize it.

15 | Thursday, October 07

15.1 The Euler Class

Remark 15.1.1: Let $E \to X$ be an orientable bundle, then

$$
\begin{array}{ccc}
H^n(\mathbb{D}E, \mathbb{S}E; \mathbb{Z}) & & H^n(\mathbb{D}E; \mathbb{Z}) \cong H^n(X; \mathbb{Z}) \\
u_{-E} = -u_E & \longmapsto & e(E)
\end{array}
$$

Link to Diagram

Lemma 15.1.2 *(?).*
$e(-E) = -e(C)$.

Proof (?).
Using naturality:

$$
\begin{array}{ccc}
E = f^*(\mathbb{R}^n) & \longrightarrow & \mathbb{R}^n \\
\big\downarrow & \lrcorner & \big\downarrow \pi \\
X & \xrightarrow[f]{} & \mathrm{pt}
\end{array}
$$

Link to Diagram

Then $e(\mathbb{R}^n \to \mathrm{pt}) = 0$, and $0 = f^*(0) = e(f^*(\mathbb{R}^n \to \mathrm{pt})) = e(E)$. ∎

Lemma 15.1.3 *(?).*
$e(E) = 0$ if E is the trivial bundle.

Proof (?).
Using the exact sequence for the pair $(\mathbb{D}E, \mathbb{S}E)$:

$$\begin{array}{ccccc} u_E & \longmapsto & e & & \\ H^n(\mathbb{D}E, \mathbb{S}E; \mathbb{Z}) & \longrightarrow & H^n(\mathbb{D}E; \mathbb{Z}) & \longrightarrow & H^n(\mathbb{S}E; \mathbb{Z}) \\ & & \cong H^n(X \times \mathbb{D}^n; \mathbb{Z}) \cong H^n(X) \otimes H^n(\mathbb{D}; \mathbb{Z}) & & \cong H^n(X \times S^{n-1}; \mathbb{Z}) \\ & & H^n(X) & \hookrightarrow & H^n(X) \oplus H^1(X) \\ & & \alpha & \longrightarrow & (\alpha, 0) \end{array}$$

Link to Diagram

So the second map is the inclusion $\alpha \mapsto (\alpha, 0)$, which is thus injective. The first map is 0, so $e(E) = 0$.

■

Example 15.1.4*(?)*: Compute $e(\mathbf{T}S^2)$.

Lemma 15.1.5*(?)*.
If E is an odd dimensional vector bundle, then $2e(E) = 0$

Remark 15.1.6: Check that $\alpha \smile \beta = (-1)^{ij} \beta \smile \alpha$.

Proof (First way).
The antipodal map will give an isomorphism $E \cong -E$, so $e(E) = e(-E) = -e(E)$ and thus $2e(E) = 0$.

■

Proof (Second way).
Use the map:

$$\begin{array}{ccc} H^n(\mathbb{D}E, \mathbb{S}E; \mathbb{Z}) & \xrightarrow{u_E \smile (-)} & H^{2n}(\mathbb{D}E, \mathbb{S}E; \mathbb{Z}) \\ u_E & \longmapsto & u_E \smile u_E = -u_E \smile u_E \end{array}$$

Link to Diagram

Then $2u_E \smile u_E = 0$, making $2u_E = 0$ and thus $2e(E) = 0$.

■

Lemma 15.1.7 *(?).*
Given $X \to X$ and $F \to Y$, then we can form the bundle $E \times F \to X \times Y$. Writing $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$, we have

$$e(E \times F) = \pi_1^*(e(E)) \smile \pi_2^*(e(F)) = e(E) \otimes e(F).$$

Exercise 15.1.8 (?)
Prove this using the Kunneth formula.

Corollary 15.1.9 *(?).*
Let $E, E' \to X$ be two vector bundles. Then $e(E \oplus E') = e(E) \smile e(E')$.

Proof (?).
Consider the diagonal $\Delta : X \to X^{\times^2}$, then take the pullback:

$$\begin{array}{ccc} \Delta^*(E \times E') \cong E \oplus E' & \longrightarrow & E \times E' \\ \big\downarrow & \lrcorner & \big\downarrow \\ X^{\times^2} & \xrightarrow{\Delta} & X \end{array}$$

Link to Diagram

Then

$$\begin{aligned} e(E \oplus E') &= \Delta^*(e(E \times E')) \\ &= \Delta^*(\pi_1^*(e(E)) \smile \pi_2^*(e(E'))) \\ &= \Delta^*\pi_1^*(e(E)) \smile \Delta^*\pi_2^* e(E') \\ &= e(E) \smile e(E'). \end{aligned}$$

■

Corollary 15.1.10 *(?).*
If $E' = L \oplus E$ for L a trivial line bundle, $e(E') = 0$.

Proof (?).

$$e(E') = e(L) \smile e(E) = 0.$$

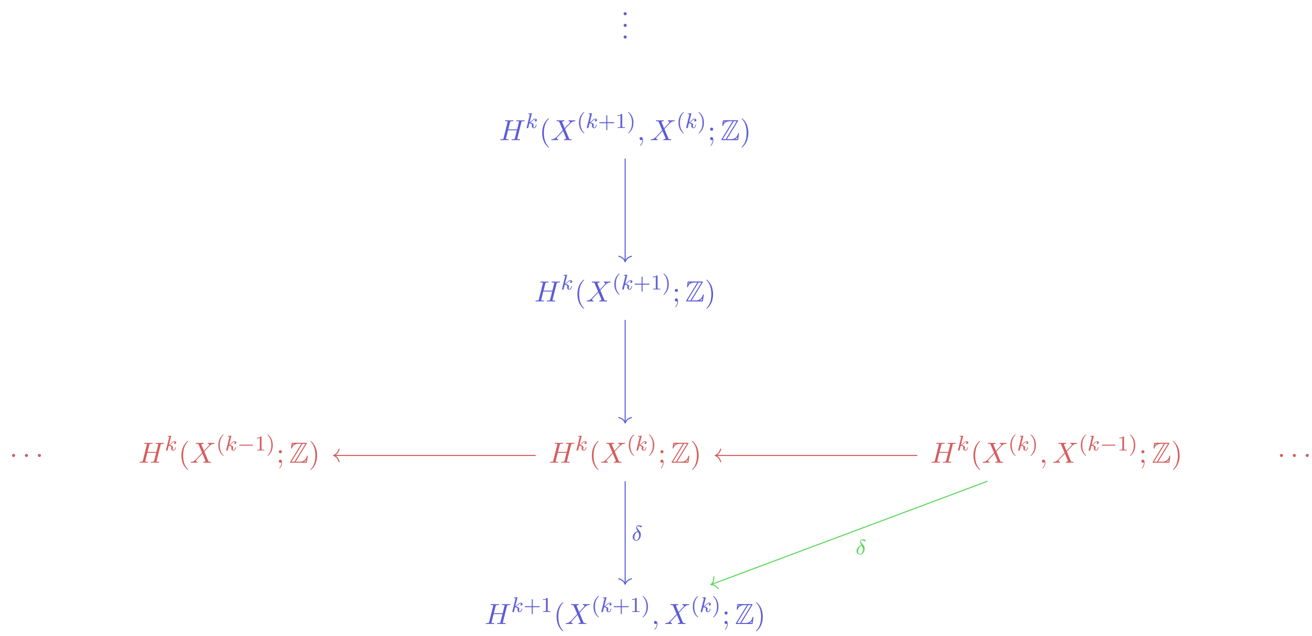
■

15.2 Obstruction Theory

Remark 15.2.1: The Euler class is the **obstruction** to finding a nonvanishing section over the n-skeleton of X. Note that if we have a line bundle, we automatically have a section. Conversely, a nonvanishing section will produce a line bundle summand.

Exercise 15.2.2 (?)
Describe the correspondence between line subbundles and nonvanishing global section.

Remark 15.2.3: Some review: let $X \in \mathsf{CW}$ and write $H^*(X;\mathbb{Z})$ for cellular homology. Then $C^k_{\text{cell}}(X) = \text{Hom}(H_k(X^{(k)}, X^{(k-1)}), \mathbb{Z}) = \text{Hom}(C^{\text{cell}}_k, \mathbb{Z})$. To produce the differential (green), use that the LES for the pairs (red and blue) are intertwined:

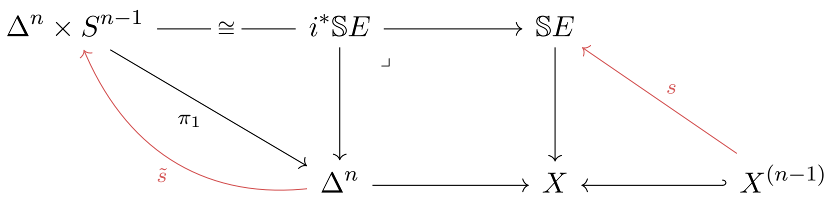


Link to Diagram

Thus we have a formula

$$\left\langle \delta\alpha,\ [\Delta^{k+1}] \right\rangle = \left\langle \alpha,\ \partial[\Delta^{k+1}] \right\rangle.$$

Remark 15.2.4: Recall that if the fibers of a bundle were n-connected, we could construct sections on $X^{(n-1)}$. Given any section, we can use a Riemannian metric to project onto norm 1 elements to get a section for the sphere bundle, say $s : X^{(n-1)} \to \mathbb{S}E$. Note that $\pi_i(S^{n-1}) = 0$ for $i \leq n-2$, so we can construct such a section. Let $i : \Delta^n \to X$ be the attaching map for some n-cell, then form the following pullback to get a trivial bundle, where we can pull back the section:



Link to Diagram

So s yields a map $\partial\Delta^n \cong S^{n-1} \to S^{n-1}$, so it has a degree. Then a cocycle can be defined as $[\Delta^n] \xrightarrow{e_s} \deg(\partial\Delta^n \to S^{n-1}) \in \mathbb{Z}$.

Lemma 15.2.5 *(?).*
e_S is independent of the choice of trivialization for the pullback.

Proof (?).
Since Δ^n is contractible, any two trivializations are homotopic. Thus the sections $s_T, s_{T'}$: $\partial\Delta^n \to S^{n-1}$ corresponding to two trivializations are homotopic. ■

Lemma 15.2.6 *(?).*
If s' is a different section over $X^{(n-1)}$, then $e_{s'} - e_s = \delta(\alpha)$ for some $\alpha \in C^{n-1}_{\text{cell}}(X)$.

Proof (?).
See phone notes. ■

16 | Tuesday, October 12

16.1 More Euler Class

Review how to construct the Euler and Thom classes.

Remark 16.1.1: Recall that the Euler class is the obstruction to finding a nowhere vanishing section on the n-skeleton. Given $E \to X \in \mathsf{Bun}(\mathrm{GL}_r)(X)^{\dim=n}$, we can form the sphere bundle $S^{n-1} \to \mathbb{S}E \to X$. Define a section $s : X \to \mathbb{S}E$ over $X^{(0)}$, then inductively if s is defined over $X^{(i-1)}$ for $i < n$, let $j : \Delta^i \to X^{(i)}$ be the attaching map for an i-cell. Note that Δ^i is contractible, and we can form a pullback

$$\begin{CD} \Delta^i \times S^{n-1} @>>> \mathbb{S}E \\ @VVV @VVV \\ \Delta^i @>{j}>> X^{(i)} \end{CD}$$

If s is defined over $X^{(i-1)} \subseteq X^{(i)}$, we have $s|_{\partial\Delta^i} : \partial\Delta^i \to S^{n-1}$. But $\partial\Delta^i \cong S^{i-1}$, and if $i \leq n-1$, $\pi_i S^{n-1} = 0$ so s extends over Δ^i. If $i = n$, note that $\pi_{n-1} S^{n-1} \cong \mathbb{Z}$, so we define $e_s(\Delta^n)$ to be the corresponding homotopy class of s, i.e. $e_s(\Delta^n)$ is the degree of this map between spheres.

We showed that

- e_s doesn't depend on the trivialization chosen, and
- If s' is another nowhere vanishing section on $X^{(n-1)}$, then $e_s - e_{s'} \in \operatorname{im} \delta$ for $\delta : C^{n-1}(X) \to C^n(X)$.
- If $\delta e_s = 0$, then $[e_s] \in H^n(X; \mathbb{Z})$. The content here is that the cochain descends to a cocycle.

For $i : \Delta^{-1} \to X^{(n-1)}$ and $\mathbb{S}E \to X^{(n-1)}$, form the pullback $i^*\mathbb{S}E$. Generally given two sections $s, s' : \Delta^{n-1} \to i^*\mathbb{S}E \cong \Delta^{n-1} \times S^{n-1}$, we can glue them to define a map $-\Delta^{n-1} \coprod_\partial \Delta^{n-1} \xrightarrow{-s \coprod s'} S^{n-1}$. The glued space is an S^{n-1}, so the degree of this map will be $\alpha^{\Delta^{n-1}}$

Lemma 16.1.2 ***(?).***
For any $\alpha \in C^{n-1}(X)$, there exists a section s' on $X^{(n-1)}$ such that $e_{s'} - e_s = \delta\alpha$.

Exercise 16.1.3 (?)
Prove this!

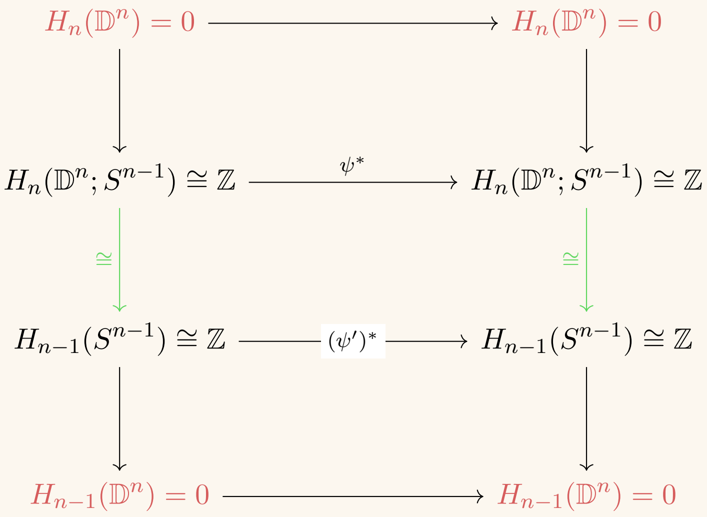
Proposition 16.1.4 ***(?).***
$[e_s] \in H^n(X)$ agrees with the Euler class.

Proof (?).
We can equivalently define e_s as follows: extend s to any section $\bar{s}$ of the disc bundle $\mathbb{D}E \to X$. Then for any n-cell Δ^n, we can form the composition:

$$\begin{array}{ccccc} & & \overset{\psi}{\dashrightarrow} & & \\ (\Delta^n, \partial\Delta^n) & \xrightarrow{\bar{s}} & (\Delta^n \times \mathbb{D}^n, (\partial\Delta^n) \times S^{n-1}) & \xrightarrow{p} & (\mathbb{D}^n, S^{n-1}) \\ \uparrow & & & & \uparrow \\ \partial\Delta^n & & \xrightarrow[\psi']{\hspace{6em}} & & S^{n-1} \end{array}$$

Link to Diagram

We can then define $e_s(\Delta^n) := \deg \psi$, and we claim that this also equals $\deg \psi'$. Look at the LES of a pair:



Link to Diagram

We may assume that X is n-dimensional since for $(X, X^{(n)})$ we have

$$H^n(X, X^{(n)}) = 0 \to H^n(X) \hookrightarrow H^n(X^n),$$

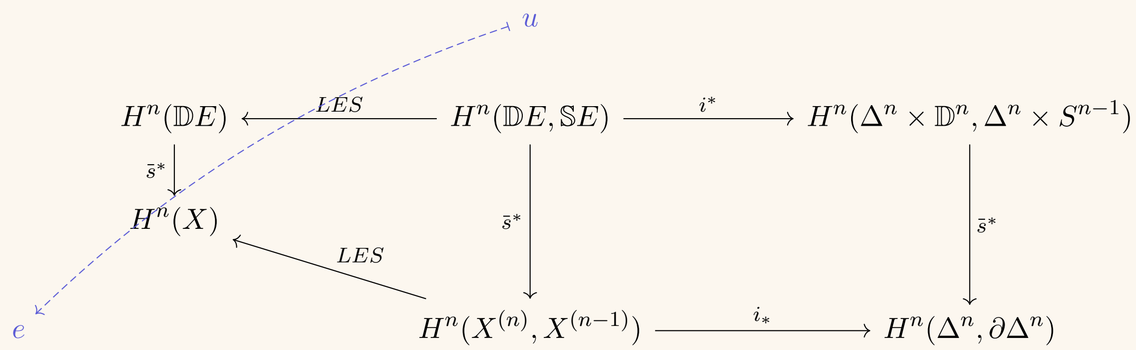
so anything equal in $H^n(X^{(n)})$ must be equal in $H^n(X)$. Fix a positive generator $\langle x \rangle = H^n(\mathbb{D}^n, S^{n-1})$ and $\langle y \rangle = H_n(\Delta^n, \partial\Delta^n)$ to be the fundamental class (positive generator). Then

$$e_s(\Delta^n) = \langle \bar{s}^* p^* x,\ y \rangle.$$

Consider the map $(\Delta^n \times \mathbb{D}^n, \partial\Delta^n \times S^{n-1}) \xrightarrow{p} (\mathbb{D}^n, S^{n-1})$, we have $p(\Delta^n \times S^{n-1}) = S^{n-1}$. Then we claim that $p^*(x) = u$ will be the Thom class. Using the attaching map $i : \Delta^n \to X^{(n)}$, we obtain

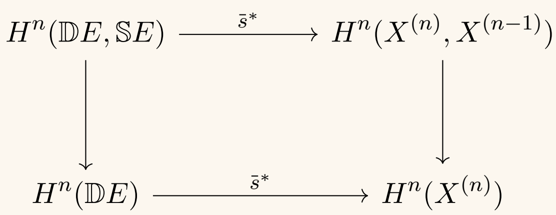
$$H^n(\mathbb{D}E, \mathbb{S}E) \xrightarrow{i^*} H^n(\Delta^n \times \mathbb{D}^n, \Delta^n \times S^{n-1}).$$

Use that $\bar{s} : X \to \mathbb{D}E$ induces an isomorphism $\bar{s}^* : H^n(\mathbb{D}E) \to H^n(X)$, inducing the same isomorphism as the zero section. So $(s')^* = \bar{s}^*$, i.e. any two sections of the disc bundle will be homotopic and thus induce equal maps in homology. Now doing exactly what we did for the Euler class, we get a diagram:



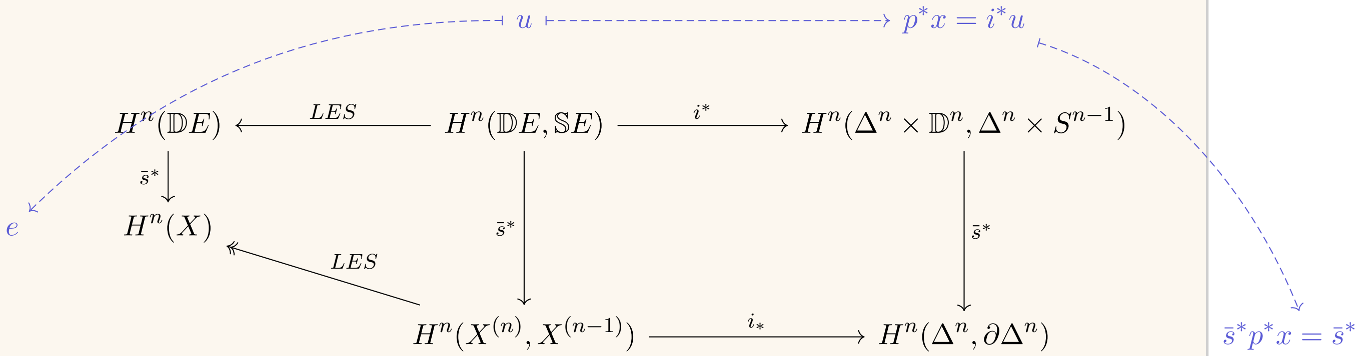
Link to Diagram

More directly, we can map the LESs to get a commutative square:



Link to Diagram

In other words, regard $e \in H^n_{\text{cell}}(X)$, so e corresponds to $[c] \in \underset{\mathbb{Z}\text{-Mod}}{\text{Hom}}(C^n(X), \mathbb{Z})$. We then claim that for $c \in H^n(X^{(n)}, X^{(n-1)})$, we have $e_s(\Delta^n) = i^* c(\Delta^n, \partial\Delta^n)$ using $i : (\Delta^n, \partial\Delta^n) \to (X^{(n)}, X^{(n-1)})$. Final diagram:



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■

16.2 Computations on Smooth Manifolds

Remark 16.2.1: Recall that a smooth structure on a manifold M is a collection (U, φ_U) where $U \subseteq M$ is open, $\varphi_U : U \to \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$, and for all U, V we have

$$\psi_{VU} := \varphi_V^{-1} \circ \varphi_U : \varphi_U(U \cap V) \to \varphi_V(U \cap V) \in C^\infty,$$

so there are all diffeomorphisms of open subsets of $\mathbb{R}^n$. A smooth atlas is maximal if not contained in any other smooth atlas, and two atlases are compatible if their union is again a smooth atlas. We say two smooth structures are equivalent if they are compatible.

Exercise 16.2.2 (?)
Show that any smooth manifold has a unique maximal smooth atlas.

Remark 16.2.3: Recall that for $f : \mathbb{R}^n \to \mathbb{R}^m$, $df \in \text{Mat}(m \times n; \mathbb{R})$ is given by $(df)_{ij} = \dfrac{\partial f_i}{\partial x_j}$. For any $p \in U \cap V$, we have $d_{\varphi_U(p)}\psi_{VU} \in \mathrm{GL}_n(\mathbb{R})$, so we get a map $d\psi_{VU} : U \cap V \to \mathrm{GL}_n(\mathbb{R})$. By the chain rule they satisfying the cocycle definition, so these glue to a vector bundle $\mathbf{T}M \to M$.

Exercise 16.2.4 (?)
Show that every other definition of $\mathbf{T}M$ coincides with this one.

17 | Thursday, October 14

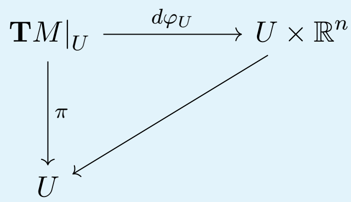
Remark 17.0.1: Some background from smooth manifolds: a map $f : M \to N$ of manifolds is **smooth** if for any smooth charts (U, φ_U) and (V, ψ_V) on M, N respectively, the transition map $\psi_V \circ f \circ \varphi_U^{-1}\mathbb{R}^m \to \mathbb{R}^n$ is smooth.

Exercise 17.0.2 (?)

1. Any smooth map $f : M \to N$ induces a bundle map $df : \mathbf{T}M \to f^*\mathbf{T}N$.
2. There is a canonical isomorphism $\mathbf{T}\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$.
3. Let (U, φ_U) be a chart on M, then show that there are trivializing charts for $\mathbf{T}M$:

$$d\varphi_U : \mathbf{T}M|_U \to \varphi_U^*|_{\mathbf{T}\mathbb{R}^n}U \cong U \times \mathbb{R}^n.$$

In particular, the following diagram commutes:



Link to Diagram

Exercise 17.0.3 (?)
Show that there is a canonical isomorphism

$$T_pM \xrightarrow{\sim} \mathrm{Der}(M, \mathbb{R}) \leq C^\infty(M, \mathbb{R}),$$

where a derivation at p is a smooth functional $v : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ such that $v(fg) = v(f)g(p) + f(p)v(g)$.

Remark 17.0.4: $N \subseteq M$ is a **smooth submanifold** if for any $p \in N$ there exists a smooth chat (U, φ) on M such that $p \in U$ and $\varphi(U \cap N) \subseteq \mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^n$.

Exercise 17.0.5 (?)
Show that this is equivalent to $N = f(\tilde{N})$ where $\tilde{N}$ is some smooth manifold and $f : \tilde{N} \to M$

is a smooth embedding and $d_p f : \mathbf{T}_p \tilde{N} \hookrightarrow F_{f(p)} M$ is injective for all $p \in \tilde{N}$.

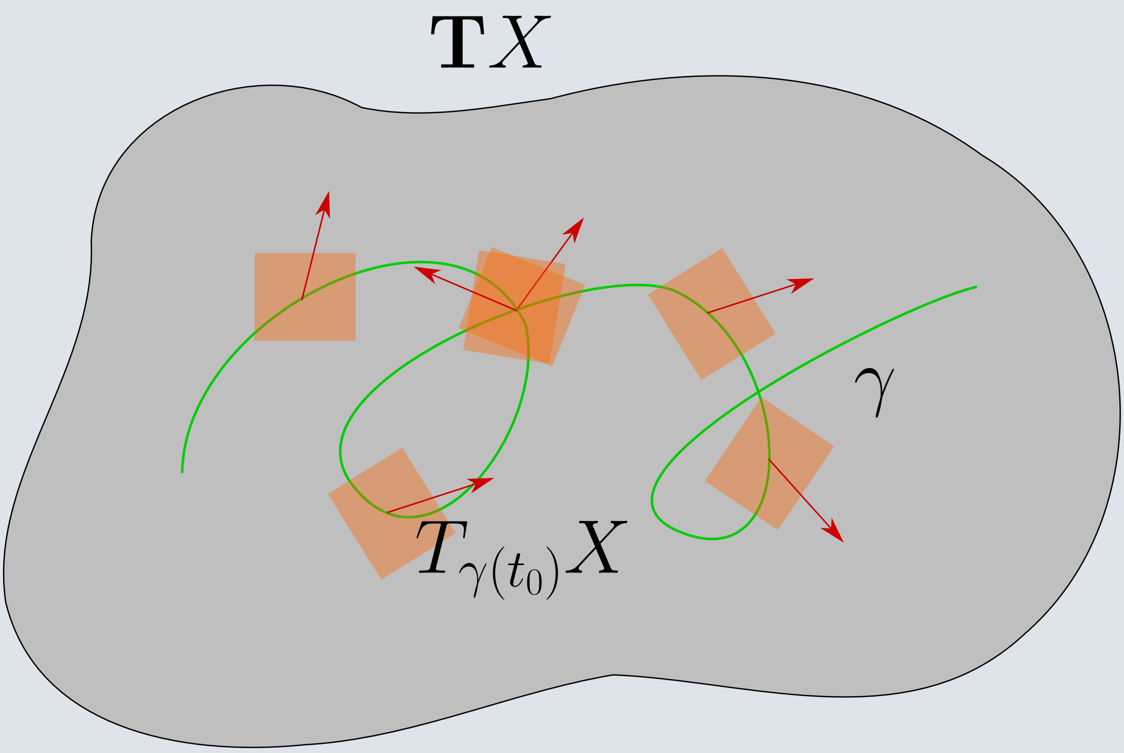
Definition 17.0.6 (?)
Given a Riemannian metric on $\mathbf{T}M$ and a smooth submanifold $N \subseteq M$, let νN denote $(\mathbf{T}N)^\perp \subseteq \mathbf{T}M|_N$ for the orthogonal complement of $\mathbf{T}N$. This is a vector bundle $\nu N \to N$.

Exercise 17.0.7 (?)
Show that up to a canonical isomorphism, this is independent of the choice of Riemannian metric.

Definition 17.0.8 (?)
Given a curve $\gamma : I \to M$, then $\gamma'(t) \in \mathbf{T}_{\gamma(t)} M$ is the following derivation:

$$\gamma'(t).f := \lim_{h \to 0} \frac{f(\gamma(t+h)) - f(\gamma(t))}{h}.$$

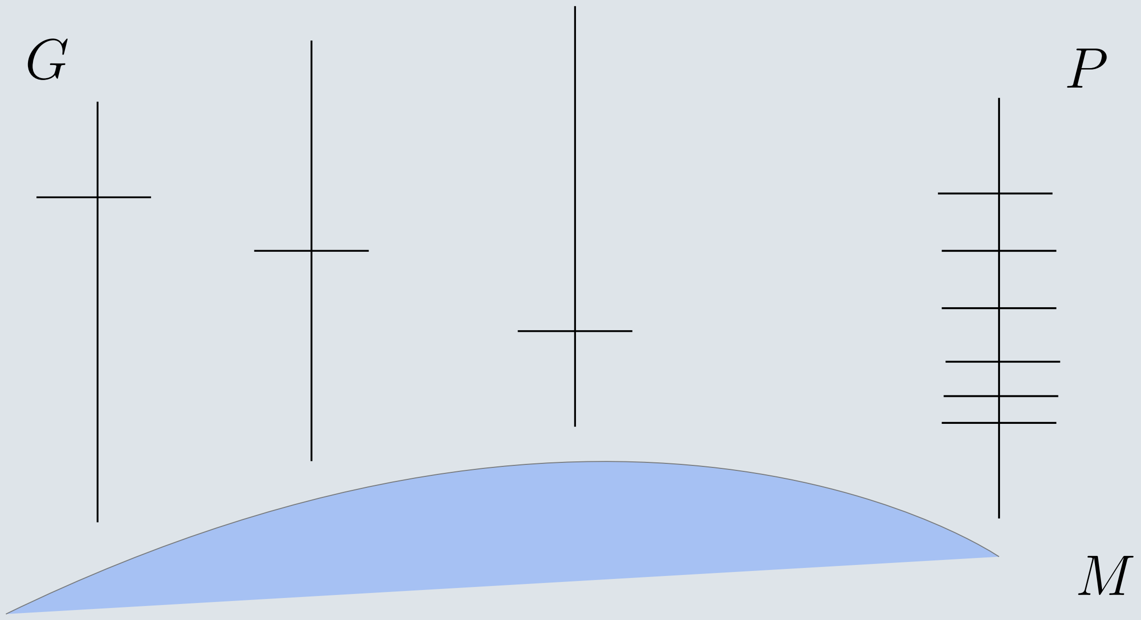
A **vector field** on M is a section of $\mathbf{T}M$, and a vector field along γ is a section of $\gamma^* \mathbf{T}M$:



Example 17.0.9*(?)*: $\gamma'(t)$ is a vector field along γ.

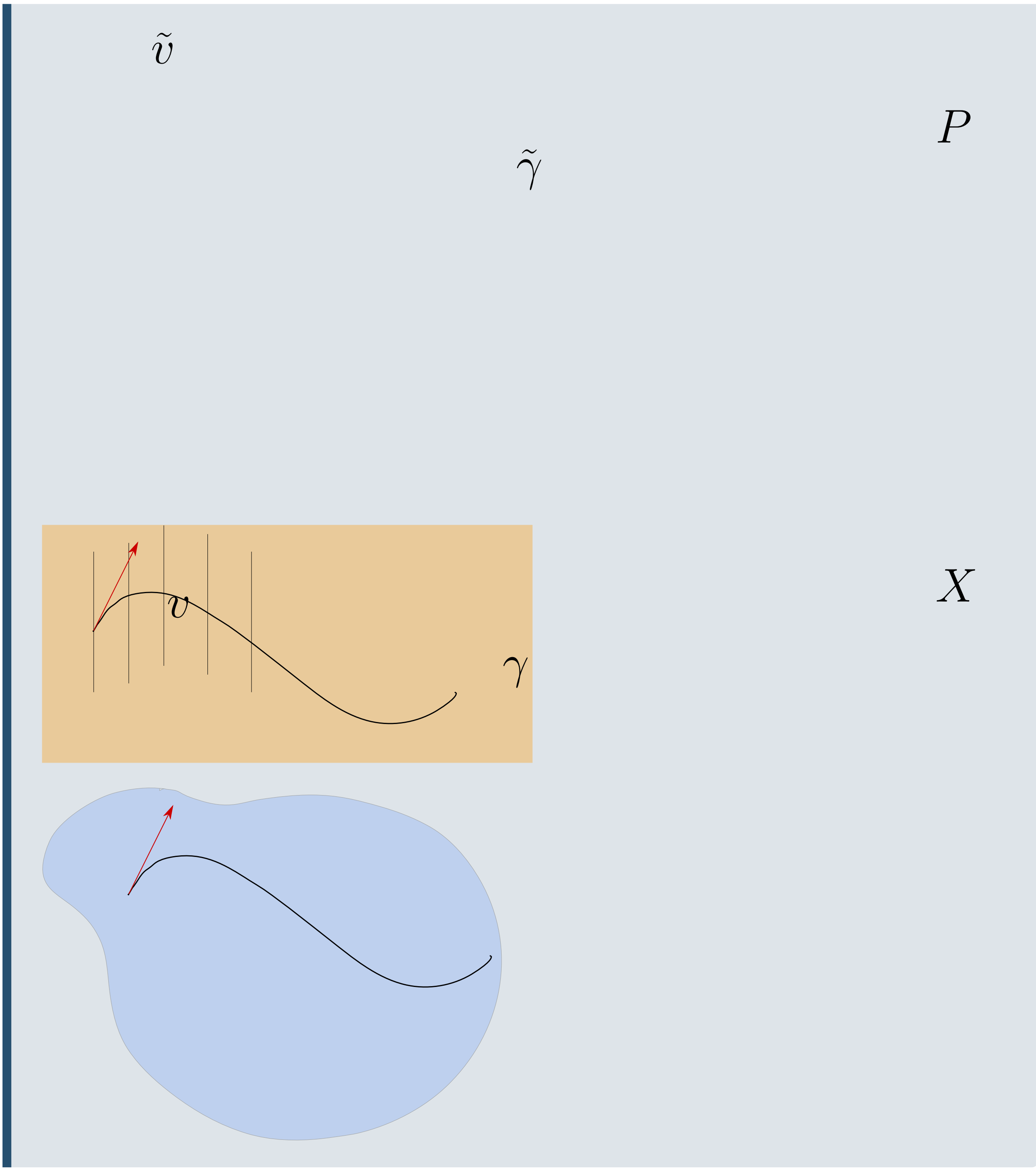
Remark 17.0.10: A **Lie group** is a group G with the structure of a smooth manifold where multiplication and inversion are smooth self-diffeomorphisms.

Definition 17.0.11 (?)
Given a Lie group G and smooth principal G-bundle $P \xrightarrow{\pi} M^n$, a **connection** on P is a choice of subspaces $\xi_p \subseteq T_pP$ for all $p \in P$ such that $d\pi : \xi_p \to \mathbf{T}_{\pi(p)}M$ is an isomorphisms for all p, where ξ_p is the horizontal subspace:



Remark 17.0.12: Given a connection, curves γ in M can be lifted to curves $\tilde{\gamma}$ in P in such a way that tangents of $\tilde{\gamma}$ are projected to tangents of γ:

Definition 17.0.13 (?)
Given a connection on P and a smooth path $\gamma : I \to M$, a horizontal lift $\tilde{\gamma}$ of γ is a path $\tilde{\gamma} : I \to P$ such that $\tilde{\gamma}'(t) \in \xi_{\tilde{\gamma}(t)}$ and $\pi \circ \tilde{\gamma} = \gamma$.



Lemma 17.0.14 ***(?).***
Given a smooth path γ and $\tilde{\gamma}(0)$, there is a unique horizontal lift $\tilde{\gamma}$ starting at $\tilde{\gamma}(0)$.

Remark 17.0.15: Consider $\text{Frame}(\mathbf{T}M)$, then recall that $\mathbf{T}M = \text{Frame}(\mathbf{T}M) \overset{\text{GL}_n(\mathbb{R})}{\times} \mathbb{R}^n$ by the mixing construction. Given a connection on $\text{Frame}(\mathbf{T}M)$, we can parallel transport vectors in $\mathbf{T}M$ along curves. This comes from taking $[F_0, v_0]$ and evolving F_0 along $F_t \coloneqq \tilde{\gamma}(t)$, choosing pairs $[F_t, v_0]$ for all t, and ending at $[F_t, v_0]$.

Exercise 17.0.16 (?)
Show that parallel transport yields a well-defined map $\mathbf{T}_{\gamma(0)}M \to \mathbf{T}_{\gamma(1))}M$.

Theorem 17.0.17 *(?).*
Given a Riemannian metric on $\mathbf{T}M$, there is a canonical connection ∇, the **Levi-Cevita** connection, which is torsionfree (i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$) for X, Y vector fields and $\nabla_X Y$ denotes parallel transporting Y along X.

Definition 17.0.18 (?)
A curve γ is a **geodesic** if $\gamma'(t)$ is parallel.

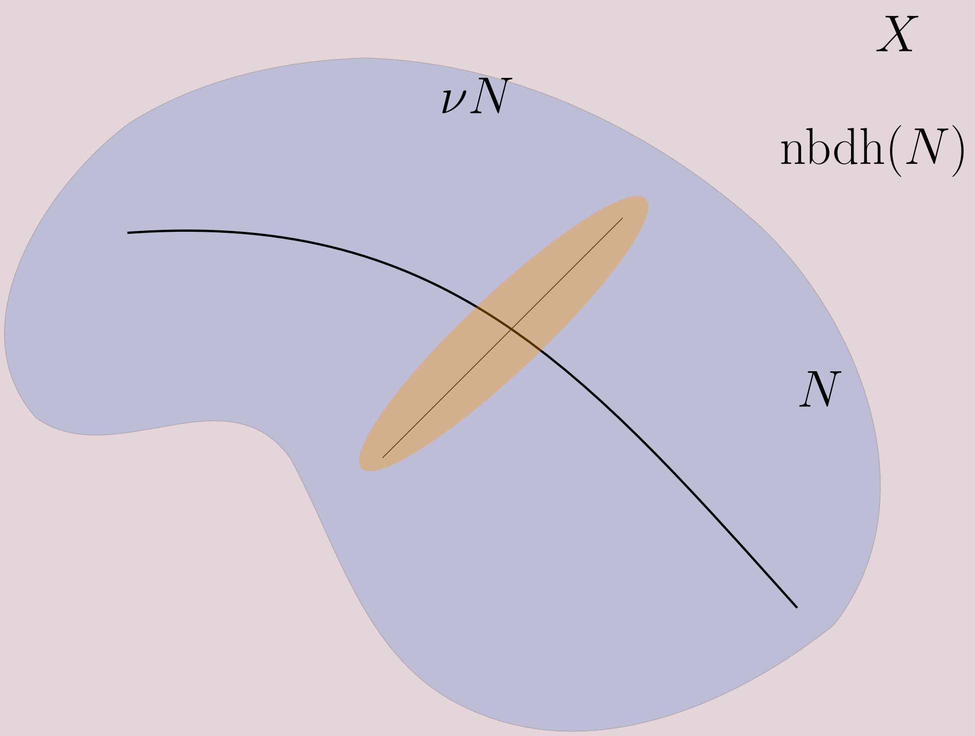
Theorem 17.0.19 *(?).*
For any $v \in \mathbf{T}M$, there is a unique geodesic γ_v with $\gamma_v'(0) = v$.

Definition 17.0.20 (Exponential map)
There is a map

$$\begin{aligned} \exp : T_p M &\to M \\ v &\mapsto \gamma_v(1). \end{aligned}$$

This is well-defined for v of small norm, and for all v if M is closed.

Theorem 17.0.21 *(?).*
If $N \leq M$ is a submanifold, then exp defines a diffeomorphism from a neighborhood of the zero section in νN to a neighborhood of N.



18 | Tuesday, October 19

Remark 18.0.1: Today: computations involving the Euler class.

Theorem 18.0.2 *(?)*.
Let $M \in \mathsf{smMfd}^{\dim = n}$ be closed and oriented, then noting that $e(\mathbf{T}M) \in H^n(M; \mathbb{Z})$, [a]

$$\langle e(\mathbf{T}M),\ [M] \rangle = \chi_{\mathsf{Top}}(M),$$

the topological Euler characteristic of M.

[a] $\mathbb{Z}$ can be replaced with $\mathbb{Z}/2$ if M is not oriented.

Theorem 18.0.3 *(?)*.
Let $M \in \mathsf{smMfd}^{\dim = n}$ closed and $E \xrightarrow{\pi} M \in \mathsf{Bun}\,(\mathrm{GL}_k)(X)$ oriented. Suppose a generic section s of E, i.e. $\operatorname{im} s := \left\{ s(x) \in E \;\middle|\; x \in M \right\} \subseteq E$, intersects $M \subseteq E$ transversally. Then

- $z := s^{-1}(0)$ is a submanifold of M is a submanifold of M
- $ds|_Z : \nu Z \xrightarrow{\sim} E|_Z$

Exercise 18.0.4 (?)
Prove this.

Corollary 18.0.5 *(?).*
$e(E) = \mathrm{PD}([Z])$, where $[Z] \in H_{n-k}(M)$.

Claim: The second theorem implies the first.

Proof (?).
Note that a section of $\mathbf{T}M$ is a vector field. Use Morse functions $f : M \to \mathbb{R}$:

- $\mathrm{crit}(f) = \left\{ p \in M \;\middle|\; df_p = 0 \right\}$.
- $p \in \mathrm{crit}(f)$ is **nondegenerate** if the Hessian $H_p(f) \coloneqq \left(\frac{\partial^2}{\partial x_i \partial x_j} \right)(p)$ is nonsingular, i.e. $\det H_p \neq 0$.
- f is **Morse** if all critical points are nondegenerate.
- If $p \in \mathrm{crit}(f)$ is nondegenerate, then $\mathrm{Ind}_p(f)$ is the number of negative eigenvalues of H_p.
- f Morse on M induces a CW complex structure with exactly one k-cell corresponding to each index k critical point $p \in \mathrm{Crit}(f)_k$. Thus we can compute $\chi(M) = \sum_k (-1)^k \dim C_k^{\mathrm{cell}} = \sum_k (-1)^k \# \mathrm{Crit}(f)_k$.

■

Proof .
f Morse induces a gradient vector field on M: picking a Riemannian metric g on M, define $df(-) \coloneqq g(\mathrm{grad}\, f, -)$ to get a section $\mathrm{grad}\, f : M \to \mathbf{T}M$. Note that if $p \in \mathrm{crit}(f)$ then $d_p f = 0$ since $\mathrm{grad}_p\, f = 0$. So the vector field df vanishes at $\mathrm{crit}(f)$ and $(\mathrm{grad}\, f)^{-1}(0)$

Exercise *(?)*
Show that the sign of a zero of the gradient vector field is $(-1)^{\mathrm{Ind}_p(f)}$.

So taking $Z = \mathrm{crit}(f)$, we can write

$$e(\mathbf{T}M) = \mathrm{PD}\left[\sum_{p \in \mathrm{crit}(f)} (-1)^{\mathrm{Ind}_p(f)} [x] \right],$$

where $[x]$ is the dual of a generator of $H_0(M)$ Then

$$\langle e(\mathbf{T}M),\ [M] \rangle = \sum_{p \in \mathrm{crit}(f)} (-1)^{\mathrm{Ind}_p(f)} = \# \mathrm{crit}_{\mathrm{even}}(f) - \# \mathrm{crit}_{\mathrm{odd}}(f) = \chi_{\mathsf{Top}}(M).$$

■

Remark 18.0.7: Given $N^n \hookrightarrow M^m$ an oriented closed submanifold with M closed, we can consider Thom class of the disc/sphere bundles. We identify the disc bundle $\mathbb{D}\nu N$ with a tubular neighborhood of N in M and apply excision to get the following:

$$\begin{array}{ccccc}
u_{\nu N} \in H^{m-n}(\mathbb{D}\nu N, \mathbb{S}\nu n) & \xrightarrow{\cong} & H^{m-n}(M, M \setminus N) & \longrightarrow & H^{m-n}(M) \\
& \searrow^{\cong} & \big\downarrow \text{excision} \cong & & \\
& & H^{m-n}(\mathbb{D}\nu N, \mathbb{D}\nu N \setminus N) & &
\end{array}$$

Link to Diagram

We can also consider the composition:

$$[N] \in H_n(N) \xrightarrow{i_*} H_n(M) \xrightarrow{\text{PD}} H^{m-n}(M)$$

Link to Diagram

Claim: These two classes are equal in $H^{m-n}(M)$.

Proof (?).
Consider the following:

$$\begin{array}{ccccc}
u_{\nu N} & \dashrightarrow & & & \\
\langle u_{\nu N} \rangle = H^{m-n}(\mathbb{D}\nu n, \mathbb{S}\nu n) & \xleftrightarrow{\text{PD}, \cong} & H_n(\mathbb{D}\nu N) & & \\
\big\downarrow \cong & & \big\downarrow \cong & \searrow & \\
H^{m-n}(M, M \setminus N) & & H_n(N) = \langle [N] \rangle & & \pm [N] \\
\big\downarrow & & \big\downarrow & & \\
H^{m-n}(M) & \xleftrightarrow{\text{PD}(-) = [M] \frown (-), \cong} & H_m(M) & &
\end{array}$$

Link to Diagram

The claim is that $u_{\nu N}$ is mapped to the *positive* generator, so the both paths from $H^{m-n}(\mathbb{D}\nu N, \mathbb{S}\nu N) \to H^{m-n}(M)$ agree. ■

Remark 18.0.8: An aside on cup/cap products. The cap product is a map:

$$\frown : H_m(X) \otimes_{\mathbb{Z}} H^i(X) \to H_{m-i}(X)$$

.

On chains, it's given by

$$\begin{aligned} \frown C_m(X) \otimes_{\mathbb{Z}} C^i(X) &\to C_{m-i}(X) \\ (\sigma, \varphi) &\mapsto \varphi\left(\sigma|_{[v_0, \cdots, v_i]}\right) \sigma|_{v_{i+1}, \cdots, v_m}. \end{aligned}$$

Then $\mathrm{PD}(-) \coloneqq [X] \frown (-)$.

The cup product is a map

$$\begin{aligned} \smile : H^i(X) \otimes_{\mathbb{Z}} H^j(X) &\to H^{i+j}(X) \\ C^i(X) \otimes_{\mathbb{Z}} C^j(X) &\to C^{i+j}(X) \\ (\varphi, \psi) &\mapsto \left(\sigma \mapsto \varphi\left(\sigma|_{[v_0, \cdots, v_i]}\right) \psi\left(\sigma|_{[v_{i+1}, \cdots, v_{i+j}]}\right)\right). \end{aligned}$$

Then fixing any element yields a map $\alpha \smile (-) : C^j(X) \to C^{i+j}(X)$, which is induced by a map $\varphi \frown (-) : C_{i+j}(X) \to C_i(X)$.

19 | Thursday, October 21

Remark 19.0.1: Recall that if $N^n \leq M^n$ is a submanifold, we have the following diagram:

$$\begin{array}{ccccc} u_{\nu N} \quad H^{m-n}(\mathbb{D}\nu N, \mathbb{S}\nu N) & \xrightarrow[\cong,\ \text{excision}]{} & H^{m-n}(M, M \setminus N) & \xrightarrow[j^*]{} & H^{m-n}(M) \\ \uparrow \text{PD} & & & & \uparrow \text{PD} \\ H_n(\mathbb{D}\nu N) & \xrightarrow{\cong} & H_n(N) & \xrightarrow{i_*} & H_n(M) \\ & & [N] & & \end{array}$$

Link to Diagram

Then the Thom class $u_{\nu N} \in H^{m-n}(\mathbb{D}\nu N, \mathbb{S}\nu N) \to H^{m-n}(\mathbb{D}\nu N_x, \mathbb{S}\nu N_x)$ is mapped to the generator specified by the orientations on fibers.

Theorem 19.0.2 *(?).*
For $A^i \pitchfork B^j \leq X$ are smooth oriented submanifolds intersecting transversally, then

$$\begin{gathered} \mathrm{PD}([A]) \smile \mathrm{PD}([B]) = \mathrm{PD}([A \frown B]) \\ H^{n-i}(X) \times H^{n-j}(X) \to H^{2n-i-j}(X). \end{gathered}$$

Remark 19.0.3: Then

$$[(\mathbb{D}\nu N_x, \mathbb{S}\nu N_x) \frown N] = [\{x\}],$$

which is the positive generator. So

$$\mathrm{PD}[(\mathbb{D}\nu N_x, \mathbb{S}\nu N_x)] \smile \mathrm{PD}[N] = \mathrm{PD}[x].$$

Now we can cap this to obtain

$$\begin{aligned}[\mathbb{D}\nu N, \mathbb{S}\nu N] \frown (\mathrm{PD}[(\mathbb{D}\nu N_x, \mathbb{S}\nu N_x)] \smile \mathrm{PD}[N]) &= ([\mathbb{D}\nu N, \mathbb{S}\nu N] \frown \mathrm{PD}[\mathbb{D}\nu N_x, \mathbb{S}\nu N_x]) \frown \mathrm{PD}[N] \\ &= [\mathbb{D}\nu N, \mathbb{S}\nu N] \frown \mathrm{PD}[x] \\ &= 1,\end{aligned}$$

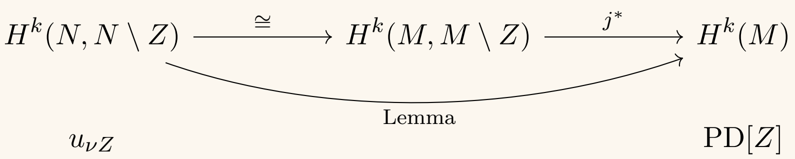
where we've used that $\langle \mathrm{PD}[x], [\mathbb{D}\nu N, \mathbb{S}\nu N]\rangle$. So $\mathrm{PD}[N]$ is j^* of the Thom class of νN.

Theorem 19.0.4*(?)*.
Let $M \in \mathsf{smMfd}^n$ be closed and oriented and $E \xrightarrow{\pi} M$ a k-dimensional oriented vector bundle. Consider a generic section s of E, so $\mathrm{im}(s) \pitchfork M$ in E. Then

$$e(E) = \mathrm{PD}[Z] \qquad Z := \mathrm{im}(s) \cap M = s^{-1}(0).$$

Remark 19.0.5: Recall that $s^{-1}(0) \leq M$ is a smooth submanifold, and $ds|_Z : \nu Z \xrightarrow{\sim} E|_Z$, and since this orients νZ this orients Z as well.

Proof (?).
Let N be a tubular neighborhood of Z in M, such that $N \cong \mathbb{D}\nu Z$. By the lemma, we have two maps



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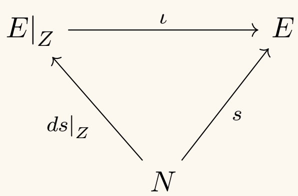
on the other hand, since $Z \overset{i}{\hookrightarrow} M$, we get



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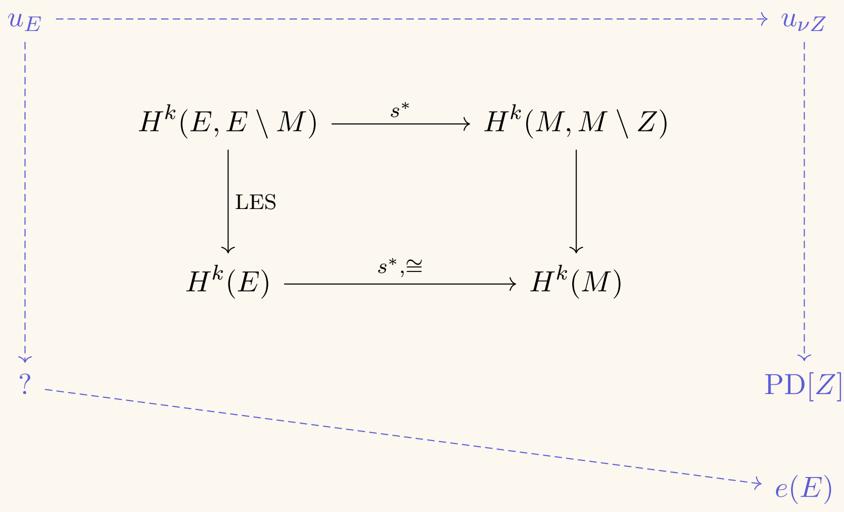
Exercise *(?)*
Show $ds|_Z \simeq s$ are homotopic sections.

Then $(ds|_Z)^* \circ i^*(u_E) = u_{\nu Z}$, so $u_{\nu Z} = s^* u_E$, and we have



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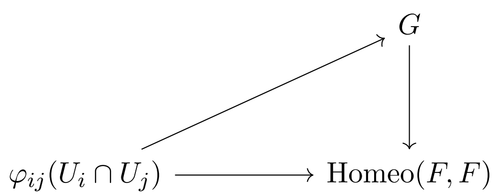
Since the diagram commutes, we get $e(E) = \mathrm{PD}[Z]$.

■

Remark 19.0.7: The Euler class for a bundle $E \to X \in \mathsf{CW}$ is the obstruction to finding a nowhere vanishing section on $X^{(n)}$ for $n := \dim E$, and $e(E) = 0$ iff there is a nowhere vanishing section on $X^{(n)}$. For a smooth manifold M and $E \xrightarrow{\pi} M$ with $\dim M = n, \dim E = k$ and $s : M \to E$ a section, then $\mathrm{PD}[s^{-1}(0)] = e[E]$ since $\dim s^{-1}(0) = n - k$. If $E = \mathbf{T}M$, then $e(\mathbf{T}M) = 0$ implies $\chi(M) = 0$ and there exists a nowhere vanishing vector field.

20 | Tuesday, October 26

Remark 20.0.1: For $F \to E \xrightarrow{\pi} B$ a fiber bundle with a G-structure, we have maps:



Link to Diagram

A vector bundle was an $\mathbb{R}^n$ bundle with a GL_n-structure, having a Riemannian metric meant having an O_n structure, and orientability meant having a GL_n^+ structure. A **principal** G-bundle was $P \to B$ where P has a G-action acting freely and transitively on each fiber. Taking the frame bundle sent vector bundles to principal GL_n-bundles.

We had a construction sending fiber bundles to principal G-bundles, namely the **clutching** construction: given $\mathcal{U} \rightrightarrows X$ and $\varphi_{ij} : U_i \cap U_j \to G$ satisfying the cocycle condition $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$, we get a fiber bundle with fiber F and transition functions for any $F \in G\text{-}\mathsf{Spaces}$.

Remark 20.0.2: On universal bundles and classifying spaces: for $X \in \mathsf{CW}$,

$$\begin{aligned} \mathrm{Prin}\,\mathsf{Bun}\,(G)(X) &\xrightarrow{\sim} [X, \mathbf{B}G] \\ f^*EG &\leftarrow f. \end{aligned}$$

We noted

- G discrete implies $\mathbf{B}G \simeq K(G,1)$
- $K(C_2, 1) = \mathbb{RP}^\infty$ with $EC_2 = S^\infty$
- $\mathbf{B}U_1 = \mathbb{CP}^\infty$ with $EU_1 = S^\infty$
- $\mathbf{B}\mathrm{O}_n = \mathrm{Gr}_n(\mathbb{R}^\infty)$, with $\mathsf{EO}_n = V_n(\mathbb{R}^\infty)$ the Stiefel manifold of n-dimensional frames.
- $\mathbf{B}\mathrm{SO}_n$ are oriented n-planes in $\mathbb{R}^\infty$.
- For $H \leq G$, $EH = EG$ and $\mathbf{B}H = EG/H$.

We had a canonical bundle $\gamma_n \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ whose fiber above $W \leq \mathbb{R}^\infty$ was exactly W, so $\gamma_n = \{(W, w \in W) \subseteq \mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty\}$. Every vector bundle $E \to X$ is of the form $f^*\gamma_n \to X$ for some $f \in [X, \mathrm{Gr}_n(\mathbb{R}^\infty)]$, and similarly O_n bundles are pullbacks of $V_n(\mathbb{R}^\infty) \to \mathrm{Gr}_n(\mathbb{R}^\infty)$.

Example 20.0.3*(?)*: A useful application: characteristic classes. For any $c \in H^d(\mathbf{B}\mathrm{O}_n)$ can be pulled back:

$$\begin{aligned} H^d(X) &\to \mathsf{Vect}_n(X) \\ f^*c &\leftarrow f \in [X, \mathbf{B}\mathrm{O}_n]. \end{aligned}$$

Noting that $\mathrm{BU}_1 = \mathbb{CP}^\infty$ and $H^2(\mathbb{CP}^\infty) = \mathbb{Z}\langle c_1 \rangle$, so any line bundle $L \to X$ $f : X \to \mathrm{BU}_1$ yields $c_1(L) \coloneqq f^*c_1 \in H^2(X)$, the **first Chern class**. Noting $H^2(X; \mathbb{Z}) \cong [X, K(\mathbb{Z}, 2)] \cong [X, \mathbb{CP}^\infty]$.

Remark 20.0.4: Note: why $c_1 = 0$ in symplectic settings, related to Maslov index and ensures that the dimension of the relevant moduli space is zero.

21 | Thursday, October 28

Remark 21.0.1: The Euler class is *natural* in the following sense: for $E \to X$ with $\dim E = n$ and $X \in \mathsf{CW}$, we can write

$$\begin{array}{ccc} E \cong f^*\gamma_n & \longrightarrow & \gamma_n \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{f} & \mathbf{BO}_n \end{array}$$

Link to Diagram

Then naturality is the following equality:

$$e(E) = e(f^*\gamma_n) = f^*(e(\gamma_n)).$$

Recall the Thom isomorphism theorem: for an oriented $E \in \mathsf{Bun}(\mathrm{GL}_r)^n$, we have a disk bundle $\mathbb{D}E$ and sphere bundle $\mathbb{S}E$, and

$$H^j(\mathbb{D}E; \mathbb{S}E) \cong \begin{cases} 0 & j < n \\ \mathbb{Z} & j = n \\ H^{j-n}(\mathbb{D}E) & j > n \end{cases},$$

noting that we can take $H^n(\mathbb{D}E, \mathbb{S}E) \to H^n(\mathbb{D}E_x, \mathbb{S}E_x) \xrightarrow{\sim} H^n(D^n, S^{n-1})$, where the target has a canonical positive generator. The preimage of this generator is u_E, the Thom class. The isomorphisms in the range $j > n$ are given by $- \smile u_E$.

We had a claim:

$$\begin{aligned} H^n(\mathbb{D}E, \mathbb{S}E) &\to H^n(\mathbb{D}E) \cong H^n(X) \\ u_E &\mapsto e(E). \end{aligned}$$

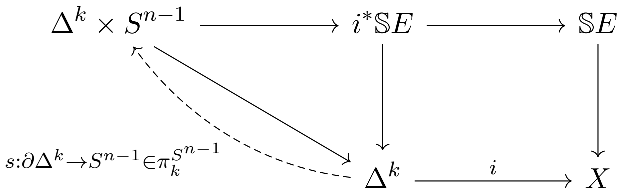
Remark 21.0.2: On the **Gysin sequence**: use the bundle $S^{n-1} \hookrightarrow \mathbb{S}E \to X$ and the LES

$$\cdots \to H^{j-1}(\mathbb{S}E) \xrightarrow{\delta} H^{j-n}(X) \xrightarrow{(-)\smile e(E)} H^j(X) \to H^j(\mathbb{S}E) \to \cdots.$$

The connecting map δ comes from the Thom isomorphisms $H^j(\mathbb{D}E, \mathbb{S}E) \xrightarrow{\sim} H^{j-n}(\mathbb{D}E)$ and splicing the LES of the pair $(\mathbb{D}E, \mathbb{S}E)$.

Proposition 21.0.3 *(?)*.
If E is odd dimensional, then $2e(E) = 0$. Note that $e(E_1 \oplus E_2) - e(E_1) \smile e(E_2)$, and if E has a nonvanishing section then $e(E) = 0$.

Remark 21.0.4: So $e(E)$ is the obstruction to finding a nonvanishing section over the n-skeleton, where $\dim E = n$. Given a nonvanishing section over $X^{(k-1)}$, consider extending it over $X^{(k)}$. We can use the cellular attaching maps to write



Link to Diagram

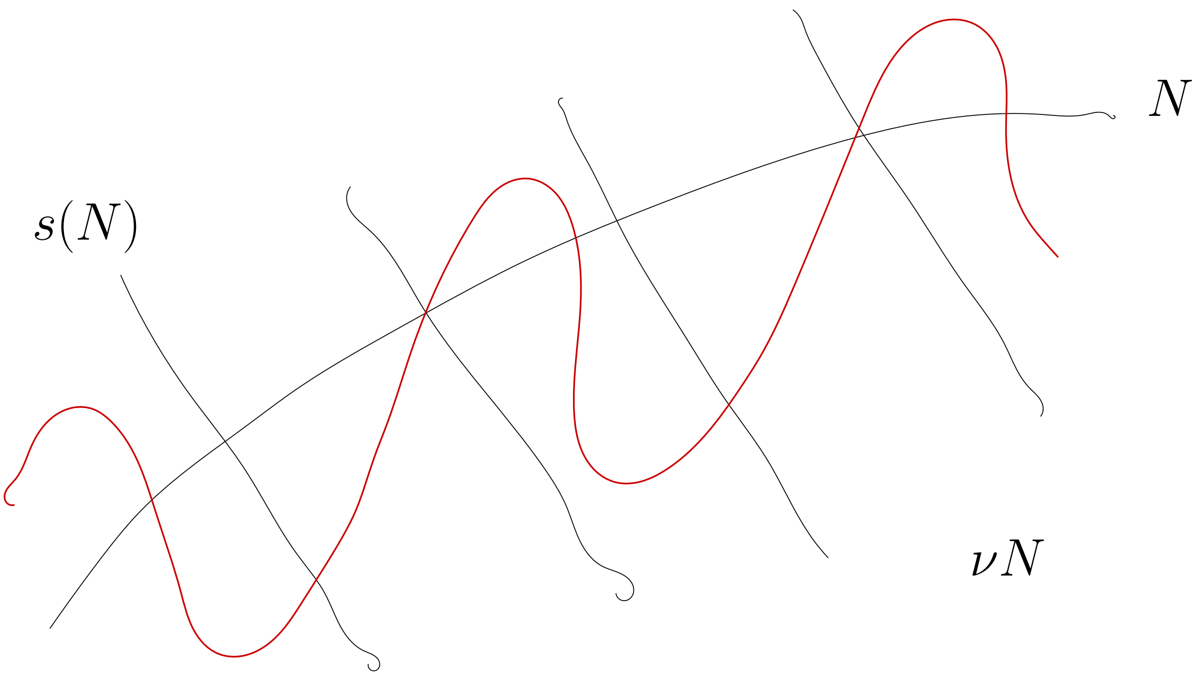
- If $s \in \pi_k(S^n)$ and $k < n$, it is nullhomotopic and can thus be extended from $\partial\Delta^k$ to Δ^k.
- If $k = n$, we get $s_* : H_{n-1}(\partial\Delta^n) \to H_{n-1}(S^{n-1})$ where $a \mapsto na$ with $n := \deg(s_*)$. This yields an cellular k-cochain $C^k \to \mathbb{Z}$ where $\Delta^n \mapsto \deg s_*$, and this represents the Euler class.

Remark 21.0.5: For smooth manifolds, we have natural bundles $\mathbf{T}X$ and νX to work with. Then

$$\langle e(\mathbf{T}M),\ [M]\rangle = \chi(M),$$

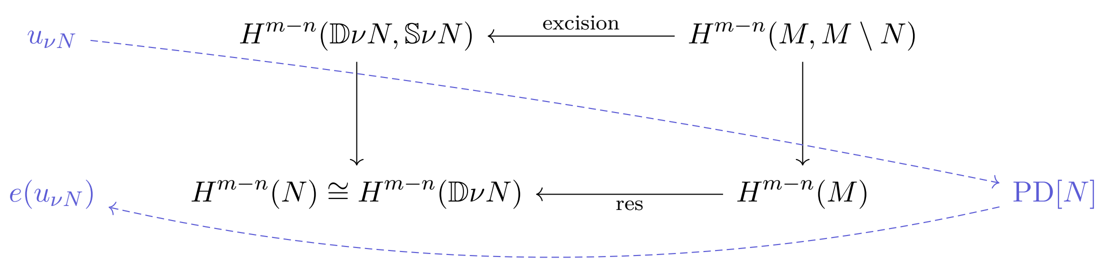
using that $e(\mathbf{T}M) \in H^n(M) \cong \mathbb{Z}$. In general, so $E \to M^n$ with a generic section s, writing $Z := s^{-1}(0) \leq E$ a submanifold of dimension $n - k$ when $\dim Z = k$, we have $[Z] \in H_{n-k}(M)$ and $\mathrm{PD}[Z] = e(E)$.

Example 21.0.6 ***(?):*** Consider $N^n \hookrightarrow M^{2n}$, then $\nu N \to N \in \mathsf{Bun}\left(\mathrm{GL}_r\right)^n(N)$. To compute $e(\nu N)$, pick a generic section:



Then $e(\nu N) = N \cdot N \in H_n(M)$ is the self-intersection number.

Remark 21.0.7: For $N^n \subseteq M^m$, we have



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So we can compute $e(E)$ for $\mathbf{T}M$ and νN.

Remark 21.0.8: Next topics: Chern and Stiefel-Whitney classes.

Theorem 21.0.9 *(?)*.
For $E \to X$ a real vector bundle, there are characteristic classes $w_i(E) \in H^i(X; C_2)$ and $w := 1 + w_1(E) + w_2(E) + \cdots \in H^*(E)$, the **Stiefel-Whitney classes**, satisfying the following properties:

1. Naturality: $w_i(f^*(E)) = f^*(w_i(E))$
2. $w(E_1 \bigoplus E_2) = w(E_1) \smile w(E_2)$
3. $w_i(E) = 0$ if $i > \dim_{\mathbb{R}} E$.
4. If $E \to \mathbb{RP}^\infty$ is the canonical line bundle, then $w_1(E) = \alpha$ for $\langle \alpha \rangle = H^1(\mathbb{RP}^\infty; C_2)$.

Moreover, these properties characterize $w_i(E)$ uniquely. For complex vector bundles, there are $c_i(E) \in H^{2i}(X; \mathbb{Z})$ and $c_{(}E)$, the **Chern classes**, which satisfy the same properties with $\mathbb{C}$ instead of $\mathbb{R}$ and $\langle \alpha \rangle = H^2(\mathbb{CP}^\infty; \mathbb{Z})$.

Remark 21.0.10: Next time: existence and uniqueness.

22 Stiefel-Whitney and Chern Classes (Tuesday, November 02)

Remark 22.0.1: The Stiefel-Whitney classes $w_i \in H^i(X; C_2)$ will be defined for $\mathsf{Bun}(\mathrm{GL}_r)(X)_{/\mathbb{R}}$ while Chern classes $c_i \in H^{2i}(X; \mathbb{Z})$ will be defined for $\mathsf{Bun}(\mathrm{GL}_r)(X)_{/\mathbb{C}}$. We setting $w(E) := \sum_i w_i(E) \in H^*(X; C_2)$, we mentioned several properties:

- $w_i(f^*E) = f^* w_i(E)$
- $w(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$

- $w_{>\dim E} E = 0$
- $w_1(\gamma)$ is the generator of $H^1(\mathbb{RP}^\infty; C_2) = C_2$ where $\gamma \to \mathbb{RP}^\infty$ is the canonical line bundle. For complex bundles, $c_1(\gamma)$ is the positive generator of $H^2(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}$.

These properties characterize the w_i and c_i uniquely.

Remark 22.0.2: On why we need C_2 coefficients: we're pulling back $\gamma \to \mathbf{BO}_1$ and $\mathbf{BO}_1 \simeq \mathbb{RP}^\infty$ where $H^1(\mathbb{RP}^\infty; \mathbb{Z}) = 0$! For line bundles, we'll automatically have $w_{\geq 2} = 0$ for any such class, so we only have w_1 the work with, and this we need to pull back something nonzero to get anything interesting.

Corollary 22.0.3 *(?).*
Some immediate consequences:

- $w_i(E \oplus \mathbb{R}) = w_i(E)$, this if E is trivial then $w_i(E) = 0$ and $w(E) = 1$. Moreover $w(E \oplus \mathbb{R}) = w(E)w(\mathbb{R}) = w(E)$ since $w(\mathbb{R}) = 1$.
- If E has k linearly independent nonvanishing sections, then there is a splitting $E = E' \oplus R^{\oplus^k}$ where $\dim E' = n - k$. Thus $w_i(E) = 0$ for $i > n - k$.
- If $E_1 \oplus E_2$ is trivial, then $w(E_1) \smile w(E_2) = 1$.

Example 22.0.4 *(?):* If $M \subseteq \mathbb{R}^N$, then $w(\mathbf{T}M) \smile w(\nu M) = 1$. If $S^n \subseteq \mathbb{R}^{n+1}$, then $w(\mathbf{T}S^n) = 1$ since $w(\nu S^n) = 1$. So for example $w(\mathbf{T}S^2) = 1$, but $\mathbf{T}S^2$ has no nonvanishing sections.

Remark 22.0.5: Consider inverting a formal power series $\sum_{i \geq 0} a_i$:

$$(1 + a_1 + a_2 + a_3 + \cdots)(1 - a_1 + (a_1^2 + a_2) + (a_1^3 + 2a_1a_2 - a_3) + \cdots) = 1.$$

So if $w(E_1)w(E_2) = 1$, we can solve for $w(E_1)$ in terms of $w(E_2)$.

Proposition 22.0.6 *(?).*
Note that $H^*(\mathbb{RP}^n; C_2) = \mathbb{F}_2[a] / \left\langle a^{n+1} \right\rangle$ where $H^1(\mathbb{RP}^n; C_2) = \langle a \rangle$. Claim: $w(\mathbf{T}\mathbb{RP}^n) = (1 + a)^{n+1}$

Proof (?).
Let $\gamma \to \mathbb{RP}^n$ be the canonical line bundle, which is a pullback of $\gamma^\infty \to \mathbb{RP}^\infty$ and $\langle w_1(\gamma^\infty) \rangle = H^1(\mathbb{RP}^\infty; C_2)$. By the naturality property, $w_1(\gamma) = \alpha$ in $H^1(\mathbb{RP}^n; C_2)$.

Lemma 22.0.7 *(?).*

$$\mathbf{T}\mathbb{RP}^n = \mathrm{Hom}(\gamma, \gamma^\perp).$$

Recall that $\gamma \subseteq \mathbb{RP}^n \times \mathbb{R}^{n+1}$ as a subbundle over $\mathbb{RP}^n$, and $\gamma = \left\{ ([\mathbf{x}], \lambda \mathbf{x}) \;\middle|\; \lambda \in \mathbb{R} \right\} = \pi^{-1}[\mathbf{x}]$.

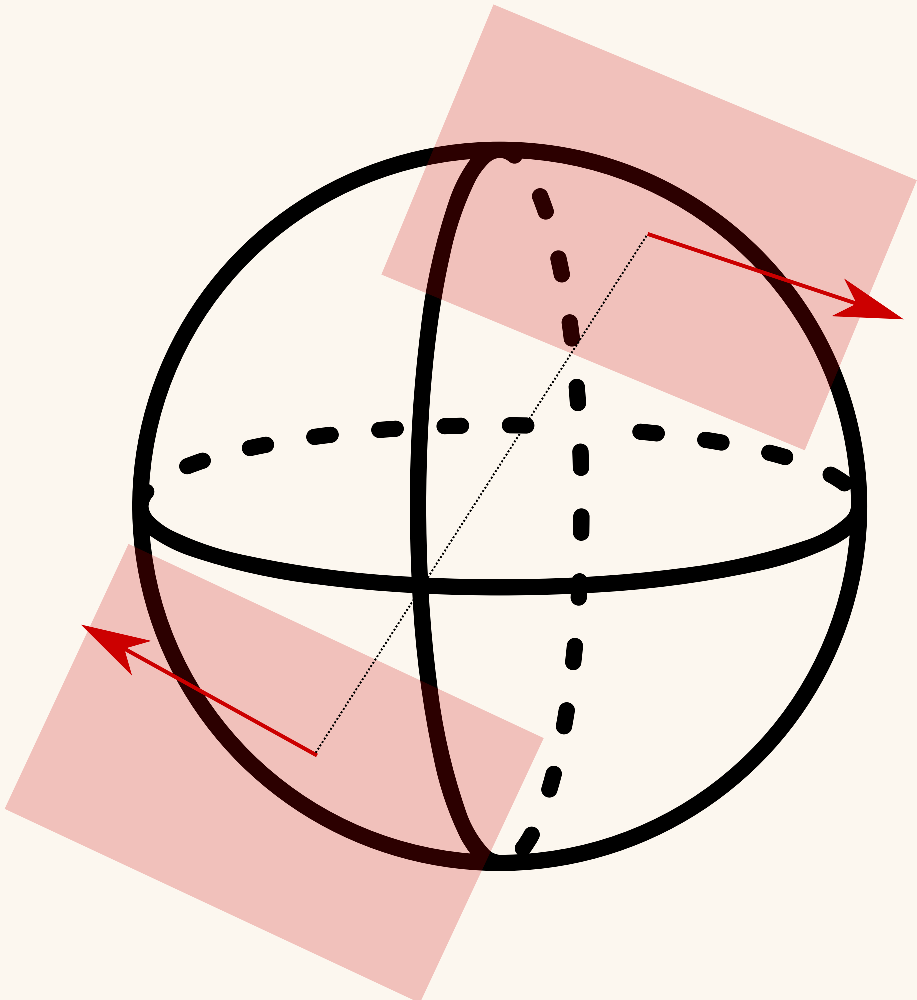
We can write $\mathbf{T}\mathbb{RP}^n = \mathbf{T}S^n/(\mathbf{x}, \mathbf{v}) \sim (-\mathbf{x}, -\mathbf{v})$, so

$$\mathbf{T}\mathbb{RP}^n = \left\{ [(x, v), (-\mathbf{x}, -\mathbf{v})] \;\middle|\; v \in \mathbf{T}_x S^n \right\} = \left\{ [(x, v), (-\mathbf{x}, -\mathbf{v})] \;\middle|\; v \in \mathbf{T}_x S^n \iff \langle x,\ v \rangle = 0 \right\}.$$

So define a map

$$\begin{aligned} \mathbf{T}\mathbb{RP}^n &\to \operatorname{Hom}(\gamma, \gamma^\perp) \\ [(\mathbf{x}, \mathbf{v}), (-\mathbf{x}, \mathbf{v})] &\mapsto \left(\lambda \mathbf{x} \xrightarrow{\ell} \lambda \mathbf{v} \right), \end{aligned}$$

and one can check that this is well-defined.



We can thus write

$$\begin{aligned}\mathbf{T}\mathbb{RP}^n \oplus \mathbb{R} &= \mathrm{Hom}(\gamma, \gamma^\perp) \oplus \mathrm{Hom}(\gamma, \gamma) \\ &= \mathrm{Hom}(\gamma, \gamma^\perp \oplus \gamma) \\ &= \mathrm{Hom}(\gamma, \mathbb{R}^{n+1}) \\ &= \mathrm{Hom}(\gamma, \mathbb{R})^{\oplus(n+1)}.\end{aligned}$$

Note that $\mathrm{Hom}(\gamma, \mathbb{R}) \cong \gamma$, so $w(\mathbf{T}\mathbb{RP}^n \oplus \mathbb{R}) = w(\gamma)^{n+1} = (1+a)^{n+1}$ and $w(\mathbf{T}\mathbb{RP}^n \oplus \mathbb{R}) = w(\mathbf{T}\mathbb{RP}^n)$.

■

Corollary 22.0.8 *(?).*
$w(\mathbf{T}\mathbb{RP}^n) = 1$ iff $n+1 = 2^r$ for some r.

Proof (?).
$\Longleftarrow$: By induction, $(1+a)^{2^r} = 1 + a^{2^r}$, using that $(1+a)^2 = 1 + a^2 + 2a$ and $2a = 0$ when we have C_2 coefficients. Now write

$$(1+a)^{2^r} = ((1+a)^{2^{r-1}})^2 \overset{IH}{=} (1+a^{2^{r-1}})^2 = 1 + a^{2^r}.$$

Now use that $\dim \mathbf{T}\mathbb{RP}^n = n$, so $(1+a)^{n+1} = 1$ since $a^{2^r} = a^{n+1} = 0$.
$\Longrightarrow$: Suppose $n+1 = m2^r$ with m odd, then

$$\begin{aligned}(1+a)^{m2^r} &= ((1+a)^{2^r})^m \\ &= (1+a^{2^r})^m \\ &= 1 + ma^{2^r} + \cdots,\end{aligned}$$

where the first nontrivial term doesn't vanish since $a^{2^r} \neq 0$.

■

Lemma 22.0.9 *(?).*
If $\mathbb{RP}^{2^r}$ admits an immersion into $\mathbb{R}^N$, then $N \geq 2^{r+1} - 1$.

Proof (?).
Why?

$$\begin{aligned}w(\mathbf{T}\mathbb{RP}^{2^r}) = (1+a)^{2^r+1} &= (1+a)^{2^r}(1+a) \\ &= (1+a^{2^r})(1+a) \\ &= 1 + a + a^{2^r} + a^{2^r+1} \\ &= 1 + a + a^{2^r}.\end{aligned}$$

Try to invert this: let $n \coloneqq 2^r$, then

$$(1+a+a^n)(1+a+a^2+\cdots+a^{n-1}) = 1.$$

Then

$$w(\nu\mathbb{RP}^n) = \sum_{0 \le i \le n-1} a^i,$$

so $\dim \nu\mathbb{RP}^n \geq n-1$. ■

23 | Thursday, November 04

Remark 23.0.1: Recall that $w(\mathbf{T}\mathbb{RP}^n) = (1+a)^{n+1}$ where $\langle a \rangle = H^1(\mathbb{RP}^n; C_2)$, and as an application, if $\mathbb{RP}^{2^r} \hookrightarrow \mathbb{R}^N$, then $N \geq 2^{r+1} - 1$. Similarly, $c(\mathbf{T}\mathbb{CP}^n) = (1+a)^{n+1}$ with $\langle a \rangle = H^2(\mathbb{CP}^\infty; \mathbb{Z})$ the positive generator.

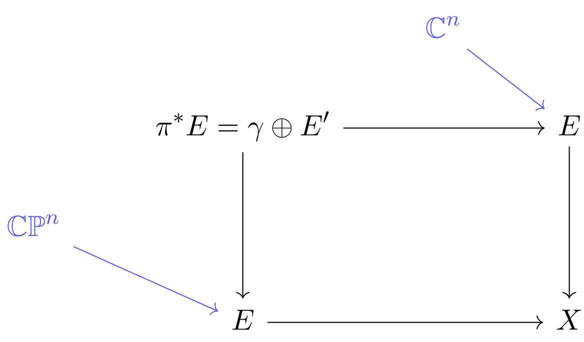
Theorem 23.0.2 ***(Leray-Hirsch).***
Fix a commutative ring R and a fiber bundle $F \hookrightarrow E \to B$ such that

1. $H^j(F; R) \in \mathsf{R}\text{-}\mathsf{Mod}$ is finitely generated and free,
2. For any j, there exist $c_{j_k} \in H^j(E; R)$ such that the restrictions of $\{c_{j_k}\}_{k \geq 0}$ is a basis for $H^j(E_x; R) = H^j(F; R)$.

Then there is an isomorphism

$$\begin{aligned} \Phi : H^*(B; R) \otimes_R H^*(F; R) &\xrightarrow{\sim} H^*(E; R) \\ b_i \otimes i^*(c_{j_k}) &\mapsto \pi^*(b_i) \smile c_{j_k}. \end{aligned}$$

Example 23.0.3 ***(?):*** Let $\mathbb{P}(E)$ be the projectivization of E, so each vector space fiber V is replaced with $\mathbb{P}(V)$. Let $\gamma \subseteq \pi^*(E)$ be the canonical over $\mathbb{P}(E)$, so the fibers are $\gamma_{(\gamma, [\gamma])} = \left\{ w \in E_x \;\middle|\; w \in [v] \right\}$. For $E \in \mathsf{Bun}\,(\mathrm{GL}_n)(\mathbb{C})_{/X}$, pick a metric on E, and define $E' := \gamma^\perp$. Now take the pullback:



Link to Diagram

So we have a bundle $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{P}(E) \to X$, we'll now try to apply the theorem. Note that $H^*(\mathbb{CP}^{n-1}; \mathbb{Z}) = \mathbb{Z}[u]/\langle u^n \rangle$, so H^i is generated by u^i, fulfilling condition 1.

Let $\iota_x : E_x \hookrightarrow E$ be the inclusion of a fiber. If we restrict $\mathbb{P}(E)$ to a fiber $\mathbb{P}(E)_x$, this yields the canonical over $\mathbb{CP}^{n-1}$:

$$\begin{array}{ccc} \gamma_x & \hookrightarrow & \gamma \\ \downarrow & & \downarrow \\ \mathbb{P}(E)_x = \mathbb{CP}^{n-1} & \xhookrightarrow{\iota_x^*} & \mathbb{P}(E) \end{array}$$

Link to Diagram

Then $\iota_x^* c_1(\gamma) = c_1(\gamma_x) = e(\gamma_x) = u$, so we have condition 2.

Exercise 23.0.4 *(?)*
For $L \in \mathsf{Bun}(\mathrm{GL}_1)(\mathbb{C})$,

$$c_1(L^\vee) = c_1(L).$$

So we have classes $1, c_1(\gamma), c_1(\gamma)^2, \cdots, c_1(\gamma)^{n-1}$, and we can take duals to obtain $1, c_1(\gamma^\vee), c_1(\gamma^\vee)^2, \cdots, c_1(\gamma^\vee)^{n-1}$
There exist $c_1(E), c_2(E), \cdots, c_n(E) \in H^*(X)$ such that we can write the former as a linear combination of the latter:

$$c_1(\gamma^\vee)^n + c_1(E)c_1(\gamma^\vee)^{n-1} + \cdots + c_n(E) = 0,$$

and the $c_i(E)$ are called the **Chern classes**.

Remark 23.0.5: Write $c(E) = 1 + c_1(E) + \cdots + c_n(E)$, so

$$\begin{aligned} \pi^* c(E) &= c(\gamma)c(E') = (1 + c_1(\gamma))(1 + c_1(E') + \cdots + c_{n-1}(E')) \\ &= 1 + (c_1(\gamma) + c_1(E')) + (c_1(\gamma)c_1(E') + c_2(E')) + \cdots + (c_1(\gamma)c_{i-1}(E') + c_1(E')). \end{aligned}$$

Plugging this into the LHS above yields

$$c_1(\gamma^\vee)^n + (c_1(\gamma) + c_1(E'))c_1(\gamma^\vee)^{n-1} + (c_1(\gamma)c_1(E') + c_2(E'))c_1(\gamma^\vee)^{n-2} + \cdots + c_1(\gamma)c_{n-1}(E') = (c_1(\gamma) + c_1(\gamma^\vee$$

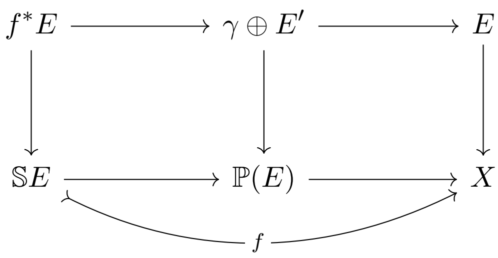
Since $\gamma^\vee$ and γ are dual, the first term is zero, so this entire expression is zero.

Exercise 23.0.6 (?)
Show that $c_1(E^\vee) = -c_1(E)$.

Hint: consider an explicit description in terms of transition functions.

Exercise 23.0.7 (?)
Show that given $b_1 = 1, b_2 = x + a_1, b_3 = xa_1 + a_2, \cdots$, then $b_1(-x)^n + b_2(-x)^{n-1} + \cdots = 0$.

Remark 23.0.8: Consider the following pullbacks:



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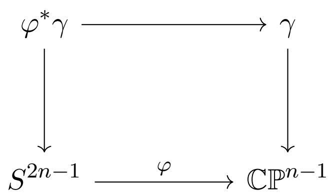
What is f^*E? There is a pullback



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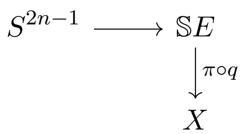
Here $\varphi^*\gamma$ will be trivial: we have $\gamma \subseteq \mathbb{CP}^{n-1} \times \mathbb{C}^n$, so $\varphi^*\gamma \subseteq S^{2n-1} \times \mathbb{C}^n$ and $\dim_{\mathbb{R}} \varphi^*\gamma = 2$. Then the fibers are $\varphi^*\gamma_x = \left\{(x, v) \;\middle|\; v = \lambda x, \lambda \in \mathbb{C}\right\}$. It turns out that $\varphi^*E = \mathbb{C} \oplus q^*E'$ where $\mathbb{S}E \xrightarrow{q} \mathbb{P}(E)$. Writing $\pi : \mathbb{P}(E) \to X$, we have

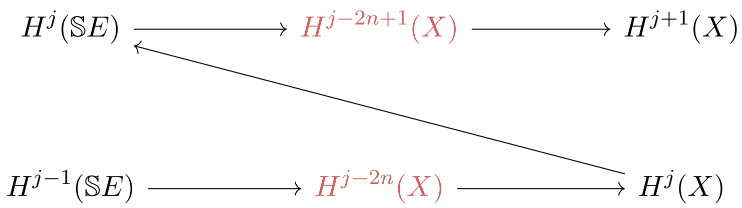
$$(\pi \circ q)^* c_i(E) = c_i(E) \qquad i < n - 1,$$

and $c_n(E) = e(E)$. To find this we'll need $\pi \circ q$ to be injective.

Consider



By the Gysin sequence, for $j < 2n - 1$ we'll have $H^j(X) \cong H^j(\mathbb{S}E)$ since the red terms vanish:



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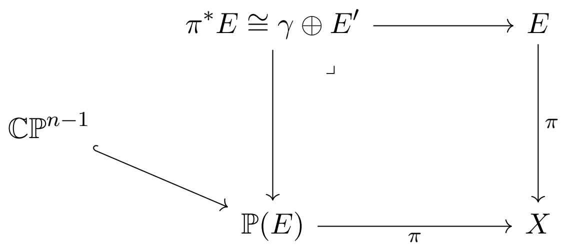
This uses that for $j < 2n$ that $H^{j-2n}(X) = 0$. For $i \leq n-1$, $2i < 2n - i < 2n-1$, and so the Chern classes for X and $\mathbb{S}E$ are isomorphic via $(\pi \circ q)^* : H^j(X) \to H^j(\mathbb{S}E)$ in the LES. By induction, we can define $c_i(E)$ for $i \leq n-1$.

Remark 23.0.9: Idea: pull back to split off a line bundle, pull back further to split off another line bundle, and continue. This is why we only need c_1 to determine a line bundle.

24 | Tuesday, November 09

Remark 24.0.1: Today: proving/checking the axioms for Chern classes. Recall that given $E \xrightarrow{\pi} X$, we can form a pullback:



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Here $\gamma^\vee$ is the dual of γ and $c_1(\gamma^\vee) = -_1(\gamma)$. By Leray-Hirsch, we have

$$1 + c_1(\gamma^\vee) + c_1(\gamma^\vee)^2 + \cdots + c_1(\gamma^\vee)^{n-1} \mid (c_1(\gamma^\vee))^n$$

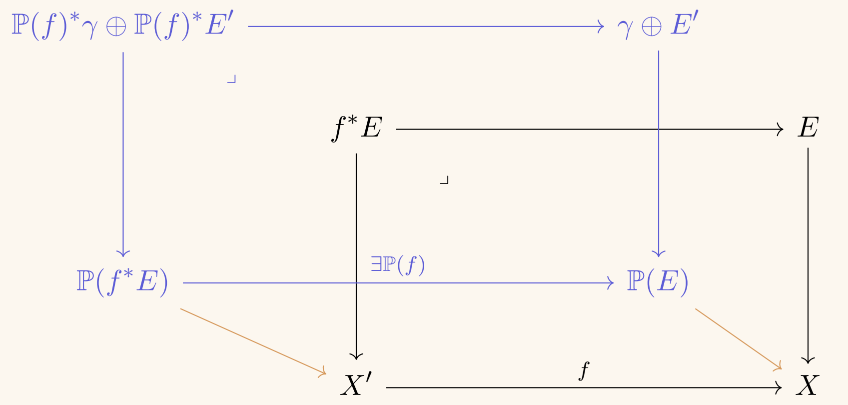
and for $c_i(E)$ the ith Chern class of E,

$$c_1(\gamma^\vee)^n + c_1(E) c_1(\gamma^\vee)^{n-1} + \cdots + c_n(E) = 0.$$

Proposition 24.0.2 *(?).*
These satisfy some axioms:

1. Naturality: $c_1(f^*E) = f^*c_1(E)$
2. Homomorphism: $c_(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$.
3. $c_{>\dim E}(E) = 0$.
4. For $\gamma \to \mathbb{CP}^\infty$ the canonical, $c_1(\gamma) \in H^2(\mathbb{CP}^\infty)$ is the positive generator.

Proof (?).
Here (3) is clear by definition, since we don't even define c_{n+1} or higher if $\dim E = n$. Number (4) isn't bad either, since $c_1(\gamma) = e(\gamma)$, using that $c(\mathcal{L}) = c_1(\mathcal{L})$ for line bundles. For (1), consider a pullback:



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We then use that on fibers, $\gamma_{\mathbb{P}(f)^*E} \cong \mathbb{P}(f)^*\gamma_{\mathbb{P}(E)}$, so the canonical pulls back to something isomorphic to the canonical. Then $c_1(\gamma^\vee_{\mathbb{P}(f)^*E}) = \mathbb{P}(f)^*c_1(\gamma^\vee_{\mathbb{P}(E)})$, so

$$(c_1\gamma^\vee_{\mathbb{P}(E)})^n + c_1(E)\cdot(c_1\gamma^\vee_{\mathbb{P}(E)})^{n-1} + c_2(E)\cdot(c_1\gamma^\vee_{\mathbb{P}(E)})^{n-2} + \cdots + c_n(E),$$

and applying $\mathbb{P}(f)^*$ to this entire expression yields zero. Thus

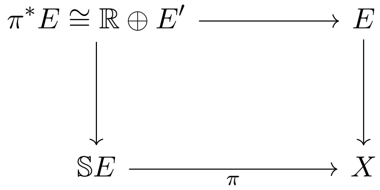
$$(c_1\gamma^\vee_{\mathbb{P}(f)^*E})^n\mathbb{P}(f)^*c_1(E)\cdot(c_1\gamma^\vee_{\mathbb{P}(f)^*E})^{n-1}\mathbb{P}(f)^*c_2(E)\cdot(c_1\gamma^\vee_{\mathbb{P}(f)^*E})^{n-2} + \cdots + +\mathbb{P}(f)^*c_n(E) = 0,$$

and we can note that by definition this equals

$$(c_1\gamma^\vee_{\mathbb{P}(f)^*E})^n c_1 f^*E\cdot(c_1\gamma^\vee_{\mathbb{P}(f)^*E})^{n-1}c_2 f^*E\cdot c_1\gamma^\vee_{\mathbb{P}(f)^*E})^{n-2} + \cdots + c_n f^*E = 0.$$

■

Remark 24.0.3: Recall the alternative description of Chern classes:



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By the Gysin sequence, we obtained $H^j(X) \cong H^j(\mathbb{S}E)$ for $j \leq n-2$, and there are $c_1(E'), \cdots, c_{n-2}(E')$ living in $H^2(\mathbb{S}E), H^4(\mathbb{S}E), \cdots, H^{2n-2}(\mathbb{S}E)$. So we defined $c_i(E)$ such that $\pi^* c_i(E) = c_i(E')$ for $i \leq n-1$, so $c_n(E) = e(E)$.

24.1 Splitting Principle

Proposition 24.1.1 *(?).*
Given an $E \in \mathsf{Bun}(\mathrm{GL}_n)(\mathbb{R})_{/X}$, there exists $Y \in \mathsf{Top}$ with $f : Y \to X$ such that

1. $f^*E = L_1 \oplus \cdots \oplus L_n$ with L_i line bundles.
2. $f^* : H^*(X) \hookrightarrow H^*(Y)$ is injective, so classes in $H^*(Y)$ can be uniquely pulled back.

Proof (?).
By induction, it suffices to find Y where $f^*E \cong E' \oplus E''$ splits nontrivially and f^* is injective. Taking the projectivization does exactly this, so take $Y := \mathbb{P}(E)$ and $f : \mathbb{P}(E) \to X$ to be the projection. Then $f^*E = \gamma \oplus E'$, so checking the 2nd condition in Leray-Hirsch, we get that $H^*(Y)$ is generated over $H^*(X)$ by $1, c_1(\gamma), c_1(\gamma)^2, \cdots, c_1(\gamma)^{n-1}$. Equivalently, the following map is injective:

$$\begin{aligned} f^* : H^*(X) &\to H^*(Y) \\ \alpha &\mapsto 1 \cdot \alpha. \end{aligned}$$

Here being generated means that

$$\beta \in H^*(Y) \implies \beta = \sum_{k=0}^{n-1} c_1(\gamma)^k \alpha_k, \quad \alpha_k \in H^*(X).$$

■

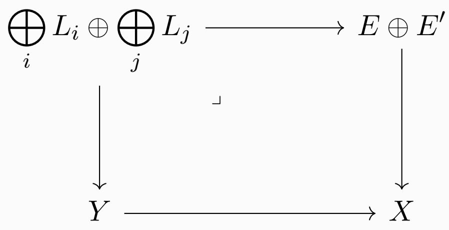
Corollary 24.1.2 *(?).*
If a polynomial identity holds for Chern classes in $E := \bigoplus_i L_i$ for L_i line bundles, then it holds for any bundle.

Lemma 24.1.3 *(?).*
For line bundles $L_1, \cdots, L_n$,

$$c(\bigoplus L_i) = \prod c(L_i) = \prod (1 + c_1(L_i)) = 1 + \left(\sum c_1(L_i)\right) + \cdots + \left(\prod c_1(L_i)\right),$$

so for $E := \bigoplus L_i$, $c_i(E)$ is the ith symmetric polynomial in the $c_1(L)$. In other words, writing $\sigma_i(x_1, \cdots, x_n)$ as the ith symmetric polynomial, $c_i(E) = \sigma_i(c_1(L_1), c_1(L_2), \cdots, c_1(L_n))$.

Corollary 24.1.4 *(?).*
The direct sum formula $c(E \oplus E') = c(E)c(E')$. Take a pullback



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Then

$$c(\bigoplus_i L_i \oplus \bigoplus_j L_j) = \left(\prod_i c(L_i)\right) \cdot \left(\prod_j c(L_j)\right) = c\left(\bigoplus_i L_i\right) \cdot c\left(\bigoplus L_j\right) = f^* c(E) \cdot f^* c(E'),$$

so $f^*c(E \oplus E') = f^*c(E) \cdot f^*c(E') = f^*(c(E) \cdot c(E'))$, and by injectivity, $c(E \oplus E') = c(E)c(E')$.

Proof (of lemma).
Write $E = L_1 \oplus \cdots \oplus L_n$ where $E \to X$. Then pull back along $\pi : \mathbb{P}E \to X$ to get $\pi^* E = \pi^* L_1 \oplus \cdots \oplus \pi^* L_n$. Note that $\gamma \subseteq \pi^* E$.

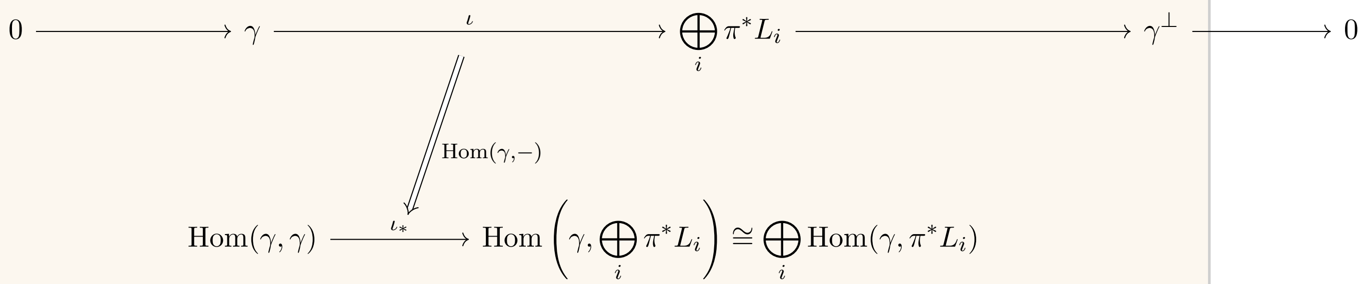
Claim:

$$\prod_{i=1}^n (c_1(\gamma^\vee) + c_1(\pi^* L_i)) = 0.$$

Using the claim, we get

$$0 = c_1(\gamma^\vee)^n + \left(\sum c_1 \pi^* L_i\right)(c_1(\gamma^\vee))^{n-1} + \cdots + \prod c_1(\pi^* L_i) \cdot c_1(\gamma^\vee)^1 = c_1(\gamma^\vee)^n + \pi^* c_1(E) \cdot (c_1(\gamma^\vee))^{n-1} + \cdots$$

which proves e.g. that $c_1(E) = \sigma_1(c_1 L_1, \cdots, c_1 L_n)$. The idea is that e.g. for the first term, $c_1(\gamma^\vee) = -c_1(\gamma)$ and $c_1 \pi^* L_1 = c_1(\gamma)$, yielding zero. If we can cover E by opens where this happens for at least one term, then the entire product must be zero on E.
Take a SES and apply $\mathrm{Hom}(\gamma, -)$:



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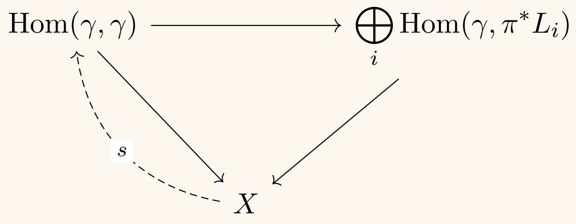
Now note that we can get a splitting:



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So take $X = \mathbb{P}E$ and pick a nonvanishing section s such that $\iota_* \circ s = (s_1, s_2, \cdots s_n)$ is a sum of sections. Then let $U_i = \left\{ x \in \mathbb{P}E \;\middle|\; s_i(x) \neq 0 \right\} \subseteq \mathbb{P}E$ by the complement of $Z(s_i)$, then $\bigcup_i U_i = \mathbb{P}E$ since s is nonvanishing. So $\text{Hom}(\gamma, \pi^* L_i)|_{U_i}$ is trivial, this yields $\gamma|_{U_i} \cong \pi^* L_i|_{U_i}$, making the restrictions of $c(-)$ equal, so $c_1(\pi^* L_i) - c_1(\gamma) = 0$ on U_i, and this is equal to $c_1(\pi^* L_i) + c_1(\gamma^\vee)$. Since $\{U_i\} \rightrightarrows \mathbb{P}E$, this concludes the proof.

■

25 Chern and Stiefel-Whitney classes (Thursday, November 11)

Remark 25.0.1: Recall that we were proving the splitting principle: given $E \to X$ of dimension n, there is a space Y with $f : Y \to X$ where

- $f^* E = \bigoplus L_i$ splits as a direct sum of line bundles.
- $f^* : H^*(X) \to H^*(Y)$ is injective.

Lemma 25.0.2 *(?).*
Assuming the exercise from last time that $c_1(L^\vee) = -c_1(L)$

$$c_i(E^\vee) = (-1)^i c_i(E).$$

Proof (?).
It suffices to show this for $E = \bigoplus L_i$. Write $E^\vee = \bigoplus L_i{}^\vee$, so

$$\begin{aligned} c(E^\vee) = \prod c(L_i{}^\vee) = \prod (1 - c_1(L_i)) \\ &= 1 - \left(\sum c_1(L_i) \right) + \sigma_2(c_1 L_1, \cdots, c_1 L_n) - \cdots \pm \prod c_1(L_i) \\ &= 1 - c_1(E) + c_2(E) \cdots \pm c_1(L_n). \end{aligned}$$

■

Theorem 25.0.3 *(?).*

For $E \in \mathsf{Bun}\,(\mathrm{GL}_n)(\mathbb{C})_{/X}$, the top Chern class equals the Euler class:

$$c_n(E) = e(E).$$

Similarly, for $E \in \mathsf{Bun}\,(\mathrm{GL}_n)(\mathbb{R})_{/X}$,

$$w_n(E) \equiv e(E) \bmod 2.$$

Proof (?).
Both classes satisfy the Whitney sum formula, so

$$\begin{aligned} E = \bigoplus L_i \implies c(E) &= \prod c(L_i) \\ \implies e(E) &= \prod e(L_i) \\ &= \prod (1 + c_1(L_i)) \\ &= 1 + \sigma_1(c_1 L_1, \cdots, c_1 L_n) + \cdots + \prod c_1(L_i) \\ &:= 1 + \sigma_2(c_1 L_1, \cdots, c_1 L_n) + \cdots + c_n(E). \end{aligned}$$

Then $e(E) = \prod e(L_i) = \prod c_1(L_i) = c_n(E)$.
Now the claim follows from the splitting principle and naturality of characteristic classes. ■

Exercise 25.0.4 (?)
Show that for any $E \in \mathsf{Bun}\,(\mathrm{GL}_n)(\mathbb{C})_{/E}$,

$$c_i(E) \equiv w_{2i}(E) \bmod 2.$$

Lemma 25.0.5 ***(?)*.**
For $E \in \mathsf{Bun}\,(\mathrm{GL}_n)(\mathbb{R})_{/X}$,
Then

$$c_1(E) = c_1(\bigwedge^n E),$$

noting that $\bigwedge^n E$ is a line bundle. A similar claim holds for c_1.

Fact 25.0.6

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).$$

:::{.remark} ETS this holds for $E = \bigoplus L_i$, in which case $\bigwedge^n E = \bigotimes_{i=1}^n L_i$. By the above fact, $c_1(\bigwedge^n E) = \sum c_1(L_i)$. Since $c_i(E) = \sigma_i(c_1 L_1, \cdots, c_1 L_n)$, we have

$$c_1(E) = \sigma_1(c_1 L_1, \cdots, c_1 L_n) = \sum c_1 L_i.$$

::

Remark 25.0.7: Recall that the Euler class is the obstruction to finding an obstruction to extending a section over the n-skeleton.

25.1 Obstruction Theory

Lemma 25.1.1 *(?).*
$w_1 E$ is the obstruction to orienting E in the following sense: for any $S^1 \xrightarrow{f} E$, write $f_*[S]$ for the image of the fundamental class, then

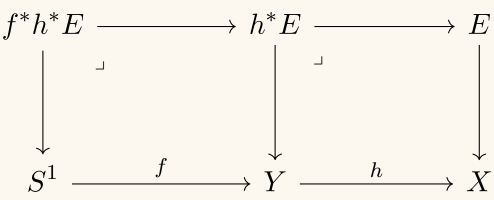
$$\left\langle w_1(E),\ f_*[S^1] \right\rangle = \chi_{f^*E \text{ is orientable}} = \begin{cases} 1 & f^*E \text{ is nonorientable} \\ 0 & f^*E \text{ is orientable.} \end{cases}.$$

Remark 25.1.2: Why? For example, consider $E \to S^1$. Trivialize over $S^1 \setminus \{\text{pt}\}$, then glue the ends by some element $A \in \mathrm{GL}_n(\mathbb{R})$. If $\det(A) > 0$, this will be orientable, and $\det(A) < 0$ will be nonorientable.

Proof (?).
If E is a line bundle, define $\tilde{w}_1(E) \in H^1(X)$ by the following

$$\left\langle \tilde{w}_1(E),\ f_*[S^1] \right\rangle = \chi_{f^*E \text{ nonorientable}}.$$

This is natural under pullback, consider the following diagram:

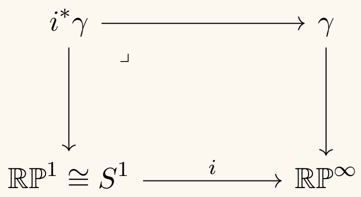


Link to Diagram

We have

$$\begin{aligned} \left\langle \tilde{w}_1 h^*E,\ f_*[S^1] \right\rangle &= \left\langle \tilde{w}_1 h^*E,\ (h \circ f)_*[S^1] \right\rangle \\ &= \left\langle h^*\tilde{w}_1 h^*E,\ f_*[S^1] \right\rangle \\ &= \tilde{w}_1(h^*E) \\ &= h^*\tilde{w}_1(E). \end{aligned}$$

We need to show $\tilde{w}_1(\gamma) = w_1(\gamma)$ where $\gamma \to \mathbb{RP}^\infty$ is the canonical. Write $H^1(\mathbb{RP}^\infty; C_2) = C_2 = \langle a \rangle$, so $w_1(\gamma) = a$. Take the pullback:



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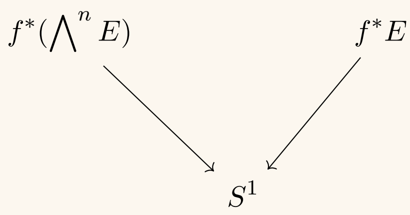
Then $i^*\gamma$ is not orientable since locally this looks like a Mobius band, so we have

$$\left\langle \tilde{w}_1(\gamma),\ i_*[\mathbb{RP}^1] \right\rangle = 1.$$

Now $w_1(E) = w_1(\bigwedge^n E)$ and we have to bundles:



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These are either simultaneously orientable or simultaneously nonorientable, by considering $\det A_1, \det A_2$ for the gluing maps between trivializations. So $w_1(E) = \left\langle \bigwedge^n E,\ f_*[S^1] \right\rangle = \chi_{f^* \bigwedge^n E \text{ nonorientable}}$.

■

Theorem 25.1.3 ***(?).***
$w_k(E)$ is the mod 2 reduction of the obstruction to extending $n - k + 1$ linearly independent sections of E over $X^{(k)}$ when $\dim_{\mathbb{R}} E = n$.

Remark 25.1.4: For $k = n$, $w_n(E)$ corresponds to 1 linearly independent section of $X^{(n)}$ since $e(E) \equiv w_n(E) \bmod 2$. For $k = 1$, $w_1(E)$ corresponds to n linearly independent sections over $X^{(1)}$.

26 Pontryagin Classes (Tuesday, November 23)

26.1 Complexification

Question 26.1.1
How to we get integral cohomology classes when we don't have a complex structure?

Definition 26.1.2 (Complexification)
Given a real vector bundle E, its **complexification** is defined as

$$E \otimes_{\mathbb{R}} \mathbb{C} \coloneqq \coprod_{x \in X} E_x \otimes_{\mathbb{R}} \mathbb{C}.$$

Then $\dim_{\mathbb{R}} E \otimes_{\mathbb{R}} \mathbb{C} = 2 \dim_{\mathbb{R}} E$.

Remark 26.1.3: In terms of transition functions (i.e. Čech cohomology data), this is the inclusion $\mathrm{GL}_n(\mathbb{R}) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$. We can now consider the Chern classes $c_i(E \otimes \mathbb{C})$.

Definition 26.1.4 (Conjugate bundle)
Given a complex vector bundle E, there is a **conjugate bundle** $\overline{E}$ with the complex structure $i_{\overline{E}} \cdot V \coloneqq -i_E \cdot V$. This corresponds to conjugating Čech cohomology data, i.e. replacing transition functions f_{ij} with $\overline{f_{ij}}$.

Lemma 26.1.5 *(?).*
For any real vector bundle E,

$$E \otimes \mathbb{C} \cong \overline{\mathbb{E} \otimes \mathbb{C}}.$$

Proof (?).
The transition functions for the former are in $\mathrm{GL}_n(\mathbb{R})$, which is fixed under conjugation. ■

Remark 26.1.6: Note that $E \otimes \mathbb{C} \cong E \oplus E$, and we can map

$$\mathbb{R}^{\oplus^2} \otimes \mathbb{C} \xrightarrow{\sim} \mathbb{C}^{\oplus^2}$$
$$(x \oplus y) \otimes \langle 1, i \rangle \mapsto (x, ix) \oplus (y, iy).$$

Exercise 26.1.7 (?)
Show that if E is a complex vector bundle,

$$E \otimes \mathbb{C} \cong E \oplus \overline{E},$$

and

$$c_i(\overline{E}) = (-1)^i c_i(E).$$

26.2 Pontryagin Classes

Lemma 26.2.1 ***(?).***
For i odd, $c_i(E \otimes \mathbb{C})$ is 2-torsion.

Proof (?).
On one hand,

$$E \otimes \mathbb{C} \cong \overline{E \otimes \mathbb{C}} \implies c_i(E \otimes \mathbb{C}) = c_i(\overline{E \otimes \mathbb{C}}) = (-1)^i c_i(E \otimes \mathbb{C}).$$

So if i is odd, $c_i = -c_i$ and thus $2c_i = 0$. ∎

Definition 26.2.2 (Pontryagin classes)
The i**th Pontryagin class** is

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(X).$$

Remark 26.2.3: Suppose R is a coefficient ring where 2 is invertible (or not a zero divisor). Some properties of the p_i:

a. p_i is natural, so $p_i(f^* E) = f^* p_i(E)$.

b. $p_{i > \dim(E)/2}$.

c. $p(E \bigoplus E') = p(E) \smile p(E')$. Why? Note $p(E) = 1 - c_2 + c_4 - c_6 + \cdots$, so multiplying two such things yields

$$p(E)p(E') = 1 - (c_2 + c_2') + (c_4 + c_2 c_2' + c_4') - (c_6 + c_4 c_2' + c_2 c_4' + c_6') - \cdots = p(E \oplus E').$$

d. If $\dim_{\mathbb{R}} E = 2n$ then $p_n(E) = e(E)^2$. Why?

$$p_n(E) = (-1)^n c_{2n}(E \otimes \mathbb{C}) = (-1)^n e(E \otimes \mathbb{C}) = (-1)^n (-1)^{\frac{2n(2n+1)}{2}} = (-1)^{n+n(2n+1)} e(E)^2 = (-1)^{n(2n+2)} e($$

e. If E is a complex line bundle,

$$c(E \oplus \overline{E}) = c(E \otimes \mathbb{C}) = 1 - p_1(E) + p_2(E) - \cdots = c(E)c(\overline{E}) = (1 + c_1(E) + c_2(E) + \cdots)(1 - c_1(E) + c_2$$

Example 26.2.4 ***(?):*** We can compute $p(\mathbf{T}\mathbb{CP}^5)$. Recall that $c(\mathbf{T}\mathbb{CP}^n) = (1+a)^{n+1}$ where $\langle a \rangle = H^2(\mathbb{CP}^n; \mathbb{Z})$ is the positive generator. So $c(\mathbf{T}\mathbb{CP}^5) = (1+a)^6 = 1 + \binom{6}{1} a + \binom{6}{2} a + \cdots$. Using (e) above, we have $c(E)c(\overline{E}) = (1+a)^6(1-a)^6 = (1-a^2)^6 = 1 - 6a^2 + 15a^4$. So $p(\mathbf{T}\mathbb{CP}^5) = 1 + 6a^2 + 15a^4$.

Corollary 26.2.5 *(?).*
If $\mathbb{CP}^5 \hookrightarrow \mathbb{R}^N$ immerses, then $p(\nu\mathbb{CP}^5) = \dfrac{1}{1+6a^2+15a^4} = 1 - 6a^2 + 21a^4$. So $p_2(\nu\mathbb{CP}^5) \neq 0$, so $\dim(\nu)/2 \geq 2 \implies \dim(\nu) \geq 4$ and this forces $n \geq 14$. Note that here we used that $p_i(E) = 0$ for $i > \dim E/2$.

Theorem 26.2.6 *(?).*

$$\begin{aligned} H^*(\mathbf{B}\mathrm{SO}_{2n+1}; R) &\xrightarrow{\sim} R[p_1, \cdots, p_n] \\ H^*(\mathbf{B}\mathrm{SO}_{2n}; R) &\xrightarrow{\sim} R[p_1, \cdots, p_{n-1}, e] \\ H^*(\mathbf{B}\mathrm{O}_{2n+1}; R) &\xrightarrow{\sim} H^*(\mathbf{B}\mathrm{O}_{2n}; R) \xrightarrow{\sim} R[p_1, \cdots, p_n] \end{aligned}$$

,

where p_i are the Pontryagin classes for the respective canonical bundles.

27 | Bordism (Tuesday, November 30)

Definition 27.0.1 (Bordism)
Let $M_1, M_2 \in \mathsf{smMfd}^n$ be closed, then M_1 is **bordant** to M_2 iff $\exists W \in \mathsf{smMfd}^{n+1}$ with $\partial W = M_1 \coprod M_2$.

Remark 27.0.2: This defines an equivalence relation on smMfd^n, and yields a ring $(\prod_n \mathcal{N}^n, \coprod, \times)$.

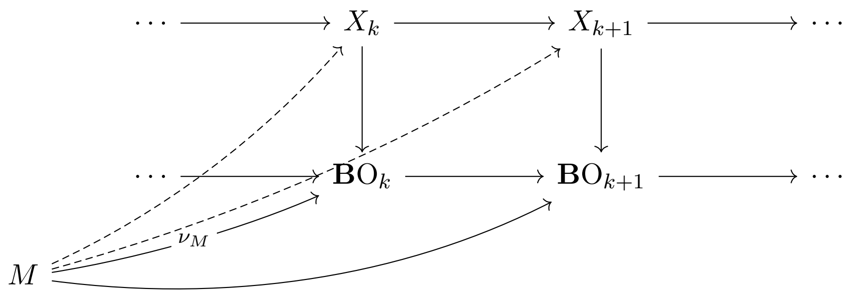
We'll add extra structure to get refinements of this ring, for which we'll need a bit about stable normal bundles and the Whitney embedding theorem.

Theorem 27.0.3 *(Whitney Embedding).*
Any $M \in \mathsf{smMfd}^n$ (possibly with boundary) embeds in $\mathbb{R}^{2n+1}$, and any two such embeddings into $\mathbb{R}^{2n+3}$ are isotopic.

Remark 27.0.4: Moreover if $i, i' : M \hookrightarrow \mathbb{R}^n$ with $n \geq 2n+3$, then the normal bundles $\nu M, (\nu M)'$ are bundle isomorphic.

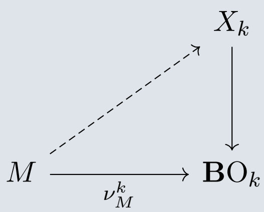
Corollary 27.0.5 *(?).*
Every such M has a stable normal bundle ν_M which is well-defined up to adding trivial line bundles.

Remark 27.0.6: Note that adding trivial line bundles yields a tower $\mathbf{B}\mathrm{O}_1 \to \mathbf{B}\mathrm{O}_2 \to \cdots$, where the classifying maps of E and $E \oplus L$ are classified by maps to $\mathbf{B}\mathrm{O}_k$ and $\mathbf{B}\mathrm{O}_{k+1}$ respectively. We can define lifts of extra structures by looking for commutative towers over X_k for e.g. $X = \mathbf{B}\mathrm{O}, \mathrm{Spin}$, etc:



Link to Diagram

Definition 27.0.7 (Structures)
An X-structure on M is a family of lifts for large k:



Link to Diagram

We also require these to be compatible with the morphism $\mathbf{BO}_k \to \mathbf{BO}_{k+1}$ and $X_k \to X_{k+1}$.

Example 27.0.8 ***(?):*** Some examples:

- $\mathbf{BSO}_k$ for orientation,
- V_k (the Stiefel manifold) for framings.

Definition 27.0.9 (X-bordism)
Given M_i with X-structures, we say N is an **X-bordism** if

- $\partial N = M_1 \coprod M_2$
- There is an X-structure on N that restricts to the X-structures on M_1 and M_2.

Remark 27.0.10: For orientations, we get a bordism ring Ω_n, and for framings we get Ω_n^{Fr}.

Theorem 27.0.11 ***(Identification of framed bordism group).***

$$\Omega_n^{\mathrm{Fr}} \cong \pi_n^s,$$

the stable homotopy groups of spheres $\varinjlim_k \pi_{n+k} S^n$ (which stabilize for $k > n+1$).

Definition 27.0.12 (Stiefel-Whitney numbers)
Given $M \in \mathsf{smMfd}^n$ and $I := \{r_1, \cdots, r_n\}$ such that $\sum_{1 \leq k \leq n} kr_k = n$, consider $W_n := \prod_{1 \leq k \leq n} w_k(\mathbf{T}M)^{r_k} \in H^n(M; C_2)$. We can evaluate this to get

$$w_I(M) := \langle W_n,\ [M] \rangle \in C_2.$$

Note that the condition on I guarantees that $W_n \in H^n$.

Remark 27.0.13: For $n = 3$, the only possibilities for I are

- $(3, 0, 0) \rightsquigarrow W_n = w_1^3(\mathbf{T}M)$
- $(1, 1, 0 \rightsquigarrow W_n = w_1(\mathbf{T}M)w_2(\mathbf{T}M))$
- $(0, 0, 1) \rightsquigarrow W_n = w_3(\mathbf{T}M)$

Definition 27.0.14 (Pontryagin number)
Given an *oriented* $M \in \mathsf{smMfd}^n$, the **Pontryagin number** of M is given by taking $I := \left\{r_1, \cdots, r_{\frac{n}{4}}\right\}$ such that $4r_1 + 8r_2 + \cdots + nr_{\frac{n}{4}} = n$, setting $P_n := \prod_k p_k(\mathbf{T}M)^{r_k}$, and evaluating

$$p_I(M) := \langle P_n,\ [M] \rangle.$$

Corollary 27.0.15 *(?).*
If any Pontryagin number is nonzero, then M can not admit an orientation reversing diffeomorphism.

Proof (?).
Use that

$$P_I(-M) = \left\langle \prod p_1 \mathbf{T}M,\ [-M] \right\rangle = -P_I(M),$$

but also

$$\begin{aligned} P_I(M) &= \left\langle \prod p_i \mathbf{T}M,\ [M] \right\rangle \\ &= \left\langle f^* \left(\prod p_i \mathbf{T}M \right),\ [M] \right\rangle \\ &= \left\langle \prod p_i \mathbf{T}M,\ [f_* M] \right\rangle \\ &= \left\langle \prod p_i \mathbf{T}M,\ [-M] \right\rangle \\ &= P_I(-M). \end{aligned}$$

■

Example 27.0.16 *(?):* Some examples:

- $P_I(\mathbb{CP}^{2n}) \neq 0$ by a computation.

- $P_I(\mathbb{CP}^{2n+1}) = 0$ since conjugating the complex structure $J \mapsto -J$ is an orientation reversing diffeomorphism.

Theorem 27.0.17 ***(Pontryagin, Thom).***
$M = 0$ in $\mathcal{N}_n$ iff $w_I(M) = 0$ for all I.

Proof (?).
$\implies$: Easy, a simple algebraic topology calculation.
$\impliedby$: Difficult and omitted! ■

Corollary 27.0.18 ***(?).***
There is an injective group morphism

$$\mathcal{N}_n(\mathcal{N}_{I_1}, \cdots, \mathcal{N}_{I_k}) \to C_2^{\times^k}.$$

Theorem 27.0.19 ***(Pontryagin-Thom).***
If $M = 0$ in Ω_n, then $P_I(M) = 0$ for all I. Conversely, if $P_I(M) = 0$ for all I then M is torsion in Ω_n, so there is some k such that $M \coprod^k = \partial W$ for some W.

Remark 27.0.20: Milnor and Wall: the only torsion in Ω_n is order 2, and an oriented manifold M is 0 in $\Omega_n \iff w_I(M), P_I(M) = 0$ for all I. For a proof (for at least the first statement), see Milnor-Stasheff.

Remark 27.0.21: We can kill torsion to get an injective map

$$\Omega_n \otimes \mathbb{Q} \xrightarrow{(P_{I_1}, \cdots, P_{i_k})} \mathbb{Z}^{\times^k}.$$

Theorem 27.0.22 ***(Pontryagin-Thom).***
The ring $\mathcal{N}_n$ is a polynomial ring over $\mathbb{F}_2$ with one generator in each dimension:

$$\mathcal{N}_n \cong \mathbb{F}_2[\left\{ v_i \;\middle|\; i \neq 2^i - 1 \right\}].$$

Theorem 27.0.23 ***(Pontryagin-Thom).***
The ring $\Omega_* \otimes \mathbb{Q}$ is polynomial ring over $\mathbb{Q}$ generated by $\mathbb{CP}^n$ for $n = 1, 2, \cdots$:

$$\Omega_* \cong\cong \mathbb{Q}[v_1, v_2, \cdots] \qquad\qquad |v_i| = i.$$

28 | Thursday, December 02

Remark 28.0.1: Today: the Hirzebruch signature theorem and exotic S^7. We saw that $\Omega_* \otimes \mathbb{Q}$ is the polynomial algebra on $\mathbb{Q}[v_1, v_2 \cdots]$ where v_2 corresponds to $\mathbb{CP}^{2n}$. This was because

$\ker\left(\Omega_n \xrightarrow{\{p_{I_n}\}} \mathbb{Z}^{\times^n}\right)$ can only be torsion.

Definition 28.0.2 (Signature)
For B a symmetric bilinear form, $\mathrm{sig}(B) = \dim V^+ - \dim V^-$, the dimensions of positive/negative definite subspaces respectively, or equivalently the difference in the number of positive and negative eigenvalues. For $M \in \mathsf{smMfd}^{4n}$, there is a middle-dimensional pairing

$$\begin{aligned} H^{2n}(M;\mathbb{Z})^{\otimes^2_{\mathbb{Z}}} &\to \mathbb{Z} \\ a \otimes b &\mapsto \langle a \smile b,\ [M] \rangle, \end{aligned}$$

and we write $\sigma(M)$ for the signature of this form.

Theorem 28.0.3 *(?).*
The signature induces a ring morphism

$$\sigma : (\Omega_*, \coprod, \times) \to (\mathbb{Z}, +, \cdot),$$

so

- $\sigma(M \coprod N) = \sigma(M) + \sigma(N)$,
- $\sigma(M \times N) = \sigma(M)\sigma(N)$,
- $\sigma(M) = 0$ if $M = \partial M'$ for some M'.

Lemma 28.0.4 *(?).*
If there is a half-dimensional subspace $L \subseteq H^{2n}(M;\mathbb{Q})$ such that $\langle a \smile, [M] \rangle = 0$ for all $a, b \in L$ then $\sigma(M) = 0$.

Proof (?).
Use Poincare duality to get an isomorphism:

$$\begin{array}{ccccc} H^{2n}(N;\mathbb{Q}) & \xrightarrow{i^*} & H^{2n}(M;\mathbb{Q}) & \xrightarrow{f} & H^{2n+1}(N,M;\mathbb{Q}) \\ & & \downarrow \cong & & \cong,\mathrm{PD} \downarrow \\ & & H_{2n}(M;\mathbb{Q}) & \xrightarrow{i_*} & H_{2n}(N;\mathbb{Q}) \end{array}$$

Link to Diagram

Note that $\operatorname{rank} i^* = \operatorname{rank} i_*$ since these spaces are dual, and this has the same rank as f since the sides are isomorphisms. Then $\dim H^{2n}(M) = \dim \operatorname{im} i^* + \dim \operatorname{coker} i^*$ and $\dim \operatorname{coker} i^* = \dim \operatorname{im} i_*$, so $\dim H^{2n}(M) = 2 \dim \operatorname{im} i^* \leq 2 \dim \operatorname{im} i^*$. Then writing $i^*\alpha = a, i^*\beta = b \in \operatorname{im} i^*$,

$$\langle i^*(a \smile b),\ [M] \rangle = \langle \alpha \smile \beta,\ i_*[M] \rangle = 0,$$

since $i_*[M] = 0$. Thus $\sigma(M) = 0$.

■

Corollary 28.0.5 *(?).*
One can compute the signature in terms of Pontryagin numbers, i.e. there is a formula for $\sigma(M)$ in terms of $\{p_{I_n}(M)\}$. The explicit formula is the **Hirzebruch signature theorem**.

Example 28.0.6 *(?):* Write $\Omega_4 \otimes \mathbb{Q} = \left\langle \mathbb{CP}^2 \right\rangle$ and consider $M \in \mathsf{smMfd}^4$. Note $\mathbb{CP}^2$ has only 1 Pontryagin number and $c(\mathbb{CP}^n) = (1+x)^{n+1}$, so we have a formula

$$\begin{aligned} 1 - p_1(\mathbb{CP}^2) &= c(\mathbb{CP}^2) \cdot c(\overline{\mathbb{CP}^2}) \\ &= (1+x)^3(1-x)^3 \\ &= (1-x^2)^3 \\ &= 1 + -3x^2 + 3x^4 - x^6 \\ &= 1 - 3x^2, \end{aligned}$$

where we've used that $H^{\geq 4} = 0$. So $p_1(\mathbb{CP}^2) = 3x^2$, which implies

$$\left\langle p_1 \mathbb{CP}^2,\ [\mathbb{CP}^2] \right\rangle = \left\langle 3x^2,\ [\mathbb{CP}^2] \right\rangle = 3.$$

Using that $\sigma(\mathbb{CP}^2) = 1$, we can deduce $\sigma(M) = p_1(M)/3$ by considering eigenvalues of the intersection pairing.

Example 28.0.7 *(?):* Consider M^8 and $\Omega_8 \otimes \mathbb{Q} = \left\langle \mathbb{CP}^4, \mathbb{CP}^2 \times \mathbb{CP}^2 \right\rangle$. Then $\Sigma(\mathbb{CP}^4) = \sigma(\mathbb{CP}^{2\times^2}) = 1$, what are the Pontryagin numbers? Write $v_2 = \mathbb{CP}^2, v_2^2 = \mathbb{CP}^2 \times \mathbb{CP}^2, v_4 = \mathbb{CP}^4$,then

$$\begin{aligned} 1 - p_1(v_4) + p_2(v_4) &= c(v_4)c(\bar{v}_4) \\ &= (1+x)^5(1-x)^5 \\ &= (1-x^2)^5 \\ &= 1 - 5x^2 + 10x^4, \end{aligned}$$

so $p_1(v_4) = 5x^2$ and $p_2(v_4) = 10x^4$. Then

$$\begin{aligned} \left\langle p_1^2 v_4,\ [v_4] \right\rangle &= \left\langle 25x^4,\ [v_4] \right\rangle = 25 \\ \langle p_2 v_4,\ [v_4] \rangle &= \left\langle 10x^4,\ [v_4] \right\rangle = 10. \end{aligned}$$

Similarly,

$$1 + p_1(v_4^2) + p_2(v_4^2) = (1+3x^2)(1+3y^2) = 1 + 3x^2 + 3y^2 + 9x^2y^2,$$

so

$$\left\langle p_1^2 v_2^2,\ [v_2^2] \right\rangle = \left\langle (3x^2+3y^2)^2,\ [v_2^2] \right\rangle = \left\langle 9x^4 + 9y^4 + 18x^2y^2,\ [v_2^2] \right\rangle = 18,$$

since the cohomology ring is $\mathbb{F}[x]/\left\langle x^2 \right\rangle \otimes \mathbb{F}[y]/\left\langle y^2 \right\rangle$, and a similar calculation shows

$$\left\langle p_2(v_2^2),\ [v_2^2] \right\rangle = 9.$$

Summarizing, where we abuse notation and identity classes with numbers,

- $p_1^2(v_4) = 25$,
- $p_2(v_4) = 10$,
- $p_1^2(v_2^2) = 18$,
- $p_2(v_2^2) = 9$.

Writing $\sigma = ap_1^2 + bp_2$,

$$1 = \sigma(v_4) = 25a + 10b$$
$$1 = \sigma(v_2^2) = 18a + 9b.$$

This yields $b = \frac{1}{9} - 2a$ and $1 = 25a + 10\left(\frac{1}{9} - 2a\right)$ yields $a = -1/45$ and $b = 7/45$. Thus

$$\sigma = \frac{1}{45}\left(2p_2 - p_1^2\right).$$

Theorem 28.0.8 ***(Thom).***

$$\Omega_7 = 0.$$

Definition 28.0.9 (?)
Given M^7 with $H^3(M) = H^4(M)$ and suppose $M = \partial B^*$. Let $i : H^4(B; M) \to H^4(B)$ and define

$$\lambda(M) := 2\left\langle (i^{-1}p_1\mathbf{T}B)^2,\ [B]\right\rangle - \sigma(B).$$

Here $\sigma(B)$ is the signature of

$$H^4(B, M; \mathbb{Q})^{\otimes_{\mathbb{Z}}^2} \to \mathbb{Z}$$
$$a \otimes b \mapsto \langle a \smile b,\ [B]\rangle.$$

Proposition 28.0.10 ***(?).***
$\lambda(M)$ is independent of B.

Remark 28.0.11: Let $C := B_1 \coprod_M B_2$ where $\partial B_i = M$, then there is a relation expressing $\left\langle p_1(C)^2,\ [C]\right\rangle$ in terms of pairings of $p(B_i)$ against $[B_i]$, and $\sigma(C) = \sigma(B_1) - \sigma(B_2)$. We have

$$45\sigma(C) + \left\langle p_1(C)^2,\ [C]\right\rangle \equiv 0 \bmod 7$$
$$\implies -40\sigma(C) + \left\langle p_1(C)^2,\ [C]\right\rangle \equiv 0 \bmod 7$$
$$\implies \sigma(C) + 2\left\langle p_1(C)^2,\ [C]\right\rangle \equiv 0 \bmod 7$$
$$\implies (\sigma(B_1) - \sigma(B_2)) - 2\left(\left\langle (i^{-1}p_1B_1)^2,\ [B_1]\right\rangle \cdots\right) = \lambda_{B_1}(M) - \lambda_{B_2}(M).$$

Remark 28.0.12: If the λs differ, the manifolds can not be diffeomorphic. Constructing S^7s: take S^3 bundles over S^4. Pick any map $S^3 \hookrightarrow \mathrm{SO}_4(\mathbb{R}) \subseteq \mathrm{Homeo}(S^3)$ to get clutching data, which are elements of $\pi_3(\mathrm{SO}_4) = \mathbb{Z}^{\times^2}$. Label these bundles with parameters $(m, n) \in \mathbb{Z}^{\times^2}$ so that $\xi_{1,0}$ corresponds to lifting $[S^3, \mathrm{SO}_3]$ to $[S^3, \mathrm{SO}_4]$ fixing the x-axis, and $\xi_{1,1}$ is the canonical for $\mathbb{HP}^1$. If M_k is the total space of $\xi_{k,1}$, then $\lambda(M_k) \equiv k^2 - 1 \bmod 7$, so aren't diffeomorphic. Then one can show that the M_k are homeomorphic by finding a Morse function with exactly 2 critical points.

29 | Problem Set 1

29.1 1

Problem 29.1.1 (?)
With the definition of a vector bundle from class, show that the vector space operations define continuous maps:

$$
\begin{aligned}
+ &: E \underset{B}{\times} E \to E \\
\times &: \mathbb{R} \times E \to E
\end{aligned}
$$

Remark 29.1.1: Definition of vector bundle: need charts (U, φ) with $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ which when restricted to a fiber F_b yields an isomorphism $F_b \xrightarrow{\sim} \mathbb{R}^n$.

What are these maps??

29.2 2.

Problem 29.2.1 (?)
Suppose you are given the following data:

- Topological spaces B and F
- A set E and a map of sets $\pi : E \to B$
- An open cover $\mathcal{U} = \{U_i\}$ of B and for each i a bijection $\varphi : \pi^{-1}(U_i) \to U_i \times F$ so that $\pi \circ \varphi_i = \pi$.

Give conditions on the maps φ_i so that there is a topology on E making $\varphi : E \to B$ into a fiber bundle with $\{(U_i, \varphi_i)\}$ as an atlas.

29.3 3.

Problem 29.3.1 (?)
An *oriented n-dimensional vector bundle* is a vector bundle $\pi : E \to B$ together with an orientation of each fiber E_b, so that these orientations are continuous in the following sense. For each $b \in B$ there is a chart (U, φ) with $b \in U$ and $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ so that for all $b' \in U$,

$$\varphi|_{E_{b'}} : E_{b'} \to \mathbb{R}^n$$

is orientation-preserving.
Show that given an oriented n-dimensional vector bundle there is an induced principal $GL_+(\mathbb{R}^n)$-bundle (the "bundle of oriented frames"), and conversely given a principal $GL_+(\mathbb{R}^n)$-bundle there is an induced oriented n-plane bundle.

29.4 4.

Problem 29.4.1 (?)
A Riemannian metric on a vector bundle $\pi : E \to B$ is an inner product $\langle \cdot, \cdot \rangle_b$ on each fiber E_b of E, which is continuous in the sense that the induced map $E \oplus E = E \times_B E \to \mathbb{R}$ is continuous.
Show that given a Riemannian metric on a vector bundle, there is an induced principal $O(n)$-bundle (the "bundle of orthonormal frames"), and conversely given a principal $O(n)$-bundle there is an induced vector bundle with Riemannian metric.

29.5 5 .

Problem 29.5.1 (?)
What operation on principal $O(n)$-bundles corresponds to dualizing a vector bundle? What about the direct sum of vector bundle?

29.6 6.

Problem 29.6.1 (?)
For nice spaces X (e.g. CW complexes) and abelian groups G, there is a canonical isomorphism

$$\check{H}^i(X;G) \cong H^i(X;G)$$

between Čech and singular cohomology of X with coefficients in G.

A nice, readable proof can be found in Frank Warner's Foundations of Differential Manifolds and Lie Groups, Chapter 5. In the rest of this problem, cohomology either means Čech cohomology or singular cohomology after applying this isomorphism.

(a) Let $\pi : E \to B$ be an n-dimensional vector bundle, or equivalently, a principal $GL(n,\mathbb{R})$-bundle, given by a Čech cocycle $\varphi \in H^1(B; GL(n,\mathbb{R}))$. Show that the sign of the determinant

$$\operatorname{sgn}\det : \mathrm{GL}_n(\mathbb{R}) \to \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$$

induces a map

$$\check{H}^1(B; GL(n,\mathbb{R})) \to \check{H}^1(B; \mathbb{Z}/2\mathbb{Z}),$$

and so φ induces an element $w_1(E) \in H^1(B; \mathbb{Z}/2\mathbb{Z})$.

(b) Compute w_1 for the trivial line bundle (1-dimensional vector bundle) over the circle and for the Möbius band.

(c) Prove that (for nice spaces) a line bundle $\pi : E \to B$ is trivial if and only if $w_1(E) = 0 \in H^1(B; \mathbb{Z}/2\mathbb{Z})$

29.7 7.

Problem 29.7.1 (?)
Show that the exact sequence of abelian topological groups

$$0 \to \mathbb{Z} \to \mathbb{R} \to S^1 = GL(1,\mathbb{C}) \to 0$$

induces an exact sequence in Čech cohomology

$$\check{H}^1(B,\mathbb{Z}) \to \check{H}^1(B,\mathbb{R}) \to \check{H}^1\left(B; S^1\right) \xrightarrow{\delta} \check{H}^2(B;\mathbb{Z})$$

Given a complex line bundle (principal $GL(1,\mathbb{C})$-bundle) $\pi : E \to B$ coming from the cocycle data $\varphi \in H^1(B; GL(1,\mathbb{C}))$, let $c_1(E) = \delta(\varphi)$. Compute $c_1(E)$ for some complex line bundle over S^2.

ToDos

List of Todos

Definitions

Theorems

Exercises

Figures

List of Figures