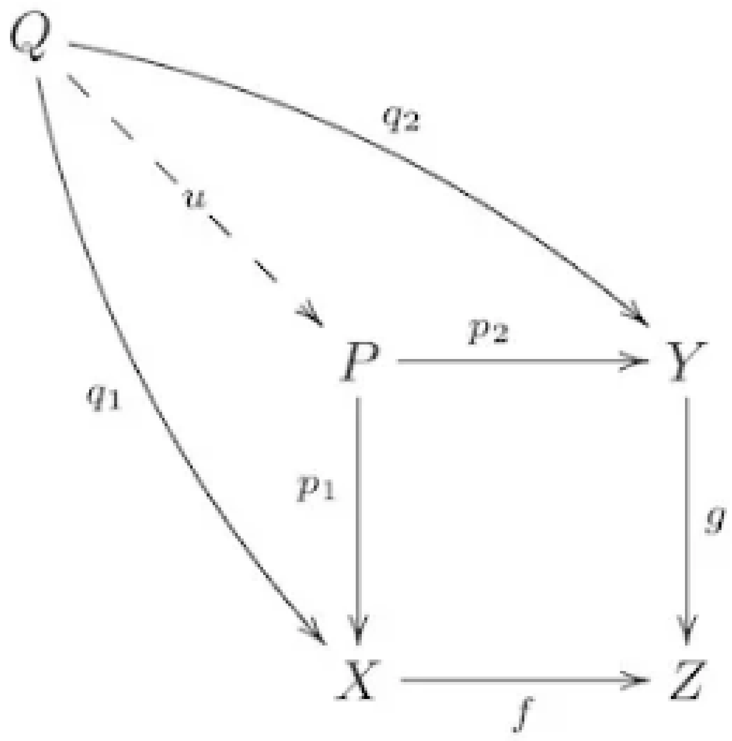

Notes: These are notes live-tex'd from a graduate course in Schemes taught by Phil Engel at the University of Georgia in Fall 2021, with material based on Hartshorne. Any errors or inaccuracies are almost certainly my own.

Schemes

Lectures by Phil Engel. University of Georgia, Fall 2021

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1 | Wednesday, August 18: Sheaves

Remark 1.0.1: We'll be covering Hartshorne, chapter 2:

- Sections 1-5: Fundamental, sheaves, schemes, morphisms, constant sheaves.
- Sections 6-9: Divisors, linear systems of differentials, nonsingular varieties.

Note that most of the important material of this book is contained in the exercises!

Remark 1.0.2: Recall that a **topological space** X is collection of *open* sets $\mathcal{U} = \{U_i \subseteq X\}$ which is closed under arbitrary unions and finite intersections, where $X, \emptyset \in \mathcal{U}$.

Definition 1.0.3 (Presheaf)
A **presheaf of abelian groups** $\mathcal{F}$ on X a topological space is an assignment to every open $U \subseteq X$ an abelian group $\mathcal{F}(U)$ and restriction morphisms $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for every inclusion $V \subseteq U$ satisfying

- $\mathcal{F}(\emptyset) = 0$
- $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ is $\mathrm{id}_{\mathcal{F}(U)}$.
- If $W \subseteq V \subseteq U$ are opens, then

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV}.$$

We'll write $\mathcal{F}(U)$ to be the **sections of $\mathcal{F}$ over** U, also notated $\Gamma(U; \mathcal{F})$ and write the restrictions as $s|_v = \rho_{UV}(s)$ for $V \subseteq U$.

Example 1.0.4 ***(Presheaf of continuous functions):*** Let $X := \mathbb{R}$ with the standard topology and take $\mathcal{F} = C^0(-; \mathbb{R})$ (continuous real-valued functions) as the associated presheaf. So for $U \subset \mathbb{R}$ open, the sections are $\mathcal{F}(U) := \{f : U \to \mathbb{R} \text{ continuous}\}$.

For restriction maps, given $U \subseteq V$ take the actual restriction of functions $C^0(V; \mathbb{R}) \to C^0(U; \mathbb{R})$. One needs to check the 3 conditions, but we can declare $C^0(\emptyset; \mathbb{R}) = \{0\} = 0$, and the others follow right away.

Example 1.0.5 ***(Constant presheaves):*** The **constant presheaf** associated to $A \in \mathsf{Ab}$ on $X \in \mathsf{Top}$ is denote $F = \underline{A}$, where

$$\underline{A}(U) := \begin{cases} A & U \neq \emptyset \\ 0 & U = \emptyset. \end{cases}$$

and

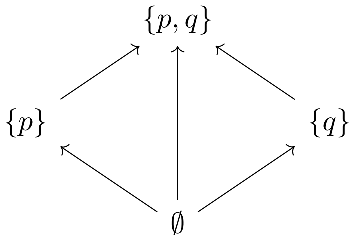
$$\rho_{UV} := \begin{cases} \mathrm{id}_A & V \neq \emptyset \\ 0 & V = \emptyset. \end{cases}.$$

Warning 1.0.6
The constant sheaf is not the sheaf of constant functions! Instead these are *locally* constant functions.

Remark 1.0.7: Let $\mathsf{Open}_{/X}$ denote the category of open sets of X, defined

- $\mathrm{Ob}(\mathsf{Open}_{/X}) := \{U_i\}$, so each object is an open set.
- $\mathsf{Open}_{/X}(U, V)$ is empty when $V \not\subset U$ and is the singleton inclusion $\{\iota : U \hookrightarrow V\}$ otherwise.

Example 1.0.8 ***(Of*** $\mathsf{Open}_{/X}$ ***):*** Take $X := \{p, q\}$ with the discrete topology to obtain a category with 4 objects:



Link to Diagram

Similarly, the indiscrete topology yields $\emptyset \to \{p, q\}$, a category with two objects.

Remark 1.0.9: Then a presheaf is a contravariant functor $\mathcal{F} : \mathsf{Open}_{/X} \to \mathsf{Ab}$ which sends the cofinal/initial object $\{\emptyset\} \in \mathsf{Open}_{/X}$ to the final/terminal object $0 \in \mathsf{Ab}$. More generally, we can replace Ab with any category C admitting a final object:

- $\mathsf{C} := \mathsf{CRing}$ the category of commutative rings, which we'll use to define schemes.
- $\mathsf{C} = \mathsf{Grp}$, the full category of (potentially nonabelian) groups.
- $\mathsf{C} := \mathsf{Top}$, arbitrary topological spaces.

Example 1.0.10 ***(of presheaves):*** Let $X \in \mathsf{Var}_{/k}$ a variety over $k \in \mathsf{Field}$ equipped with the Zariski topology, so the opens are complements of vanishing loci. Given $U \subseteq X$, define a presheaf of regular functions $\mathcal{F} := \mathcal{O}$ where

- $\mathcal{O}(U)$ are the regular functions $f : U \to k$, i.e. functions on U which are locally expressible as a ratio $f = g/h$ with $g, h \in k[x_1, \cdots, x_n]$.

- Restrictions are restrictions of functions.

Taking $X = \mathbb{A}^1_{/k}$, the Zariski topology is the cofinite topology, so every open U is the complement of a finite set and $U = \{t_1, \cdots, t_m\}^c$. Then $\mathcal{O}(U) = \{\varphi : U \to k\}$ which is locally a fraction, and it turns out that these are all globally fractions and thus

$$\mathcal{O}(U) = \left\{ \frac{f}{g} \;\middle|\; f, g \in k[t], g(t) \neq 0 \;\; \forall t \in U \right\} = \left\{ \frac{f}{\prod (t - t_i)^{m_i}} \;\middle|\; f \in k[t] \right\} = k[t] \left[S^{-1} \right],$$

where $S = \left\langle \prod t - t_i \right\rangle$ is the multiplicative set generated by the factors.

This gives an abelian group since we can take least common denominators, and we have restrictions.

Warning 1.0.11
Note that there are two similar notations for localization which mean different things! For a multiplicative set S, the ring $R[S^{-1}]$ literally means localizing at that set. For $\mathfrak{p} \in \operatorname{Spec} R$, the ring $R[\mathfrak{p}^{-1}]$ means localizing at the multiplicative set $S := \mathfrak{p}^c$.

2 | Friday, August 20

Definition 2.0.1 (Sheaf)
Recall the definition of a presheaf, and the main 3 properties:

1. $F(\emptyset) = 0$,
2. $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$
3. For all $W \subseteq V \subseteq U$, a cocycle condition:

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV}.$$

Write $s_i \in \mathcal{F}(U_i)$ to be a section.

A presheaf is a **sheaf** if it additionally satisfies

4. When restrictions are compatible on overlaps, so

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j},$$

there exists a uniquely glued section $\mathcal{F}(\cup U_i)$ such that $s|_{U_i} = s_i$ for all i.

Example 2.0.2 ***(?):*** Take $C^0(-;\mathbb{R})$ the sheaf of continuous real-valued functions on a topological space. For $f_i : U_i \to \mathbb{R}$ agreeing on overlaps, there is a continuous function $f : \cup U_i \to \mathbb{R}$ restricting to f_i on each U_i by just defining $f(x) = f_i(x)$ for $x \in U_i$, which is well-defined by agreement of the f_i on overlaps.

Example 2.0.3 ***(?):*** Let X be a topological space and $A \in \mathsf{CRing}$, then take the constant sheaf $\underline{A}$ which maps to A iff $U \neq \emptyset$ and 0 otherwise. This is not a sheaf, taking $X = \mathbb{R}$ and $A = \mathbb{Z}/2$. Let $U_1 = (0,1)$ and $U_2 = (2,3)$ and take $s_1 = 0$ on U_1 and $s_2 = 1$ on U_2. Using that $U_1 \cap U_2 = \emptyset$, so they trivially agree on overlaps, but there is no constant function on $U_1 \cup U_2$ restricting to 1 on U_2 and 0 on U_1

Definition 2.0.4 (Locally constant sheaves)

The **(locally) constant sheaf** $\underline{A}$ on any $X \in \mathsf{Top}$ is defined as

$$\underline{A}(U) := \left\{ f : U \to A \;\middle|\; f \text{ is locally constant} \right\}.$$

Remark 2.0.5: As a general principle, this is a sheaf since this property can be verified locally.

Example 2.0.6 ***(?):*** Let C^0_{bd} be the presheaf of bounded continuous functions on S^1. This is not a sheaf, but one needs to go to infinitely many sets: take the image of $[\frac{1}{n}, \frac{1}{n+1}]$ with (say) $f_n(x) = n$ for each n. Then each f_n is bounded (it's just constant), but the full collection is unbounded, so these can not glue to a bounded function.

Definition 2.0.7 (Stalks)
Let $\mathcal{F} \in \underset{\mathsf{pre}}{\mathsf{Sh}}(X)$ and $p \in X$, then the **stalk** of $\mathcal{F}$ at p is defined as

$$\mathcal{F}_p(U) := \lim_{U \ni p} := \left\{ (s, U) \;\middle|\; U \ni p \text{ open, } s \in \mathcal{F}(U) \right\} / \sim,$$

where $(s, U) \sim (t, V)$ iff there exists a $W \ni p$ with $W \subset U \cap V$ with $s|_W = t|_W$. An equivalence class $[(s, U)] \in \mathcal{F}_p$ is referred to as a **germ**.

Example 2.0.8 ***(?):*** Let $C^\omega(-; \mathbb{R})$ be the sheaf of analytic functions, i.e. those locally expressible as convergent power series. This is a sheaf because this condition can be checked locally. What is the stalk C^ω_0 at zero? An example of a function in this germ is $[(f(x) = \frac{1}{1-x}, (-1, 1))$. A first guess is $\mathbb{R}[t]$, but the claim is that this won't work.

Note that there is an injective map $C^\omega_0 \hookrightarrow \mathbb{R}[t]$ because f, g have analytic power series expansions at zero, and if these expressions are equal then $f|_I = g|_I$ for some I containing zero. This map won't be surjective because there are power series with a non-positive radius of convergence, for example taking $f(t) := \sum_{k=0}^{\infty} kt^k$ which only converges at $t = 0$. So the answer is that $C^\omega_0 \leq \mathbb{R}[t]$ is the subring of power series with positive radius of convergence.

Definition 2.0.9 (Local ring of the structure sheaf, $\mathcal{O}_p$)
Let $X \in \mathsf{AlgVar}$ and $\mathcal{O}$ its sheaf of regular functions. For $p \in X$, the stalk $\mathcal{O}_p$ is the **local ring** of X at p.

Example 2.0.10 ***(?):*** For $X := \mathbb{A}^1_{/k}$ for $k = \bar{k}$, the opens are cofinite sets and $\mathcal{O}(U) = \left\{ f/g \;\middle|\; f, g \in k[t] \right\}$. Consider the stalk $\mathcal{O}_p$. Applying the definition, we have

$$\mathcal{O}_p := \left\{ (f/g, U) \;\middle|\; p \in U,\, g \neq 0 \text{on } U \right\} / \sim .$$

Given any $g \in k[t]$ with $g(p) \neq 0$, there is a Zariski open set $U = V(g)^c = D_g$, the distinguished open associated to g, where $g \neq 0$ on U by definition. Thus $p \in U$, and so any $f/g \in \operatorname{ff} k[t]$ with $p \neq 0$ defines an element $(f/g, D_g) \in \mathcal{O}_p$. Concretely:

$$f/g|_W = f/g|_{W'} \implies f/g = f'/g' \in \operatorname{ff} k[t] = k(t),$$

and $fg' = f'g$ on the cofinite set W, making them equal as polynomials. We can thus write

$$\mathcal{O}_p = \left\{ f/g \in k(t) \;\middle|\; g(p) \neq 0 \right\} = k[t]\left[\langle t-p \rangle^{-1}\right], \quad \langle t-p \rangle \in \operatorname{mSpec} k[t],$$

recalling that $k[t]\left[\mathfrak{p}^{-1}\right] = \left\{ f/g \;\middle|\; g \notin \mathfrak{p} \right\}$.

Note that for $X \in \mathsf{AffVar}$, so $X = V(f_i) = V(I)$ for I reduced, we have the coordinate ring $k[X] = k[x_1, \cdots, x_n]/I = R$, then $\mathcal{O}_p = R\left[\mathfrak{m}_p{}^{-1}\right]$ where $\mathfrak{m}_p := \left\{ f \in R \;\middle|\; f(p) = 0 \right\}$.

⚠ **Warning 2.0.11**
This doesn't quite hold for non-algebraically closed fields. Take $f(x)x^p - x \in \mathbb{F}_p[x]$, then $f(x) \equiv 0$ since every element in $\mathbb{F}_p$ is a root.

Remark 2.0.12: Next time: morphisms of sheaves/presheaves, and isomorphisms can be checked on stalks for sheaves.

3 | Monday, August 23

Remark 3.0.1: Recall that the **stalk** of a presheaf $\mathcal{F}$ at p is defined as

$$\mathcal{F}_p := \underset{U \ni p}{\varinjlim} \mathcal{F}(U) = \left\{ (s, U) \;\middle|\; s \in \mathcal{F}(U) \right\}_{/\sim}.$$

Definition 3.0.2 (Morphisms of presheaves)
Let $\mathcal{F}, \mathcal{G} \in \underset{\mathsf{pre}}{\mathsf{Sh}}(X)$, then a **morphism** $\varphi : \mathcal{F} \to \mathcal{G}$ is a collection $\{\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)\}$ of morphisms of abelian groups for all $U \in \mathsf{Open}(X)$ such that for all $V \subset U$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \big\downarrow {\scriptstyle \operatorname{res}(UV)} & & \big\downarrow {\scriptstyle \operatorname{res}'(UV)} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

Link to Diagram

An **isomorphism** is a morphism with a two-sided inverse.

Remark 3.0.3: Note that if we regard a sheaf as a contravariant functor, a morphism is then just a natural transformation.

Remark 3.0.4: A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ defines a morphisms on stalks $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$.

Example 3.0.5 ***(of a nontrivial morphism of sheaves):*** Let $X := \mathbb{C}^\times$ with the classical topology, making it into a real manifold, and take $C^0(-;\mathbb{C}) \in \mathsf{Sh}(X, \mathsf{Ab})$ be the sheaf of continuous functions and let $C^0(-;\mathbb{C})^\times$ the sheaf of of nowhere zero continuous continuous functions. Note that this is a sheaf of abelian groups since the operations are defined pointwise. There is then a morphism

$$\begin{aligned} \exp(-) : C^0(-;\mathbb{C}) &\to C^0(-;\mathbb{C})^\times \\ f &\mapsto e^f \end{aligned} \qquad \text{on open sets } U \subseteq X.$$

Since exponentiating and restricting are operations done pointwise, the required square commutes, yielding a morphism of sheaves.

Definition 3.0.6 ((co)kernel and image sheaves)
Let $\varphi : \mathcal{F} \to \mathcal{G}$ be morphisms of presheaves, then define the presheaves

$$\begin{aligned} \ker(\varphi)(U) &:= \ker(\varphi(U)) \\ \operatorname{coker}^{\mathsf{pre}}(\varphi)(U) &:= \mathcal{G}(U)/\varphi(\mathcal{F}(U)) \\ \operatorname{im}(\varphi)(U) &:= \operatorname{im}(\varphi(U)) \end{aligned}$$
.

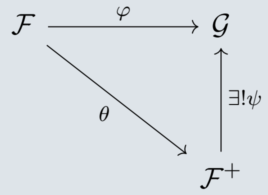
⚠ **Warning 3.0.7**
If $\mathcal{F}, \mathcal{G} \in \mathsf{Sh}(X)$, then for a morphism $\varphi : \mathcal{F} \to \mathcal{G}$, the image and cokernel presheaves need not be sheaves!

Example 3.0.8 ***(?):*** Consider $\ker\exp$ where $\exp : C^0(-;\mathbb{C}) \to C^0(-;\mathbb{C})^\times \in \mathsf{Sh}(\mathbb{C}^\times)$. One can check that $\ker\exp = 2\pi i\underline{\mathbb{Z}}(U)$, and so the kernel is actually a sheaf.

We also have $\operatorname{coker}^{\mathsf{pre}}\exp(U) := C^0(U;\mathbb{C})/\exp(C^0(U;\mathbb{C})^\times)$. On opens, $\operatorname{coker}^{\mathsf{pre}}\exp(U) = \{1\} \iff$ every nonvanishing continuous function g on U has a continuous logarithm, i.e. $g = e^f$ for some f. Examples of opens with this property include any contractible (or even just simply connected) open set in $\mathbb{C}^\times$. Consider $U := \mathbb{C}^\times$ and $z \in C^0(\mathbb{C}^\times;\mathbb{C})^\times$, which is a nonvanishing function. Then the equivalence class $[z] \in \operatorname{coker}^{\mathsf{pre}}\exp(\mathbb{C}^\times)$ is nontrivial, since $z \neq e^f$ for any $f \in C^0(\mathbb{C}^\times;\mathbb{C})$, since any attempted definition of $\log(z)$ will have monodromy.

on the other hand we can cover $\mathbb{C}^\times$ by contractible opens $\{U_i\}_{i\in I}$ where $[z]|_{U_i} = 1 \in \operatorname{coker}^{\mathsf{pre}}\exp(U_i)$ and similarly $1|_{\mathrm{id}} = 1 \in \operatorname{coker}^{\mathsf{pre}}\exp(U_i)$, showing that the cokernel fails the unique gluing axiom and is not a sheaf.

Definition 3.0.9 (Sheafification)
Given any $\mathcal{F} \in \mathsf{Sh}_{\mathsf{pre}}(X)$ there exists an $\mathcal{F}^+ \in \mathsf{Sh}(X)$ and a morphism of presheaves $\theta : \mathcal{F} \to \mathcal{F}^+$ such that for any $\mathcal{G} \in \mathsf{Sh}(X)$ with a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ there exists a unique $\psi : \mathcal{F}^+ \to \mathcal{G}$ making the following diagram commute:



Link to Diagram

The sheaf $\mathcal{F}^+ \in \mathsf{Sh}(X)$ is called the **sheafification** of $\mathcal{F}$. This is an example of an adjunction of functors:

$$\underset{\substack{\mathsf{Sh}(X) \\ \mathsf{pre}}}{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}^{\mathsf{pre}}) \cong \underset{\mathsf{Sh}(X)}{\mathrm{Hom}}(\mathcal{F}^+, \mathcal{G}),$$

where we use the forgetful functor $\mathcal{G} \to \mathcal{G}^{\mathsf{pre}}$. This yields an adjoint pair

$$c \underset{b}{\overset{a}{\underset{\perp}{\rightleftarrows}}} d.$$

Proof (of existence of sheafification).
We construct it directly as $\mathcal{F}^+ := \left\{ s : U \to \coprod_{p \in U} \mathcal{F}_p \right\}$ such that

1. $s(p) \in \mathcal{F}_p$,
2. The germs are compatible locally, so for all $p \in U$ there is a $V \supseteq p$ such that for some $t \in \mathcal{F}(V)$, $s(p) = t_p$ for all p in V.

Slogan
Collections of germs that are locally compatible.

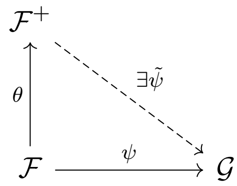
So about any point, there should be an actual function specializing to all germs in an open set.

■

Remark 3.0.11: The point will be that $\operatorname{coker} \exp$ will be zero as a sheaf, since it'll be zero on a sufficiently small set.

4 | Wednesday, August 25

Remark 4.0.1: Recall the definition of sheafification: let $\mathcal{F} \in \underset{\mathsf{pre}}{\mathsf{Sh}}(X; \mathsf{AbGrp})$. Construct a sheaf $\mathcal{F}^+ \in \mathsf{Sh}(X, \mathsf{AbGrp})$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ of presheaves satisfying the appropriate universal property:



Link to Diagram

So any presheaf morphism to a sheaf factors through the sheafification uniquely (via θ). Note that this is a instance of a general free/forgetful adjunction.

We can construct it as

$$\mathcal{F}^+(U) := \left\{ s : U \to \coprod_{p \in U} \mathcal{F}_p, \quad s(p) \in \mathcal{F}_p, \cdots \right\}.$$

where the addition condition is that for all $q \in U$ there exists a $V \nu q$ and $t \in \mathcal{F}(V)$ such that $t_p = s(p)$ for all $p \in V$. Note that θ is defined by $\theta(U)(s) = \{s : p \to s_p\}$, the function assigning points to germs with respect to the section s. Idea: this is like replacing an analytic function on an interval with the function sending a point p to its power series expansion at p.

Example 4.0.2 ***(?):*** Recall $\exp : C^0 \to (C^0)^\times$ on $\mathbb{C}^\times$, then $\operatorname{coker}^{\mathsf{pre}}(\exp)(U) = \{1\}$ on contractible U, using that one can choose a logarithm on such a set. However $\operatorname{coker}^{\mathsf{pre}}(\exp)(\mathbb{C}^\times) \neq \{1\}$ since $[z] \in (C^0)^\times(\mathbb{C}^\times)/\exp(C^0(\mathbb{C}^\times))$.

Remark 4.0.3: Letting $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphisms of sheaves, then we defined $\operatorname{coker}(\varphi) := (\operatorname{coker}^{\mathsf{pre}}(\varphi))^+$ and $\operatorname{im}(\varphi) := (\operatorname{im}^{\mathsf{pre}}(\varphi))^+$. Then

$$\begin{aligned} \operatorname{coker}^{\mathsf{pre}}(\exp) &\to \operatorname{coker}(\exp) \\ s \in \mathcal{F}(U) &\mapsto s(p) = s_p. \end{aligned}$$

The claim is that $[z]_p = 1$ for all $p \in \mathbb{C}^\times$, since we can replace $[([z], \mathbb{C}^\times)]$ with $([z]_U, U)$ for U contractible.

Example 4.0.4 ***(?):*** A useful example to think about: $X = \{p, q\}$ with

- $\mathcal{F}(p) = A$
- $\mathcal{F}(q) = B$
- $\mathcal{F}(X) = 0$

Then local sections don't glue to a global section, so this isn't a sheaf, but it is a presheaf. The sheafification satisfies $\mathcal{F}^+(X) = A \times B$.

4.1 Subsheaves

Definition 4.1.1 (Subsheaves, injectivity, surjectivity)
$\mathcal{F}'$ is a **subsheaf** of $\mathcal{F}$ if

- $\mathcal{F}'(U) \leq \mathcal{F}(U)$ for all U,
- $\overset{\prime}{\text{Res}}(U, V) = \text{Res}(U, V)|_{\mathcal{F}'(U)}$.

$\varphi : \mathcal{F} \to \mathcal{G}$ is **injective** iff $\ker \varphi = 0$, **surjective** if $\text{im}(\varphi) = \mathcal{G}$ or $\text{coker}\, \varphi = 0$.

Exercise 4.1.2 (?)
Check that $\ker \varphi$ already satisfies the sheaf property.

Definition 4.1.3 (Exact sequences of sheaves)
Let $\cdots \to \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \to \cdots$ be a sequence of morphisms in $\mathsf{Sh}(X)$, this is **exact** iff $\ker \varphi^i = \text{im}\, \varphi^{i-1}$.

Lemma 4.1.4 *(?).*
$\ker \varphi$ is a sheaf.

Proof (?).
By definition, $\ker(\varphi)(U) := \ker(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$, satisfying part (a) in the definition of presheaves. We can define restrictions $\text{Res}(U, V)|_{\ker(\varphi)(U)} \subseteq \ker(\varphi)(V)$. Use the commutative diagram for the morphism $\varphi : \mathcal{F} \to \mathcal{G}$.
Now checking gluing: Let $s_i \in \ker(\varphi)(U_i)$ such that $\text{Res}(s_i, U_i \cap U_j) = \text{Res}(s_j, U_i \cap U_j)$ for all i, j. This holds by viewing $s_i \in \mathcal{F}(U_i)$, so $\exists! s \in \mathcal{F}(\bigcup_i U_i)$ such that $\text{Res}(s, U_i) = s_i$. We want to show $s \in \ker(\varphi)\left(\bigcup U_i\right)$, so consider

$$t := \varphi\left(\bigcup_i U_i\right)(s) \in \mathcal{G}\left(\bigcup U_i\right),$$

which is zero. Now

$$\text{Res}(t, U_i) = \varphi(U_i)(\text{Res}(s, U_i)) = \varphi(U_i)(s_i) = 0$$

by assumption, using the commutative diagram. By unique gluing for $\mathcal{G}$, we have $t = 0$, since 0 is also a section restricting to 0 everywhere.

■

Definition 4.1.5 (Quotients)
For $\mathcal{F}' \leq \mathcal{F}$ a subsheaf, define the **quotient** $\mathcal{F}/\mathcal{F}' := ((\mathcal{F}/\mathcal{F}')^{\mathsf{pre}})^+$ where

$$(\mathcal{F}/\mathcal{F}')^{\mathsf{pre}}(U) := \mathcal{F}(U)/\mathcal{F}'(U).$$

5 | Friday, August 27

Theorem 5.0.1 *(Sheaf isomorphism iff isomorphism on stalks).*
Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathsf{Sh}(X)$, then φ is an isomorphism iff $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for all $p \in X$.

Proof ($\implies$).
Suppose φ is an isomorphism, so there exists a $\psi : \mathcal{G} \to \mathcal{F}$ which is a two-sided inverse for φ. Then ψ_p is a two-sided inverse to φ_p, making it an isomorphism. This follows directly from the formula:

$$\begin{aligned}\varphi_p : \mathcal{F}_p &\to \mathcal{G}_p \\ (s, U) &\mapsto (\varphi(U)(s), U).\end{aligned}$$

■

Proof ($\impliedby$).
It suffices to show $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for all U. This is because we could define $\psi(U) : \mathcal{G}(U) \to \mathcal{F}(U)$ and set $\varphi^{-1}(U) := \psi(U)$, then reversing the arrows in the diagram for a sheaf morphism again yields a commutative diagram.

Claim: $\varphi(U)$ is injective.

For $s \in \mathcal{F}(U)$, we want to show $\varphi(U)(s) = 0$ implies $s = 0$. Consider the germs $(s, U) \in \mathcal{F}_p$ for $p \in U$, we have $\varphi_p(s, U) = (0, U) = 0 \in \mathcal{F}_p$. So $S_p = 0$ for all $p \in U$. Since we have a germ, there exists $V_p \ni p$ open such that $s|_{V_p} = 0$. Noting that $\left\{ V_p \mid p \in U \right\} \rightrightarrows U$, by unique gluing we get an s where $s|_{V_p} = 0$ for all V_p, so $s \equiv 0$ on U.

Claim: $\varphi(U)$ is surjective.

Let $t \in \mathcal{G}(U)$, and consider germs $t_p \in \mathcal{G}_p$. There exists a unique $s_p \in \mathcal{F}_p$ with $\varphi_p(s_p) = t_p$, since φ_p is an isomorphism of stalks by assumption. Use that s_p is a germ to get an equivalence class (s_p, V) where $V \subseteq U$. We have $\varphi(V)(s(p), V) \sim (t, U)$, noting that s depends on p. Having equivalent germs means there exists a $W(p) \subseteq V$ with $p \in W$ with $\varphi(W(p)) \left(s(p)|_W \right) = t|_{W(p)}$. We want to glue these $\left\{ s(p)|_{W(p)} \mid p \in U \right\}$ together. It suffices to show they agree on intersections. Taking $p, q \in U$, both $s(p)|_{W(p) \cap W(q)}$ and $s(q)|_{W(p) \cap W(q)}$ map to $t|_{W(p) \cap W(q)}$ under $\varphi(W(p) \cap W(q))$. Injectivity will force these to be equal, so $\exists! s \in \mathcal{F}(U)$ with $s|_{W(p)} = s(p)$. We want to now show that $\varphi(U)(s) = t$. Using commutativity of the square, we have $\varphi(U)(s)|_{W(p)} = \varphi(W(p)) \left(s|_{W(p)} \right)$. This equals $\varphi(W(p))(s(p)) = t|_{W(p)}$. Therefore $\varphi(U)(s)$ and t restrict to sections $\left\{ w(p) \mid p \in U \right\}$. Using unique gluing for $\mathcal{G}$ we get $\varphi(U)(s) = t$.

■

Remark 5.0.2: Note: we only needed to check overlaps because of exactness of the following sequence:

$$0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i<j} \mathcal{F}(U_{ij}) \to \cdots .$$

Definition 5.0.3 (?)
Let $f \in \mathsf{Top}(X, Y)$, let $\mathcal{F} \in \mathsf{Sh}(X)$ and define the **pushforward sheaf** $f_* \mathcal{F} \in \mathsf{Sh}(Y)$ by

$$f_* \mathcal{F}(V) := \mathcal{F}(f^{-1}(V)).$$

The **inverse image** sheaf is define as

$$(f^{-1}\mathcal{F})(U) = \lim_{V \ni f(U) \text{ open}} F(V).$$

Remark 5.0.4: The inverse image sheaf generalizes stalks, recovering $\mathcal{F}_p$ when $f(U) = p$. Note that $f(U)$ need not be open, unless f is an open map. In this case $f^{-1}\mathcal{F}(f(U))$.

⚠ **Warning 5.0.5**
f^* is not the pullback!

Exercise 5.0.6 (?)
Show that $f_* \mathcal{F}$ makes sense precisely because f is continuous. Check that $f_* \mathcal{F}$ satisfies the sheaf axioms. Use that the set of opens of the form $f^{-1}(V)$ are e.g. closed under intersections, and thus inherit all of the sheaf axioms from $\mathcal{F}>$

6 | Monday, August 30

Remark 6.0.1: Let R be a commutative unital ring in which $0 \neq 1$ unless $R = 0$. The goal is to define a space X such that R is the ring of functions on X, imitating the correspondence between X a manifold and $C^0(X; \mathbb{R})$. Recall that an ideal $\mathfrak{p} \in \mathrm{Id}(R)$ is **prime** iff $\mathfrak{p} \subset A$ is a proper subset and $fg \in \mathfrak{p} \implies f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.

Definition 6.0.2 (Spectrum of a ring)
For A a ring, $\mathrm{Spec}(A)$ is the set of prime ideals. Topologize this by setting the closed sets to be of the form $V(I) = \left\{ \mathfrak{p} \in \mathrm{Spec}(A) \;\middle|\; \mathfrak{p} \supseteq I \right\}$.

Remark 6.0.3: Ideals are "contagious" under multiplication, so prime ideals have reverse contagion! It remains to prove that $\mathrm{Spec}(A)$ forms a topological space.

Example 6.0.4 ***(?):*** For A a field, $\mathrm{Spec}(A) = \{\langle 0 \rangle\}$, since any other nonzero element would be a unit and put 1 in the ideal.

Example 6.0.5 ***(?):*** For k an algebraically closed field, $\operatorname{Spec} k[t] = \left\{ \langle 0 \rangle, \langle t-a \rangle \;\middle|\; a \in A \right\}$. This is a PID, so every ideal is of the form $I = \langle f \rangle$, so

$$V(\langle f \rangle) = \begin{cases} \operatorname{Spec} k[t] & f = 0 \\ \langle x - a_1, \cdots, a - a_k \rangle & f = \prod_{i=1}^{k} (x - a_i) \end{cases}.$$

Note that this is not the cofinite topology, since $f = 0$ is a generic point.

7 | Wednesday, September 01

Example 7.0.1 ***(?):*** Let $k = \bar{k}$ be algebraically closed, then

$$\operatorname{Spec} k[x] = \left\{ \langle x - a \rangle \;\middle|\; a \in k \right\} \cup \langle 0 \rangle.$$

Similarly,

$$\operatorname{Spec} k[x, y] = \left\{ \langle x - a, y - b \rangle \;\middle|\; a, b \in k \right\} \cup \left\{ \langle f \rangle \;\middle|\; f \text{ irreducible} \right\} \cup \langle 0 \rangle.$$

Note that both have non-closed, generic points.

Example 7.0.2 ***(?):*** Consider $X := \operatorname{Spec} \mathbb{Z}_p$ and $Y := \operatorname{Spec} \mathbb{C}[t]$, then $\operatorname{Spec}(X) = \{\langle p \rangle, \langle 0 \rangle\}$ and $\operatorname{Spec}(Y) = \{\langle t \rangle, \langle 0 \rangle\}$. Both are two point spaces, with open points $\langle 0 \rangle$ and closed points $\langle p \rangle$ and $\langle f \rangle$ respectively. This spaces are homeomorphic, and later we'll see that we can distinguish them as ringed spaces.

Proposition 7.0.3 ***(Prime spectra of rings).***
Let $A \in \mathsf{CRing}$, then $\operatorname{Spec} A$ with the closed sets declared to be those of the form $V(I) = \left\{ p \in \operatorname{Spec}(A) \;\middle|\; p \supseteq I \right\}$.

Lemma 7.0.4 ***(?).***
$V(IJ) = V(I) \cup V(J)$, so if a prime ideal p contains IJ then $p \supseteq I$ or $p \supseteq J$.

Proof (?).
$\Longleftarrow$: If $I \subseteq P$ or $J \subseteq P$, then $IJ \subseteq I$ and $IJ \subseteq J$, so $IJ \subset p$.
$\Longrightarrow$: Suppose $IJ \subset p$ but $J \not\subset p$, so pick $j \in J \setminus p$. Then for all $i \in I$, we have $ij \in IJ \subseteq p$, forcing $i \in p$.

■

Lemma 7.0.5 ***(?).***
An arbitrary intersection satisfies $\bigcap_i V(I_i) = V(\sum_i I_i)$.

Proof (?).
$\implies$: For $p \in \operatorname{Spec}(A)$, we want to show that $p \supseteq \sum I_i$ iff $p \supseteq I_i$ for all i, so $I_i \subseteq \sum I_i \subset P$.
$\impliedby$: Ideals are additive groups, regardless of whether or not they're prime! ■

Proof (of proposition).

- $\emptyset$ is closed, since $\emptyset = V(A)$
- X is closed, since $X = V(0)$ and O is contained in every prime ideal.
- Closure under finite unions: by induction, it's enough to show that $V(I) \cup V(J)$ is closed. This follows from the 1st lemma above.
- Closure under arbitrary unions: this follows from the 2nd lemma.

■

Proposition 7.0.6 *(?).*
$V(I) = V(\sqrt{I})$. The proof is simple: prime ideals are radical.

Example 7.0.7 *(?)*: Note that $\operatorname{Spec} \mathbb{Z} = \left\{ \langle p \rangle, \langle 0 \rangle \;\middle|\; p \text{ is prime} \right\}$. Note that maximal ideals are always closed points, and $\langle 0 \rangle$ is not a closed point. This is homeomorphic to, say $\operatorname{Spec} \overline{\mathbb{Q}}[t]$.

Definition 7.0.8 (?)
Suppose $p \subseteq A$ is a prime ideal, then the **localization** of A at p, $A\left[(p^c)^{-1}\right]$ (or A_p) is defined as

$$A\left[(p^c)^{-1}\right] := \left\{ a/f \;\middle|\; f \notin p \right\}_{/\sim} \qquad \frac{a}{f} \sim \frac{b}{g} \iff \exists h \in A \text{ s.t. } h(ag - bf) = 0.$$

This makes the elements of p^c invertible, and is a local ring with residue field $\kappa = \operatorname{ff}(A/p)$ and maximal ideal pA_p. Ideals of A_p biject with ideals of A contained in p.

Remark 7.0.9: Idea: A_p should look like germs of functions at the point p. Note that localizing at the ideal p is like deleting $\operatorname{cl}_X(V(p))$, which is also useful. We now want to construct a sheaf $\mathcal{O} = \mathcal{O}_{\operatorname{Spec} A}$ which has stalks A_p. We'll construct something that's obviously a sheaf, at the cost of needing to work hard to prove things about it!

Definition 7.0.10 (Structure sheaf)
For $U \in \operatorname{Spec}(A)$ open, so $U = V(I)^c$, define the **structure sheaf of** X as the sheaf given

$$\mathcal{O}(U) := \left\{ s : U \to \coprod p \in U A_p \;\middle|\; s(p) \in A_p, \text{ and } s \text{ is locally a fraction} \right\}.$$

Here *locally a fraction* means that for all $p \in U$ there is an open $p \in V \subseteq U$ and elements $a, f \in A$ such that

1. $f \notin Q$ for any $Q \in V$ and
2. $s(Q) = a/f$ for all $Q \in V$.

Restriction is defined for $V \subseteq U$ as honest function restriction on $\mathcal{O}(U) \to \mathcal{O}(V)$.

Remark 7.0.11: Note that this is sheafifying the presheaf $U = D_f \mapsto A_f$.

Example 7.0.12*(?)*: Let $k \in \mathsf{Field}$, then $X := \operatorname{Spec}(k) = \{\langle 0 \rangle\}$ and $\mathcal{O}_X$ is determined by

$$\Gamma(X; \mathcal{O}_X) = \left\{ s : \operatorname{Spec} k \to k \;\middle|\; \cdots \right\} = k,$$

since the conditions are vacuous here.

Example 7.0.13*(?)*: Let $X = \operatorname{Spec} \mathbb{C}[t] = \{\langle 0 \rangle, \langle 1 \rangle\}$ and $\mathcal{O}_X(X) = \mathbb{C}[t]$ and $\mathcal{O}_X(\langle 0 \rangle) = \mathbb{C}(t)$.

8 | Friday, September 03

Remark 8.0.1: Last time: we defined $\operatorname{Spec} A$ as a topological space and $\mathcal{O}_{\operatorname{Spec} A}$, a sheaf of rings on $\operatorname{Spec} A$ which evidently satisfied the gluing condition:

$$\mathcal{O}_{\operatorname{Spec} A}(U) := \left\{ s : U \to \coprod_{p \in U} A\left[p^{-1}\right] \;\middle|\; s(p) \in A\left[p^{-1}\right] \forall p,\; s \text{ is locally a fraction} \right\}.$$

Example 8.0.2*(?)*: Set $X := \mathbb{A}^1_{/k} := \operatorname{Spec} k[t]$ for $k = \bar{k}$. Take a point $\langle t \rangle = \langle t - 0 \rangle \in \operatorname{Spec} k[t]$ corresponding to 0, then $\mathcal{O}_X(X \setminus \{0\}) = k[t, t^{-1}]$, i.e. functions of the form f/t^k for any k. Generally for $p = \langle t - a_i \rangle$ we get $s_p \in k[t]\left[\{t-a_i\}^{-1}\right]$. Note that for $p = \langle 0 \rangle$, we get $s_p \in k(t)$.

Claim: s is determined by $s_{\langle 0 \rangle}$, so there is an injective map

$$\begin{aligned} \mathcal{O}(k[t] \setminus \{0\}) &\to k(t) \\ s &\mapsto s_{\langle 0 \rangle}. \end{aligned}$$

Proof (?).
Note that $\langle 0 \rangle$ is in every open set, so locally near p there exists a $P \in V$ and a, f with $f \notin Q$ for all Q and $s_Q = a/f$ for all $Q \in V$. Since $\langle 0 \rangle \in V$, we have $s_{\langle 0 \rangle} = a/f \in k(t)$ and $s_p = a/f \in A_p$. Since $A_p \subseteq k(t)$, we get $s_p = s_{\langle 0 \rangle}$ under this inclusion. ■

Claim: $\mathcal{O}(\operatorname{Spec} k[t] \setminus \{0\}) = k[t, t^{-1}]$.

Proof (?).
We showed that the LHS is a subset of $k(t)$, so which subsets can be written as things that are locally fractions on the complement of zero.
$\supseteq$: This can clearly be done in $k[t, t^{-1}]$ since every element is locally the fraction f/t^k.
$\subseteq$: Suppose f/g with f, g coprime (this is a PID!) with a pole away from zero, so $g \in Q$ for some $Q \neq \langle 0 \rangle$. But then f/g isn't in A_Q.

■

Remark 8.0.3: Note that $X := \operatorname{mSpec} k[t] \subseteq X' := \operatorname{Spec} k[t]$ as the set of closed points, and restricting $\mathcal{O}_{X'}$ to X yields the sheaf of regular functions. Having the extra generic point was useful!

Exercise 8.0.4 (?)
Show that the maximal ideals in $\operatorname{mSpec} A$ correspond precisely to closed points.

Example 8.0.5 ***(?):*** Of a function that is locally but not globally a fraction. Take $A := k[x, y, z, w]/\langle xy - zw\rangle$, which is the cone over a smooth quadric surface and $X := \operatorname{Spec} A$. Then take $U = \operatorname{Spec}(A) \setminus V(y, w) = V(y)^c \cap V(w)^c$ and consider the section

$$s(p) := \begin{cases} x/w & p \in V(w)^c \\ z/y & p \in V(y)^c. \end{cases}$$

For $p \in U$, it makes sense to consider x/w and z/y. Are they equal? The answer is yes because $xy - zw = 0$. Check that this can't be a global fraction, which is a consequence of this random open set not being the complement of localizing at a prime ideal.

Definition 8.0.6 (?)
Given $f \in A$, the **distinguished open** $D(f)$ corresponding to f is defined as

$$D(f) = V(\langle f\rangle)^c := \left\{ p \in \operatorname{Spec}(A) \;\middle|\; f \in p \right\}^c = \left\{ p \in \operatorname{Spec} A \;\middle|\; f \notin p \right\},$$

i.e. the points of $\operatorname{Spec}(A)$ where f doesn't vanish.

Remark 8.0.7: The sets $\left\{ D(f) \;\middle|\; f \in A \right\}$ for a basis for the topology on $\operatorname{Spec}(A)$. This follows from writing $V(I)^c = \bigcup_{f \in I} D(f)$.

Theorem 8.0.8 ***(Hartshorne Prop 2.2).***
Let $A \in \mathsf{CRing}$ be unital with $1 \neq 0$ unless $A = 0$ and consider $(\operatorname{Spec} A, \mathcal{O})$. Then

a. For any $p \in \operatorname{Spec} A$, the stalk $\mathcal{O}_p \cong A\left[p^{-1}\right]$.
b. For any $f \in A$, $\mathcal{O}(D(f)) = A\left[f^{-1}\right]$.
c. Taking $f = 1$, $\Gamma(\operatorname{Spec} A, \mathcal{O}) = A$.

Remark 8.0.9: Note that (b) gives the values of $\mathcal{O}$ on a basis of opens, which determines the sheaf.

Proof (of a).
Define a map

$$\begin{aligned} f_p : \mathcal{O}_p &\to A_p \\ (U, s) &\mapsto s(p). \end{aligned}$$

This is well-defined since $p \in W$ for any $W \subseteq U \cap V$ for equivalent germs $(U, s) \sim (V, t)$.
Surjectivity: given $x = a/g \in A_p$, we want $(U, s) \in \mathcal{O}_p$ such that $f_p(U, s) = a/g$, so just take $U = D(g)$ and $s = a/g$ (which makes sense!) and evidently maps to a/g.
Injectivity: supposing $f_p(U, s) = 0$ in A_p, we want $(U, s) = 0$. If $s(p) = 0$, then there exists some $h \in P$ with $h \cdot s(p) = 0$. Since $s(p)$ is locally a fraction, we can find $p \in V \subseteq U$ with $s = a/g$ on V with $g \neq 0$ on V, so consider $V \cap D(h)$. The claim is that s is 0 here, which follows from $h \cdot (a/g) = 0$.

■

9 | Wednesday, September 08

Remark 9.0.1: Recall that we defined a first version of *affine schemes*, namely pairs $(\operatorname{Spec} A, \mathcal{O}_A)$ where for $U \subseteq \operatorname{Spec} A$ open we have $s \in \mathcal{O}(U)$ locally represented by $s|_V = a/f$ for $V \subseteq U$ where $a, f \in A$ and $V(f) \cap V = \emptyset$, so f doesn't vanish on V. Note that the $D(f)$ form a topological basis for $\operatorname{Spec} A$, and the gluing condition is difficult, i.e. $\mathcal{O}(U)$ may be hard to compute. We proved that $OO_{\mathfrak{p}} = A\left[\mathfrak{p}^{-1}\right]$ last time, and today we're showing

- $\mathcal{O}(D(f)) = A\left[f^{-1}\right]$,
- $\Gamma\left(()\operatorname{Spec} A, \mathcal{O}\right) \cong A$.

Proof (of b and c).
$b \implies c$: Take $f = 1 \in A$, then $\mathcal{O}(\operatorname{Spec} A) = \mathcal{O}(D(1)) = A$ using (b), so the only hard part is showing (b).
To prove (b), by definition of $\mathcal{O}$ there is a ring morphism

$$\begin{aligned} \psi : A\left[f^{-1}\right] &\to \mathcal{O}(D(f)) \\ \frac{a}{f^n} &\mapsto \frac{a}{f^n}. \end{aligned}$$

Note that this is just a careful statement, since the morphisms on stalks $\psi_{\mathfrak{p}} : A\left[f^{-1}\right] \to A\left[\mathfrak{p}^{-1}\right]$ by not be injective in general.

Claim: ψ is bijective.

Proof (of injectivity).
Suppose $\psi(s) = 0$, we then want to show $s = 0$. Write $s = a/f^n$, then for all $\mathfrak{p} \in D(f)$ we know $a/f^n = 0 \in A[\mathfrak{p}^{-1}]$. So for each $\mathfrak{p}$ there is some $h_\mathfrak{p} \notin \mathfrak{p}$ where

$$h_\mathfrak{p}(a \cdot 1 - f^n \cdot 0) = 0 \qquad \text{in } A$$

in A. Consider the ideal $\mathfrak{a} := \operatorname{Ann}(a) := \left\{ b \in A \;\middle|\; ab = 0 \in A \right\} \ni h_\mathfrak{p}$. So take the closed subset $V(\mathfrak{a})$, which does not contain $\mathfrak{p}$ since $\mathfrak{a} \not\subseteq \mathfrak{p}$. Now iterating over all $\mathfrak{p} \in D(f)$, we get $V(\mathfrak{a}) \cap D(f) = \emptyset$. So $V(\mathfrak{a}) \subseteq V(f) = D(f)^c$, thus $f \in \sqrt{\mathfrak{a}}$ and $f^m a = 0$ for some m. Then $f^m(a \cdot 1 - f^n \cdot 0) = 0$ in A, so $a/f^n = 0$ in $A[f^{-1}]$.

■

Proof (of surjectivity).
Step 1: Expressing $s \in \mathcal{O}(D(f))$ nicely locally.
By definition of $\mathcal{O}_{D(f)}$, there exist V_i with $s|_{V_i} = a_i/g_i$ for $a_i, g_i \in A$. We'd like $g_i = h_i^{m_i}$ for some m_i, so g is a power of h_i, but this may not be true a priori. Fix $V_i = D(h_i)$, then $a_i/g_i \in \mathcal{O}(D(h_i))$ implies that $g_i \notin \mathfrak{p}$ for any $\mathfrak{p} \in D(h_i)$. This implies that $D(h_i) \subseteq D(g_i)$, and taking complements yields $V(h_i) \supseteq V(g_i)$, and $h_i \in \sqrt{\langle g_i \rangle}$ and $h_i^n = g_i$. Writing $g_i = h_i^n/c$ we have $a_i/g_i = ca_i/h_i^n$. Note that $D(h_i) = D(h_i^n)$. Now replace a_i with ca_i and g_i with h_i to get

$$s|_{D(h_i)} = a_i/h_i.$$

Step 2: Quasicompactness of $D(f)$.
Note that $\{D(h_i)\}_{i \in I} \rightrightarrows D(f)$, so take a finite subcover $\{D(h_i)\}_{i \leq m}$.
Proof of quasicompactness: since $D(f) \supseteq \bigcup_{i \in I} D(h_i)$, we get

$$V(f) \subseteq \bigcap_{i \in I} V(h_i) = V\left(\sum h_i\right).$$

So $f^u \in \sum h_i$, and up to reordering we can conclude $f^u = \sum_{i \leq m} b_i h_i$ for some $b_i \in A$. Then $D(f) \subseteq \bigcup_{i \leq m} D(h_i)$.

Remark 9.0.2: Since we can write $\operatorname{Spec} A = D(1)$, it is quasicompact.

Step 3: Showing surjectivity.
Next time.

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10 | Friday, September 10

10.1 Sections and Stalks of $\mathcal{O}_{\operatorname{Spec} A}$ as Localizations

Remark 10.1.1: Last time: any affine scheme is quasicompact, so for each ring A and an open cover $\mathcal{U} \rightrightarrows D(f)$ then there is a finite subcover $\{D(h_i)\} \rightrightarrows D(f)$. We were looking at proposition: for the ringed space $(\operatorname{Spec} A, \mathcal{O}_A)$,

- $\mathcal{O}_{\mathfrak{p}} \cong A\left[\mathfrak{p}^{-1}\right]$ for all $\mathfrak{p} \in \operatorname{Spec} A$,
- $\mathcal{O}(d(f)) \cong A\left[f^{-1}\right]$ for all $f \in A$,
- $\Gamma\left(\operatorname{Spec} A; \mathcal{O}_A\right) \cong A$.

Note that $\mathcal{O}_A$ is uniquely characterized by these properties.

Remark 10.1.2: We can write $D(1) = \operatorname{Spec} A$ and write $\{D(h_i)\} \rightrightarrows \operatorname{Spec} A$ to obtain $1^n = \sum b_i h_i$. This is analogous to a partition of unity, where $b_i h_i$ vanishes on $D(h_i)^c = V(h_i)$

Proof (of 2.2b).
Let $\psi : A\left[f^{-1}\right] \hookrightarrow \mathcal{O}(D(f))$ where we just take restrictions of functions. We know this is injective, and we want to show surjectivity.
Step 1: Let $s \in \mathcal{O}(D(f))$. For each open $D(h_i)$, write $s|_{D(h_i)} = a_i/h_i$ for some $a_i \in A$.
Step 2: By quasicompactness, write $f^n = \sum_{1 \leq i \leq m} b_i h_i$.
Step 3: Glue the a_i/h_i into an element a/f of $A\left[f^{-1}\right]$.
Part 1: For any $1 \leq i \neq j \leq m$, $D(h_i h_j) = D(h_i) \cap D(h_j)$. Note that $a_i/h_i = a_j/h_j$ in $\mathcal{O}(D(h_i h_j))$, and these are elements of $A\left[h_i h_j^{-1}\right]$ since $a_i/h_i = a_i h_j / h_i h_j$. Using injectivity of ψ for $h_i h_j$, we get an equality $a_i/h_i = a_j/h_j$ in $A_{h_i h_j}$. Then for ℓ large enough, $(h_i h_j)^\ell (a_i h_j - a_j h_i) = 0 \in A$. Regrouping yields $h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j a_j) = 0$, so

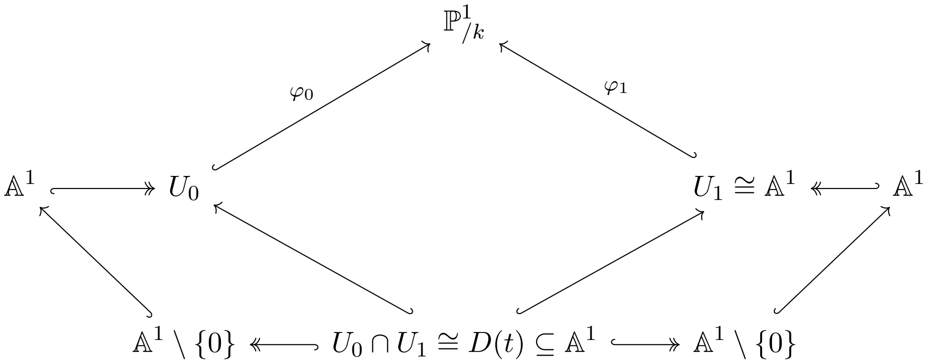
$$\frac{a_i h_i^n}{h_i^{n+1}} = \frac{a_j h_j^r}{h_j^{n+1}} := \frac{\tilde{a}_i}{\tilde{h}_i} = \frac{\tilde{a}_j}{\tilde{h}_j},$$

noting that $D(\tilde{h}_i) = D(\tilde{h}_i)$ since $\tilde{h}_i$ is a power of h_i.
Now use POU gluing to write $f^n = \sum_i \tilde{b}_i \tilde{h}_i$ and $a := \sum \tilde{a}_i \tilde{h}_i \in A$ be a global function on $D(f)$. Then (claim) $a_j/f^n = \tilde{a}_j/\tilde{h}_j$ on $D(\tilde{h}_j)$. We can rewrite

$$\tilde{h}_j a = \sum_i \tilde{b}_i \tilde{a}_i \tilde{h}_j = \sum_i \tilde{b}_i \tilde{a}_j \tilde{h}_i.$$

But then $a/f^n = s|_{D(\tilde{h}_i)}$, so $s = a/f^n$. ∎

Example 10.1.3 ***(?):*** Consider $\mathbb{P}^1_{/k}$ as a scheme – we know the space, and the claim is that we can glue sheaves of affines to obtain a structure sheaf for it. Cover $\mathbb{P}^1$ by $U_0 = \mathbb{P}^1 \setminus \{\infty\} \cong \mathbb{A}^1$ and $U_1 = \mathbb{P}^1 \setminus \{0\} \cong \mathbb{A}^1$. The gluing data is the following:



Link to Diagram

Here the transition maps are

$$\begin{aligned}\varphi_1 \circ \varphi_0^{-1} : \varphi_0(U_0 \cap U_1) &\to \varphi_1(U_0 \cap U_1) \\ t &\mapsto t^{-1}.\end{aligned}$$

What is the map on sheaves? We need a map $\mathcal{O}|_{U_0 \setminus \{0\}} \xrightarrow{\sim} \mathcal{O}|_{U_1 \setminus \{\infty\}}$.

Definition 10.1.4 (Ringed Space)
A **ringed space** $(X, \mathcal{O}_X) \in \mathsf{Top} \times \mathsf{Sh}(X, \mathsf{Ring})$ as a topological space with a sheaf of rings. A morphism is a pair $(f, f^\#) \in C^0(X, Y) \times \in \operatorname{Mor}_{\mathsf{Sh}}(\mathcal{O}_Y, f_* \mathcal{O}_X)$.

Example 10.1.5 ***(?):*** The scheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is a ringed space.

Example 10.1.6 ***(?):*** Consider $\mathbb{R}$ in the Euclidean topology, then $(\mathbb{R}, C^0(-, \mathbb{R}))$ with the sheaf of continuous functions is a ringed space. Then consider the morphism given by projection onto the first coordinate of $\mathbb{R}^2$:

$$\begin{aligned}(\pi, \pi^\#) : (\mathbb{R}^2, C^0(-, \mathbb{R})) &\to (\mathbb{R}, C^\infty(-, \mathbb{R})) \\ (x, y) &\mapsto x.\end{aligned}$$

For $\pi^\#$, we can consider $\pi_* C^0(-, \mathbb{R})(U) := C^0(\pi^{-1}(U)) = C^0(U \times \mathbb{R})$, so

$$\begin{aligned}\pi^\# : C^\infty(U, \mathbb{R}) &\to C^0(U \times \mathbb{R}) \\ f &\mapsto f \circ \pi,\end{aligned}$$

which is a well-defined map of rings since smooth functions are continuous.

⚠ **Warning 10.1.7**
Not every scheme is built out of affine opens!

11 | Monday, September 13

11.1 Affine Schemes

Definition 11.1.1 (Restricted sheaves)
Let $(X, \mathcal{O}_X) \in \mathsf{RingSp}$ and $U \subseteq X$ be open, then for $V \subseteq U$ open, define the restricted sheaf $\mathcal{O}_X|_V(V) := \mathcal{O}_X(V)$.

Warning 11.1.2

$$\mathsf{Sh}_{/X} \ni \mathcal{O}_X|_U \neq \mathcal{O}_X(U) \in \mathsf{Ring}.$$

Remark 11.1.3: Recall the definition of a ringed space $(X, \mathcal{O}_X)$. The quintessential example: X a smooth manifold and $\mathcal{O}_X := C^\infty(-; \mathbb{R})$ the sheaf of smooth functions. For defining morphisms, consider a map $f : X \to Y$, then an alternative way of defining f to be smooth is that there is a pullback

$$\begin{aligned} f^* : C^0(V, \mathbb{R}) &\to C^0(U, \mathbb{R}) \\ g &\mapsto g \circ f \end{aligned}$$

for $U \subseteq X, V \subseteq Y$, and that f^* in fact restricts to $f^* : C^\infty(V; \mathbb{R}) \to C^\infty(U; \mathbb{R})$, i.e. preserving smooth functions.

Definition 11.1.4 (Morphisms of ringed spaces)
A morphism of ringed spaces is a pair

$$(M, \mathcal{O}_M) \xrightarrow{(\varphi, \varphi^\#)} (N, \mathcal{O}_N).$$

where $\varphi \in C^0(M, N)$ and $\varphi^\# \in \operatorname{Mor}_{\mathsf{Sh}_{/N}}(\mathcal{O}_N, \varphi_* \mathcal{O}_M)$.
This is an **isomorphism** of ringed spaces if

1. φ is a homeomorphism, and
2. $\varphi^\#$ is an isomorphism of sheaves.

Remark 11.1.5: In the running example,

$$\varphi^\#(U) : \mathcal{O}_N(U) \to \varphi_* \mathcal{O}_M(M) = \mathcal{O}_M(\varphi^{-1}(U)).$$

This implies that maps of ringed spaced here induce smooth maps, and so there is an embedding $\mathsf{smMfd}_{/\mathbb{R}} \hookrightarrow \mathsf{RingSp}$.

Remark 11.1.6: We'll try to set up schemes the same way one sets up smooth manifolds. A topological manifold is a space locally homeomorphic to $\mathbb{R}^n$, and a smooth manifold is one in which it's locally isomorphic as a ringed space to $(\mathbb{R}^n, C^\infty(-; \mathbb{R}))$ with its sheaf of smooth functions.

Definition 11.1.7 (Smooth manifolds, alternative definition)
A **smooth manifold** is a ringed space $(M, \mathcal{O}_M)$ that is locally isomorphic to $(\mathbb{R}^d, C^\infty(-;\mathbb{R}))$, i.e. there is an open cover $\mathcal{U} \rightrightarrows M$ such that

$$(U_i, \mathcal{O}_M|_{U_i}) \cong (\mathbb{R}^n, C^\infty(-;\mathbb{R})).$$

Example 11.1.8 *(?)*: An example of a morphism of ringed spaces that is not an isomorphism: take $(\mathbb{R}, C^0) \to (\mathbb{R}, C^\infty)$ given by $(\mathrm{id}, \mathrm{id}^\#)$ where $\mathrm{id}^\# : C^\infty \to \mathrm{id}_* C^0$ is given by $\mathrm{id}^\#(U) : C^\infty(U) \to C^0(U)$ is the inclusion of continuous functions into smooth functions.

Remark 11.1.9: We'll define schemes similarly: build from simpler pieces, namely an open cover with isomorphisms to affine schemes. A major difference is that there may not exist a *unique* isomorphism type among all of the local charts, i.e. the affine scheme can vary across the cover.

Remark 11.1.10: Recall that for A a ring we defined $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, where $\operatorname{Spec} A := \{\text{Prime ideals } \mathfrak{p} \trianglelefteq A\}$, equipped with the Zariski topology generated by closed sets $V(I) := \left\{\mathfrak{p} \trianglelefteq A \;\middle|\; I \supseteq \mathfrak{p}\right\}$. We then defined

$$\mathcal{O}_{\operatorname{Spec} A}(U) := \left\{ s : U \to \coprod_{\mathfrak{p} \in U} A\left[\mathfrak{p}^{-1}\right] \;\middle|\; s(\mathfrak{p}) \in A\left[\mathfrak{p}^{-1}\right], s \text{ locally a fraction} \right\}.$$

We saw that

1. We can identify stalks: $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} = A\left[\mathfrak{p}^{-1}\right]$
2. We can identify sections on distinguished opens:

$$\mathcal{O}_{\operatorname{Spec} A}(D_f) = A\left[f^{-1}\right] = \left\{ a/f^k \;\middle|\; a \in A, k \in \mathbb{Z}_{\geq 0} \right\},$$

   where $D_f := V(f)^c = \left\{\mathfrak{p} \in \operatorname{Spec} A \;\middle|\; f \notin \mathfrak{p}\right\}$.

As a corollary, we get $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A$, noting $\operatorname{Spec} A = d_1$ and $A\left[1^{-1}\right] = A$. Thus we can recover the ring A from the ringed space $(X, \mathcal{O}_X) := (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ by taking global sections, i.e. $\Gamma(\operatorname{Spec} A; \mathcal{O}_{\operatorname{Spec} A}) = A$.

11.2 Affine Varieties

Remark 11.2.1: Let $k = \bar{k}$ and set $\mathbb{A}^n_{/k} = k^n$ whose regular functions are given by $k[x_1, \cdots, x_n]$, regarded as maps to k.

Definition 11.2.2 (Affine variety)
An affine variety is any set of the form

$$X := V(f_1, \cdots, f_n) = \left\{ p \in \mathbb{A}^n_{/k} \;\middle|\; f_1(p) = \cdots = f_m(p) = 0 \right\}$$

for $f_i \in k[x_1, \cdots, x_n]$,

Remark 11.2.3: Writing $I = \langle f_1, \cdots, f_m \rangle$, we have $X = V(\sqrt{I})$. Letting $I(S) = \left\{ f \in k[x_1, \cdots, x_n] \;\middle|\; f|_S = 0 \right\}$ then by the Nullstellensatz, $IV(I) = \sqrt{I}$. This gives a bijection between affine varieties in $\mathbb{A}^n_{/k}$ and radical ideals $I \trianglelefteq k[x_1, \cdots, x_n]$.

Definition 11.2.4 (Coordinate rings of affine varieties)
The **coordinate ring** of an affine variety X is $k[X] \coloneqq k[x_1, \cdots, x_n]/I(X)$, regarded as polynomial functions on X.

Remark 11.2.5: We quotient here because if the difference of functions is in $I(X)$, these functions are equal when restricted to X. For example, $y = x$ in $k[x, y]/\langle x - y \rangle$, which are different functions where for $X \coloneqq \Delta$, we have $x|_\Delta = y|_\Delta$.

Remark 11.2.6: As an application of the Nullstellensatz, there is a correspondence

$$\{\text{Points } p \in X\} \underset{V(-)}{\overset{I(-)}{\rightleftarrows}} \operatorname{mSpec} k[X]$$

Remark 11.2.7: Why is an affine variety X an example of an affine scheme $\operatorname{Spec} k[X]$? These don't have the same underlying topological space:

$$\tau(X) \coloneqq \left\{ V(I) \coloneqq \left\{ p \in X \;\middle|\; f_i(p) = 0 \; \forall f_i \in I \right\} \;\middle|\; I \trianglelefteq k[X] \right\}$$
$$\tau(\operatorname{mSpec} k[X]) \coloneqq \left\{ V(I) \coloneqq \left\{ \mathfrak{m} \in \operatorname{mSpec} k[X] \;\middle|\; \mathfrak{m} \supseteq I \right\} \;\middle|\; I \trianglelefteq k[X] \right\}.$$

However, they are closely related:

$$\tau(\operatorname{mSpec} k[X])|_{\operatorname{Spec} k[X]} = \tau(X_{\text{zar}}),$$

i.e. the space $\operatorname{Spec} k[X]$ with the restricted topology from $\operatorname{mSpec} k[X]$ is homeomorphic to X with the Zariski topology. I.e. restricting to *closed points* recovers regular functions on X.

⚠ **Warning 11.2.8**
Defining things that are locally isomorphic to schemes would work for objects but not morphisms!

12 | Wednesday, September 15

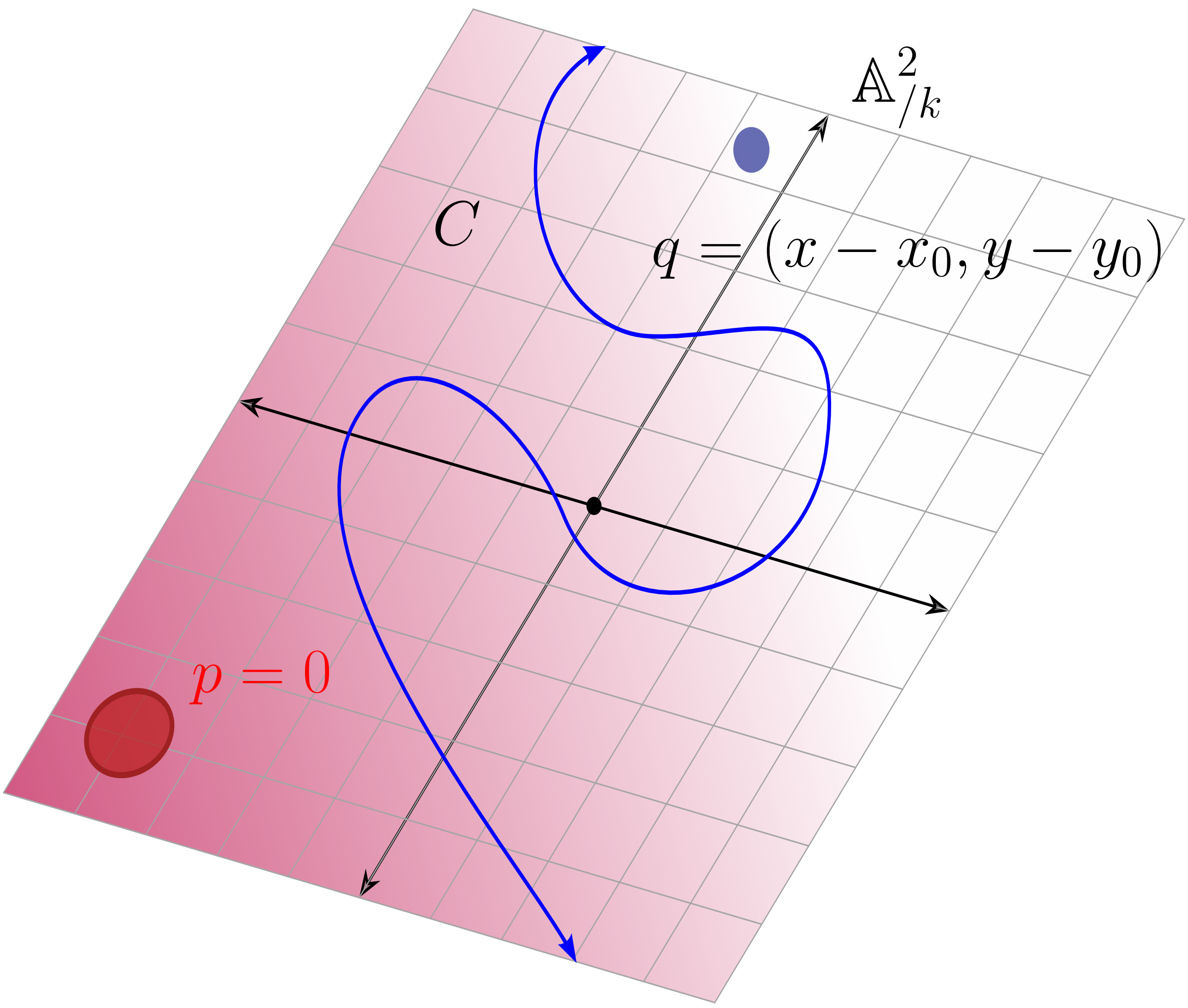
Remark 12.0.1: Last time: for AffVar we considered $X = V(I) \subseteq \mathbb{A}^n_{/k}$, and for AffSch we considered $\operatorname{Spec} k[X]$ where $k[X] \coloneqq k[x_1, \cdots, x_n]/I(X)$. Both had the Zariski topology, and

$X = \operatorname{mSpec} k[X] \subseteq \operatorname{Spec} k[X]$. We had structure sheaves $\mathcal{O}_{\operatorname{Spec} k[X]}$, and for X, we have $U' := U \cap \operatorname{mSpec} k[X]$. On $\operatorname{mSpec} k[X]$, we have the notion of a regular function, and $\mathcal{O}_X(U') = \mathcal{O}_{\operatorname{Spec} k[X]}(U')$. The previous setup only worked for rings finitely generated over a field, whereas for schemes, we can take things like $\operatorname{Spec} \mathbb{Z}$, so they're much more versatile (e.g. for number theory).

Example 12.0.2 ***(?):*** Compare $\mathbb{A}^2_{/k} \in \mathsf{AffVar}$ to $\operatorname{Spec} k[x,y]$. Note that $\langle 0 \rangle \in \operatorname{Spec} k[x,y]$, and taking its closure yields

$$\begin{aligned}
\operatorname{cl}(\langle 0 \rangle) &= \bigcap_{V(I) \supseteq \langle 0 \rangle} V(I) \\
&= \bigcap_{V(I) \ni 0} V(I) \\
&= V(0) \\
&= \operatorname{Spec} k[x,y],
\end{aligned}$$

so 0 is a dense point!



But there are prime ideals of height > 1. For example, any irreducible subvariety of $\mathbb{A}^2$ yields a generic point.

Krull's dimension theorem?

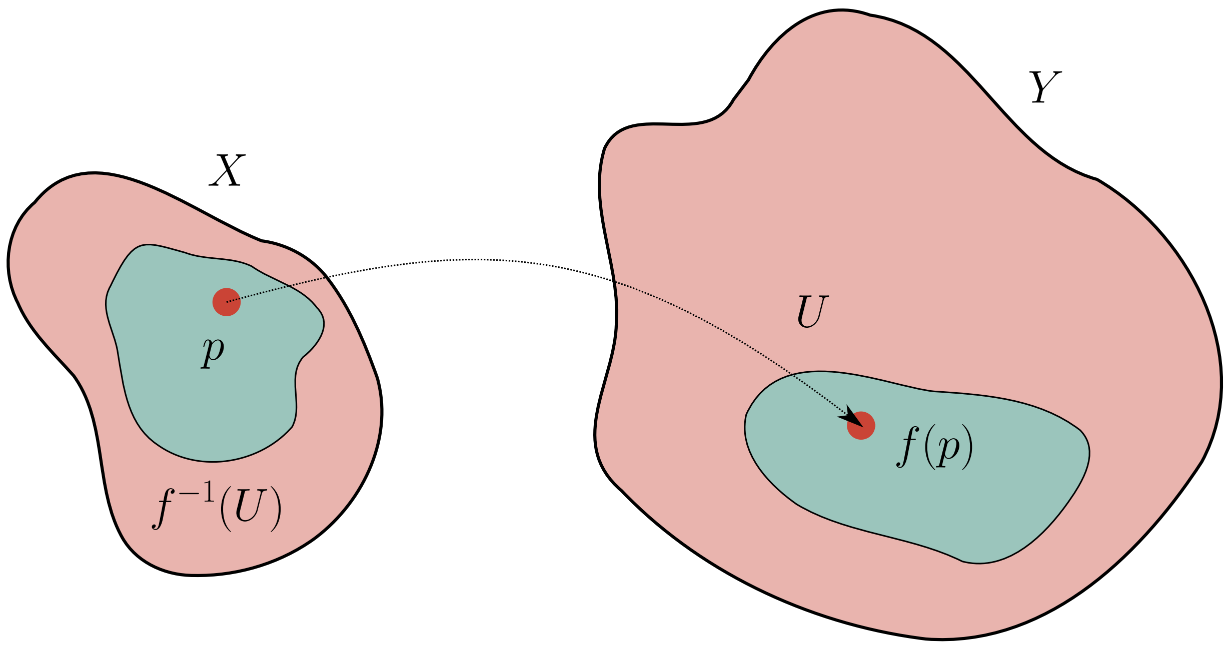
Exercise 12.0.3 (?)
Try to draw $\operatorname{Spec} \mathbb{Z}$ and $\operatorname{Spec} \mathbb{Z}[t]$.

Remark 12.0.4: We'll now try a naive definition of schemes, which we'll find won't quite work.

Definition 12.0.5 (A wrong definition of a scheme!)
A **scheme** is a ringed space $(X, \mathcal{O}_X)$ which is locally an affine scheme, i.e. there exists an open cover $\mathcal{U} \rightrightarrows X$ such that there is a collection of rings A_i with

$$(U_i, \mathcal{O}_X|_{U_i}) \xrightarrow{\sim} (\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i}).$$

Remark 12.0.6: This produces the right objects, but not the correct morphisms. This is a subtle issue! With this definition, a morphism of schemes would be a morphism of ringed spaces $(f, f^\#)$ with $f \in \mathsf{Top}(X, Y)$ and $f^\# \in \mathsf{Sh}_{/Y}(\mathcal{O}_Y, f_* \mathcal{O}_X)$, where $f^\#$ is supposed to capture "pullback of functions". The issue: $f^\#$ may not notice that $p \xrightarrow{f} f(p)$! In particular, it may not preserve the fact that $f(p) = 0$.



Hartshorne exercises for how this issue can actually arise.

Remark 12.0.7: Let $(f, f^{\#})$ be a map of ringed spaces, then there is an induced map

$$\begin{aligned} f_p^{\#} : \mathcal{O}_{Y,f(p)} &\to \mathcal{O}_{X,p} \\ (U, s) &\mapsto (f^{-1}(U), f^{\#}(U)(s)). \end{aligned}$$

Definition 12.0.8 (Locally ringed space)
A **locally ringed space** $(X, \mathcal{O}_X)$ is a ringed space for which the stalks $\mathcal{O}_{X,p} \in \mathsf{LocRing}$ are local rings, i.e. there exists a unique maximal ideal.

Example 12.0.9 ***(of a locally ringed space):*** For $(X, \mathcal{O}_X) := (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, we saw that $\mathcal{O}_{X,p} = A\left[p^{-1}\right]$, which is local.

Definition 12.0.10 (Morphisms of locally ringed spaces)
A **morphism of locally ringed spaces** is a morphism of ringed spaces

$$(f, F^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

such that $f_p^{\#} : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a homomorphism of local rings, i.e. $f^{\#}(\mathfrak{m}_{f(p)}) \subseteq \mathfrak{m}_p$.

Here we should also require that $f^{\#} \neq 0$.

Remark 12.0.11: Morally: this extra condition enforces that pulling back functions vanishing at $f(p)$ yields functions which vanish at p.

Remark 12.0.12: Alternatively one could require that $(f^{\#})^{-1}(\mathfrak{m}_p) = \mathfrak{m}_{f(p)}$, and (claim) this is equivalent to the above definition. Use that $(f^{\#})^{-1}(\mathfrak{m}_p)$ is a prime ideal containing $\mathfrak{m}_p$.

Example 12.0.13 ***(of a locally ringed space):*** Take $(X, \mathcal{O}_X) := (\mathbb{R}, C^0(-;\mathbb{R}))$. Why this is in $\mathsf{LocRingSp}$: write a stalk as

$$C_p^0 = \left\{ (f, I) \;\middle|\; I \ni p \text{ an interval}, f \in \mathsf{Top}(I, \mathbb{R}) \right\}_{/\sim}.$$

Why is this local? Take $\mathfrak{m}_p := \left\{ (f, I) \;\middle|\; f(p) = 0 \right\}$, which is maximal since $C_p^0/\mathfrak{m} \cong \mathbb{R}$ is a field. Then $\mathfrak{m}_p^c = \left\{ (f, I) \;\middle|\; f(p) \neq 0 \right\}$, and any continuous function that isn't zero at p is nonzero in some neighborhood $J \supseteq I$, so $f|_J \neq 0$ anywhere. Then $(f, I) \sim (f|_J, J)$, which is invertible in the ring, so any element in the complement is a unit.

Example 12.0.14 ***(?):*** Consider

$$(\mathbb{R}, C^0(-;\mathbb{R})) \xrightarrow{(f, f^{\#})} (\mathbb{R}, C^\infty(-;\mathbb{R})).$$

Take $f = \mathrm{id}$ and the inclusion

$$f^{\#} : C^\infty(-;\mathbb{R}) \hookrightarrow \mathrm{id}_* C^0(-;\mathbb{R}) = C^0(-;\mathbb{R}).$$

Then

$$f_p^{\#} : C_p^{\infty}(-;\mathbb{R}) \to C_p^0(-;\mathbb{R}).$$

satisfies $f_p^{\#}(\mathfrak{m}_{\mathrm{id}(p)}^{\infty}) = \mathfrak{m}_p^0$. We also have $(f_p^{\#})^{-1}(\mathfrak{m}_p^0) = \mathfrak{m}_p^{\infty}$, since the germs are just equal.

Definition 12.0.15 (Scheme)
A **scheme** $(X, \mathcal{O}_X)$ is a locally ringed space which is locally isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ in $\mathsf{LocRingSp}$. A **morphism of schemes** is a morphism in $\mathsf{LocRingSp}$.

Remark 12.0.16: Note that we can restrict to opens, since this doesn't change the stalks.

Remark 12.0.17: As a proof of concept that this is a good notion, we'll see that there's a fully faithful contravariant functor $\operatorname{Spec} : \mathsf{CRing} \to \mathsf{Sch}$, so

$$\operatorname{Spec}(\operatorname*{Mor}_{\mathsf{Ring}}(B, A)) = \operatorname*{Mor}_{\mathsf{Sch}}(\operatorname{Spec} A, \operatorname{Spec} B).$$

13 | Appendix

Remark 13.0.1: A bunch of stuff I always forget!

Definition 13.0.2 (Classical AG)

- A **section** is just an element $s \in \mathcal{F}(U)$.
- A **stalk** of a (pre)sheaf $\mathcal{F}$ at a point p is defined as

$$\mathcal{F}_p := \underset{p \ni U_i}{\underrightarrow{\operatorname{colim}}}(\mathcal{F}(U_i), \operatorname{res}_{ij}).$$

- A **germ** $\tilde{f}_p$ at a point p is an element in a stalk $\mathcal{F}_p$. It can concretely be described as

$$\tilde{f}_p = [(U \ni p, s \in \mathcal{F}(U))]/\sim \qquad (U, s) \sim (V, t) \iff \exists W \subseteq U \cap V,\ s|_W = t|_W.$$

Definition 13.0.3 (Colimit of a diagram)
Given a diagram J in a category C, regard it as a functor $F : \mathsf{J} \to \mathsf{C}$ where J is the diagram category of J. Then the **colimit** of J is defined as the initial object in the category of co-cones over F.

- A **co-cone** of F is an $N \in \operatorname{Ob}(\mathsf{C})$ and a family of morphisms $\left\{\psi_X : F(X) \to N \,\middle|\, X \in \operatorname{Ob}(\mathsf{J})\right\}$.
- The **category of co-cones** over F is the comma category $F \downarrow \Delta$, where $\Delta : \mathsf{C} \to \mathsf{Fun}(\mathsf{J}, \mathsf{C})$

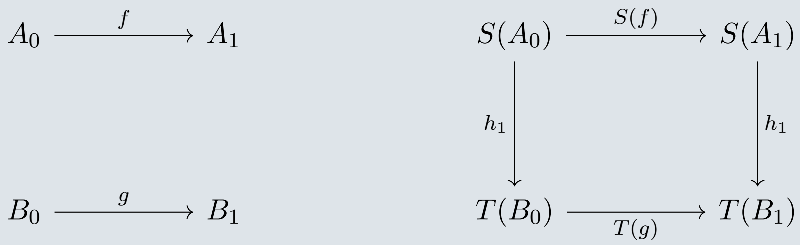
is the diagonal functor sending $N \in \mathrm{Ob}(\mathsf{C})$ to the constant functor to N:

$$\begin{aligned}\Delta(N) : \mathsf{J} &\to \mathsf{C} \\ X &\mapsto N.\end{aligned}$$

- The **comma category** generalizes slice categories: given categories and functors

$$\mathsf{A} \mapsto S\mathsf{C} \leftarrow\!\!\!\shortmid T\mathsf{B},$$

the comma category $S \downarrow T$ is given by triples $(A, B, h : S(A) \to T(B))$ making the obvious diagrams commute:



Link to Diagram

Taking $\mathsf{C} = A$, $S = \mathrm{id}_{\mathsf{A}}$, and $\mathsf{B} := \mathrm{pt}$ to be a 1-object category with only the identity morphism forces $X := T(\mathrm{pt}) \in \mathrm{Ob}(\mathsf{A})$ to be a single object and $(\mathsf{A} \downarrow X)$ is the usual slice category over X.

14 | Problem Sets

14.1 Problem Set 1

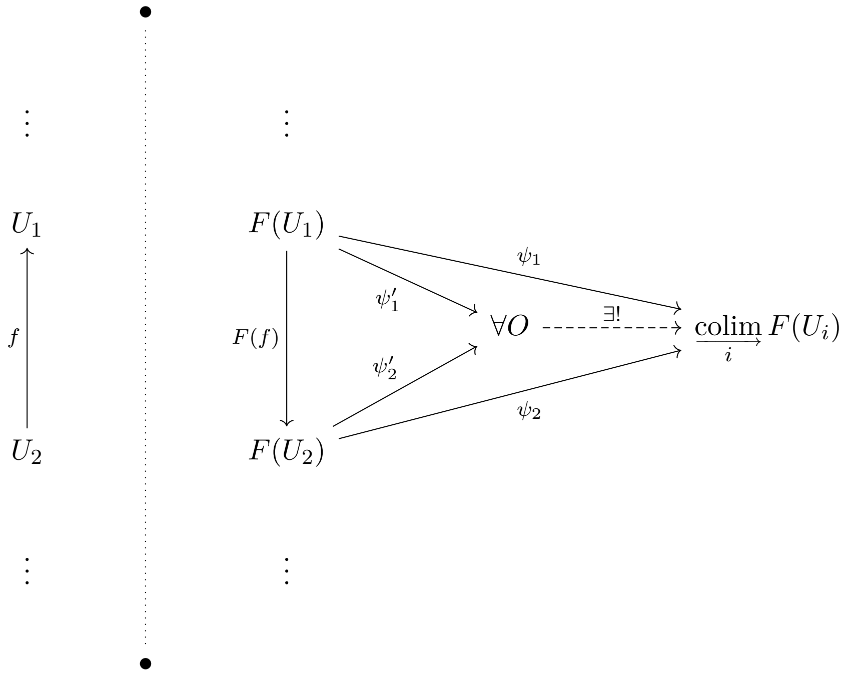
Remark 14.1.1: All problems are sourced from Hartshorne.

14.2 Chapter 2, Section 1

Remark 14.2.1: List of useful facts used:

- Morphisms of sheaves commute with restrictions: if $\varphi : \mathcal{F} \to \mathcal{G}$ then for any $s \in \mathcal{F}(U)$ and $V \subseteq U$, $\mathrm{Res}(U, V)(\varphi(s)) = \varphi(\mathrm{Res}(U, V)(s))$.
- φ is an isomorphism iff φ_p are all isomorphisms.
- Elements of stalks $\mathcal{F}_p$: equivalence classes $[U, s \in \mathcal{F}(U)]$.
- The induced map on stalks: $\varphi_p([U, s]) := [U, \varphi(U)(s)]$.

- A surjection of sheaves need not induce a surjection on sections.
- The colimit diagram:



Link to Diagram

- Colimits are initial co-cones, where I is initial if $I \to X$ for any X. AKA direct limits.

- Filtered colimits commute with finite limits.

  - In particular, monomorphisms are pullbacks, so finite limits, and stalks are filtered colimits. So injections of sheaves induce injections on stalks.

Remark 14.2.2: Recommended problems:

- 1.1
- 1.2
- 1.3
- 1.4
- 1.5

Problem 14.2.1 (1.1)
Let A be an abelian group, and define the *constant presheaf* associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \emptyset$, with restriction maps the identity.
Show that the constant sheaf $\mathcal{A}$ defined in Hartshorne is the sheafification of this presheaf.

Solution:
Let $X \in \mathsf{Top}$ be a space. Recapping the definitions, define the constant presheaf as

$$\underline{A}^{\mathsf{pre}}(U) := \begin{cases} A & U \neq \emptyset \\ 0 & \text{else.} \end{cases} \qquad \mathrm{res}^1(U,V) := \begin{cases} \mathrm{id}_A & U \neq \emptyset \\ 0 & \text{else.} \end{cases}.$$

Then define the constant *sheaf* as

$$\underline{A}(U) := \operatorname*{Hom}_{\mathsf{Top}}(U, A) \qquad \mathrm{res}^2(U,V)(f) := f|_V.$$

We're then tasked with finding a morphism of sheaves

$$\Psi : (\underline{A}^{\mathsf{pre}})^+ \xrightarrow{\sim} \operatorname*{Hom}_{\mathsf{Top}}(-, A),$$

which we'll also want to have an inverse morphism and this define an isomorphism in $\mathsf{Sh}(X)$. We'll use the implicitly stated fact in Hartshorne that $\operatorname*{Hom}_{\mathsf{Top}}(U, A) = A^{\oplus n}$ where $n := \#\pi_0(X)$ is the number of connected components of U. Suppose first that $n = 1$, so X is connected, and define the following morphism of groups:

$$\begin{aligned} \Psi_U : (\underline{A}^{\mathsf{pre}})(U) = A &\longrightarrow \operatorname*{Hom}_{\mathsf{Top}}(U, A) \\ a_0 &\mapsto \begin{cases} \varphi_{a_0} : U \to A \\ \qquad x \mapsto a_0, \end{cases} \end{aligned}$$

which maps a group element a_0 to the constant function on U sending every point to $a_0 \in A$. The claim is that the following diagram commutes in the category $\operatorname*{Sh}_{\mathsf{pre}}(X)$ (in both directions) for all U and V:

$$\begin{array}{ccccc}
 & f(U) & \longleftarrow & f & \\
 & a_0 & \longmapsto & \varphi_{a_0} : U \to A,\ x \mapsto a_0 & \\
U & (\underline{A}^{\mathsf{pre}})(U) = A & \xrightarrow{\Psi_U} & \operatorname*{Hom}_{\mathsf{Top}}(U, A) & \\
\updownarrow & \downarrow{\scriptstyle \mathrm{res}^1(U,V)} & & \downarrow{\scriptstyle \mathrm{res}^2(U,V)} & \\
V & (\underline{A}^{\mathsf{pre}})(V) = A & \xrightarrow{\Psi_V} & \operatorname*{Hom}_{\mathsf{Top}}(V, A) & \\
 & a_0 & \longmapsto & \varphi_{a_0} : V \to A,\ x \mapsto a_0 & \\
 & f(V) & \longleftarrow & f &
\end{array}$$

Link to Diagram

Here we've specified simultaneously what Ψ and Ψ^{-1} prescribe on opens U, V, and abuse notation slightly by writing $\underset{\mathsf{Top}}{\mathrm{Hom}}(-, A)$ for the sheaf it represents and its underlying presheaf.

- That this commutes follows readily, since running the diagram counterclockwise yields $\mathrm{res}^1(U,V) = \mathrm{id}_A$, so the composition

$$(A \xrightarrow{\mathrm{res}^1(U,V)} A \xrightarrow{\Psi_V} \mathrm{Hom}(V, A)) = (A \xrightarrow{\Psi_V} \mathrm{Hom}(V, A))$$

  sends an element $a_0 \in A$ to the constant function $\varphi_{a_0,V} : V \to A$. Running the diagram clockwise yields

$$(A \xrightarrow{\Psi_U} \mathrm{Hom}(U, A) \xrightarrow{\mathrm{res}^2(U,V)} \mathrm{Hom}(V, A)),$$

  which sends a_0 to the constant function $\varphi_{a_0,U} : U \to A$ sending everything to a_0, which then gets sent to $\varphi_{a_0,U}|_V : V \to A$ sending everything to a. Since $\varphi_{a_0}|_V(x) = \varphi_{a_0,V}(x) = a$ for every $x \in U$, these functions are equal.

- That the reverse maps Ψ_U^{-1} are well-defined follows from the fact that U is connected: the continuous image of a connected set is connected. Since A is given the discrete topology, any subset with 2 or more elements in disconnected, so each function $f \in \mathrm{Hom}(U, A)$ is necessarily a constant function and $f(U) = \{a\}$ is a singleton.

- Ψ_U, Ψ_U^{-1} clearly compose to the identity in either order, so Ψ_U defines an isomorphism of abelian groups.

As a consequence, we get a well-defined morphism of presheaves $\underline{A}^{\mathsf{pre}}(-) \to \mathrm{Hom}(-, A)|_{\mathsf{Sh}_{\mathsf{pre}}}$, and by the sheafification adjunction we can lift this to a morphism of sheaves:

$$\underset{\mathsf{pre}}{\mathsf{Sh}}(X) \underset{\mathcal{G} \mapsto \mathcal{F}|_{\mathsf{Sh}_{\mathsf{pre}}}}{\overset{\mathcal{F} \mapsto \mathcal{F}^+}{\rightleftarrows}} \mathsf{Sh}(X),$$

which reads

$$\underset{\mathsf{Sh}_{\mathsf{pre}}}{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}|_{\mathsf{Sh}_{\mathsf{pre}}}) \xrightarrow{\sim} \underset{\mathsf{Sh}}{\mathrm{Hom}}(\mathcal{F}^+, \mathcal{G})$$
$$\Psi \mapsto \tilde{\Psi},$$

and since Ψ was an isomorphism, so is $\tilde{\Psi}$.

It remains to handle the $n \geq 2$, case when (say) $U = U_1 \coprod U_2$ has more than 1 connected component. Actually, is it even true that adjunctions preserve isomorphisms...? Todo: help??

Alternatively, consider the map Ψ defined on presheaves – by the universal property, we get some sheaf morphism $\tilde{\Psi}$, which we can show is an isomorphism by showing its induced map on stalks is an isomorphism. This amounts to showing the following map is a group isomorphism:

$$\Psi_p : (\underline{A}^{\mathsf{pre}}(-))_p \xrightarrow{\sim} \underset{\mathsf{Top}}{\mathrm{Hom}}(-, A)_p.$$

First we identify the LHS:

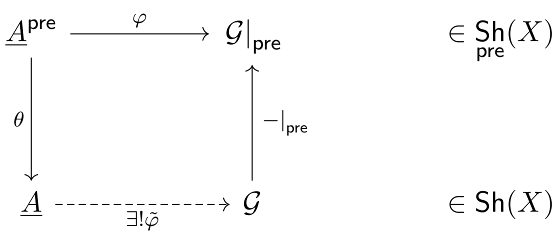
$$(\underline{A}^{\mathsf{pre}}(-))_p \coloneqq \underset{\overrightarrow{U \ni p}}{\mathrm{colim}}\, \underline{A}^{\mathsf{pre}}(U) = \underset{\overrightarrow{U \ni p}}{\mathrm{colim}}\, A = A.$$

(todo: show A satisfies the universal property for a colimit)
Identifying the RHS, we have equivalence classes $[U \ni p, s : U \to A]$

- Injectivity: that Ψ_p is injective follows from the fact that $\ker \psi_p \coloneqq \left\{ a \in A \;\middle|\; \Psi_p(a) = e \right\}$, where e is the identity in the right-hand side stalk, which is represented by the class $[U, f_e : U \to A]$ where $f_e(x) \coloneqq e_A$, the identity of A, for every $x \in U$.
- Surjectivity: that Ψ_p is surjective follows from the fact that every fixed $f : U \to A$ for A discrete is constant on connected components. Use that p is contained in a connected component $U_1 \ni p$, then $[U, f] \sim [U_1, f|_{U_1}] \coloneqq [U_1, g]$ to get that g is now a constant function of U_1. So $g(x) = a$ for some $a \in A$, so $g = \Psi_p(a)$ is in the image.

Alternatively:

- Show that $\underline{A}$ satisfies the universal property of $(\underline{A}^{\mathsf{pre}})^+$: we need to produce a morphism $\theta : (\underline{A}^{\mathsf{pre}}) \to \underline{A}$ such that for any $\mathcal{G} \in \mathsf{Sh}(X)$ and morphism of presheaves $\varphi : \underline{A}^{\mathsf{pre}} \to \mathcal{G}|_{\mathsf{pre}}$ we can produce a unique morphism $\tilde{\varphi}$ of sheaves making the following diagram commute:



Link to Diagram

- To define $\tilde{\varphi}$, it suffices to define morphisms of the form

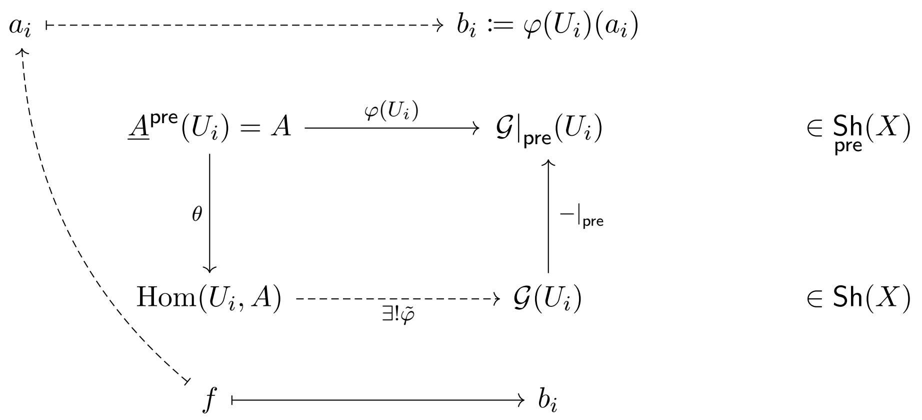
$$\begin{aligned} \tilde{\varphi}(U) : \underline{A}(U) &\to \mathcal{G}(U) \\ f &\mapsto \tilde{\varphi}(U)(f) \end{aligned}$$

- Take a map $f \in \underline{A}(U) \coloneqq \underset{\mathsf{Top}}{\mathrm{Hom}}(U, A)$. Write $U \coloneqq \coprod U_i$ as a union of connected components. Use that f is constant on connected components since A is discrete, so $f(U_i) = a_i$ for some elements $a_i \in A \in \mathsf{AbGrp}$.

- Plug the U_i into $\underline{A}^{\mathsf{pre}}$ to get morphisms

$$\varphi(U_i) : \underline{A}^{\mathsf{pre}}(U_i) = A \to \mathcal{G}|_{\mathsf{pre}}(U_i) \qquad \in \mathsf{AbGrp}$$

- Write $b_i := \varphi(U_i)(a_i) \in \mathcal{G}|_{\mathsf{pre}}(U_i) = \mathcal{G}(U_i)$.



Link to Diagram

- Since $\mathcal{G}$ is in fact a sheaf, by unique gluing there exists a unique element $b \in \mathcal{G}(U)$ such that $b|_{U_i} = b_i$. So define $\tilde{\varphi}(U)(f) := b$.
- Now define the map $\theta : \underline{A}^{\mathsf{pre}}(U_i) \to \mathrm{Hom}(U_i, A)$ sending a_i to the constant function $f_i(x) := a_i$. Since $\underline{A}$ is a sheaf, there is a well defined $F \in \mathrm{Hom}(U, A)$ such that $F|_{U_i} = f_i$. So for $a \in \underline{A}^{\mathsf{pre}}(U)$ set $\theta(a) = F \in \underline{A}(U)$.
- This makes the relevant diagram commute: if $a \in A = \underline{A}^{\mathsf{pre}}(U)$, then $b := \varphi(U)(a) \in \mathcal{G}(U)$. On the other hand, $\theta(a)$ is the constant function $f_a : x \mapsto a$ (on every connected component of U), and setting $F := \tilde{\varphi}(f_a) \in \mathcal{G}(U)$, we have $F := b$.

Problem 14.2.2 (1.2) (a) For any morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, show that for each point p that $\ker(\varphi)_p = \ker(\varphi_p)$ and $\operatorname{im}(\varphi)_p = \operatorname{im}(\varphi_p)$.

(b) Show that φ is injective (resp. surjective) if and only if the induced map on the stalks φ_p is injective (resp. surjective) for all p.

(c) Show that a sequence of sheaves and morphisms

$$\dots \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \to \dots$$

is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

Proof (of 1, kernels).

- Write $K \in \mathsf{Sh}(X)$ for the kernel sheaf $U \mapsto \ker\left(\mathcal{F}(U) \xrightarrow{\varphi(U)} \mathcal{G}(U)\right)$,
- We then want to show $K_p = \ker\left(\mathcal{F}_p \xrightarrow{\varphi_p} \mathcal{G}_p\right)$, an equality of sets in AbGrp. So we just do it!
  - Addendum: this works because both are subsets of the same abelian group, $\mathcal{F}_p$.
- We can write

$$\begin{aligned} \varphi_p : \mathcal{F}_p &\to \mathcal{G}_p \\ [U, s] &\mapsto [U, \varphi(U)(s)], \end{aligned}$$

and note that the zero element in a stalk is the equivalence class $[U, 0]$ where $0 \in \mathsf{AbGrp}$ is the zero object. Thus

$$\begin{aligned} \ker \varphi_p &:= \left\{x \in \mathcal{F}_p \;\middle|\; \varphi_p(x) = 0 \in \mathcal{G}_p\right\} \\ &= \left\{[U, s] \in \mathcal{F}_p \;\middle|\; [U, \varphi(U)(s)] = [U, 0]\right\} \\ &= \left\{[U, s] \in \mathcal{F}_p \;\middle|\; \varphi(U)(s) = 0\right\} \\ &= \left\{[U, s] \in \mathcal{F}_p \;\middle|\; s \in \ker \varphi(U)\right\} \\ &= \left\{[U, s] \;\middle|\; s \in \ker\left(\mathcal{F}(U) \xrightarrow{\varphi(U)} \mathcal{G}(U)\right)\right\} \\ &:= \left\{[U, s] \;\middle|\; s \in K(U)\right\} \\ &:= K_p. \end{aligned}$$

■

Proof (of 1, images).

- Write $\mathcal{I}$ for the sheaf $\operatorname{im} \varphi$ which sends $U \mapsto \operatorname{im}\left(\mathcal{F}(U) \xrightarrow{\varphi(U)} \mathcal{G}(U)\right)$.
- We want to show $\mathcal{I}_p = \operatorname{im}\left(\mathcal{F}_p \xrightarrow{\varphi_p} \mathcal{G}_p\right)$, where both are subsets of $\mathcal{G}_p$.
- So we show set equality:

$$\begin{aligned} \operatorname{im}(\varphi_p) &= \left\{y \in \mathcal{G}_p \;\middle|\; \exists x \in \mathcal{F}_p,\, \varphi_p(x) = y\right\} \\ &= \left\{[U, t] \in \mathcal{G}_p \;\middle|\; \exists [U, s] \in \mathcal{F}_p,\, \varphi_p([U, s]) = [U, t]\right\} \\ &= \left\{[U, t] \in \mathcal{G}_p \;\middle|\; \exists s \in \mathcal{F}(U),\, \varphi(U)(s) = t\right\} \\ &= \left\{[U, t] \in \mathcal{G}_p \;\middle|\; t \in \operatorname{im}\left(\mathcal{F}(U) \xrightarrow{\varphi(U)} \mathcal{G}(U)\right)\right\} \\ &:= \left\{[U, t] \in \mathcal{G}_p \;\middle|\; t \in \mathcal{I}(U)\right\} \\ &:= \mathcal{I}_p. \end{aligned}$$

■

Proof (of 2, injectivity).
$\implies$:

- Use that injectivity of a morphism φ of sheaves is *defined* to hold exactly when $\ker \varphi = 0$ is the constant zero sheaf.

- Now use (1):

$$0 = \ker(\varphi) \implies 0 = \ker(\varphi)_p = \ker(\varphi_p) \qquad \forall p.$$

- If $\ker \varphi = 0$, so φ is injective, then $\ker \varphi_p = 0$ for all p, so $\ker \varphi_p$ is injective.

$\impliedby$:

- Conversely, suppose $\ker \varphi_p = 0$ for all p; we want to show $\ker \varphi(U) = 0$ for all U.
- So fix $U \ni p$, we want to show

$$s \in K(U) := \ker\left(\mathcal{F}(U) \xrightarrow{\varphi(U)} \mathcal{G}(U)\right) \implies s = 0 \in \mathcal{F}(U).$$

- We have $\varphi(U)(s) = 0$, so

$$\varphi_p([U, s]) := [U, \varphi(U)(s)] = [U, 0] \in \mathcal{G}_p \implies [U, s] \in \ker(\varphi_p).$$

- By injectivity of φ_p, we have $[U, s] = 0 \in \mathcal{F}_p$.
- So there is some open U_p with $U \supseteq U_p \ni p$ and $\mathrm{Res}(U, U_p)(s) = 0$ in $\mathcal{F}(U_p)$.
- Then $\{U_p\}_{p \in U} \rightrightarrows U$, and since $\mathcal{F}$ is a sheaf, by existence of gluing these glue to an $F \in \mathcal{F}(U)$ with $\mathrm{Res}(U, U_p)(F) = 0$ for each p. By uniqueness of gluing, $0 = F = s$.

■

Proof (of 2, surjectivity).
$\implies$:

- Suppose φ is surjective, then by definition $\operatorname{im} \varphi = \mathcal{G}$ is an equality of sheaves.
- So $(\operatorname{im} \varphi)(U) = \mathcal{G}(U)$ for all U.
- Let $[U, t] \in \mathcal{G}_p$, so $t \in \mathcal{G}(U)$.
- Then $t \in (\operatorname{im} \varphi)(U)$, so there exists an $s \in \mathcal{F}(U)$ such that $\varphi(U)(s) = t$.
- Then $[U, s] \mapsto [U, \varphi(U)(s)] = [U, t]$, under φ_p, making φ_p surjective.

$\impliedby$:

- Suppose φ_p is surjective for all p, fix U, and let $t \in \mathcal{G}(U)$. We want to produce an $s \in \mathcal{F}(U)$ such that $\varphi(U)(s) = t$.

- For $p \in U$, the image of t in the stalk of $\mathcal{G}$ is of the form $[U_p, t_p] \in \mathcal{G}_p$ where $t_p \in \mathcal{G}(U_p)$.

- Since $\varphi_p : \mathcal{F}_p \twoheadrightarrow \mathcal{G}_p$, we can find some pair $[U_p, s_p]$ mapping to $[U_p, t_p]$ under φ_p, so $\varphi(U_p)(s_p) = t_p$.
  - Note that $\mathrm{Res}(U, U_p)(t) = t_p$.

- Note: may need to pull back to some $\tilde{U}_p$, then take a common refinement in both germs?

- Now $\{U_p\}_{p\in U} \rightrightarrows U$, so using existence of gluing for $\mathcal{F}$ we have some $s \in \mathcal{F}(U)$ with $\operatorname{Res}(U, U_p)(s) = s_p$ for all p.
- Claim: $\varphi(U)(s) = t$.

$$\begin{aligned}\operatorname{Res}(U, U_p)(\varphi(s)) &= \varphi(\operatorname{Res}(U, U_p)(s)) \\ &= \varphi(s_p) \\ &= t_p \\ &= \operatorname{Res}(U, U_p)(t) \qquad \forall p \in U,\end{aligned}$$

so $\varphi(s) = t$ by uniqueness of gluing of $\mathcal{G}$.

■

Proof (of 3, exactness).
$\Longrightarrow$: Assuming exactness of sheaves,

$$\ker(\mathcal{F}^{i+1}) = \operatorname{im}(\mathcal{F}^i) \iff \ker(\mathcal{F}^{i+1})_p = \operatorname{im}(\mathcal{F}^i)_p \qquad \forall p.$$

$\Longleftarrow$: Assuming exactness on stalks, write

$$\begin{aligned}\ker(\mathcal{F}^{i+1})_p &= \ker(\mathcal{F}_p^{i+1}) && \text{by 1} \\ &= \operatorname{im}(\mathcal{F}_p^i) && \text{exactness, by assumption} \\ &= \operatorname{im}(\mathcal{F}^i)_p && \text{by 1.}\end{aligned}$$

■

Problem 14.2.3 (1.3) (a) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds:

For every open set $U \subseteq X$, and for every $s \in \mathcal{G}(U)$, there is a cover $\{U_i\}$ of U and elements $t_i \in \mathcal{F}(U_i)$ such that $\varphi(t_i) = s|_{U_i}$ for all i.

(b) Give an example of a surjective morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, and an open set U such that $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is not surjective.

Proof (of 1).
$\Longrightarrow$:

- If $\varphi : \mathcal{F} \twoheadrightarrow \mathcal{G}$, then $\varphi_p : \mathcal{F}_p \twoheadrightarrow \mathcal{G}_p$ for all p, since $\operatorname{im}(\varphi_p) = (\operatorname{im}\varphi)_p = \mathcal{G}_p$, using problem 1.2.
- Fix $U \subseteq X$ and $s \in \mathcal{G}(U)$, we want
  - To produce a cover $\{U_i\} \rightrightarrows U$,
  - To find $t_i \in \mathcal{F}(U_i)$, and

- To show that $\varphi(t_i) = \text{Res}(U, U_i)(s)$ for all i.

- Fix p, and take the image of s in the stalk of $\mathcal{G}$ to get $[U_p, s_p] \in \mathcal{G}_p$ with $s_p \in \mathcal{G}(U_p)$ and $\text{Res}(U, U_p)(s) = s_p$. Note that $\{U_p\}_{p \in U} \rightrightarrows U$.
- By surjectivity on stalks, these pull back to $[U_p, t_p] \in \mathcal{F}_p$ with $t_p \in \mathcal{F}(U_p)$ and $\varphi_p([U_p, t_p]) \coloneqq [U_p, \varphi(U_p)(t_p)] = [U_p, s_p]$.
- Then $s_p \in \text{im}(\mathcal{F}(U_p) \xrightarrow{\varphi(U_p)} \mathcal{G}(U_p))$ and $\varphi(t_p) = s_p = \text{Res}(U, U_p)(s)$.

$\Longleftarrow$:

- If $\{U_i\} \rightrightarrows U$ with $\varphi(t_i) = \text{Res}(U, U_i)(s)$ for all i, then the t_i glue to a unique section $t \in \mathcal{F}(U)$ since $\mathcal{F}$ is a sheaf.
- Moreover $\text{Res}(U, U_i)(\varphi(t)) = \varphi(\text{Res}(U, U_i)(t)) = \varphi(t_i) = \text{Res}(U, U_i)(s)$ for all i, and by unique gluing for $\mathcal{G}$ we have $\varphi(t) = s$.
- So $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all U, making $\text{im}(\varphi(U)) = \mathcal{G}(U)$
- So $\text{im}\,\varphi = \mathcal{G}$ as sheaves since they make the same assignment to every open set U, making $\varphi : \mathcal{F} \to \mathcal{G}$ surjective by definition.

■

Proof (of 2).

- Take $X \coloneqq \{a, b, c\}$ a 3-point space with the topology $\tau_X \coloneqq \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$.

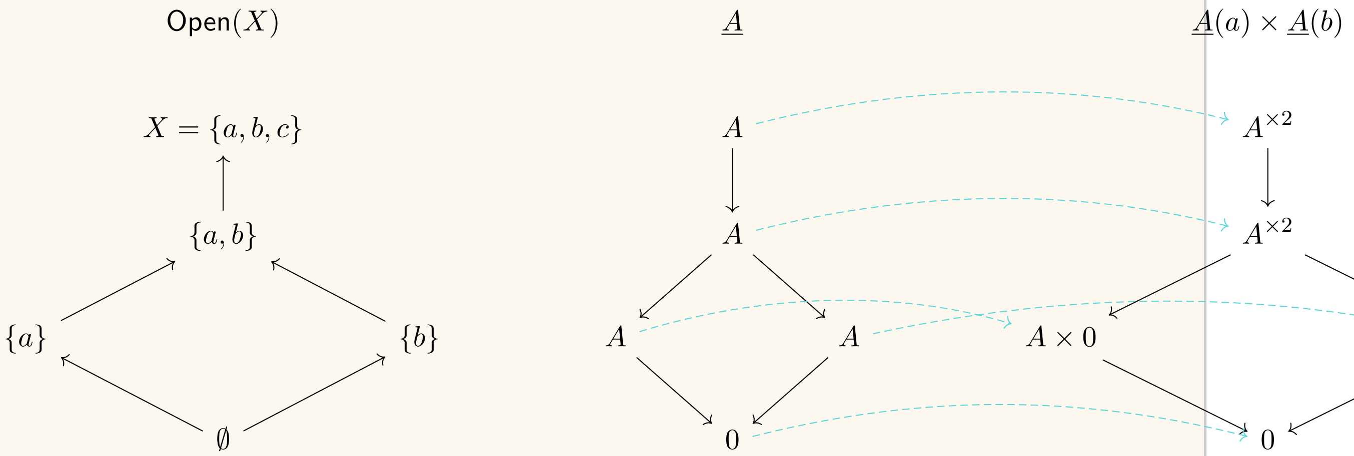
- Take $\mathcal{F} \coloneqq \underline{A}$ for some nontrivial $A \in \mathsf{AbGrp}$. We have the stalks
  - $\mathcal{F}_a = A$
  - $\mathcal{F}_b = A$
  - $\mathcal{F}_c = A$

- Take $\mathcal{G} \coloneqq \underline{A}(a) \times \underline{A}(b)$, the skyscraper sheaves at a and b respectively, where

$$\underline{A}(x)(U) \coloneqq \begin{cases} A & x \in U \\ 0 & \text{else.} \end{cases}$$

  Note that the stalks are given by $\underline{A}(x)_x = A$ and $\underline{A}(x)_y = 0$ for $y \neq x$, so
  - $\mathcal{G}_a = A \times 0$
  - $\mathcal{G}_b = 0 \times A$
  - $\mathcal{G}_c = 0 \times 0$.

- Now define $\varphi : \mathcal{F} \to \mathcal{G}$ by specifying $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for all U in the following way:



Link to Diagram

- Note that the induced maps on stalks are surjective, since $\varphi_p : A \to A, 0$ is either the identity or the zero map. But e.g. for $\{a, b\}$ we have $A \mapsto A^{\times 2}$, which can not be surjective.

Question: what is this map? Apparently its image is the diagonal...?

■

Problem 14.2.4 (1.4) (a) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves such that $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for each U. Show that the induced map $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}^+$of associated sheaves is injective.

(b) Use part (a) to show that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then $\operatorname{im} \varphi$ can be naturally identified with a subsheaf of $\mathcal{G}$, as mentioned in the text.

Proof (of a).

- $\varphi : \mathcal{F} \to \mathcal{G}$ is injective iff $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is injective for all p.
- Sheafification induces a map $\varphi^+ : \mathcal{F}_p^+ \to \mathcal{G}_p^+$
- The sheafification has the same stalks, so $\mathcal{F}_p^+ = \mathcal{F}_p$ and $\mathcal{G}_p^+ = \mathcal{G}_p$.
- So in fact $\varphi_p^+ = \varphi_p$. Since φ_p^+ is thus injective on all stalks, φ^+ is injective on sheaves.

■

Proof (of b).

- Noting that on opens $(\operatorname{im} \varphi)(U) \subseteq \mathcal{G}(U)$ is an inclusion of abelian groups, so define a morphism of sheaves by $\iota(U) : (\operatorname{im} \varphi)(U) \to \mathcal{G}(U)$ using this inclusion.
  - By definition, it suffices to show $\ker \iota = 0$ as a sheaf.
  - By 1.2.2, it suffices to show $(\ker \iota)_p = 0$ on all stalks.

– By 1.2.1, $(\ker \iota)_p = \ker(\iota_p)$, so it suffices to show ι_p is injective for all p.

- Now use that

$$\ker(\iota_p) = \varinjlim_{U \ni p} (\ker \varphi)(\iota(U)) = \varinjlim_{U \ni p} 0 = 0,$$

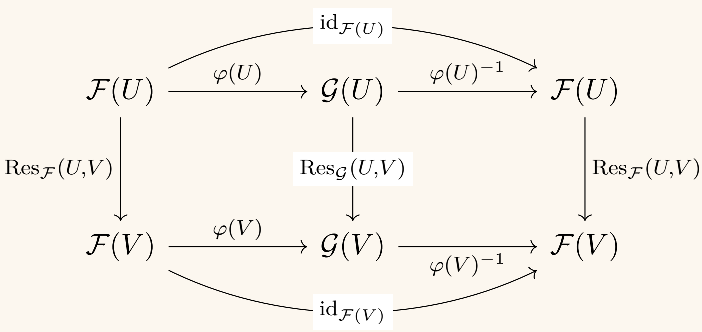
since all of the $\iota(U)$ are injective, so 0 satisfies the universal property for this colimit. So we're done.

■

Problem 14.2.5 (1.5)
Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.

Proof (?).
Problem: surjections of sheaves don't induce surjections ons sections!

- $\varphi : \mathcal{F} \to \mathcal{G}$ being injective means that $(\ker \varphi) = 0$ as sheaves, and surjective means $(\operatorname{im} \varphi) = \mathcal{G}$.
- Thus $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective, since $(\ker \varphi)(U) = 0(U) = 0$, and surjective since $\operatorname{im}(\varphi(U)) = (\operatorname{im} \varphi)(U) = \mathcal{G}(U)$. This $\varphi(U)$ is an isomorphism in abelian groups, and has an left and right inverse $\varphi^{-1}(U) : \mathcal{G}(U) \to \mathcal{F}(U)$.
- So we have a diagram:



Link to Diagram

- Both squares form a morphism of sheaves, so the right square assembles to $\varphi^{-1} : \mathcal{G} \to \mathcal{F}$
- Moreover $(\varphi^{-1} \circ \varphi)(\mathcal{F})(U) = \mathrm{id}_{\mathcal{F}(U)}$ and similarly in the other order, so $\varphi^{-1} \circ \varphi = \mathrm{id}_{\mathcal{F}}$ Similarly $(\varphi \circ \varphi^{-1})(\mathcal{G})(U) = \mathrm{id}_{\mathcal{G}(U)}$ and $(\varphi^{-1} \circ \varphi) = \mathrm{id}_{\mathcal{G}}$.
- Then by definition an isomorphism of sheaves is a morphism with a two-sided inverse, so we're done.

■

14.3 Problem Set 2

14.4 II.1

Exercise 14.4.1 (II.1.8)
For any open $U \subseteq X$ show that the functor

$$\Gamma(U, -) : \mathsf{Sh}(X) \to \mathsf{AbGrp}$$

is left-exact, but need not be exact.

Exercise 14.4.2 (II.1.14)
Let $\mathcal{F} \in \mathsf{Sh}(X)$ and $s \in \mathcal{F}(U)$ be a section, and define

$$\operatorname{supp} s := \left\{ p \in U \;\middle|\; s_p \neq 0 \right\} \subseteq U$$
$$\operatorname{supp} \mathcal{F} := \left\{ p \in X \;\middle|\; \mathcal{F}_p \neq 0 \right\} \subseteq U,$$

where s_p denotes the germ of s in the stalk $\mathcal{F}_p$. Show that $\operatorname{supp} s$ is closed in U but $\operatorname{supp} \mathcal{F}$ need not be.

Exercise 14.4.3 (II.1.17)
Let $X \in \mathsf{Top}, A \in \mathsf{AbGrp}, p \in X$ and define the skyscraper sheaf as

$$\iota_p(A)(U) := \begin{cases} A & p \in U \\ 0 & \text{else.} \end{cases}.$$

Show that the stalk $\iota_p(A)_q = A$ when $q \in \operatorname{cl}_X(\{p\})$ and 0 otherwise, and that there is an equality of sheaves $\iota_p(A) = \iota_*(\underline{A})$ where $\iota : \operatorname{cl}_X(\{p\}) \hookrightarrow X$ is the inclusion.

14.5 II.2

Exercise 14.5.1 (II.2.1)
Let $A \in \mathsf{Ring}$ and $X := \operatorname{Spec}(A)$, and for $f \in A$ let $D(f) := V(\langle f \rangle)^c$. Show that there is an isomorphism of ringed spaces

$$(D(f), \mathcal{O}_X|_{D(f)}) \xrightarrow{\sim} \operatorname{Spec}(A_f).$$

Exercise 14.5.2 (II.2.3)
Note that $(X, \mathcal{O}_X) \in \mathsf{Sch}$ is **reduced** iff $\mathcal{O}_X(U)$ has no nilpotents, and for $A \in \mathsf{Ring}$ define $A^{\text{red}} := A/\sqrt{0}$ to be A modulo its ideal of nilpotents.

a. Show that X is reduced iff for every $p \in X$, the local ring $\mathcal{O}_{X,p}$ has no nilpotents.

b. Let $\mathcal{O}_X^{\text{red}}$ be the sheafification of $U \mapsto \mathcal{O}_X(U)^{\text{red}}$. Show that $X_{\text{red}} := (X, \mathcal{O}_X^{\text{red}})$ is a scheme, and there is a morphism of schemes $X_{\text{red}} \xrightarrow{\text{red}} X$ which induces a homeomorphism $|X_{\text{red}}| \to |X|$ on underlying topological spaces.

c. Let $X \xrightarrow{f} Y \in \mathsf{Sch}$ with X reduced. Show that there is a unique morphism $X \xrightarrow{g} Y_{\text{red}}$ such that f is the composition

$$(X \xrightarrow{f} Y) = (X \xrightarrow{g} Y_{\text{red}} \xrightarrow{\text{red}} Y).$$

Exercise 14.5.3 (II.2.5)
Describe $\operatorname{Spec} \mathbb{Z}$ and show it is terminal in Sch, i.e. each $X \in \mathsf{Sch}$ admits a unique morphism $X \to \operatorname{Spec} \mathbb{Z}$.

Exercise 14.5.4 (II.2.7)
Let $X \in \mathsf{Sch}$ and for $x \in X$ let $\mathcal{O}_x$ be the local ring at x and $\mathfrak{m}_x$ its maximal ideal. Let $\kappa(x) := \mathcal{O}_x/\mathfrak{m}_x$ be the residue field at x.
Then for k any field, show that giving a morphism $\operatorname{Spec}(k) \to X \in \mathsf{Sch}$ is equivalent to giving a point $x \in X$ and an inclusion $\kappa(x) \hookrightarrow k$.

$$x^2 - y^q = 1 \qquad x^p - y^2 = 1.$$

$$x^2 - y^q = 1 \qquad x^p - y^2 = 1.$$

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