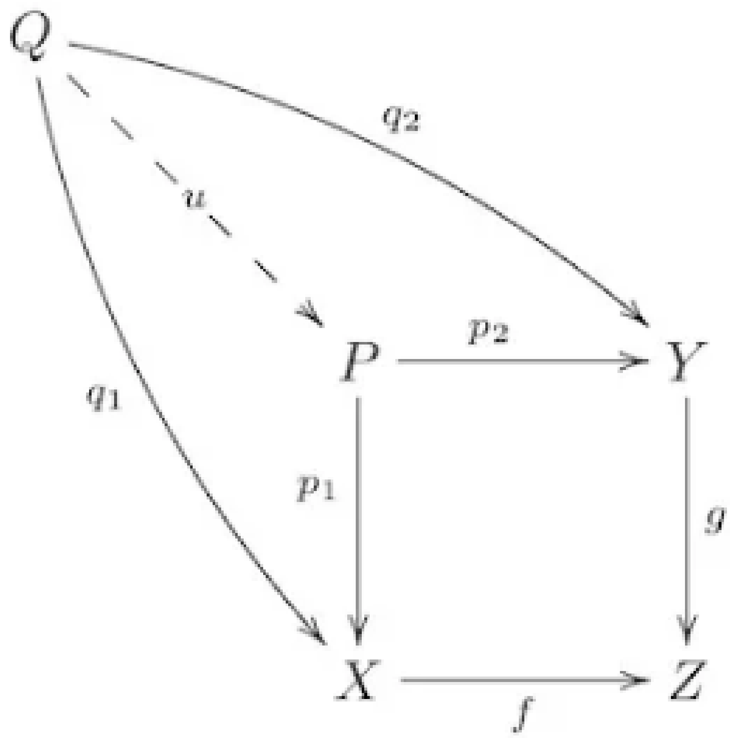

Notes: These are notes live-tex'd from a graduate course in Schemes taught by Phil Engel at the University of Georgia in Fall 2021, with material based on Hartshorne. Any errors or inaccuracies are almost certainly my own.

Schemes

Lectures by Phil Engel. University of Georgia, Fall 2021

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1 | Presheaves (Wednesday, August 18)

1.1 Definitions and Examples

Remark 1.1.1: We'll be covering Hartshorne, chapter 2:

- Sections 1-5: Fundamental, sheaves, schemes, morphisms, constant sheaves.
- Sections 6-9: Divisors, linear systems of differentials, nonsingular varieties.

Note that most of the important material of this book is contained in the exercises!

Remark 1.1.2: Recall that a **topological space** X is collection of *open* sets $\mathcal{U} = \{U_i \subseteq X\}$ which is closed under arbitrary unions and finite intersections, where $X, \emptyset \in \mathcal{U}$.

Definition 1.1.3 (Presheaf)
A **presheaf of abelian groups** $\mathcal{F}$ on X a topological space is an assignment to every open $U \subseteq X$ an abelian group $\mathcal{F}(U)$ and restriction morphisms $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for every inclusion $V \subseteq U$ satisfying

1. $\mathcal{F}(\emptyset) = 0$
2. $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ is $\mathrm{id}_{\mathcal{F}(U)}$.
3. If $W \subseteq V \subseteq U$ are opens, then

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV}.$$

We'll refer to $\mathcal{F}(U)$ as the **sections of** $\mathcal{F}$ **over** U, also denoted $\Gamma(U; \mathcal{F})$ and write the restrictions as $s|_v = \rho_{UV}(s)$ for $V \subseteq U$.

Example 1.1.4*(Presheaf of continuous functions):* Let $X := \mathbf{R}^1$ with the standard topology and take $\mathcal{F} = C^0(-; \mathbf{R}^1)$ (continuous real-valued functions) as the associated presheaf. For any open $U \subset \mathbf{R}$, the group of sections is

$$\mathcal{F}(U) := \left\{ f : U \to \mathbf{R}^1 \;\middle|\; f \text{ is continuous} \right\}.$$

For restriction maps, given $U \subseteq V$ take the actual restriction of functions

$$\begin{aligned} C^0(V; \mathbf{R}^1) &\to C^0(U; \mathbf{R}^1) \\ f &\mapsto f|_U. \end{aligned}$$

We can declare $C^0(\emptyset; \mathbf{R}^1) = \{0\} = 0 \in \mathsf{Grp}$, and the remaining conditions in the definition above follow immediately.

1.2 Constant Presheaves

Definition 1.2.1 (Constant presheaves)
The **constant presheaf** associated to $A \in \mathsf{Ab}$ on $X \in \mathsf{Top}$ is denoted $\underline{A}$, where

$$\underline{A}(U) := \begin{cases} A & U \neq \emptyset \\ 0 & U = \emptyset. \end{cases}$$

and

$$\rho_{UV} := \begin{cases} \mathrm{id}_A & V \neq \emptyset \\ 0 & V = \emptyset. \end{cases}.$$

⚠ **Warning 1.2.2**
The constant sheaf is not the sheaf of constant functions! Instead these are *locally* constant functions.

Remark 1.2.3: Let $\mathsf{Open}_{/X}$ denote the category of open sets of X, defined by

- Objects: $\mathrm{Ob}(\mathsf{Open}_{/X}) := \{U_i\}$, so each object is an open set.
- Morphisms:

$$\mathsf{Open}_{/X}(U,V) := \begin{cases} \emptyset & V \not\subset U \\ \text{The singleton } \left\{ U \overset{\iota}{\hookrightarrow} V \right\} & \text{otherwise.} \end{cases}.$$

Example 1.2.4 ***(Of*** $\mathsf{Open}_{/X}$ ***):*** Take $X := \{p, q\}$ with the discrete topology to obtain a category with 4 objects:

$$\begin{tikzcd}
 & \{p,q\} & \\
\{p\} \arrow[ur] & & \{q\} \arrow[ul] \\
 & \emptyset \arrow[ul] \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

Link to Diagram

Similarly, the indiscrete topology yields $\emptyset \to \{p,q\}$, a category with two objects.

Remark 1.2.5: A presheaf is a contravariant functor $\mathcal{F} : \mathsf{Open}_{/X} \to \mathsf{Ab}$ which sends the cofinal/initial object $\emptyset \in \mathsf{Open}_{/X}$ to the final/terminal object $\{\mathrm{pt}\} \in \mathsf{Ab}$. More generally, we can replace Ab with any category C admitting a final object:

- $\mathsf{C} := \mathsf{CRing}$ the category of commutative rings, which we'll use to define schemes.
- $\mathsf{C} = \mathsf{Grp}$, the category of (potentially nonabelian) groups.
- $\mathsf{C} := \mathsf{Top}$, the category of (arbitrary) topological spaces.

Example 1.2.6 ***(of presheaves):*** Let $X \in \mathsf{Var}_{/k}$ a variety over $k \in \mathsf{Field}$ equipped with the Zariski topology, so the opens are complements of vanishing loci. Given $U \subseteq X$, define a presheaf of regular functions $\mathcal{F} := \mathcal{O}$ where

- $\mathcal{O}(U)$ are the regular functions $f : U \to k$, i.e. functions on U which are locally expressible as a ratio $f = g/h$ with $g, h \in k[x_1, \cdots, x_n]$.

- Restrictions are restrictions of functions.

Taking $X = \mathbf{A}^1_{/k}$, the Zariski topology is the cofinite topology, so every open U is the complement of a finite set and $U = \{t_1, \cdots, t_m\}^c$. Then $\mathcal{O}(U) = \{\varphi : U \to k\}$ which is locally a fraction, and it turns out that these are all globally fractions and thus

$$\begin{aligned}
\mathcal{O}(U) &= \left\{ \frac{f(t)}{g(t)} \;\middle|\; f, g \in k[t], \quad g(t) \neq 0 \;\; \forall t \in U \right\} \\
&= \left\{ \frac{f(t)}{\prod_{i=1}^m (t - t_i)^{m_i}} \;\middle|\; f \in k[t] \right\} \\
&= k[t] \left[S^{-1} \right],
\end{aligned}$$

where $S = \langle \prod_{i=1}^m t - t_i \rangle$ is the multiplicative set generated by the factors. This forms an abelian group since we can take least common denominators, and we have restrictions.

⚠ **Warning 1.2.7**
Note that there are two similar notations for localization which mean different things! For a multiplicative set S, the ring $R\left[S^{-1}\right]$ literally means localizing at that set. For $\mathfrak{p} \in \operatorname{Spec} R$, the ring $R\left[\mathfrak{p}^{-1}\right]$ means localizing at the multiplicative set $S := \mathfrak{p}^c$.

2 | Sheaves, Stalks, Local Rings (Friday, August 20)

2.1 Sheaves

Definition 2.1.1 (Sheaf)
Recall the definition of a presheaf, and the main 3 properties:

1. $\mathcal{F}(\emptyset) = \{\mathrm{pt}\}$ where $\{\mathrm{pt}\} = 0 \in \mathsf{Ab}$,

2. $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$

3. For all $W \subseteq V \subseteq U$, a cocycle condition:

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV}.$$

   Write $s_i \in \mathcal{F}(U_i)$ to be a section.

   A presheaf is a **sheaf** if it additionally satisfies

4. When restrictions are compatible on overlaps, so

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j},$$

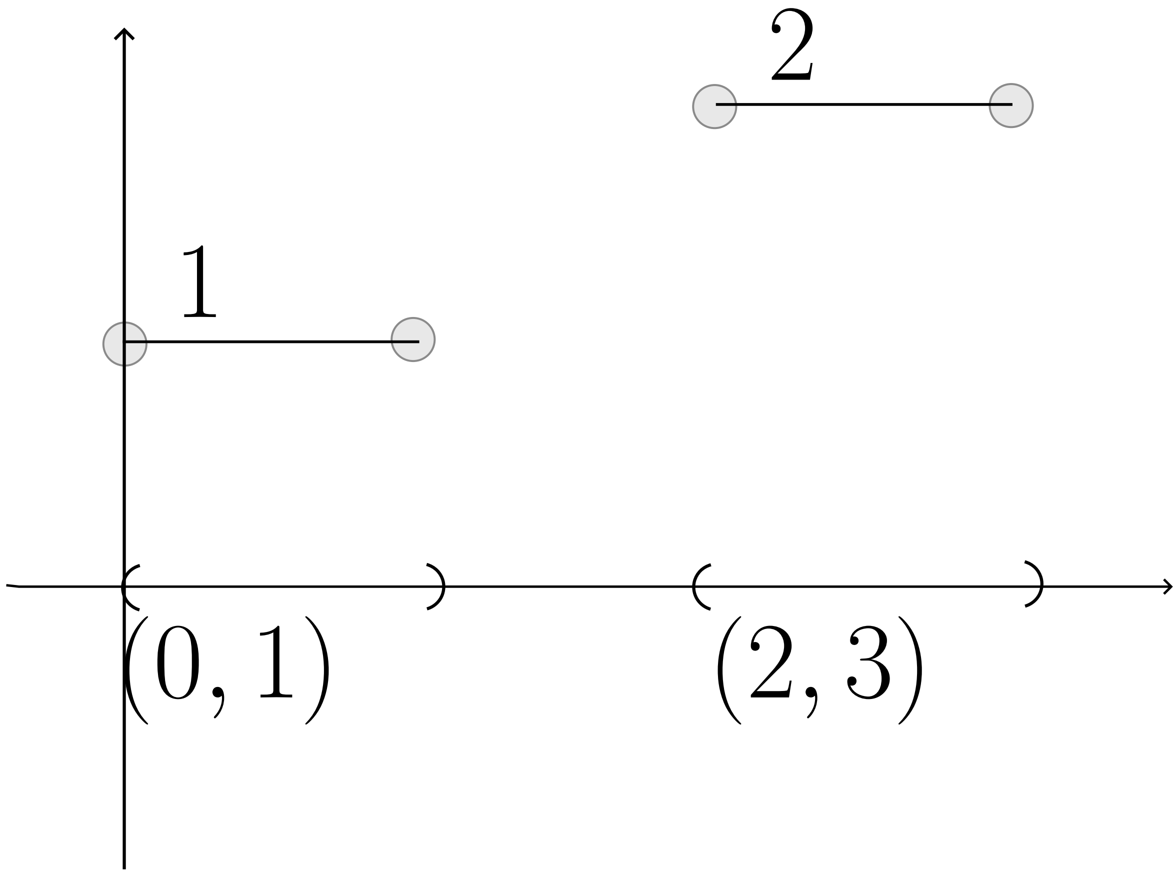
   there exists a uniquely glued section $\mathcal{F}(\cup U_i)$ such that $s|_{U_i} = s_i$ for all i.

Example 2.1.2 ***(?):*** Take $C^0(-;\mathbf{R})$ the sheaf of continuous real-valued functions on a topological space. For $f_i : U_i \to \mathbf{R}$ agreeing on overlaps, there is a continuous function $f : \cup U_i \to \mathbf{R}$ restricting to f_i on each U_i by just defining $f(x) \coloneqq f_i(x)$ for $x \in U_i$ and assembling these into a piecewise function, which is well-defined by agreement of the f_i on overlaps.

Example 2.1.3 ***(A presheaf which is not a sheaf):*** Let X be a topological space and $A \in \mathsf{CRing}$, then take the constant sheaf

$$\underline{A}(U) \coloneqq \begin{cases} A & U \neq \emptyset \\ 0 & \text{else.} \end{cases}.$$

This is not a sheaf – let $X = \mathbf{R}$ and $A = \mathbf{Z}/2$, let $U_1 = (0,1)$ and $U_2 = (2,3)$, and take $s_1 = 0$ on U_1 and $s_2 = 1$ on U_2. Since $U_1 \cap U_2 = \emptyset$, the sections trivially agree on overlaps, but there is no constant function on $U_1 \cup U_2$ restricting to 1 on U_2 and 0 on U_1



Definition 2.1.4 (Locally constant sheaves)
The **(locally) constant sheaf** $\underline{A}$ on any $X \in \mathsf{Top}$ is defined as

$$\underline{A}(U) := \left\{ f : U \to A \;\middle|\; f \text{ is locally constant} \right\}.$$

Remark 2.1.5: As a general principle, this is a sheaf since the defining property can be verified locally.

Example 2.1.6 *(?)*: Let C^0_{bd} be the presheaf of bounded continuous functions on S^1. This is not a sheaf, but one needs to go to infinitely many sets: take the image of $[\frac{1}{n}, \frac{1}{n+1}]$ with (say) $f_n(x) = n$ for each n. Then each f_n is bounded (it's just constant), but the full collection is unbounded, so these can not glue to a bounded function.

2.2 Stalks and Local Rings

Definition 2.2.1 (Stalks)
Let $\mathcal{F} \in \underset{\text{pre}}{\mathsf{Sh}}(X)$ and $p \in X$, then the **stalk** of $\mathcal{F}$ at p is defined as

$$\mathcal{F}_p(U) := \lim_{U \ni p} := \left\{ (s, U) \;\middle|\; U \ni p \text{ open}, \, s \in \mathcal{F}(U) \right\} /_{/\sim},$$

where $(s, U) \sim (t, V)$ iff there exists a $W \ni p$ with $W \subset U \cap V$ with $s|_W = t|_W$. An equivalence class $[(s, U)] \in \mathcal{F}_p$ is referred to as a **germ**.

Example 2.2.2 ***(Stalks of sheaves of analytic functions):*** Let $C^\omega(-; \mathbf{R})$ be the sheaf of analytic functions, i.e. those locally expressible as convergent power series. This is a sheaf because this condition can be checked locally. What is the stalk C_0^ω at zero? An example of a function in this germ is $[(f(x) = \frac{1}{1-x}, (-1, 1))$. A first guess is $\mathbf{R}[\![t]\!]$, but the claim is that this won't work.

Note that there is an injective map $C_0^\omega \hookrightarrow \mathbf{R}[\![t]\!]$ because f, g have analytic power series expansions at zero, and if these expressions are equal then $f|_I = g|_I$ for some I containing zero. This map won't be surjective because there are power series with a non-positive radius of convergence, for example taking $f(t) := \sum_{k=0}^{\infty} kt^k$ which only converges at $t = 0$. So the answer is that $C_0^\omega \leq \mathbf{R}[\![t]\!]$ is the subring of power series with positive radius of convergence.

Definition 2.2.3 (Local ring of the structure sheaf, $\mathcal{O}_p$)
Let $X \in \mathsf{AlgVar}$ and $\mathcal{O}$ its sheaf of regular functions. For $p \in X$, the stalk $\mathcal{O}_p$ is the **local ring** of X at p.

Example 2.2.4 ***(Local rings of affine space):*** For $X := \mathbf{A}^1_{/k}$ for $k = \bar{k}$, the opens are cofinite sets and $\mathcal{O}(U) = \left\{ f/g \;\middle|\; f, g \in k[t] \right\}$. Consider the stalk $\mathcal{O}_p$ for some fixed $p \in \mathbf{A}^1_{/k}$. Applying the definition, we have

$$\mathcal{O}_p := \left\{ (f/g, U) \;\middle|\; p \in U, \, g \neq 0 \text{ on } U \right\} /_{/\sim}.$$

Given any $g \in k[t]$ with $g(p) \neq 0$, there is a Zariski open set $U = V(g)^c = D_g$, the distinguished open associated to g, where $g \neq 0$ on U by definition. Thus $p \in U$, and so any $f/g \in \mathrm{ff}(k[t]) = k(t)$ with $p \neq 0$ defines an element $(f/g, D_g) \in \mathcal{O}_p$. Concretely:

$$f/g|_W = f/g|_{W'} \implies f/g = f'/g' \in k(t),$$

and $fg' = f'g$ on the cofinite set W, making them equal as polynomials. We can thus write

$$\mathcal{O}_p = \left\{ f/g \in k(t) \;\middle|\; g(p) \neq 0 \right\} = k[t]\left[\langle t-p \rangle^{-1}\right], \quad \langle t - p \rangle \in \operatorname{mSpec} k[t],$$

recalling that $k[t]\left[\mathfrak{p}^{-1}\right] := \left\{ f/g \;\middle|\; f, g \in k[t], \; g \notin \mathfrak{p} \right\}$.

Remark 2.2.5: Note that for $X \in \mathsf{AffVar}$, writing $X = V(f_i) = V(I)$ for a radical ideal I, we have the coordinate ring

$$k[X] \coloneqq k[x_1, \cdots, x_n]/I = R \implies \mathcal{O}_p = R\left[\mathfrak{m}_p{}^{-1}\right], \quad \mathfrak{m}_p \coloneqq \left\{ f \in R \,\middle|\, f(p) = 0 \right\}.$$

We thus have the following:

Slogan 2.2.6
The local ring at p is the localization at the maximal ideal of all functions in the coordinate ring vanishing at p.

⚠ **Warning 2.2.7**
This doesn't quite hold for non-algebraically closed fields:

$$f(x) \coloneqq x^p - x \in \mathbf{F}_p[x] \implies f(x) = 0 \quad \forall x \in \mathbf{F}_p \implies f \equiv 0 \in \mathbf{F}_p[x].$$

DZG: I missed something here, so I'm not sure ***what*** *isn't supposed to hold!*

Remark 2.2.8: Next time: morphisms of sheaves/presheaves, and isomorphisms of sheaves can be checked on stalks.

3 | More Sheaves (Monday, August 23)

3.1 Morphisms of Presheaves

Remark 3.1.1: Recall that the **stalk** of a presheaf $\mathcal{F}$ at p is defined as

$$\mathcal{F}_p \coloneqq \underset{U \ni p}{\underrightarrow{\operatorname{colim}}}\, \mathcal{F}(U) = \left\{ (s, U) \,\middle|\, s \in \mathcal{F}(U) \right\}_{/\sim}.$$

Definition 3.1.2 (Morphisms of presheaves)
Let $\mathcal{F}, \mathcal{G} \in \mathsf{Sh}_{\mathsf{pre}}(X)$, then a **morphism** $\varphi : \mathcal{F} \to \mathcal{G}$ is a collection $\{\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)\}$ of morphisms of abelian groups for all $U \in \mathsf{Open}(X)$ such that for all $V \subset U$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\mathrm{res}(UV)"] & \mathcal{G}(U) \arrow[d, "\mathrm{res}'(UV)"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{G}(V) \end{tikzcd}$$

Link to Diagram

An **isomorphism** is a morphism with a two-sided inverse.

Remark 3.1.3: Note that if we regard a sheaf as a contravariant functor, a morphism is then just a natural transformation.

Remark 3.1.4: A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ defines a morphisms on stalks $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$.

Example 3.1.5 ***(of a nontrivial morphism of sheaves):*** Let $X := \mathbf{C}^\times$ with the classical topology, making it into a real manifold, and take $C^0(-; \mathbf{C}) \in \mathsf{Sh}(X, \mathsf{Ab})$ be the sheaf of continuous functions and let $C^0(-; \mathbf{C})^\times$ the sheaf of of nowhere zero continuous continuous functions. Note that this is a sheaf of abelian groups since the operations are defined pointwise. There is then a morphism

$$\begin{aligned} \exp(-) : C^0(-; \mathbf{C}) &\to C^0(-; \mathbf{C})^\times \\ f &\mapsto e^f \qquad\qquad \text{on open sets } U \subseteq X. \end{aligned}$$

Since exponentiating and restricting are operations done pointwise, the required square commutes, yielding a morphism of sheaves.

3.2 Kernel and cokernel sheaves

Definition 3.2.1 ((co)kernel and image sheaves)
Let $\varphi : \mathcal{F} \to \mathcal{G}$ be morphisms of presheaves, then define the presheaves

$$\begin{aligned} \ker(\varphi)(U) &:= \ker(\varphi(U)) \\ \operatorname{coker}^{\mathsf{pre}}(\varphi)(U) &:= \mathcal{G}(U)/\varphi(\mathcal{F}(U)) \\ \operatorname{im}(\varphi)(U) &:= \operatorname{im}(\varphi(U)) \end{aligned}$$

.

⚠ **Warning 3.2.2**
If $\mathcal{F}, \mathcal{G} \in \mathsf{Sh}(X)$, then for a morphism $\varphi : \mathcal{F} \to \mathcal{G}$, the image and cokernel presheaves need not be sheaves!

Example 3.2.3 ***(of why the cokernel presheaf is not a sheaf):*** Consider $\ker \exp$ where

$$\exp : C^0(-;\mathbf{C}) \to C^0(-;\mathbf{C})^\times \qquad \in \mathsf{Sh}(\mathbf{C}^\times).$$

One can check that $\ker \exp = 2\pi i \underline{\mathbf{Z}}(U)$, and so the kernel is actually a sheaf. We also have

$$\operatorname{coker}^{\mathsf{pre}} \exp(U) := \frac{C^0(U;\mathbf{C})}{\exp(C^0(U;\mathbf{C})^\times)}.$$

On opens, $\operatorname{coker}^{\mathsf{pre}} \exp(U) = \{1\} \iff$ every nonvanishing continuous function g on U has a continuous logarithm, i.e. $g = e^f$ for some f. Examples of opens with this property include any contractible (or even just simply connected) open set in $\mathbf{C}^\times$. Consider $U := \mathbf{C}^\times$ and $z \in C^0(\mathbf{C}^\times;\mathbf{C})^\times$, which is a nonvanishing function. Then the equivalence class $[z] \in \operatorname{coker}^{\mathsf{pre}} \exp(\mathbf{C}^\times)$ is nontrivial – note that $z \neq e^f$ for any $f \in C^0(\mathbf{C}^\times;\mathbf{C})$, since any attempted definition of $\log(z)$ will have monodromy.

On the other hand, we can cover $\mathbf{C}^\times$ by contractible opens $\{U_i\}_{i\in I}$ where $[z]|_{U_i} = 1 \in \operatorname{coker}^{\mathsf{pre}} \exp(U_i)$ and similarly $1|_{\mathrm{id}} = 1 \in \operatorname{coker}^{\mathsf{pre}} \exp(U_i)$, showing that the cokernel fails the unique gluing axiom and is not a sheaf.

3.3 Sheafification

Definition 3.3.1 (Sheafification)
Given any $\mathcal{F} \in \underset{\mathsf{pre}}{\mathsf{Sh}}(X)$ there exists an $\mathcal{F}^+ \in \mathsf{Sh}(X)$ and a morphism of presheaves $\theta : \mathcal{F} \to \mathcal{F}^+$ such that for any $\mathcal{G} \in \mathsf{Sh}(X)$ with a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ there exists a unique $\psi : \mathcal{F}^+ \to \mathcal{G}$ making the following diagram commute:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[dr, "\theta"'] & \mathcal{G} \\
 & \mathcal{F}^+ \arrow[u, "\exists ! \psi"']
\end{tikzcd}$$

Link to Diagram

The sheaf $\mathcal{F}^+ \in \mathsf{Sh}(X)$ is called the **sheafification** of $\mathcal{F}$. This is an example of an adjunction of functors:

$$\underset{\underset{\mathsf{pre}}{\mathsf{Sh}(X)}}{\operatorname{Hom}}(\mathcal{F}, \mathcal{G}^{\mathsf{pre}}) \cong \underset{\mathsf{Sh}(X)}{\operatorname{Hom}}(\mathcal{F}^+, \mathcal{G}),$$

where we use the forgetful functor $\mathcal{G} \to \mathcal{G}^{\mathsf{pre}}$. This can be expressed as the adjoint pair

$$\begin{tikzcd}
\underset{\mathsf{pre}}{\mathsf{Sh}}(X) \arrow[r, shift left, "(-)^+"] \arrow[r, phantom, "\perp"] & \mathsf{Sh}(X). \arrow[l, shift left, "(-)^{\mathsf{pre}}"]
\end{tikzcd}$$

Proof (of existence of sheafification).
We construct it directly as $\mathcal{F}^+ := \left\{ s : U \to \coprod_{p \in U} \mathcal{F}_p \right\}$ such that

1. $s(p) \in \mathcal{F}_p$,
2. The germs are compatible locally, so for all $p \in U$ there is a $V \supseteq p$ such that for some $t \in \mathcal{F}(V)$, $s(p) = t_p$ for all p in V.

So about any point, there should be an actual function specializing to all germs in an open set.

■

Slogan 3.3.2
The sheafification is constructed from collections of germs which are locally compatible.

Remark 3.3.3: This process will make coker exp zero as a sheaf, since it will be zero on a sufficiently small set.

4 Exactness for Sheaves (Wednesday, August 25)

4.1 Some examples

Remark 4.1.1: Recall the definition of sheafification: let $\mathcal{F} \in \underset{\mathsf{pre}}{\mathsf{Sh}}(X; \mathsf{AbGrp})$. Construct a sheaf $\mathcal{F}^+ \in \mathsf{Sh}(X, \mathsf{AbGrp})$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ of presheaves satisfying the appropriate universal property:

$$\begin{tikzcd}
\mathcal{F}^+ \arrow[dr, dashed, "\exists \tilde{\psi}"] & \\
\mathcal{F} \arrow[u, "\theta"] \arrow[r, "\psi"] & \mathcal{G}
\end{tikzcd}$$

Link to Diagram

So any presheaf morphism to a sheaf factors through the sheafification uniquely (via θ). Note that this is a instance of a general free/forgetful adjunction.

We can construct it as

$$\mathcal{F}^+(U) := \left\{ s : U \to \coprod_{p \in U} \mathcal{F}_p, \quad s(p) \in \mathcal{F}_p, \cdots \right\}.$$

where the addition condition is that for all $q \in U$ there exists a $V \nu q$ and $t \in \mathcal{F}(V)$ such that $t_p = s(p)$ for all $p \in V$. Note that θ is defined by $\theta(U)(s) = \{s : p \to s_p\}$, the function assigning points to germs with respect to the section s. Idea: this is like replacing an analytic function on an interval with the function sending a point p to its power series expansion at p.

Example 4.1.2*(?)*: Recall $\exp : C^0 \to (C^0)^\times$ on $\mathbf{C}^\times$, then $\operatorname{coker}^{\mathsf{pre}}(\exp)(U) = \{1\}$ on contractible U, using that one can choose a logarithm on such a set. However $\operatorname{coker}^{\mathsf{pre}}(\exp)(\mathbf{C}^\times) \neq \{1\}$ since $[z] \in (C^0)^\times(\mathbf{C}^\times)/\exp(C^0(\mathbf{C}^\times))$.

Remark 4.1.3: Letting $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphisms of sheaves, then we defined $\operatorname{coker}(\varphi) := (\operatorname{coker}^{\mathsf{pre}}(\varphi))^+$ and $\operatorname{im}(\varphi) := (\operatorname{im}^{\mathsf{pre}}(\varphi))^+$. Then

$$\begin{aligned} \operatorname{coker}^{\mathsf{pre}}(\exp) &\to \operatorname{coker}(\exp) \\ s \in \mathcal{F}(U) &\mapsto s(p) = s_p. \end{aligned}$$

The claim is that $[z]_p = 1$ for all $p \in \mathbf{C}^\times$, since we can replace $[([z], \mathbf{C}^\times)]$ with $([z]_U, U)$ for U contractible.

Example 4.1.4*(?)*: A useful example to think about: $X = \{p, q\}$ with

- $\mathcal{F}(p) = A$
- $\mathcal{F}(q) = B$
- $\mathcal{F}(X) = 0$

Then local sections don't glue to a global section, so this isn't a sheaf, but it is a presheaf. The sheafification satisfies $\mathcal{F}^+(X) = A \times B$.

4.2 Subsheaves

Definition 4.2.1 (Subsheaves, injectivity, surjectivity)
$\mathcal{F}'$ is a **subsheaf** of $\mathcal{F}$ if

- $\mathcal{F}'(U) \leq \mathcal{F}(U)$ for all U,
- $\operatorname{Res}'(U, V) = \operatorname{Res}(U, V)|_{\mathcal{F}'(U)}$.

$\varphi : \mathcal{F} \to \mathcal{G}$ is **injective** iff $\ker \varphi = 0$, **surjective** if $\operatorname{im}(\varphi) = \mathcal{G}$ or $\operatorname{coker} \varphi = 0$.

Exercise 4.2.2 (?)
Check that $\ker \varphi$ already satisfies the sheaf property.

4.3 Exact Sequences of sheaves

Definition 4.3.1 (Exact sequences of sheaves)
Let $\cdots \to \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \to \cdots$ be a sequence of morphisms in $\mathsf{Sh}(X)$, this is **exact** iff $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$.

Lemma 4.3.2 *(?).*
$\ker \varphi$ is a sheaf.

Proof (?).
By definition, $\ker(\varphi)(U) \coloneqq \ker\left(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)\right)$, satisfying part (a) in the definition of presheaves. We can define restrictions $\operatorname{Res}(U, V)|_{\ker(\varphi)(U)} \subseteq \ker(\varphi)(V)$. Use the commutative diagram for the morphism $\varphi : \mathcal{F} \to \mathcal{G}$.
Now checking gluing: Let $s_i \in \ker(\varphi)(U_i)$ such that $\operatorname{Res}(s_i, U_i \cap U_j) = \operatorname{Res}(s_j, U_i \cap U_j)$ for all i, j. This holds by viewing $s_i \in \mathcal{F}(U_i)$, so $\exists! s \in \mathcal{F}(\bigcup_i U_i)$ such that $\operatorname{Res}(s, U_i) = s_i$. We want to show $s \in \ker(\varphi)\left(\bigcup U_i\right)$, so consider

$$t \coloneqq \varphi\left(\bigcup_i U_i\right)(s) \in \mathcal{G}\left(\bigcup U_i\right),$$

which is zero. Now

$$\operatorname{Res}(t, U_i) = \varphi(U_i)(\operatorname{Res}(s, U_i)) = \varphi(U_i)(s_i) = 0$$

by assumption, using the commutative diagram. By unique gluing for $\mathcal{G}$, we have $t = 0$, since 0 is also a section restricting to 0 everywhere.

■

Definition 4.3.3 (Quotients)
For $\mathcal{F}' \leq \mathcal{F}$ a subsheaf, define the **quotient** $\mathcal{F}/\mathcal{F}' \coloneqq ((\mathcal{F}/\mathcal{F}')^{\mathsf{pre}})^+$ where

$$(\mathcal{F}/\mathcal{F}')^{\mathsf{pre}}(U) \coloneqq \mathcal{F}(U)/\mathcal{F}'(U).$$

5 Passing to stalks, pushforward/inverse image (Friday, August 27)

5.1 Isomorphism $\iff$ isomorphism on stalks

Theorem 5.1.1 *(Sheaf isomorphism $\iff$ isomorphism on stalks).*
Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathsf{Sh}(X)$, then φ is an isomorphism $\iff$ $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for all $p \in X$.

Proof ($\implies$).
Suppose φ is an isomorphism, so there exists a $\psi : \mathcal{G} \to \mathcal{F}$ which is a two-sided inverse for φ. Then ψ_p is a two-sided inverse to φ_p, making it an isomorphism. This follows directly from the formula:

$$\begin{aligned} \varphi_p : \mathcal{F}_p &\to \mathcal{G}_p \\ (s, U) &\mapsto (\varphi(U)(s), U). \end{aligned}$$

■

Proof ($\impliedby$).
It suffices to show $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for all U. This is because we could define $\psi(U) : \mathcal{G}(U) \to \mathcal{F}(U)$ and set $\varphi^{-1}(U) := \psi(U)$, then reversing the arrows in the diagram for a sheaf morphism again yields a commutative diagram.

Claim: $\varphi(U)$ is injective.

For $s \in \mathcal{F}(U)$, we want to show $\varphi(U)(s) = 0$ implies $s = 0$. Consider the germs $(s, U) \in \mathcal{F}_p$ for $p \in U$, we have $\varphi_p(s, U) = (0, U) = 0 \in \mathcal{F}_p$. So $S_p = 0$ for all $p \in U$. Since we have a germ, there exists $V_p \ni p$ open such that $s|_{V_p} = 0$. Noting that $\left\{ V_p \;\middle|\; p \in U \right\} \rightrightarrows U$, by unique gluing we get an s where $s|_{V_p} = 0$ for all V_p, so $s \equiv 0$ on U.

Claim: $\varphi(U)$ is surjective.

Let $t \in \mathcal{G}(U)$, and consider germs $t_p \in \mathcal{G}_p$. There exists a unique $s_p \in \mathcal{F}_p$ with $\varphi_p(s_p) = t_p$, since φ_p is an isomorphism of stalks by assumption. Use that s_p is a germ to get an equivalence class (s_p, V) where $V \subseteq U$. We have $\varphi(V)(s(p), V) \sim (t, U)$, noting that s depends on p. Having equivalent germs means there exists a $W(p) \subseteq V$ with $p \in W$ with $\varphi(W(p)) \left(s(p)|_W \right) = t|_{W(p)}$. We want to glue these $\left\{ s(p)|_{W(p)} \;\middle|\; p \in U \right\}$ together. It suffices to show they agree on intersections. Taking $p, q \in U$, both $s(p)|_{W(p) \cap W(q)}$ and $s(q)|_{W(p) \cap W(q)}$ map to $t|_{W(p) \cap W(q)}$ under $\varphi(W(p) \cap W(q))$. Injectivity will force these to be equal, so $\exists ! s \in \mathcal{F}(U)$ with $s|_{W(p)} = s(p)$. We want to now show that $\varphi(U)(s) = t$. Using commutativity of the square, we have $\varphi(U)(s)|_{W(p)} = \varphi(W(p)) \left(s|_{W(p)} \right)$. This equals $\varphi(W(p))(s(p)) = t|_{W(p)}$.

Therefore $\varphi(U)(s)$ and t restrict to sections $\left\{w(p) \;\middle|\; p \in U\right\}$. Using unique gluing for $\mathcal{G}$ we get $\varphi(U)(s) = t$.

■

Remark 5.1.2: Note: we only needed to check overlaps because of exactness of the following sequence:

$$0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i<j} \mathcal{F}(U_{ij}) \to \cdots .$$

5.2 Inverse image and pushforward

Definition 5.2.1 (Pushforward and inverse image sheaves)
Let $f \in \mathsf{Top}(X, Y)$, let $\mathcal{F} \in \mathsf{Sh}(X)$ and define the **pushforward sheaf** $f_*\mathcal{F} \in \mathsf{Sh}(Y)$ by

$$f_*\mathcal{F}(V) \coloneqq \mathcal{F}(f^{-1}(V)).$$

The **inverse image** sheaf is define as

$$(f^{-1}\mathcal{F})(U) \coloneqq \lim_{\text{open } V \supseteq f(U)} F(V).$$

Remark 5.2.2: The inverse image sheaf generalizes stalks, recovering $\mathcal{F}_p$ when $f(U) = p$. Note that $f(U)$ need not be open unless f is an open map, and checking that $f(U)$ is (co?)final in the system $\{\text{open } V \supseteq f(U)\}$ yields

$$(f^{-1}\mathcal{F})(U) = \mathcal{F}(f(U)).$$

⚠ **Warning 5.2.3**
We will have a notion of f^*, but this will not generally be the pullback!

Exercise 5.2.4 (?)
Show that $f_*\mathcal{F}$ makes sense precisely because f is continuous. Check that $f_*\mathcal{F}$ satisfies the sheaf axioms. Use that the set of opens of the form $f^{-1}(V)$ are e.g. closed under intersections, and thus inherit all of the sheaf axioms from $\mathcal{F}$.

6 | Spec A as a space (Monday, August 30)

6.1 The prime spectrum Spec A

Remark 6.1.1: Let $R \in \mathsf{CRing}$ be a commutative unital ring in which $0 \neq 1$ unless $R = 0$. The goal is to define a space X such that R is the ring of functions on X, imitating the correspondence between $X \in \mathsf{Mfd}$ and $R := C^0(X; \mathbf{R})$. Recall that an ideal $\mathfrak{p} \in \operatorname{Id}(R)$ is **prime** iff $\mathfrak{p} \subset A$ is a proper subset and $fg \in \mathfrak{p} \implies f \in \mathfrak{p}$ or $g \in \mathfrak{p}$, or equivalently $R/\mathfrak{p}$ is a field.

Slogan 6.1.2
Ideals are "contagious" under multiplication, and *prime* ideals have "reverse contagion".

Definition 6.1.3 (Spectrum of a ring)
For $A \in \mathsf{CRing}$ as above,

$$\operatorname{Spec} A := \left\{ \mathfrak{p} \trianglelefteq A \,\middle|\, \mathfrak{p} \text{ is a prime ideal} \right\} \qquad \in \mathsf{Set}.$$

We topologize $\operatorname{Spec} A$ by defining a topology of closed sets as follows:

$$\tau(A) := \left\{ V(I) \,\middle|\, I \trianglelefteq A \right\}, \qquad V(I) := \left\{ \mathfrak{p} \in \operatorname{Spec}(A) \,\middle|\, \mathfrak{p} \supseteq I \right\}.$$

Exercise 6.1.4 (The topology is really a topology)
Prove that $(\operatorname{Spec} A, \tau(A))$ yields a well-defined topological space.

Example 6.1.5 ***(Spec of a field):*** For A a field, $\operatorname{Spec}(A) = \{\langle 0 \rangle\}$ is a point – any other nonzero element $\mathfrak{p} \in \operatorname{Spec} A$ would contain a unit u, in which case $u^{-1}u = 1 \in \mathfrak{p} \implies \mathfrak{p} = A$.

Example 6.1.6 ***(Spec of a polynomial ring):*** For k an algebraically closed field,

$$\operatorname{Spec} k[t] = \left\{ \langle 0 \rangle, \langle t - a \rangle \,\middle|\, a \in k \right\}.$$

This is a PID, so every ideal is of the form $I = \langle f \rangle$, and one can check that

$$V(\langle f \rangle) = \begin{cases} \operatorname{Spec} k[t] & f = 0 \\ \langle x - a_1, \cdots, a - a_k \rangle & f(x) = \prod_{i=1}^{k} (x - a_i) \end{cases}.$$

Note that this is **not** the cofinite topology on $\operatorname{Spec} A$, since $f = 0$ defines a generic point $\eta := \langle 0 \rangle$.

7 | The structure sheaf $\mathcal{O}$ (Wednesday, September 01)

7.1 Ringed spaces are finer than topological spaces

Example 7.1.1 ***(Polynomial rings):*** Let $k = \bar{k}$ be algebraically closed, then

$$\operatorname{Spec} k[x] = \left\{ \langle x - a \rangle \;\middle|\; a \in k \right\} \cup \langle 0 \rangle .$$

Similarly,

$$\operatorname{Spec} k[x, y] = \left\{ \langle x - a, y - b \rangle \;\middle|\; a, b \in k \right\} \cup \left\{ \langle f \rangle \;\middle|\; f \in k[x, y] \text{ irreducible} \right\} \cup \langle 0 \rangle .$$

Note that both have non-closed, generic points $\eta = \langle 0 \rangle$.

Example 7.1.2 ***(Distinct ringed spaces which are homeomorphic):*** Consider $X \coloneqq \operatorname{Spec} \mathbf{Z}_{\widehat{p}}$ and $Y \coloneqq \operatorname{Spec} \mathbf{C}[\![t]\!]$, then

$$X = \{ \langle p \rangle , \langle 0 \rangle \} , \qquad Y = \{ \langle t \rangle , \langle 0 \rangle \} .$$

Both are two point spaces, with exactly one open/generic point $\langle 0 \rangle$ and one closed point ($\langle p \rangle$ and $\langle t \rangle$ respectively). These spaces are isomorphic as topological spaces (i.e. there is a homeomorphism between them), but later we'll see that they can be distinguished as ringed spaces.

7.2 Properties of V

Remark 7.2.1: Recall that for $A \in \mathsf{CRing}$, we defined $\operatorname{Spec} A$ to have closed sets of the form

$$V(I) = \left\{ p \in \operatorname{Spec}(A) \;\middle|\; p \supseteq I \right\} \qquad \forall I \trianglelefteq A.$$

Lemma 7.2.2 ***(V sends finite products to unions).***

$$V(IJ) = V(I) \cup V(J),$$

Corollary 7.2.3 ***(?).***
If a prime ideal p contains IJ then $p \supseteq I$ or $p \supseteq J$.

Proof (of lemma).
$\Longleftarrow$: If $I \subseteq P$ or $J \subseteq P$, then $IJ \subseteq I$ and $IJ \subseteq J$, so $IJ \subset p$.
$\Longrightarrow$: Suppose $IJ \subset p$ but $J \not\subset p$, so pick $j \in J \setminus p$. Then for all $i \in I$, we have $ij \in IJ \subseteq p$, forcing $i \in p$.
■

Lemma 7.2.4 ***(V sends arbitrary sums to intersections).***
An arbitrary intersection satisfies

$$V\left(\sum_{i \in J} I_i\right) = \bigcap_{i \in J} V(I_i).$$

Proof (of lemma).
$\Longrightarrow$: For $p \in \operatorname{Spec}(A)$, we want to show that $p \supseteq \sum I_i$ iff $p \supseteq I_i$ for all i, so $I_i \subseteq \sum I_i \subset P$.
$\Longleftarrow$: Ideals are additive groups, regardless of whether or not they're prime!
■

Proof (of proposition).

- $\emptyset$ is closed, since $\emptyset = V(A)$
- X is closed, since $X = V(0)$ and O is contained in every prime ideal.
- Closure under finite unions: by induction, it's enough to show that $V(I) \cup V(J)$ is closed. This follows from the 1st lemma above.
- Closure under arbitrary unions: this follows from the 2nd lemma.

■

Proposition 7.2.5 ***($V(I) = V(\sqrt{I})$).***

$$V(I) = V(\sqrt{I}).$$

Proof (?).
The proof is simple: prime ideals are radical.
■

Example 7.2.6 ***(?):*** Note that

$$\operatorname{Spec} \mathbf{Z} = \{\langle 0 \rangle\} \cup \left\{ \langle p \rangle \;\middle|\; p \trianglelefteq \mathbf{Z} \text{ is prime} \right\}.$$

In general, maximal ideals are always closed points, and $\langle 0 \rangle$ is not a closed point. This is homeomorphic to e.g.

$$\operatorname{Spec} \overline{\mathbf{Q}}[t] = \{\langle 0 \rangle\} \cup \left\{ \langle t - a \rangle \;\middle|\; a \in \overline{\mathbf{Q}} \right\},$$

since both are comprised of countably many closed points and a single open point.

7.3 Localization and the structure sheaf

Definition 7.3.1 (Localization)
Suppose $p \subseteq A$ is a prime ideal, then the **localization** of A at p, is defined as

$$A_p := A\left[(p^c)^{-1}\right] := \left\{ \frac{a}{f} \;\middle|\; a, f \in A,\ f \notin p \right\}_{/\sim}$$

$$\frac{a}{f} \sim \frac{b}{g} \iff \exists\, h \in A \text{ s.t. } h(ag - bf) = 0.$$

This makes the elements of p^c invertible, and is a local ring with residue field $\kappa = \operatorname{ff}(A/p)$ and maximal ideal pA_p. Ideals of A_p biject with ideals of A contained in p.

Remark 7.3.2: Idea: A_p should look like germs of functions at the point p. Note that localizing at the ideal p is like deleting $\operatorname{cl}_X(V(p))$, which is also useful. We now want to construct a sheaf $\mathcal{O} = \mathcal{O}_{\operatorname{Spec} A}$ which has stalks A_p. We'll construct something that's obviously a sheaf, at the cost of needing to work hard to prove things about it!

Definition 7.3.3 (Structure sheaf)
For $U \in \operatorname{Spec}(A)$ open, so $U = V(I)^c$, define the **structure sheaf of** X as the sheaf given

$$\mathcal{O}(U) := \left\{ s : U \to \coprod p \in U A_p \;\middle|\; s(p) \in A_p,\ \text{and } s \text{ is locally a fraction} \right\}.$$

Here *locally a fraction* means that for all $p \in U$ there is an open $p \in V \subseteq U$ and elements $a, f \in A$ such that

1. $f \notin Q$ for any $Q \in V$ and
2. $s(Q) = a/f$ for all $Q \in V$.

Restriction is defined for $V \subseteq U$ as honest function restriction on $\mathcal{O}(U) \to \mathcal{O}(V)$.

Remark 7.3.4: Note that this is sheafifying the presheaf that sends $U = D_f$ for $f \in A$ to the ring A_f.

Example 7.3.5 ***(Structure sheaf of a field):*** Let $k \in \mathsf{Field}$ and $X := \operatorname{Spec}(k) = \{\langle 0 \rangle\}$. Then $\mathcal{O}_X$

is determined by

$$\Gamma(X; \mathcal{O}_X) = \left\{ s : \operatorname{Spec} k \to k \;\middle|\; \text{conditions above} \right\} = k,$$

since the conditions are vacuous here.

Example 7.3.6 ***(Structure sheaf of formal power series rings):*** Let $X = \operatorname{Spec} \mathbf{C}[\![t]\!] = \{\langle 0 \rangle, \langle t \rangle\}$. Then

$$\mathcal{O}_X(X) = \mathbf{C}[\![t]\!] \qquad \mathcal{O}_X(\langle 0 \rangle) = \mathbf{C}(t).$$

8 Sections of the structure sheaf (Friday, September 03)

8.1 Sections of puncturing at zero

Remark 8.1.1: Last time: we defined $\operatorname{Spec} A$ as a topological space and $\mathcal{O}_{\operatorname{Spec} A}$, a sheaf of rings on $\operatorname{Spec} A$ which evidently satisfied the gluing condition:

$$\mathcal{O}_{\operatorname{Spec} A}(U) := \left\{ s : U \to \coprod_{p \in U} A_p \;\middle|\; s(p) \in A_p \, \forall p \text{ and } s \text{ is locally a fraction} \right\}.$$

Example 8.1.2 ***(?):*** Set $X := \mathbf{A}^1_{/k} := \operatorname{Spec} k[t]$ for $k = \bar{k}$. Take a point $\langle t \rangle = \langle t - 0 \rangle \in \operatorname{Spec} k[t]$ corresponding to $0 \in X$, then

$$\mathcal{O}_X(X \setminus \{0\}) = k[t, t^{-1}] = \left\{ \frac{f(t)}{t^\ell} \;\middle|\; f \in k[t], \ell \geq 0 \right\}.$$

Generally for $p = \langle t - a_1, \cdots, t - a_m \rangle$ we get $s_p \in k[t] \left[{\{t-a_i\}_{1 \leq i \leq m}}^{-1} \right]$. Note that for $p = \langle 0 \rangle$, we get $s_p \in k(t)$.

Claim: A section s is determined by s_p for $p = \langle 0 \rangle$, so there is an injective map

$$\begin{aligned} \mathcal{O}_{\operatorname{Spec} k[t]}(\operatorname{Spec} k[t] \setminus \{0\}) &\to k(t) \\ s &\mapsto s_{\langle 0 \rangle}. \end{aligned}$$

Proof (?).
Note that $\langle 0 \rangle$ is in every open set, so locally near p there exists a $P \in V$ and a, f with $f \notin Q$ for all Q and $s_Q = a/f$ for all $Q \in V$. Since $\langle 0 \rangle \in V$, we have $s_{\langle 0 \rangle} = a/f \in k(t)$ and $s_p = a/f \in A_p$. Since $A_p \subseteq k(t)$, we get $s_p = s_{\langle 0 \rangle}$ under this inclusion.

■

Claim:

$$\mathcal{O}_{\operatorname{Spec} k[t]}(\operatorname{Spec} k[t] \setminus \{0\}) = k[t, t^{-1}] = k[t]\left[{\{t^\ell\}_{\ell \geq 0}}^{-1} \right].$$

Proof (?).
We showed that the LHS is a subset of $k(t)$, so which subsets can be written as things that are locally fractions on the complement of zero.
$\supseteq$: This can clearly be done in $k[t, t^{-1}]$ since every element is locally the fraction f/t^k.
$\subseteq$: Suppose f/g with f, g coprime (this is a PID!) with a pole away from zero, so $g \in Q$ for some $Q \neq \langle 0 \rangle$. But then f/g isn't in A_Q.

■

Remark 8.1.3: Note that $X \coloneqq \operatorname{mSpec} k[t] \subseteq X' \coloneqq \operatorname{Spec} k[t]$ as the set of closed points, and restricting $\mathcal{O}_{X'}$ to X yields the sheaf of regular functions. Having the extra generic point was useful!

Exercise 8.1.4 (?)
Show that the maximal ideals $m \trianglelefteq A$ correspond precisely to closed points of $X = \operatorname{Spec} A$.

Example 8.1.5 ***(of a function that is locally but not globally a fraction):*** Take $A \coloneqq k[x, y, z, w]/\langle xy - zw \rangle$, which is the cone over a smooth quadric surface and $X \coloneqq \operatorname{Spec} A$. Then take $U = \operatorname{Spec}(A) \setminus V(y, w) = V(y)^c \cap V(w)^c$ and consider the section

$$s(p) \coloneqq \begin{cases} x/w & p \in V(w)^c \\ z/y & p \in V(y)^c. \end{cases}$$

For $p \in U$, it makes sense to consider x/w and z/y. Are they equal? The answer is yes because $xy - zw = 0$. Check that this can't be a global fraction, which is a consequence of this random open set not being the complement of localizing at a prime ideal.

8.2 Distinguished opens

Definition 8.2.1 (Distinguished open sets)
Given $f \in A$, the **distinguished open** $D(f)$ corresponding to f is defined as

$$D(f) = V(\langle f \rangle)^c \coloneqq \left\{ p \in \operatorname{Spec}(A) \;\middle|\; f \in p \right\}^c = \left\{ p \in \operatorname{Spec} A \;\middle|\; f \notin p \right\},$$

i.e. the points of $\operatorname{Spec}(A)$ where f doesn't vanish.

Remark 8.2.2: The sets $\left\{D(f) \;\middle|\; f \in A\right\}$ form a basis for the topology on $\operatorname{Spec}(A)$. This follows from writing $V(I)^c = \bigcup_{f \in I} D(f)$.

8.3 The fundamental theorem of $\mathcal{O}_{\operatorname{Spec} A}$ (Hartshorne Proposition 2.2)

Theorem 8.3.1 ***(Hartshorne Prop 2.2).***
Let $A \in \mathsf{CRing}$ be unital with $1 \neq 0$ unless $A = 0$ and consider $(\operatorname{Spec} A, \mathcal{O})$. Then

a. For any $p \in \operatorname{Spec} A$, the stalk $\mathcal{O}_p \cong A_p$.
b. For any $f \in A$, $\mathcal{O}(D(f)) = A_f$.
c. Taking $f = 1$, $\Gamma(\operatorname{Spec} A, \mathcal{O}) = A$.

Remark 8.3.2: Note that (b) gives the values of $\mathcal{O}$ on a basis of opens, which determines the sheaf.

Proof (of a).
Define a map

$$
\begin{aligned}
f_p : \mathcal{O}_p &\to A_p \\
(U, s) &\mapsto s(p).
\end{aligned}
$$

This is well-defined since $p \in W$ for any $W \subseteq U \cap V$ for equivalent germs $(U, s) \sim (V, t)$.
Surjectivity: given $x = a/g \in A_p$, we want $(U, s) \in \mathcal{O}_p$ such that $f_p(U, s) = a/g$, so just take $U = D(g)$ and $s = a/g$ (which makes sense!) and evidently maps to a/g.
Injectivity: supposing $f_p(U, s) = 0$ in A_p, we want $(U, s) = 0$. If $s(p) = 0$, then there exists some $h \in P$ with $h \cdot s(p) = 0$. Since $s(p)$ is locally a fraction, we can find $p \in V \subseteq U$ with $s = a/g$ on V with $g \neq 0$ on V, so consider $V \cap D(h)$. The claim is that s is 0 here, which follows from $h \cdot (a/g) = 0$.

■

9 The fundamental theorem of $\mathcal{O}_{\operatorname{Spec} A}$ (Wednesday, September 08)

Remark 9.0.1: Recall that we defined a first version of *affine schemes*, namely pairs $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ where for $U \subseteq \operatorname{Spec} A$ open we have $s \in \mathcal{O}_{\operatorname{Spec} A}(U)$ locally represented by $s|_V = a/f$ for $V \subseteq U$ where $a, f \in A$ and $V(f) \cap V = \emptyset$, so f doesn't vanish on V. Note that the $D(f)$ form a topological basis for $\operatorname{Spec} A$, and the gluing condition is difficult, i.e. $\mathcal{O}_{\operatorname{Spec} A}(U)$ may be hard to compute, even given an open cover $\mathcal{V} \rightrightarrows U$. We proved that $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} = A_{\mathfrak{p}}$ last time, and today we're showing

- $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$,
- $\Gamma\left(\operatorname{Spec} A; \mathcal{O}_{\operatorname{Spec} A}\right) \cong A$.

9.1 Proof of the fundamental theorem

Proof (of b and c).
$b \implies c$: Take $f = 1 \in A$, then $\mathcal{O}(\operatorname{Spec} A) = \mathcal{O}(D(1)) = A$ using (b), so the only hard part is showing (b).
To prove (b), by definition of $\mathcal{O}$ there is a ring morphism

$$\begin{aligned} \psi : A_f &\to \mathcal{O}(D(f)) \\ \frac{a}{f^n} &\mapsto \frac{a}{f^n}. \end{aligned}$$

Note that this is just a careful statement, since the morphisms on stalks $\psi_{\mathfrak{p}} : A_f \to A_{\mathfrak{p}}$ by not be injective in general. The proof will follow if ψ is both injective and surjective. ■

Claim: ψ is bijective.

Proof (of injectivity).
Suppose $\psi(s) = 0$, we then want to show $s = 0$. Write $s = a/f^n$, then for all $\mathfrak{p} \in D(f)$ we know $a/f^n = 0 \in A_{\mathfrak{p}}$. So for each $\mathfrak{p}$ there is some $h_{\mathfrak{p}} \notin \mathfrak{p}$ where

$$h_{\mathfrak{p}}(a \cdot 1 - f^n \cdot 0) = 0 \qquad\qquad \text{in } A$$

in A. Consider the ideal $\mathfrak{a} := \operatorname{Ann}(a) := \left\{ b \in A \;\middle|\; ab = 0 \in A \right\} \ni h_{\mathfrak{p}}$. So take the closed subset $V(\mathfrak{a})$, which does not contain $\mathfrak{p}$ since $\mathfrak{a} \not\subseteq \mathfrak{p}$. Now iterating over all $\mathfrak{p} \in D(f)$, we get $V(\mathfrak{a}) \cap D(f) = \emptyset$. So $V(\mathfrak{a}) \subseteq V(f) = D(f)^c$, thus $f \in \sqrt{\mathfrak{a}}$ and $f^m a = 0$ for some m. Then $f^m(a \cdot 1 - f^n \cdot 0) = 0$ in A, so $a/f^n = 0$ in A_f. ■

Proof (of surjectivity).
Step 1: Expressing $s \in \mathcal{O}(D(f))$ nicely locally.
By definition of $\mathcal{O}_{D(f)}$, there exist V_i with $s|_{V_i} = a_i/g_i$ for $a_i, g_i \in A$. We'd like $g_i = h_i^{m_i}$ for some m_i, so g is a power of h_i, but this may not be true a priori. Fix $V_i = D(h_i)$, then $a_i/g_i \in \mathcal{O}(D(h_i))$ implies that $g_i \notin \mathfrak{p}$ for any $\mathfrak{p} \in D(h_i)$. This implies that $D(h_i) \subseteq D(g_i)$, and taking complements yields $V(h_i) \supseteq V(g_i)$, and $h_i \in \sqrt{\langle g_i \rangle}$ and $h_i^n = g_i$. Writing $g_i = h_i^n/c$ we have $a_i/g_i = ca_i/h_i^n$. Note that $D(h_i) = D(h_i^n)$. Now replace a_i with ca_i and g_i with h_i to get

$$s|_{D(h_i)} = a_i/h_i.$$

Step 2: Quasicompactness of $D(f)$.
Note that $\{D(h_i)\}_{i \in I} \rightrightarrows D(f)$, so take a finite subcover $\{D(h_i)\}_{i \le m}$.
Proof of quasicompactness: since $D(f) \supseteq \bigcup_{i \in I} D(h_i)$, we get

$$V(f) \subseteq \bigcap_{i \in I} V(h_i) = V\left(\sum h_i\right).$$

So $f^u \in \sum h_i$, and up to reordering we can conclude $f^u = \sum_{i\leq m} b_i h_i$ for some $b_i \in A$. Then $D(f) \subseteq \bigcup_{i\leq m} D(h_i)$.
Remark 9.1.1: Since we can write $\operatorname{Spec} A = D(1)$, it is quasicompact.

Step 3: Showing surjectivity.
Next time.

■

10 | Friday, September 10

10.1 Sections and Stalks of $\mathcal{O}_{\operatorname{Spec} A}$ as Localizations

Remark 10.1.1: Last time: any affine scheme is quasicompact, so for each ring A and an open cover $\mathcal{U} \rightrightarrows D(f)$ then there is a finite subcover $\{D(h_i)\} \rightrightarrows D(f)$. We were looking at proposition: for the ringed space $(\operatorname{Spec} A, \mathcal{O}_A)$,

- $\mathcal{O}_\mathfrak{p} \cong A\left[\mathfrak{p}^{-1}\right]$ for all $\mathfrak{p} \in \operatorname{Spec} A$,
- $\mathcal{O}(d(f)) \cong A\left[f^{-1}\right]$ for all $f \in A$,
- $\Gamma\left(\operatorname{Spec} A; \mathcal{O}_A\right) \cong A$.

Note that $\mathcal{O}_A$ is uniquely characterized by these properties.

Remark 10.1.2: We can write $D(1) = \operatorname{Spec} A$ and write $\{D(h_i)\} \rightrightarrows \operatorname{Spec} A$ to obtain $1^n = \sum b_i h_i$. This is analogous to a partition of unity, where $b_i h_i$ vanishes on $D(h_i)^c = V(h_i)$

Proof (of 2.2b).
Let $\psi : A\left[f^{-1}\right] \hookrightarrow \mathcal{O}(D(f))$ where we just take restrictions of functions. We know this is injective, and we want to show surjectivity.
Step 1: Let $s \in \mathcal{O}(D(f))$. For each open $D(h_i)$, write $s|_{D(h_i)} = a_i/h_i$ for some $a_i \in A$.
Step 2: By quasicompactness, write $f^n = \sum_{1\leq i\leq m} b_i h_i$.
Step 3: Glue the a_i/h_i into an element a/f of $A\left[f^{-1}\right]$.
Part 1: For any $1 \leq i \neq j \leq m$, $D(h_i h_j) = D(h_i) \cap D(h_j)$. Note that $a_i/h_i = a_j/h_j$ in $\mathcal{O}(D(h_i h_j))$, and these are elements of $A\left[{h_i h_j}^{-1}\right]$ since $a_i/h_i = a_i h_j / h_i h_j$. Using injectivity of ψ for $h_i h_j$, we get an equality $a_i/h_i = a_j/h_j$ in $A_{h_i h_j}$. Then for ℓ large enough, $(h_i h_j)^\ell (a_i h_j - a_j h_i) = 0 \in A$. Regrouping yields $h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j a_j) = 0$, so

$$\frac{a_i h_i^n}{h_i^{n+1}} = \frac{a_j h_j^r}{h_j^{n+1}} := \frac{\tilde{a}_i}{\tilde{h}_i} = \frac{\tilde{a}_j}{\tilde{h}_j},$$

noting that $D(\tilde{h}_i) = D(\tilde{h}_i)$ since $\tilde{h}_i$ is a power of h_i.
Now use POU gluing to write $f^n = \sum_i \tilde{b}_i \tilde{h}_i$ and $a := \sum \tilde{a}_i \tilde{h}_i \in A$ be a global function on $D(f)$.

Then (claim) $a_j/f^n = \tilde{a}_j/\tilde{h}_j$ on $D(\tilde{h}_j)$. We can rewrite

$$\tilde{h}_j a = \sum_i \tilde{b}_i \tilde{a}_i \tilde{h}_j = \sum_i \tilde{b}_i \tilde{a}_j \tilde{h}_i.$$

But then $a/f^n = s|_{D(\tilde{h}_i)}$, so $s = a/f^n$. ∎

Example 10.1.3 ***(?):*** Consider $\mathbf{P}^1_{/k}$ as a scheme – we know the space, and the claim is that we can glue sheaves of affines to obtain a structure sheaf for it. Cover $\mathbf{P}^1$ by $U_0 = \mathbf{P}^1 \setminus \{\infty\} \cong \mathbf{A}^1$ and $U_1 = \mathbf{P}^1 \setminus \{0\} \cong \mathbf{A}^1$. The gluing data is the following:

$$\begin{tikzcd}
 & & & \mathbf{P}^1_{/k} & & & \\
\mathbf{A}^1 \arrow[rr, hook, two heads] & & U_0 \arrow[ur, "\varphi_0"] & & U_1 \cong \mathbf{A}^1 \arrow[ul, "\varphi_1"'] & & \mathbf{A}^1 \arrow[ll, hook, two heads] \\
 & \mathbf{A}^1 \setminus \{0\} \arrow[ul] & & U_0 \cap U_1 \cong D(t) \subseteq \mathbf{A}^1 \arrow[ll, hook, two heads] \arrow[rr, hook, two heads] \arrow[ul] \arrow[ur] & & \mathbf{A}^1 \setminus \{0\} \arrow[ur] &
\end{tikzcd}$$

Link to Diagram

Here the transition maps are

$$\begin{aligned} \varphi_1 \circ \varphi_0^{-1} : \varphi_0(U_0 \cap U_1) &\to \varphi_1(U_0 \cap U_1) \\ t &\mapsto t^{-1}. \end{aligned}$$

What is the map on sheaves? We need a map $\mathcal{O}|_{U_0 \setminus \{0\}} \xrightarrow{\sim} \mathcal{O}|_{U_1 \setminus \{\infty\}}$.

Definition 10.1.4 (Ringed Space)
A **ringed space** $(X, \mathcal{O}_X) \in \mathsf{Top} \times \mathsf{Sh}(X, \mathsf{Ring})$ as a topological space with a sheaf of rings. A morphism is a pair $(f, f^\#) \in C^0(X, Y) \times \in \mathrm{Mor}_{\mathsf{Sh}}(\mathcal{O}_Y, f_* \mathcal{O}_X)$.

Example 10.1.5 ***(?):*** The scheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is a ringed space.

Example 10.1.6 ***(?):*** Consider $\mathbf{R}$ in the Euclidean topology, then $(\mathbf{R}, C^0(-, \mathbf{R}))$ with the sheaf of continuous functions is a ringed space. Then consider the morphism given by projection onto the first coordinate of $\mathbf{R}^2$:

$$\begin{aligned} (\pi, \pi^\#) : (\mathbf{R}^2, C^0(-, \mathbf{R})) &\to (\mathbf{R}, C^\infty(-, \mathbf{R})) \\ (x, y) &\mapsto x. \end{aligned}$$

For $\pi^{\#}$, we can consider $\pi_* C^0(-, \mathbf{R})(U) := C^0(\pi^{-1}(U)) = C^0(U \times \mathbf{R})$, so

$$\begin{aligned} \pi^{\#} : C^\infty(U, \mathbf{R}) &\to C^0(U \times \mathbf{R}) \\ f &\mapsto f \circ \pi, \end{aligned}$$

which is a well-defined map of rings since smooth functions are continuous.

Warning 10.1.7
Not every scheme is built out of affine opens!

11 | Monday, September 13

11.1 Affine Schemes

Definition 11.1.1 (Restricted sheaves)
Let $(X, \mathcal{O}_X) \in \mathsf{RingSp}$ and $U \subseteq X$ be open, then for $V \subseteq U$ open, define the restricted sheaf $\mathcal{O}_X|_V(V) := \mathcal{O}_X(V)$.

Warning 11.1.2

$$\mathsf{Sh}_{/X} \ni \mathcal{O}_X|_U \neq \mathcal{O}_X(U) \in \mathsf{Ring}.$$

Remark 11.1.3: Recall the definition of a ringed space $(X, \mathcal{O}_X)$. The quintessential example: X a smooth manifold and $\mathcal{O}_X := C^\infty(-; \mathbf{R})$ the sheaf of smooth functions. For defining morphisms, consider a map $f : X \to Y$, then an alternative way of defining f to be smooth is that there is a pullback

$$\begin{aligned} f^* : C^0(V, \mathbf{R}) &\to C^0(U, \mathbf{R}) \\ g &\mapsto g \circ f \end{aligned}$$

for $U \subseteq X, V \subseteq Y$, and that f^* in fact restricts to $f^* : C^\infty(V; \mathbf{R}) \to C^\infty(U; \mathbf{R})$, i.e. preserving smooth functions.

Definition 11.1.4 (Morphisms of ringed spaces)
A morphism of ringed spaces is a pair

$$(M, \mathcal{O}_M) \xrightarrow{(\varphi, \varphi^{\#})} (N, \mathcal{O}_N).$$

where $\varphi \in C^0(M, N)$ and $\varphi^{\#} \in \mathrm{Mor}_{\mathsf{Sh}_{/N}}(\mathcal{O}_N, \varphi_* \mathcal{O}_M)$.
This is an **isomorphism** of ringed spaces if

1. φ is a homeomorphism, and

2. $\varphi^{\#}$ is an isomorphism of sheaves.

Remark 11.1.5: In the running example,

$$\varphi^{\#}(U) : \mathcal{O}_N(U) \to \varphi_* \mathcal{O}_M(M) = \mathcal{O}_M(\varphi^{-1}(U)).$$

This implies that maps of ringed spaced here induce smooth maps, and so there is an embedding $\mathsf{smMfd}_{/\mathbf{R}} \hookrightarrow \mathsf{RingSp}$.

Remark 11.1.6: We'll try to set up schemes the same way one sets up smooth manifolds. A topological manifold is a space locally homeomorphic to $\mathbf{R}^n$, and a smooth manifold is one in which it's locally isomorphic as a ringed space to $(\mathbf{R}^n, C^\infty(-;\mathbf{R}))$ with its sheaf of smooth functions.

Definition 11.1.7 (Smooth manifolds, alternative definition)
A **smooth manifold** is a ringed space $(M, \mathcal{O}_M)$ that is locally isomorphic to $(\mathbf{R}^d, C^\infty(-;\mathbf{R}))$, i.e. there is an open cover $\mathcal{U} \rightrightarrows M$ such that

$$(U_i, \mathcal{O}_M|_{U_i}) \cong (\mathbf{R}^n, C^\infty(-;\mathbf{R})).$$

Example 11.1.8*(?)*: An example of a morphism of ringed spaces that is not an isomorphism: take $(\mathbf{R}, C^0) \to (\mathbf{R}, C^\infty)$ given by $(\mathrm{id}, \mathrm{id}^{\#})$ where $\mathrm{id}^{\#} : C^\infty \to \mathrm{id}_* C^0$ is given by $\mathrm{id}^{\#}(U) : C^\infty(U) \to C^0(U)$ is the inclusion of continuous functions into smooth functions.

Remark 11.1.9: We'll define schemes similarly: build from simpler pieces, namely an open cover with isomorphisms to affine schemes. A major difference is that there may not exist a *unique* isomorphism type among all of the local charts, i.e. the affine scheme can vary across the cover.

Remark 11.1.10: Recall that for A a ring we defined $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, where $\operatorname{Spec} A := \{\text{Prime ideals } \mathfrak{p} \trianglelefteq A\}$, equipped with the Zariski topology generated by closed sets $V(I) := \left\{\mathfrak{p} \trianglelefteq A \;\middle|\; I \supseteq \mathfrak{p}\right\}$. We then defined

$$\mathcal{O}_{\operatorname{Spec} A}(U) := \left\{ s : U \to \coprod_{\mathfrak{p} \in U} A\left[\mathfrak{p}^{-1}\right] \;\middle|\; s(\mathfrak{p}) \in A\left[\mathfrak{p}^{-1}\right],\, s \text{ locally a fraction} \right\}.$$

We saw that

1. We can identify stalks: $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} = A\left[\mathfrak{p}^{-1}\right]$
2. We can identify sections on distinguished opens:

$$\mathcal{O}_{\operatorname{Spec} A}(D_f) = A\left[f^{-1}\right] = \left\{ a/f^k \;\middle|\; a \in A, k \in \mathbf{Z}_{\geq 0} \right\},$$

   where $D_f := V(f)^c = \left\{ \mathfrak{p} \in \operatorname{Spec} A \;\middle|\; f \notin \mathfrak{p} \right\}$.

As a corollary, we get $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A$, noting $\operatorname{Spec} A = d_1$ and $A\left[1^{-1}\right] = A$. Thus we can recover the ring A from the ringed space $(X, \mathcal{O}_X) := (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ by taking global sections, i.e. $\Gamma\left(\operatorname{Spec} A; \mathcal{O}_{\operatorname{Spec} A}\right) = A$.

11.2 Affine Varieties

Remark 11.2.1: Let $k = \bar{k}$ and set $\mathbf{A}^n_{/k} = k^n$ whose regular functions are given by $k[x_1, \cdots, x_n]$, regarded as maps to k.

Definition 11.2.2 (Affine variety)
An affine variety is any set of the form

$$X \coloneqq V(f_1, \cdots, f_n) = \left\{ p \in \mathbf{A}^n_{/k} \;\middle|\; f_1(p) = \cdots = f_m(p) = 0 \right\}$$

for $f_i \in k[x_1, \cdots, x_n]$,

Remark 11.2.3: Writing $I = \langle f_1, \cdots, f_m \rangle$, we have $X = V(\sqrt{I})$. Letting $I(S) = \left\{ f \in k[x_1, \cdots, x_n] \;\middle|\; f|_S = 0 \right\}$ then by the Nullstellensatz, $IV(I) = \sqrt{I}$. This gives a bijection between affine varieties in $\mathbf{A}^n_{/k}$ and radical ideals $I \trianglelefteq k[x_1, \cdots, x_n]$.

Definition 11.2.4 (Coordinate rings of affine varieties)
The **coordinate ring** of an affine variety X is $k[X] \coloneqq k[x_1, \cdots, x_n]/I(X)$, regarded as polynomial functions on X.

Remark 11.2.5: We quotient here because if the difference of functions is in $I(X)$, these functions are equal when restricted to X. For example, $y = x$ in $k[x, y]/\langle x - y \rangle$, which are different functions where for $X \coloneqq \Delta$, we have $x|_\Delta = y|_\Delta$.

Remark 11.2.6: As an application of the Nullstellensatz, there is a correspondence

$$\{\text{Points } p \in X\} \underset{V(-)}{\overset{I(-)}{\rightleftarrows}} \operatorname{mSpec} k[X]$$

Remark 11.2.7: Why is an affine variety X an example of an affine scheme $\operatorname{Spec} k[X]$? These don't have the same underlying topological space:

$$\begin{aligned} \tau(X) &\coloneqq \left\{ V(I) \coloneqq \left\{ p \in X \;\middle|\; f_i(p) = 0 \ \forall f_i \in I \right\} \;\middle|\; I \trianglelefteq k[X] \right\} \\ \tau(\operatorname{mSpec} k[X]) &\coloneqq \left\{ V(I) \coloneqq \left\{ \mathfrak{m} \in \operatorname{mSpec} k[X] \;\middle|\; \mathfrak{m} \supseteq I \right\} \;\middle|\; I \trianglelefteq k[X] \right\}. \end{aligned}$$

However, they are closely related:

$$\tau(\operatorname{mSpec} k[X])|_{\operatorname{Spec} k[X]} = \tau(X_{\text{zar}}),$$

i.e. the space $\operatorname{Spec} k[X]$ with the restricted topology from $\operatorname{mSpec} k[X]$ is homeomorphic to X with the Zariski topology. I.e. restricting to *closed points* recovers regular functions on X.

Warning 11.2.8
Defining things that are locally isomorphic to schemes would work for objects but not morphisms!

12 | Wednesday, September 15

12.1 asdsadas

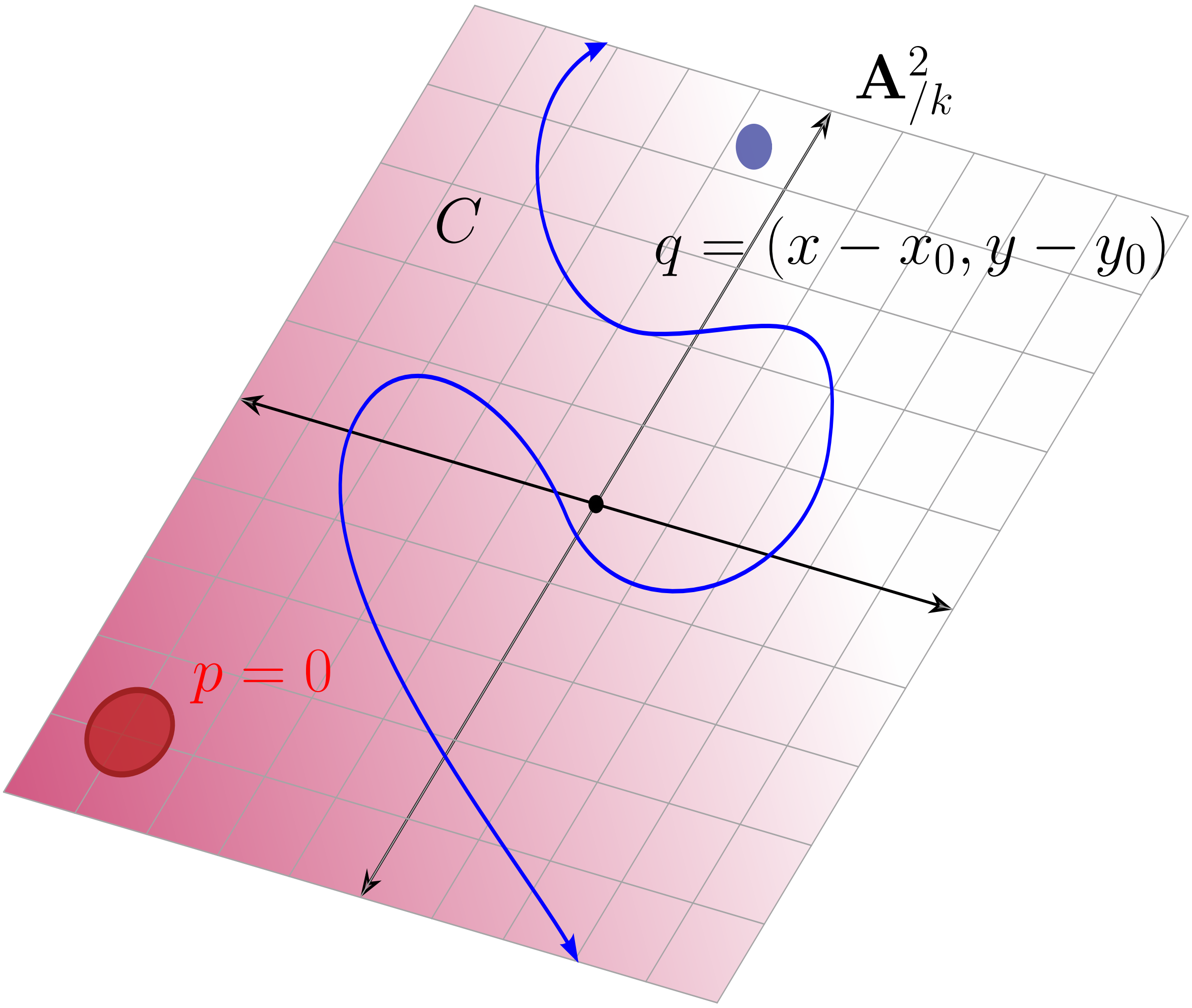
Remark 12.1.1: Last time: for AffVar we considered $X = V(I) \subseteq \mathbf{A}^n_{/k}$, and for AffSch we considered $\operatorname{Spec} k[X]$ where $k[X] := k[x_1, \cdots, x_n]/I(X)$. Both had the Zariski topology, and $X = \operatorname{mSpec} k[X] \subseteq \operatorname{Spec} k[X]$. We had structure sheaves $\mathcal{O}_{\operatorname{Spec} k[X]}$, and for X, we have $U' := U \cap \operatorname{mSpec} k[X]$. On $\operatorname{mSpec} k[X]$, we have the notion of a regular function, and $\mathcal{O}_X(U') = \mathcal{O}_{\operatorname{Spec} k[X]}(U')$. The previous setup only worked for rings finitely generated over a field, whereas for schemes, we can take things like $\operatorname{Spec} \mathbf{Z}$, so they're much more versatile (e.g. for number theory).

12.2 adssads

Example 12.2.1 ***(?):*** Compare $\mathbf{A}^2_{/k} \in \mathsf{AffVar}$ to $\operatorname{Spec} k[x,y]$. Note that $\langle 0 \rangle \in \operatorname{Spec} k[x,y]$, and taking its closure yields

$$\begin{aligned}
\operatorname{cl}(\langle 0 \rangle) &= \bigcap_{V(I) \supseteq \langle 0 \rangle} V(I) \\
&= \bigcap_{V(I) \ni 0} V(I) \\
&= V(0) \\
&= \operatorname{Spec} k[x,y],
\end{aligned}$$

so 0 is a dense point!



But there are prime ideals of height > 1. For example, any irreducible subvariety of $\mathbf{A}^2$ yields a generic point.

Krull's dimension theorem?

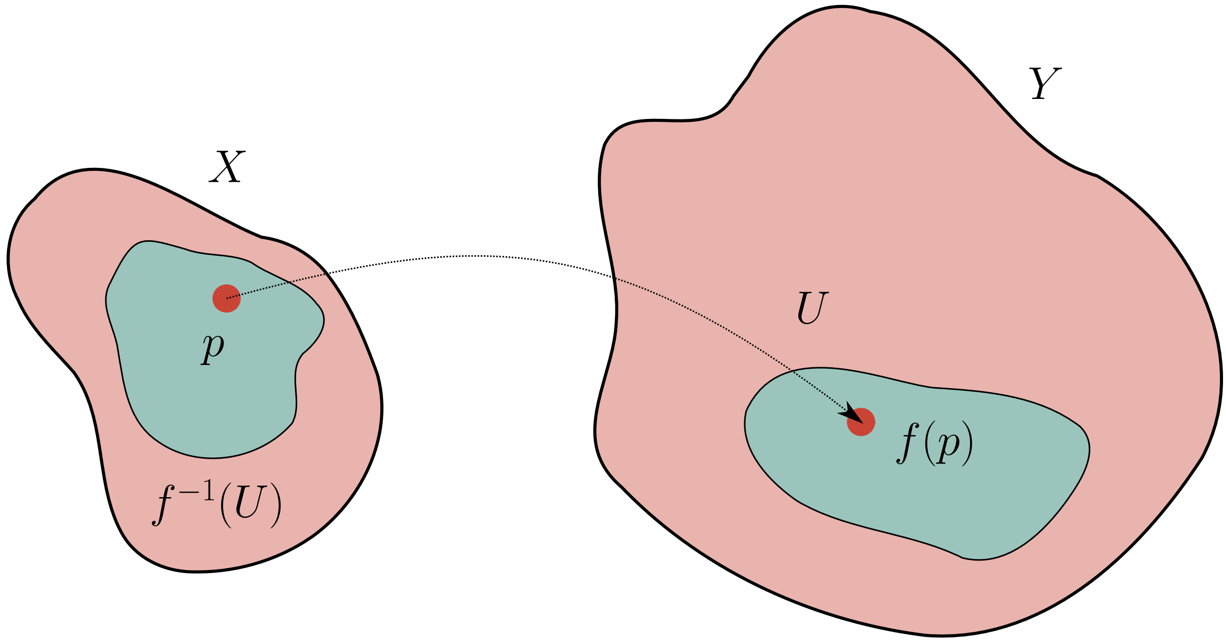
Exercise 12.2.2 (?)
Try to draw $\operatorname{Spec} \mathbf{Z}$ and $\operatorname{Spec} \mathbf{Z}[t]$.

Remark 12.2.3: We'll now try a naive definition of schemes, which we'll find won't quite work.

Definition 12.2.4 (A wrong definition of a scheme!)
A **scheme** is a ringed space $(X, \mathcal{O}_X)$ which is locally an affine scheme, i.e. there exists an open cover $\mathcal{U} \rightrightarrows X$ such that there is a collection of rings A_i with

$$(U_i, \mathcal{O}_X|_{U_i}) \xrightarrow{\sim} (\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i}).$$

Remark 12.2.5: This produces the right objects, but not the correct morphisms. This is a subtle issue! With this definition, a morphism of schemes would be a morphism of ringed spaces $(f, f^\#)$ with $f \in \mathsf{Top}(X, Y)$ and $f^\# \in \mathsf{Sh}_{/Y}(\mathcal{O}_Y, f_* \mathcal{O}_X)$, where $f^\#$ is supposed to capture "pullback of functions". The issue: $f^\#$ may not notice that $p \xrightarrow{f} f(p)$! In particular, it may not preserve the fact that $f(p) = 0$.



Hartshorne exercises for how this issue can actually arise.

Remark 12.2.6: Let $(f, f^\#)$ be a map of ringed spaces, then there is an induced map

$$\begin{aligned} f_p^\# : \mathcal{O}_{Y, f(p)} &\to \mathcal{O}_{X,p} \\ (U, s) &\mapsto (f^{-1}(U), f^\#(U)(s)). \end{aligned}$$

Definition 12.2.7 (Locally ringed space)
A **locally ringed space** $(X, \mathcal{O}_X)$ is a ringed space for which the stalks $\mathcal{O}_{X,p} \in \mathsf{LocRing}$ are local rings, i.e. there exists a unique maximal ideal.

Example 12.2.8 ***(of a locally ringed space):*** For $(X, \mathcal{O}_X) := (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, we saw that $\mathcal{O}_{X,p} = A\left[p^{-1}\right]$, which is local.

Definition 12.2.9 (Morphisms of locally ringed spaces)
A **morphism of locally ringed spaces** is a morphism of ringed spaces

$$(f, F^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

such that $f_p^\# : \mathcal{O}_{Y, f(p)} \to \mathcal{O}_{X,p}$ is a homomorphism of local rings, i.e. $f^\#(\mathfrak{m}_{f(p)}) \subseteq \mathfrak{m}_p$.

Here we should also require that $f^\# \neq 0$.

Remark 12.2.10: Morally: this extra condition enforces that pulling back functions vanishing at $f(p)$ yields functions which vanish at p.

Remark 12.2.11: Alternatively one could require that $(f^\#)^{-1}(\mathfrak{m}_p) = \mathfrak{m}_{f(p)}$, and (claim) this is equivalent to the above definition. Use that $(f^\#)^{-1}(\mathfrak{m}_p)$ is a prime ideal containing $\mathfrak{m}_p$.

Example 12.2.12 ***(of a locally ringed space):*** Take $(X, \mathcal{O}_X) \coloneqq (\mathbf{R}, C^0(-;\mathbf{R}))$. Why this is in $\mathsf{LocRingSp}$: write a stalk as

$$C_p^0 = \left\{ (f, I) \;\middle|\; I \ni p \text{ an interval}, f \in \mathsf{Top}(I, \mathbf{R}) \right\}_{/\sim}.$$

Why is this local? Take $\mathfrak{m}_p \coloneqq \left\{ (f, I) \;\middle|\; f(p) = 0 \right\}$, which is maximal since $C_p^0/\mathfrak{m} \cong \mathbf{R}$ is a field. Then $\mathfrak{m}_p^c = \left\{ (f, I) \;\middle|\; f(p) \neq 0 \right\}$, and any continuous function that isn't zero at p is nonzero in some neighborhood $J \supseteq I$, so $f|_J \neq 0$ anywhere. Then $(f, I) \sim (f|_J, J)$, which is invertible in the ring, so any element in the complement is a unit.

Example 12.2.13 ***(?):*** Consider

$$(\mathbf{R}, C^0(-;\mathbf{R})) \xrightarrow{(f, f^\#)} (\mathbf{R}, C^\infty(-;\mathbf{R})).$$

Take $f = \mathrm{id}$ and the inclusion

$$f^\# : C^\infty(-;\mathbf{R}) \hookrightarrow \mathrm{id}_* C^0(-;\mathbf{R}) = C^0(-;\mathbf{R}).$$

Then

$$f_p^\# : C_p^\infty(-;\mathbf{R}) \to C_p^0(-;\mathbf{R}).$$

satisfies $f_p^\#(\mathfrak{m}_{\mathrm{id}(p)}^\infty) = \mathfrak{m}_p^0$. We also have $(f_p^\#)^{-1}(\mathfrak{m}_p^0) = \mathfrak{m}_p^\infty$, since the germs are just equal.

Definition 12.2.14 (Scheme)
A **scheme** $(X, \mathcal{O}_X)$ is a locally ringed space which is locally isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ in $\mathsf{LocRingSp}$. A **morphism of schemes** is a morphism in $\mathsf{LocRingSp}$.

Remark 12.2.15: Note that we can restrict to opens, since this doesn't change the stalks.

Remark 12.2.16: As a proof of concept that this is a good notion, we'll see that there's a fully faithful contravariant functor $\operatorname{Spec} : \mathsf{CRing} \to \mathsf{Sch}$, so

$$\operatorname{Spec}(\operatorname*{Mor}_{\mathsf{Ring}}(B, A)) = \operatorname*{Mor}_{\mathsf{Sch}}(\operatorname{Spec} A, \operatorname{Spec} B).$$

13 | Friday, September 17

Remark 13.0.1: Recall from last time: a *locally ringed space* $(X, \mathcal{O}_X)$ is a ringed space (so $X \in \mathsf{Top}, \mathcal{O}_X \in \mathsf{Sh}(X, \mathsf{Ring})$) such that the stalks $\mathcal{O}_{X,p} \in \mathsf{LocRing}$ for all points $p \in X$. Some examples:

- $(M, \mathcal{O}_M)$ with M a manifold and $\mathcal{O}_M$ any sheaf of reasonable functions,
- $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$

We defined a scheme as a locally ringed space which is locally isomorphic in $\mathsf{LocRingSp}$ to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. Recall that **locally** meant there exists an open cover $\mathcal{U}$ with the property holding for every element in the cover. Note that most "local" conditions from commutative algebra (that can be checked at all localizations) will correspond to properties that hold on *all* open covers.

There are generally two ways to define properties of schemes: either it holds for every affine open cover, or there exists an affine open cover.

Proposition 13.0.2 *(?).*

a. If $A \in \mathsf{Ring}$, then $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \in \mathsf{LocRingSp}$.
b. If $f \in \mathsf{CRing}(A, B)$ is a ring morphism, it induces a morphism $(f, f^{\#}) \in \mathsf{LocRingSp}(\operatorname{Spec} B, \operatorname{Spec} A)$.
c. Moreover, every $(f, f^{\#}) \in \mathsf{LocRingSp}(\operatorname{Spec} B, \operatorname{Spec} A)$ is induced by some $f \in \mathsf{Top}(A, B)$.

Remark 13.0.3: Note that morphisms in RingSp don't necessarily restrict to morphisms in $\mathsf{LocRingSp}$, i.e. this is not a full subcategory, since morphisms of rings need not be morphisms of local rings (i.e. those preserving the maximal ideal).

Proof (of (a) and (b)).
Part (a): This follows from the theorem last week that $\mathcal{O}_{\operatorname{Spec} A, p} = A\left[p^{-1}\right]$.
Part(b): There's only ever one thing to do! Define the set-theoretic map

$$\begin{aligned} f : \operatorname{Spec} B &\to \operatorname{Spec} A \\ p &\mapsto \varphi^{-1}(p). \end{aligned}$$

Why this is also continuous: we'll show preimages preserve closed sets. We can write

$$\begin{aligned} f^{-1}(V(I)) &= f^{-1}\left(\left\{ Q \;\middle|\; Q \supseteq I \right\}\right) \\ &= \left\{ \mathfrak{p} \;\middle|\; \varphi^{-1}(\mathfrak{p}) \supseteq I \right\} \\ &= \left\{ \mathfrak{p} \;\middle|\; \mathfrak{p} \supseteq I \right\} \\ &= V(\langle \varphi(I) \rangle), \end{aligned}$$

using that $f^{-1}(Q) := \left\{ \mathfrak{p} \;\middle|\; \varphi^{-1}(\mathfrak{p}) = Q \right\}$.
Now localize to get $\varphi_p : A\left[\varphi^{-1}(p)^{-1}\right] \to B\left[p^{-1}\right]$. We now need a sheaf map:

$$f^{\#} : \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B},$$

i.e. an assignment $f^{\#}(V) : \mathcal{O}_{\operatorname{Spec} A}(V) \to \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(V))$ for all $V \subseteq \operatorname{Spec} A$ open. We can write

$$\mathcal{O}_{\operatorname{Spec} A}(V) := \left\{ s \in \mathsf{Top}(V, \prod_{p \in V} A\left[p^{-1}\right]) \;\middle|\; s(p) \in A\left[p^{-1}\right], s \text{ locally a fraction} \right\} \longrightarrow \mathcal{O}_{\operatorname{Spec} B}(f^{-1}V) := \left\{ t \in \mathsf{Top}\right.$$

$$(s_p)_{p \in V} \mapsto (p_q(s_p))_{q \in f^{-1}(V)}.$$

But then $q \in f^{-1}(p)$ for some $p \in V$ iff $p \in \varphi^{-1}(q)$. So using the map on stalks gives a map on sections, and it preserves the property of locally being a fraction, so this yields a morphism of sheaves of rings, and it remains to check that it's a local morphism.

Note that you can get this by composing $f^{-1}(V) \xrightarrow{f} V \xrightarrow{s} \prod A\left[p^{-1}\right] \xrightarrow{\prod \varphi_p} \prod B\left[\varphi(p)^{-1}\right]$.

We now claim $f^{\#}$ is a local homeomorphism. This is clear: we can write $f^{\#}_p : A\left[f(p)^{-1}\right] \to B\left[p^{-1}\right]$, and $f^{\#}_p = \varphi_p$ by construction, which is a local morphism of rings. So $f^{\#}$ is a morphism in $\mathsf{LocRingSp}$.

■

Proof (of (c)).
Let $(f, f^{\#}) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ be a morphism in $\mathsf{LocRingSp}$. Goal: define $\varphi \in \mathsf{CRing}(A, B)$, inducing $(f, f^{\#})$ in the sense of part (b). Note that by definition, $f^{\#}(\operatorname{Spec} A) : \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \to \mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B)$. By the previous theorem, global sections recovers rings on affines, so $f^{\#}(\operatorname{Spec} A) : A \to B$.

Claim: $\varphi^{-1}(p) = f(p)$.
For any $p \in \operatorname{Spec} B$, we can localize $f^{\#}$ to obtain a local ring morphism

$$f^{\#}_p : \mathcal{O}_{\operatorname{Spec} A, f(p)} \to \mathcal{O}_{\operatorname{Spec} B, p}.$$

We also have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f^{\#}(\operatorname{Spec} A) = \varphi"] \arrow[d, "\text{localize}"] & B \arrow[d, "\text{localize}"] \\
A\left[f(p)^{-1}\right] \arrow[r, "f^{\#}(\operatorname{Spec} A)_p = \varphi_p"'] & B\left[p^{-1}\right]
\end{tikzcd}$$

Link to Diagram

Now we use locality in a key way to conclude $\varphi^{-1}(p) = f(p)$ by commutativity: check that $(f^{\#}_p)^{-1}(\mathfrak{m}_p) = \mathfrak{m}_{f(p)} \xrightarrow{\mathsf{loc}^{-1}} f(p)$, and $\mathsf{loc}^{-1}(\mathfrak{m}_p) = p \xrightarrow{\varphi} \varphi(p)$.

■

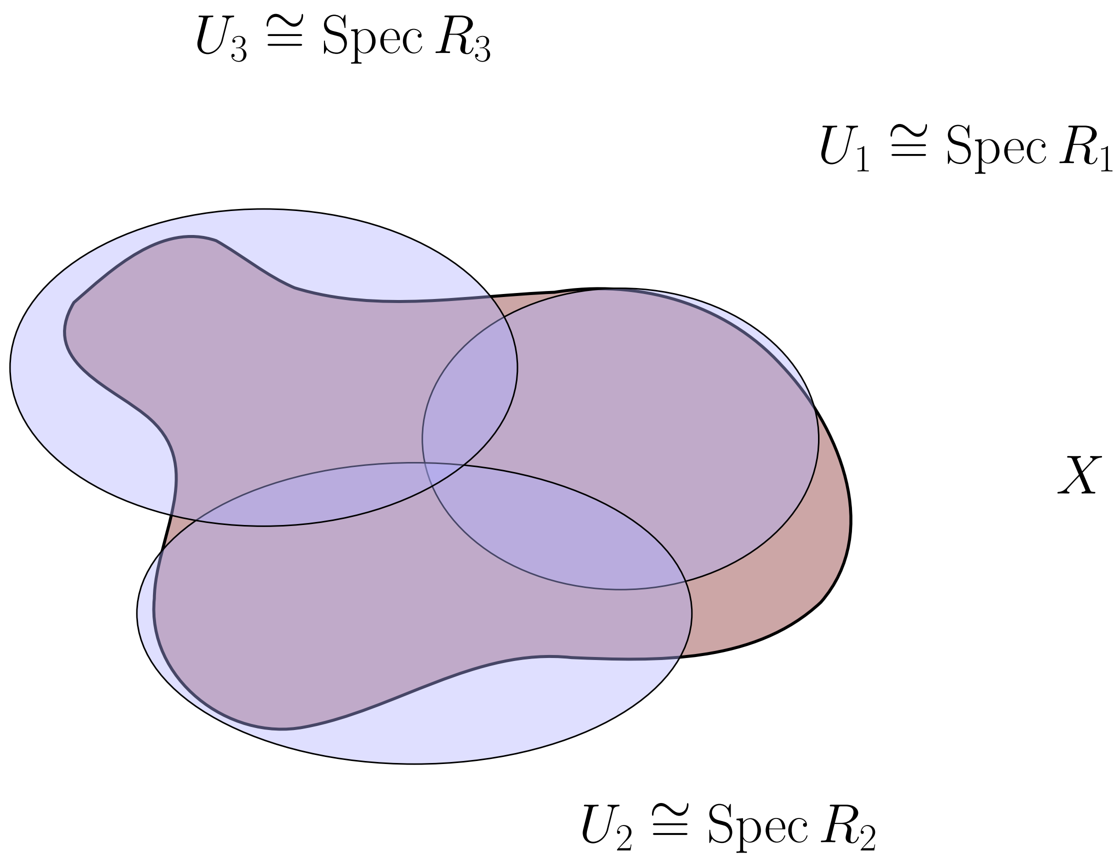
14 | Monday, September 20

Remark 14.0.1: Last time: we proved the following:

Proposition 14.0.2 *(?)*.

a. If $A \in \mathsf{Ring}$, then $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \in \mathsf{LocRingSp}$.
b. If $f \in \mathsf{CRing}(A, B)$ is a ring morphism, it induces a morphism $(f, f^{\#}) \in \mathsf{LocRingSp}(\operatorname{Spec} B, \operatorname{Spec} A)$.
c. Moreover, every $(f, f^{\#}) \in \mathsf{LocRingSp}(\operatorname{Spec} B, \operatorname{Spec} A)$ is induced by some $f \in \mathsf{Top}(A, B)$.

Remark 14.0.3: Recall that a *scheme* $(X, \mathcal{O}_X)$ is a LRS which is locally isomorphic to some affine scheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$:



Definition 14.0.4 (Complete Ring)
A ring R is **complete** with respect to $I \trianglelefteq R$ if $R = \varprojlim R/I^k$. Elements can be written as sequence $(a_k)_{k \geq 0}$ such that $a_k \equiv a_{k-1} \bmod I^{k-1}$.

Example 14.0.5 ***(?):*** A non-example: let $R = \mathbf{C} \dagger t$ and $I = \langle t \rangle$, then set

- $a_0 = 1$,
- $a_1 = 1 + t$,
- $a_2 = 1 + t + t^2$,
- $a_3 = 1 + t + t^2 + t^3$

This is an element in $\underset{\longrightarrow}{\operatorname{colim}}_n R/I^n$, but is not realized by any polynomial, since any polynomial is annihilated by quotienting by a large enough power of t. Note that $\mathbf{C}[\![t]\!] = \mathbf{C}[]t]_{\widehat{\langle t \rangle}}$.

Example 14.0.6 ***(?):*** Part (c) of the proposition would be false if we considered all ringed spaces. Let $R \in \mathsf{DVR}$, so R is local with a principal maximal idea, or equivalently equipped with a valuation $v : R \setminus \{0\} \to \mathbf{Z}_{\geq 0}$ satisfying

- $v(x + y) = v(x) + v(y)$
- $v(x + y) \geq \min v(x), v(y)$

Then $v^{-1}(\mathbf{Z}_{\geq 0})$ is the maximal ideal. Here we'll additionally require that R is **complete** with respect to its maximal ideal $\mathfrak{m}$.

An example is $R = \mathbf{C}[\![t]\!]$, with v the t-adic valuation. Another is $R = \mathbf{Z}_{\widehat{p}} = \mathbf{Z}_{\widehat{p}}$, the completion of $\mathbf{Z}$ at the prime p, given by elements $a_n \cdots a_0$ with $a_i \in \{0, \cdots, p-1\}$. This has maximal ideal $\mathfrak{m} = p\mathbf{Z}_{\widehat{p}}$.

Example 14.0.7 ***(A morphisms of ringed spaces that isn't a morphism of locally ringed spaces):*** Let $K = \operatorname{ff}(R)$ be the fraction field of R, then

$$\operatorname{ff}(\mathbf{C}[\![t]\!]) = \mathbf{C}(t) = \left\{ \sum_{k \geq -N} a_k t^k \;\middle|\; N \in \mathbf{Z}_{\geq 0} \right\}.$$

Also $\operatorname{ff}(\mathbf{Z}_{\widehat{p}}) = \mathbf{Q}_p$, and these are both examples of complete DVRs.

Consider $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ and $(\operatorname{Spec} K, \mathcal{O}_{\operatorname{Spec} K})$. Then $X_1 := \operatorname{Spec} R = \{\langle 0 \rangle, \mathfrak{m}\}$, and the closed sets are $\emptyset, X_1, V(\mathfrak{m}) = \{\mathfrak{m}\}$. For $X_2 := \operatorname{Spec} K = \{\langle 0 \rangle\}$, there is one points since it's a field. Use that $\iota : R \hookrightarrow K$, so define a morphism that does not come from a ring morphism $R \to K$:

$$(f, f^\#) : (\operatorname{Spec} K, \mathcal{O}_{\operatorname{Spec} K}) \to (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$$

$$\begin{aligned} f : \operatorname{Spec} K &\to \operatorname{Spec} R \\ 0 &\mapsto \mathfrak{m} \end{aligned}$$

$$\begin{aligned} f^\# : \mathcal{O}_{\operatorname{Spec} R} &\to f_* \mathcal{O}_{\operatorname{Spec} K} \\ \emptyset &\mapsto 0 \\ \operatorname{Spec} R &\mapsto R \\ \{\langle 0 \rangle\} &\mapsto K. \end{aligned}$$

using that $f^{-1}(\langle 0 \rangle) = \langle m \rangle$ and we can realize the last assignment as a distinguished open mapping to its stalk/localization. Then check

$$\begin{aligned} f_* \mathcal{O}_{\operatorname{Spec} K}(\emptyset) &= 0 f_* \mathcal{O}_{\operatorname{Spec} K}(\operatorname{Spec} R) && = K \\ f_* \mathcal{O}_{\operatorname{Spec} K}(\{\langle 0 \rangle\}) &= \mathcal{O}_{\operatorname{Spec} K}(f^{-1}(\langle 0 \rangle)) = \mathcal{O}_{\operatorname{Spec} K}(\emptyset) = 0 \end{aligned}$$

.

This would induces a commutative diagram, showing this is a morphism of ringed spaces:

$$\begin{tikzcd} R \arrow[r, "f^\#(X)"] \arrow[d] & K \arrow[d] \\ K \arrow[r, "f^\#(\{\langle 0 \rangle\})"] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r, "f^\#(\emptyset)"] & 0 \end{tikzcd}$$

Link to Diagram

Question 14.0.8
Is this a morphism of *locally* ringed spaces?

The answer is **no**, since the induced morphism on stalks won't be morphisms of *local* rings. We can check

$$\begin{aligned} f^\#_{\langle 0 \rangle} &: \mathcal{O}_{\operatorname{Spec} R, \mathfrak{m}} \to \mathcal{O}_{\operatorname{Spec} K, \langle 0 \rangle} \\ f^\#_{\langle 0 \rangle} &: R = R\left[\mathfrak{m}^{-1}\right] \to K\left[\langle 0 \rangle^{-1}\right] = K, \end{aligned}$$

and $(f^{\#}_{\langle 0\rangle})^{-1}(\langle 0\rangle) = \langle 0\rangle \neq \mathfrak{m}$, which is not the maximal ideal of R.

On the other hand, using part (b) of the proposition, any $\varphi \in \mathsf{Ring}(R, K)$ induces a morphism $\tilde{\varphi} : \mathsf{LocRingSp}(\operatorname{Spec} K, \operatorname{Spec} R)$. So $(f, f^{\#})$ is not induced by any such ring map φ.

Remark 14.0.9: So the functor

$$\begin{aligned} \mathsf{Ring} &\to \mathsf{RingSp} \\ A &\mapsto (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \end{aligned}$$

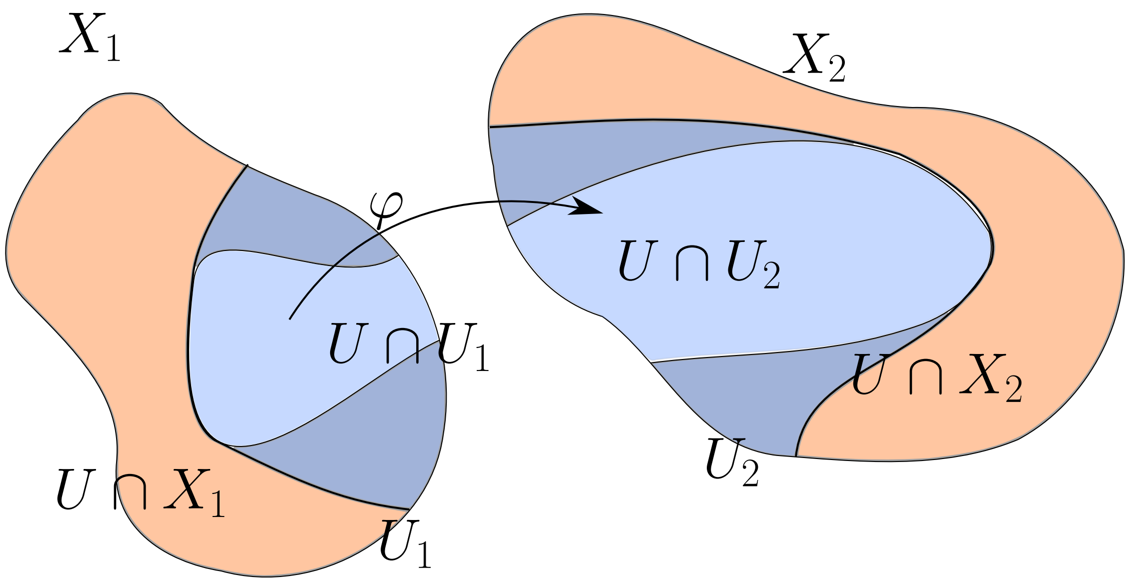
is not fully faithful, but restricting the essential image to $\mathsf{LocRingSp}$.

Remark 14.0.10: Consider the heuristic $\operatorname{Spec} \mathbf{C}[\![t]\!] \sim \mathbb{D} \subseteq \mathbf{C}$ and $\operatorname{Spec} \mathbf{C}(t) \sim \mathbb{D} \setminus \{0\}$.

15 | Wednesday, September 22

Remark 15.0.1: Today: how to build more schemes by gluing known ones together. Let $(X_1, \mathcal{O}_{X_1})$ and $(X_2, \mathcal{O}_{X_2}) \in \mathsf{Sch}$, i.e. locally ringed spaces locally isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$. Let $U_i \subseteq X_i$, and suppose we're given an isomorphism in $\mathsf{LocRingSp}$:

$$\varphi : (U_1, \mathcal{O}_{X_1}|_{U_1}) \to (U_2, \mathcal{O}_{X_2}|_{U_2}).$$



Define a locally ringed space as follows: set $X = X_1 \coprod X_2 / x \sim \varphi(x)$, and define

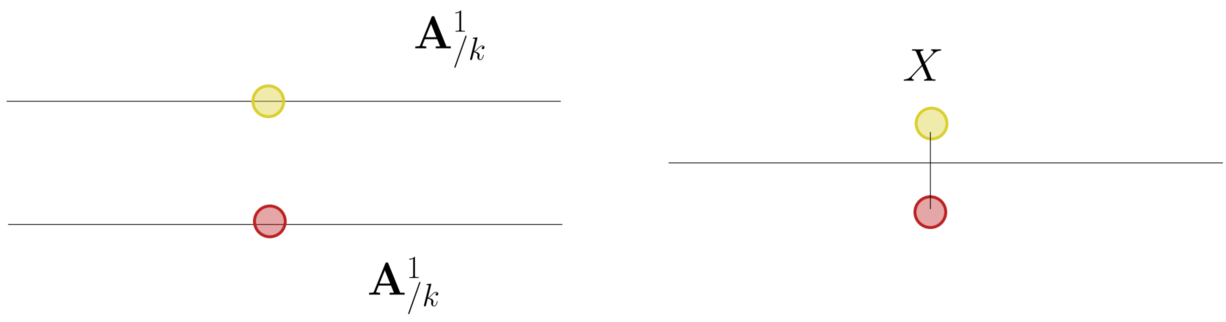
$$\mathcal{O}_X(U) := \left\{ s_1 \in \mathcal{O}_{X_1}(X_1 \cap U), s_2 \in \mathcal{O}_{X_2}(X_2 \cap U) \;\middle|\; s_1|_{U_1 \cap U} = \varphi^{\#}(U_2 \cap U)\left(s_2|_{U_2 \cap U}\right) \right\}.$$

Example 15.0.2 *(?)*: Consider $I := (0,1) \subseteq \mathbf{R}$ and take $X_1 = X_2 := (I, C^\infty(-;\mathbf{R}))$. Using these to cover the circle, we can obtain $(S^1, C^\infty(-;\mathbf{R}))$, using smooth functions that agree on the overlap (here a disjoint union of smaller intervals).

Example 15.0.3 *(A non-affine scheme)*: Let $X_1 = X_2 := \mathbf{A}^1_{/k} := \operatorname{Spec} k[x]$, and set $U_1 \subseteq X_1 := D(x)$. Then take the clear isomorphism

$$(U_1, \mathcal{O}_{X_1}|_{U_1}) \xrightarrow{\text{id}} (U_2, \mathcal{O}_{X_2}|_{U_2}),$$

since they're the same open subset of the same affine variety. Gluing yields the following:



> **Exercise 15.0.4** (?)
> Prove this is not an affine scheme. Use that regular functions are determined by their values on a Zariski open.

Claim: For $X = \operatorname{Spec} R$, recall that $D(f) := \left\{ p \in \operatorname{Spec} R \;\middle|\; p \not\ni f \right\}$. Then $(D(f), \mathcal{O}_X|_{D(f)})$ is an affine scheme.

Proof (?).
Consider $R\left[f^{-1}\right] := \left\{ r/f^k \;\middle|\; r \in R, k \geq 0 \right\} / \sim$. There's a map $R \to R\left[f^{-1}\right]$ which induces a map $\operatorname{Spec} R\left[f^{-1}\right] \to \operatorname{Spec} R$, and we claim it's the inclusion of $D(f)$.

Claim: $\operatorname{Spec} R\left[f^{-1}\right] = D(f)$ as spaces.

This uses that primes in the localization are primes in R not intersecting the inverted set. So $\operatorname{Spec} R\left[f^{-1}\right] = \left\{ p \in \operatorname{Spec} R \;\middle|\; p \cap \langle f \rangle = \emptyset \right\}$, then use that $p \cap \left\{ f^k \right\} = \emptyset \iff f \notin p$, since prime ideals are radical. We now want to show $\mathcal{O}_{\operatorname{Spec} R[f^{-1}]} = \mathcal{O}_{\operatorname{Spec} R}|_{D(f)}$. Check that

$$\mathcal{O}_{\operatorname{Spec} R[f^{-1}]} = \left\{ s : U \to \coprod_{p \in U} \left(R\left[f^{-1}\right]\right)\left[p^{-1}\right] \right\}$$

and

$$\mathcal{O}_{\operatorname{Spec} R}|_{D(f)}(U) = \left\{ s : U \to \coprod_{p \in U} R\left[p^{-1}\right] \right\},$$

but $(R_f)_p = R_p$ since $f \notin p$. This uses that $\left(R\left[S_1{}^{-1}\right]\right)\left[S_2{}^{-1}\right] = R\left[S_2{}^{-1}\right]$ when $S_1 \subseteq S_2$. The first are functions of the form $(a/f^k)/(b/f^\ell) = f^\ell a/f^k b$, so anything locally a fraction in R_f is locally a fraction in R.

■

Example 15.0.5 ***(?):*** Let $X_1 = \mathbf{A}^1_{/k}$ with $U_1 := D(x) \subseteq X_2$ and $X_2 = \mathbf{A}^1_{/k}$ with $U_2 = D(y)$. Then

$$(U_1, \mathcal{O}_{X_1}|_{U_1}) \cong (k[x, x^{-1}], \mathcal{O}_{\operatorname{Spec} k[x,x^{-1}]}) \xrightarrow{(\varphi,\varphi^\#)} (U_2, \mathcal{O}_{X_2}|_{U_2}) \cong (k[y, y^{-1}], \mathcal{O}_{\operatorname{Spec} k[y,y^{-1}]}).$$

Then $(\varphi, \varphi^\#)$ is an isomorphism in $\mathsf{LocRingSp}$ is given by a unique isomorphism

$$\begin{aligned} \iota : k[x, x^{-1}] &\to k[y, y^{-1}] \\ y &\mapsto x \\ y^{-1} &\mapsto x^{-1}. \end{aligned}$$

Note that there is another isomorphism:

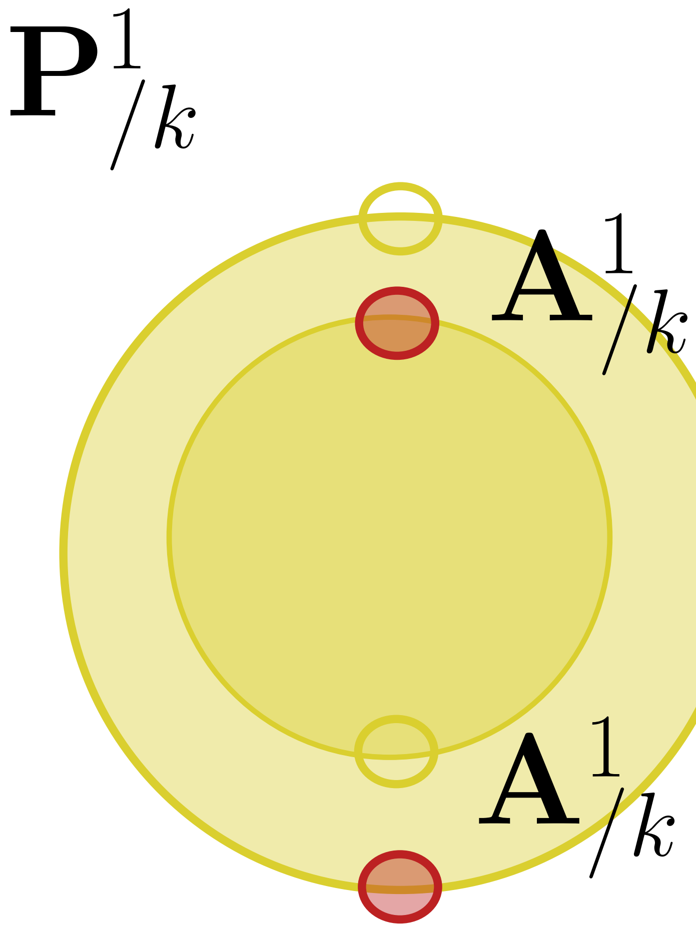
$$\begin{aligned} \iota' : k[x, x^{-1}] &\to k[y, y^{-1}] \\ y &\mapsto x^{-1} \\ y^{-1} &\mapsto x. \end{aligned}$$

So glue instead by the morphism $(\varphi, \varphi^\#)$ induced by ι'. We'll then define **projective space** as

$$\mathbf{P}^1_{/k} := \mathbf{A}^1_{/k} \coprod_{(\varphi,\varphi^\#)} \mathbf{A}^1_{/k}.$$

Note that this works over any ring!

What does this do to closed points? The closed points of $\operatorname{Spec} k[x, x^{-1}]$ are $\left\{(x - c) \;\middle|\; c \neq 0\right\}$ if $k = \bar{k}$, which corresponded to the closed points $c \in \mathbf{A}^1_{/k}$ as a variety. Under $x \mapsto y^{-1}$, we have $(x - c) \mapsto (y^{-1} - c) = (y - c^{-1})$, thus the variety point c gets sent to c^{-1}.



16 Projective Varieties (Tuesday, September 28)

16.1 Projective Space

Definition 16.1.1 (Affine space)
Let R be a ring, then the **affine space of dimension** n **over** R is defined as

$$\mathbf{A}^n_{/R} \coloneqq (\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S}) \qquad\qquad S \coloneqq R[x_1, x_2, \cdots, x_n].$$

Definition 16.1.2 (Slice schemes)
Let $S \in \mathsf{Sch}$, then $X \in \mathsf{Sch}_{/S}$ is a **scheme over** S iff X is a scheme equipped with a morphism of schemes $X \to S$. These form a category where morphisms φ are commuting triangles:

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f_X"'] & & Y \arrow[dl, "f_Y"] \\ & S & \end{tikzcd}$$

Remark 16.1.3: Since $\mathbf{Z} \in \mathsf{CRing}$ is initial, there exists a unique ring morphism $\mathbf{Z} \to \mathbf{R}$ for any $R \in \mathsf{CRing}$. Similarly, $\operatorname{Spec} \mathbf{Z} \in \mathsf{Sch}$ is final, so there exist unique morphisms $\operatorname{Spec} R \to \operatorname{Spec} \mathbf{Z}$ for every R, and thus $\mathsf{Sch}_{/\operatorname{Spec} \mathbf{Z}} \cong \mathsf{Sch}$ just recovers the category of schemes.

Remark 16.1.4: Recall that if $k = \bar{k} \in \mathsf{Field}$ is algebraically closed, then $\mathbf{P}^n_{/k} \coloneqq \mathbf{A}^{n+1}_{/k} \setminus \{0\} / \sim$ where $\mathbf{x} \sim \lambda \mathbf{x}$ for every $\lambda \in k^\times$, or equivalently $\mathrm{Gr}_k(k^{n+1})$, the space of lines through the origin in k^{n+1} (regarded as a vector space). Let $f \in k[x_1, \cdots, x_n]^{\text{homog}}_d$ be homogeneous of degree d, so $f(\lambda \mathbf{x}) = \lambda^d f(\mathbf{x})$. Then its vanishing locus in projective space is well-defined:

$$V_p(f) \coloneqq \left\{ \mathbf{x} = [x_0 : \cdots : x_n] \subseteq \mathbf{P}^n_{/k} \;\middle|\; f(\mathbf{x}) = 0 \right\} \subseteq \mathbf{P}^n_{/k}.$$

16.2 Graded Rings and Homogeneous Ideals

Definition 16.2.1 (Graded rings)
A ring $R \in \mathsf{CRing}$ is **graded** if it admits a decomposition as an abelian group $R + \bigoplus_{d \geq 0} S_d$, where S_d are the degree d pieces satisfying $S_a + S_b \subseteq S_{a+b}$.

Remark 16.2.2: Note that $S_d \in {}_{S_0}\mathsf{Mod}$ for any $S \in \mathsf{gr}\,\mathsf{CRing}$ and any degree d.

Example 16.2.3 *(?)*: $R \coloneqq k[x_1, \cdots, x_n]^{\text{homog}}$ is graded by monomial degree:

$$1 + (x_0) + (x_0^2 + x_1^2) \in R_0 + R_1 + R_2.$$

Definition 16.2.4 (Homogeneous Ideals)
Let $S \in \mathsf{gr}\,\mathsf{CRing}$ be a graded ring, then an ideal $I \trianglelefteq S$ is **homogeneous** if

1. I is generated by homogeneous elements, and
2. For all $f \in I$, all homogeneous pieces $f_i \in S_i$ such that $f = \sum_{i \leq d} f_d$ lie in I.

Example 16.2.5 *(?)*: If $f \coloneqq 1 + x_0 + x_0^2 + x_1^2 \in I$ is an element in a homogeneous ideal, then $1, x_0, x_0^2 + x_1 \in I$ as well.

Remark 16.2.6: If $I \trianglelefteq k[x_1, \cdots, x_n]$ is a homogeneous ideal, say $I = \langle f_1, f_2, \cdots, f_m \rangle$ with each f_i homogeneous of uniform degree d, then $V_p(I)$ is a well-defined projective variety.

Example 16.2.7 *(?)*:

- Take $V_p(y^2 - (x^3 + axz^2 + bz^3))$ for $a, b \in k$ and $4a^3 - 27b^2 \neq 0$. This defines an elliptic curve.
- $\mathbf{P}^n_{/k} = V_p(0)$.

Definition 16.2.8 (Irrelevant Ideal)
We define $S_+ := \bigoplus_{d \geq 1} S_d$ to be the **irrelevant ideal**.

16.3 Projective Nullstellensatz

Remark 16.3.1: We again define the Zariski topology on $X = V_p(I)$ whose closed sets are of the form $V_p(J)$ for $J \subseteq (k[x_1, \cdots, x_n]/I)^{\text{homog}}$

Theorem 16.3.2 ***(Projective Nullstellensatz).***
Let $k[X] := k[x_1, \cdots, x_n]/I$ be the projective coordinate ring of $X \subset \mathbf{P}^n_{/k}$ and $I = I(X)$. Then there is a bijection:

$$\left\{ \text{Points } x \in X \right\} \rightleftharpoons \left\{ \substack{\text{Homogeneous } I \in \text{mSpec } S \\ \text{with } I \not\supseteq S_+} \right\}$$
$$x \mapsto I(x) := \left\langle f \in k[x_1, \cdots, x_n]^{\text{homog}} \;\middle|\; f(x) = 0 \right\rangle$$
$$V_p(I) \hookleftarrow I$$

Remark 16.3.3: Note I doesn't contain S_+ iff I is strictly contained in S_+.

16.4 Proj

Definition 16.4.1 (Proj)
Let $S \in \mathsf{gr\,CRing}$, then define

$$\text{Proj } S := \left\{ p \in \text{Spec } S \text{ homogeneous} \;\middle|\; p \not\supseteq S_+ \right\},$$

where $S_+ := \bigoplus_{d \geq 1} S_d$ is the irrelevant ideal.

Remark 16.4.2: We'll define $\mathcal{O}_{\text{Proj } S}$ next class.

17 | Friday, October 01

Remark 17.0.1: Recall the Proj construction: for $S = \bigoplus_{d \geq 0} S_d \in \mathsf{gr\,CRing}$ we define the irrelevant ideal $S_+ := \bigoplus_{d \geq 1} S_d$ and

$$\begin{aligned} \operatorname{Proj} S &:= \left\{ p \in \operatorname{Spec} S \text{ homog} \;\middle|\; p \not\supseteq S_+ \right\} \\ \mathcal{O}_{\operatorname{Proj} S} &:= \left\{ s : U \to \coprod_{p \in U} S\left[(p^c)^{-1}\right] \;\middle|\; s(p) \in S\left[(p^c)^{-1}\right], s \text{ locally a fraction} \right\}, \end{aligned}$$

recalling that $S\left[(p^c)^{-1}\right] = \left\{ a/f \;\middle|\; \deg a = \deg f, a, f \in S, f \notin p \right\}$. We showed this was a locally ringed space using

$$(D(f), \mathcal{O}_{\operatorname{Proj} S}|_{D(f)} \xrightarrow{\sim} (\operatorname{Spec} S\left[f^{-1}\right], \mathcal{O}_{\operatorname{Spec} S\left[f^{-1}\right]}),$$

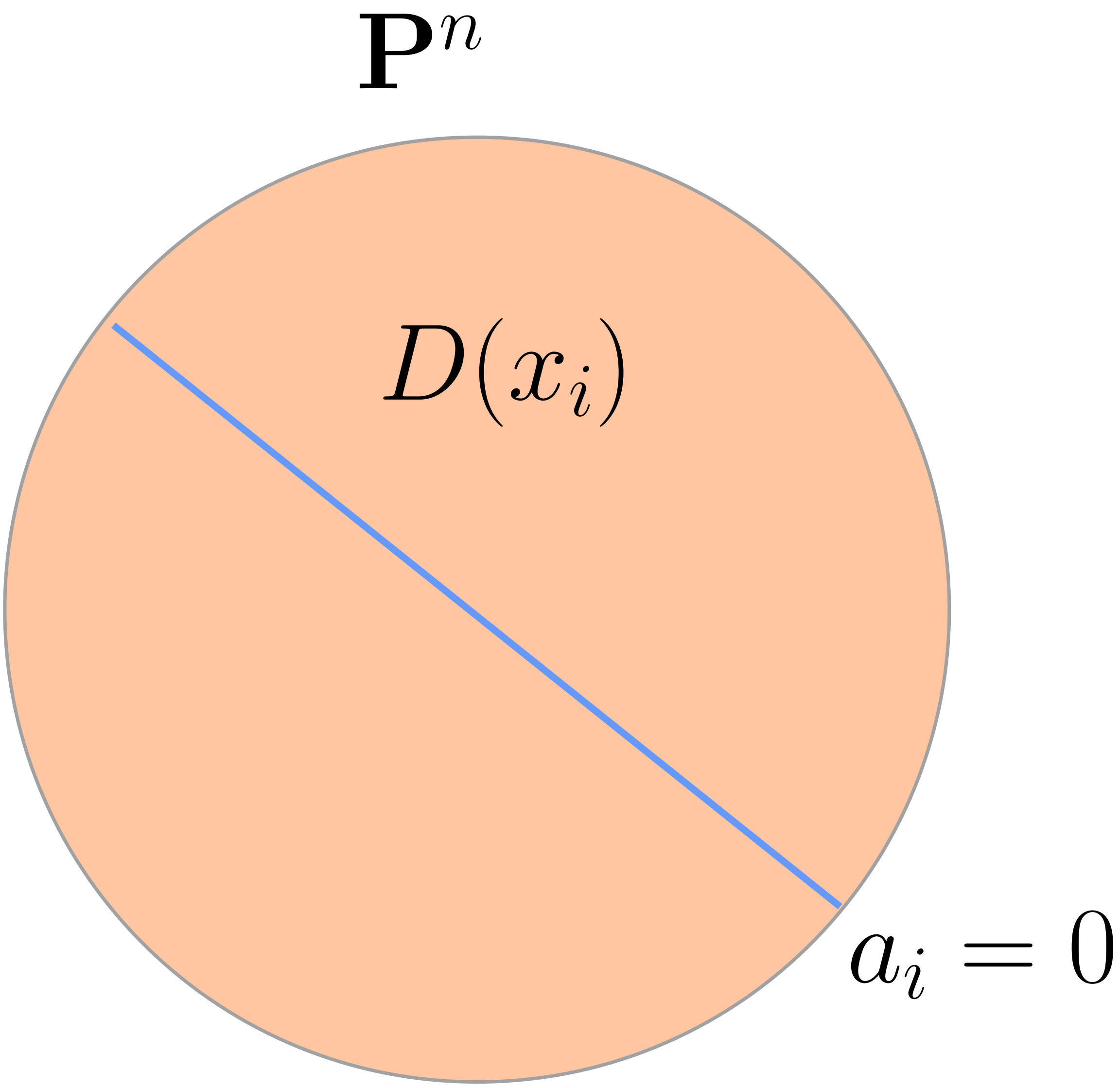
where $D(f) := \left\{ p \in \operatorname{proj} S \;\middle|\; f \notin p \right\}$, and thus $\operatorname{Proj} S \in \mathsf{Sch}$.

Exercise 17.0.2 (?)
Check that there is a natural map of schemes $\operatorname{Proj} S \to \operatorname{Spec} S_0$.

Remark 17.0.3: Consider

$$\mathbf{P}^n_{/R} := \operatorname{Proj} R[x_0, \cdots, x_n] \qquad\qquad R = k = \bar{k} \in \mathsf{Field}.$$

Then the closed points of $\mathbf{P}^n_{/k}$ are of the form $\langle a_i x_j - a_j x_i \rangle \in \operatorname{mSpec} k[x_1, \cdots, x_n]$ for points $[a_0 : \cdots : a_n] \in k^n / \sim$ where $\mathbf{a} \sim \lambda \mathbf{a}$ for $\lambda \in k^\times$. Note that $D(x_i) = \left\{ p \in \mathbf{P}^n_{/k} \;\middle|\; x_i \notin p \right\}$ – what are the closed points? We discard the hyperplane $a_i = 0$ in $\mathbf{P}^n$ to obtain



Then $x_i \in \mathfrak{m}_q$ for $q \coloneqq [a_0 : \cdots : a_n]$ iff $a_i = 0$, and

$$\begin{aligned}
D(x_i) &= \operatorname{Spec} k[x_1, \cdots, x_n]\left[x_i^{-1}\right] \\
&= \left\{ f(x_0, \cdots, x_n)/x_i^d \;\middle|\; \deg f = d \right\} \\
&= \left\{ f\left(\frac{x_0}{x_i}, \cdots, 1, \cdots, \frac{x_n}{x_i}\right) \right\} \\
&= k\left[\tfrac{x_0}{x_i}, \cdots, \tfrac{x_n}{x_i}\right] \\
&\cong \mathbf{A}^n_{/k}.
\end{aligned}$$

We claim that $\bigcup_{i \geq 0} D(x_i) = \mathbf{P}^n_{/k}$, or equivalently $\emptyset = \bigcap_{i \geq 0} V(x_i) = V(\langle x_0, \cdots, x_n \rangle)$. But this is true since $\langle x_0, \cdots, x_n \rangle = S^+$ is the irrelevant ideal.

Proposition 17.0.4 *(?).*
Let $k = \bar{k} \in \mathsf{Field}$. Then there is a fully faithful embedding of categories

$$F : \mathsf{Var}_{/k} \hookrightarrow \mathsf{Sch}_{/k}.$$

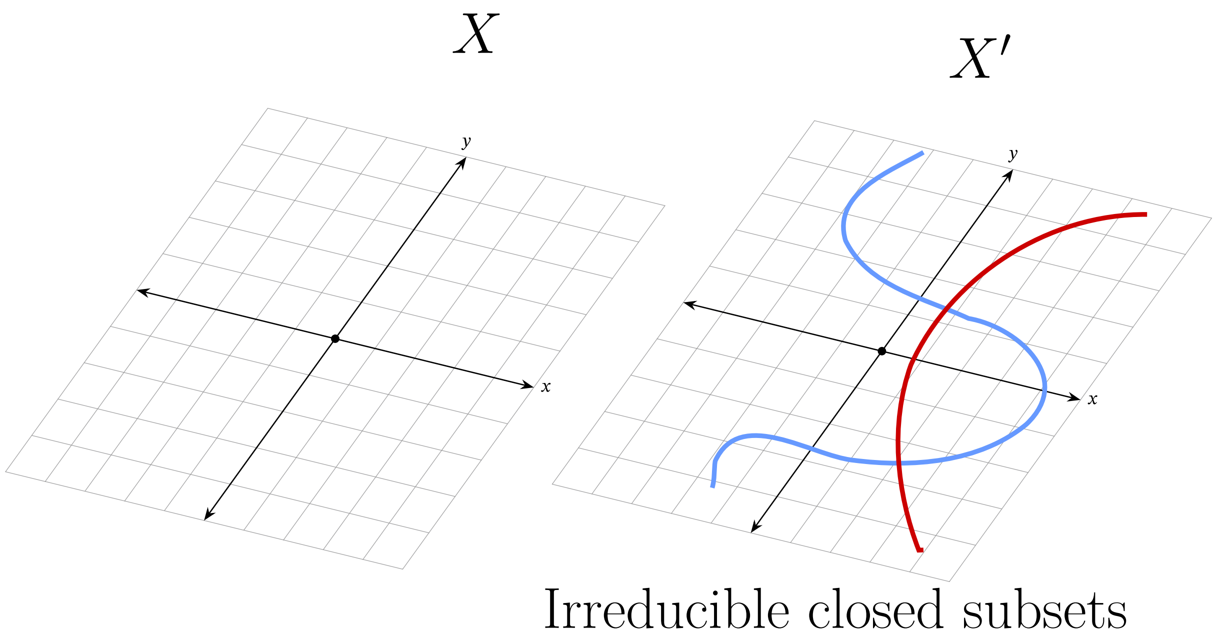
Here $\mathsf{Var}_{/k}$ are defined as topological spaces with sheaves of rings of *regular functions* which admitted an affine cover of the form $V(I) \subseteq \mathbf{A}^n_{/k}$ (viewed as a variety).

Example 17.0.5 ***(Going from a variety to a scheme):*** Consider $X := \mathbf{A}^2_{/k}$ as a variety and separately as a scheme X'. As a variety, $X := k^{\times^2}$ with the Zariski topology, while as a scheme $X' = \operatorname{Spec} k[x, y]$ with the Zariski topology. Then there is an inclusion $X \hookrightarrow X'$ which is a bijection on closed points.

More generally, for $X \in \mathsf{Top}$ any space, define $t(X)$ to be the set of irreducible closed subsets. Some facts:

- For $Y \subseteq X$ closed, $t(Y) \subseteq t(X)$,
- $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$,
- $t(\bigcap_i Y_i) = \bigcap_i t(Y_i)$.

Define a topology on $t(X)$ by declaring closed sets to be any of the form $t(Y)$ for $Y \subseteq X$ closed. Note that the scheme X' has non-closed points, i.e. irreducible subvarieties, i.e. irreducible closed subsets of X as a variety:



Then consider the map

$$\begin{aligned} \alpha : X &\to t(X) \\ p &\mapsto \{p\}, \end{aligned}$$

noting that this is only well-defined if points are closed in X.

Now let $V \in \mathsf{Var}_{/k}$ with its sheaf of regular functions $\mathcal{O}_V$ (i.e. restrictions of polynomials). Define a sheaf of rings on $t(V)$ as $\alpha_* \mathcal{O}_V$, using that α is continuous, and noting that $\alpha^{-1}(U) = U \cap \alpha(X)$.

To see this is a scheme, it suffices to check for V affine since this entire construction is compatible with restriction and we can take an affine cover. Letting $I = I(V)$ for $V \in \mathsf{AffVar}_{/k}$, then $(t(V), \alpha_* \mathcal{O}_V) \xrightarrow{\sim} \operatorname{Spec} k[V] := \operatorname{Spec} k[x_1, \cdots, x_n]/I$. There is a bijection

$$\begin{aligned} t(V) &\rightleftharpoons \operatorname{Spec} k[V] \\ Y &\mapsto I(Y) \\ V(p) &\leftarrow\!\shortmid p. \end{aligned}$$

One can check that the topology on $t(V)$ bijects with the Zariski topology on $\operatorname{Spec} k[V]$, and

$$\alpha_* \mathcal{O}_V(t(V)) = \mathcal{O}_V(V) = \mathcal{O}_{\operatorname{Spec} k[V]}(\operatorname{Spec} k[V]) = k[V].$$

> **Exercise 17.0.6** (?)
> Check this on open subsets of $t(V)$.

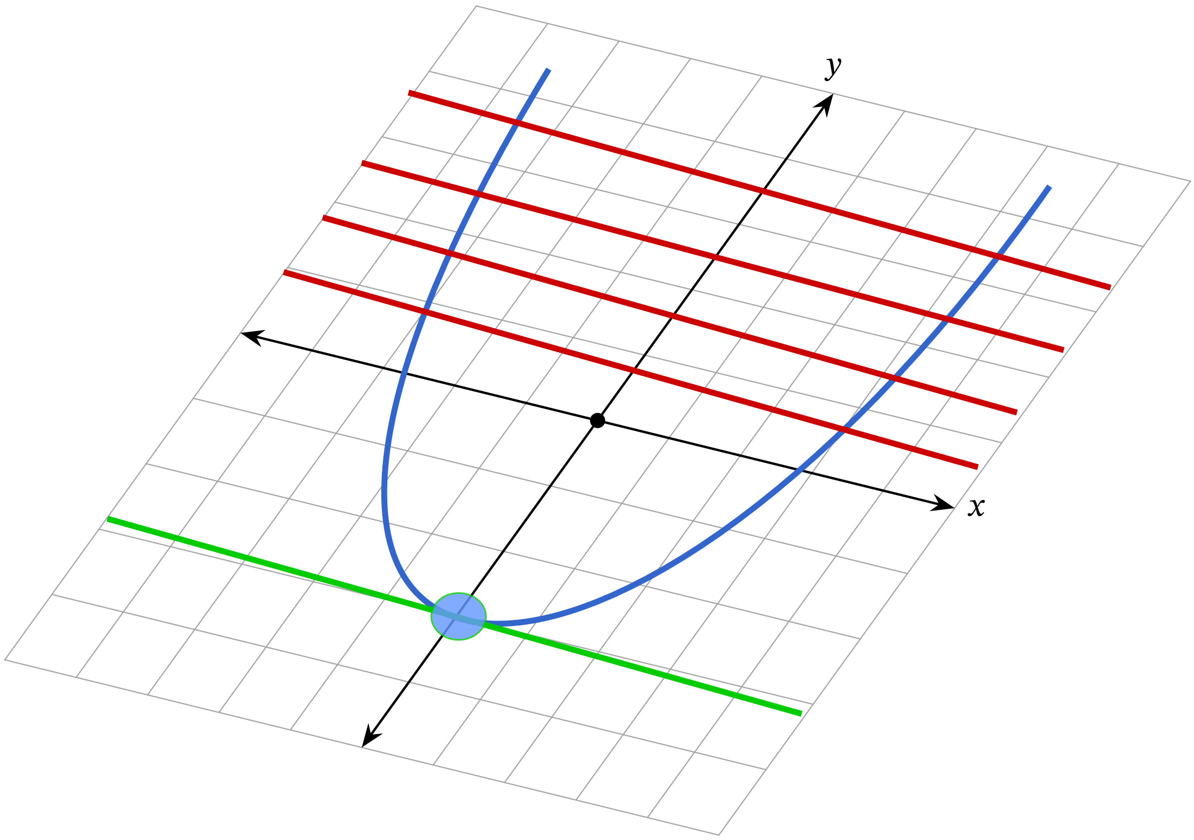
Remark 17.0.7: $\mathcal{O}_X \in \mathsf{Sh}(X, {}_k\mathsf{Alg})$ being a sheaf of k-algebras means the following diagram commutes:

$$\begin{tikzcd} k \arrow[r] \arrow[dr] & \mathcal{O}_X(U) \arrow[d, "\mathrm{res}_{UV}"] \\ & \mathcal{O}_X(V) \end{tikzcd}$$

Link to Diagram

This is the data of a morphism $(X, \mathcal{O}_X) \to \operatorname{Spec} k$.

Remark 17.0.8: What's the point of the theorem? Not everything of geometric interest is in the essential image of F. Consider $V(y - x^2) \subseteq \mathbf{A}^2_{/k}$, and consider intersecting it with lines $y = t$:



Letting $I_1 = I_1(\langle y - x^2 \rangle)$ and $I_2 = I(\langle y - t \rangle)$, intersecting in varieties yields

$$V_1 \cap V_2 = V(I_1 + I_2) = V(\sqrt{I_1 + I_2}).$$

One can check $I_1 + I_2 = (x - \sqrt{t}, y - t) \cdot (x + \sqrt{t}, y - t)$, and $\operatorname{Spec} k[x, y] / \langle y - x^2, y \rangle = \operatorname{Spec} k[x] / \langle x^2 \rangle$ when $t = 0$ (i.e. when there's a tangency with multiplicity), since the scheme intersection is $\operatorname{Spec} k[x, y] / \langle I_1 + I_2 \rangle$. Note that the regular functions on a point are just constant, so the sheaf of regular functions on a point is k itself and thus doesn't pick up the multiplicity of the intersection.

Remark 17.0.9: There can be issues for $\operatorname{Spec} R$ when R is finitely generated but not reduced!

18 | Monday, October 04

Remark 18.0.1: We'll set up some properties for schemes. A scheme can be:

- Irreducible
- Smooth
- Reduced

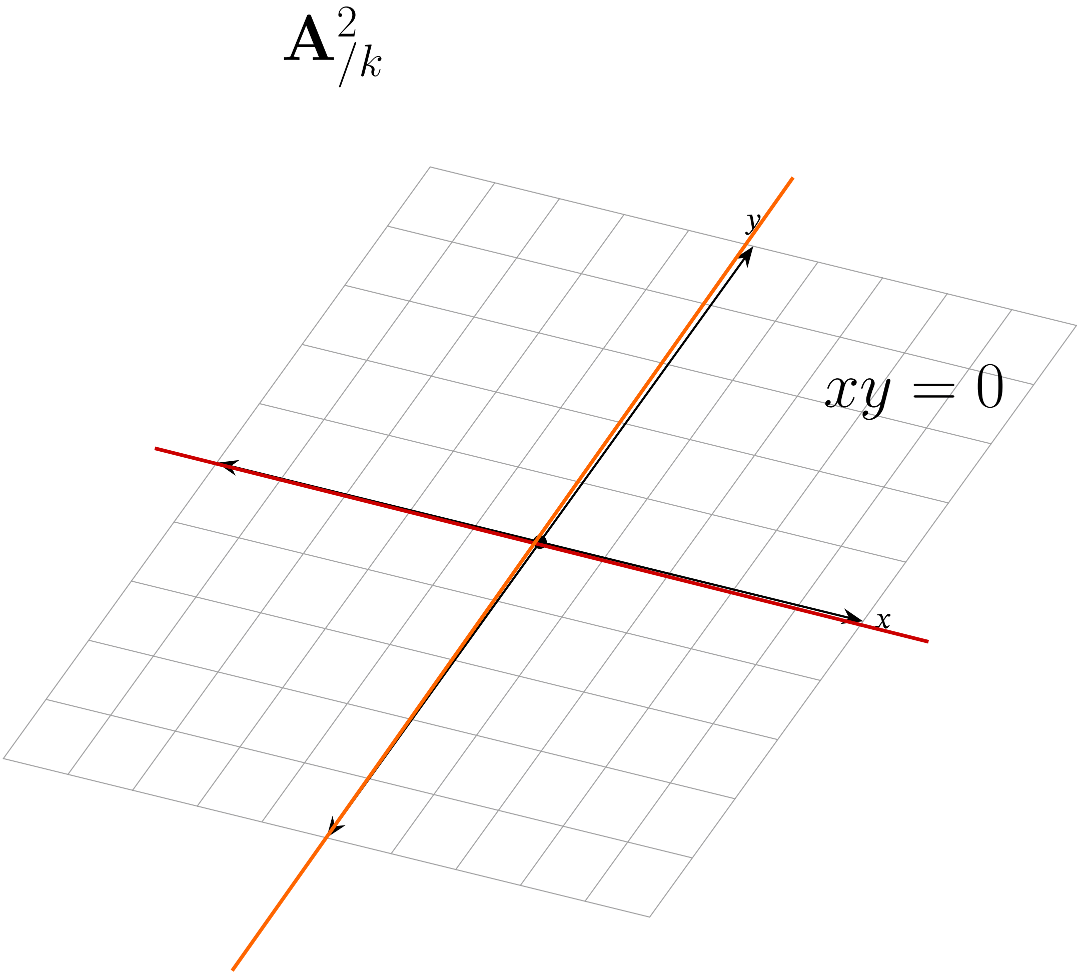
- Noetherian

Remark 18.0.2: Recall that $X \in \mathsf{Top}$ is **connected** iff whenever $X = X_1 \coprod X_2$ with X_i closed, one of $X_i = X$ and the other is empty. X is **irreducible** iff this holds in the weaker case when $X = X_1 \cup X_2$ is not necessarily disjoint. Note that irreducible implies connected.

Definition 18.0.3 (Connectedness and irreducibility for schemes)
$(X, \mathcal{O}_X) \in \mathsf{Sch}$ is **connected** (resp. **irreducible**) iff $|X| \in \mathsf{Top}$ is connected (resp. irreducible).

Example 18.0.4 ***(Connected and irreducible):*** $X := \mathbf{A}^n_{/k}$ is connected and irreducible. Write $\mathbf{A}^n = \operatorname{Spec} k[x_1, \cdots, x_n]$, whose closed sets are $V(I) := \{p \supseteq I\}$. Suppose $X = V(I) \cup V(J) = V(IJ)$, then all primes p satisfy $IJ \supseteq p$, and this $IJ \supseteq \cap_{p \in \operatorname{Spec} R} p = \sqrt{0_R} = 0$, using a fact from commutative algebra. Then $IJ = 0$ implies $I = 0$ (wlog), so $V(I) = X$.

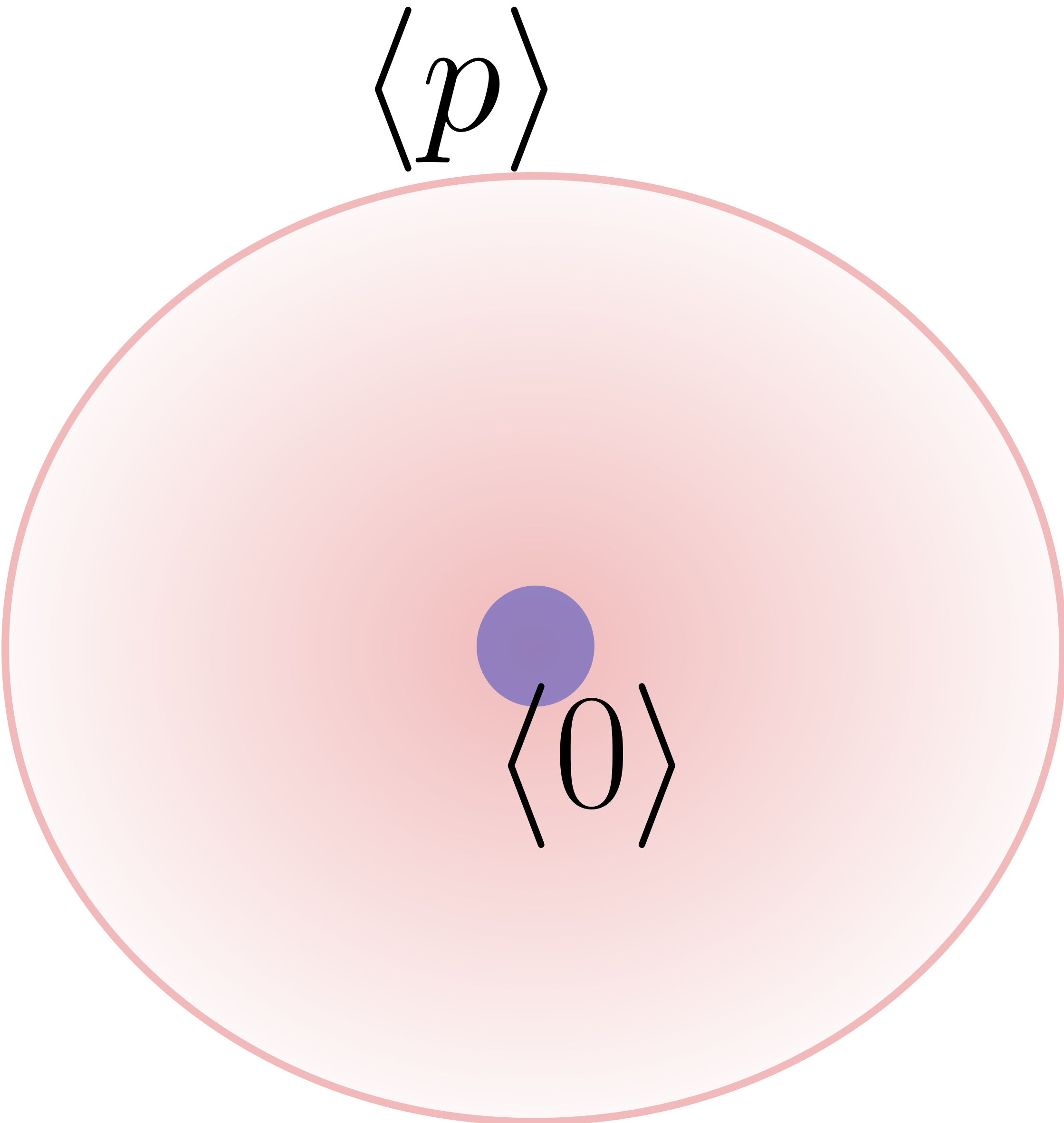
Example 18.0.5 ***(Connected but not irreducible):*** Let $X := \operatorname{Spec} k[x, y]/\langle xy \rangle$, for k not necessarily algebraically closed. This is roughly the coordinate axis in k^2, except there are generic points corresponding to irreducible closed subsets. Points $\langle x - a, y - b \rangle$ are closed and contained in X when (a, b) is on the axis. There are non-maximal primes $\langle x \rangle, \langle y \rangle$.



Note $X \supseteq V(\langle x \rangle) = \{p \in x\} = \{\langle x \rangle\} \cup \{\langle x, y-b \rangle\}$, and similarly, $V(\langle y \rangle) = \{\langle y \rangle\} \cup \{\langle x-a, y \rangle\}$. However $V(x) \cup V(y) = X$ but $V(x) \cap V(y) = \{\langle x, y \rangle\}$, making X connected but not irreducible.

Example 18.0.6 ***(Not connected):*** Let $X \coloneqq \operatorname{Spec}(k[x]/\langle x(x-1) \rangle) \cong \operatorname{Spec}(k[x]/\langle x \rangle) \oplus \operatorname{Spec}(k[x]/\langle x-1 \rangle) \cong \operatorname{Spec}(k \oplus k)$, using the Chinese remainder theorem. Note that this has two prime ideals, $\{0 \oplus k, k \oplus 0\}$, making it a discrete space and thus a disjoint union of its two closed points. Note that $0 \oplus 0$ isn't prime, consider $(a, 0) \oplus (0, b)$.

Example 18.0.7 ***(Connected and irreducible):*** Consider $X \coloneqq \operatorname{Spec} \mathbf{Z}_{\widehat{p}} = \{\langle 0 \rangle, \langle p \rangle\}$. Then $\mathrm{cl}_X(\{\langle 0 \rangle\}) = X$, since 0 is not a closed point. This is connected and irreducible, and the picture is a point with a fuzzy point:



Exercise 18.0.8 (Spec $\mathbf{Z}$ is connected and irreducible)
Show Spec $\mathbf{Z}$ is connected and irreducible.

Definition 18.0.9 (Reduced schemes)
$(X, \mathcal{O}_X) \in \mathsf{Sch}$ is **reduced** iff $\mathcal{O}_X(U)$ is a reduced ring for all open $U \subseteq X$, i.e. contains no nilpotents. Equivalently, $\mathcal{O}_{X,p}$ is reduced for all $p \in X$.

Remark 18.0.10: Note that this is a genuinely new concepts for schemes, since functions valued in a field always yields a reduced ring.

Example 18.0.11 *(Spec R is reduced for R reduced):* Let R be reduced and take $X := \operatorname{Spec} R$, then for $U \subseteq X$ with $U = V(I)^c \supseteq D(f) := V(f)$ for any $f \in I$. In fact $U = \bigcup_{f \in I} D(f)$, and $\mathcal{O}_X(U)$ by the sheaf property can be written as

$$\begin{aligned}
\mathcal{O}_X(U) &= \left\{ \varphi_f \in \mathcal{O}_X(D(f)) \;\middle|\; \varphi_f|_{D(f) \cap D(g)} = \varphi_f|_{D(f) \cap D(g)} \right\} \\
&\subseteq \prod_{f \in I} \mathcal{O}_X(D(f)) \\
&= \prod_{f \in I} R\left[f^{-1}\right],
\end{aligned}$$

by the Dan-Changho theorem, and the claim is that $R\left[f^{-1}\right]$ is reduced for all f. This follows since $(a/f^k)^d \sim 0/1 \implies f^d a^d = 0$, so $f^{dm} a^d = (f^m a)^d = 0$, so $f^m a = 0$ since R is reduced and this $a/f^k \sim 0$, so the localization has no nilpotents. Then $\mathcal{O}_X(U)$ is a subring of a reduced ring and thus reduced, and Spec R is a reduced scheme.

Definition 18.0.12 (Integral schemes)
$(X, \mathcal{O}_X) \in \mathsf{Sch}$ is **integral** iff $\mathcal{O}_X(U)$ is an integral domain for all U.

Example 18.0.13 *(Spec R is integral for R integral):* For R an integral domain, Spec R is integral. Supposing R is not an integral domain, so $xy = 0$ with $x, y \neq 0$. Then $0 \in p$ for any prime, so x or y is in any prime ideal, so $V(x) \cup V(y) = \operatorname{Spec} R$. Alternatively, one can use $\mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) = R$.

Proposition 18.0.14 *(?).*
$(X, \mathcal{O}_X)$ is integral $\iff$ it is irreducible and reduced.

Proof (of proposition).
Integral $\implies$ reduced: Just use the domains have no nilpotents.
Integral $\implies$ irreducible: For the sake of contradiction, suppose $X = X_1 \cup X_2$ with X_i closed. Then there exist $U_i = X_i^c$ open and disjoint (using $X^c = X_1^c \cap X_2^c$), so $\mathcal{O}_(U_1 \coprod U_2) = \mathcal{O}(U_1) \times \mathcal{O}(U_2)$ by the sheaf property for $\mathcal{O}$. However, this is not an integral domain.
Irreducible and reduced $\implies$ integral: Next time! ∎

19 | Wednesday, October 06

Remark 19.0.1: Recall: $X \in \mathsf{Sch}$ is integral iff X is irreducible and reduced, which are defined on sections in terms of rings.

Proof (?).
Irreducible and reduced $\implies$ integral: By contrapositive, assume $\mathcal{O}_X(U)$ is not a domain, so $fg = 0$ in $\mathcal{O}_X(U)$. A local ring need not be domain. However, the germ $f_p g_p := \mathrm{Res}(U, \mathfrak{p})(fg) = 0$ in the stalk $\mathcal{O}_{X,\mathfrak{p}}$. If $(a/s)(b/t) = 0 \in \mathfrak{p}$, then either a/s or b/t is in $\mathfrak{p}$, so f_p or g_p is in $\mathfrak{p}$. Note that $U_1 := \left\{ \mathfrak{m} \in U \;\middle|\; f_p \in \mathfrak{m} \right\} \subseteq U$ is closed, as is $U_2 := \left\{ \mathfrak{m} \in U \;\middle|\; g_p \in \mathfrak{m} \right\}$. We can write $U = U_1 \cup U_2$, so if X is irreducible then U is irreducible, so some $U_i = U$, say U_1. So take an open affine $V \subseteq U_1$ with $f|_V \neq 0$, using the sheaf property. Writing $V = \operatorname{Spec} R$, we have $f|_V \in \mathcal{O}_X(V) = R$, and the stalk $f_p \in p$ for all $p \in R$. Then $f \in p$ for all $p \in \operatorname{Spec} R$, thus in their intersection, and so $f \in \sqrt{0_R}$. Since $f \neq 0$, this contradicts that X is not reduced. $\lightning$

$\blacksquare$

Remark 19.0.2: Recall that *Noetherian rings* are those that satisfy the ACC, or equivalently that all ideals are finitely generated (e.g. a finitely generated k-algebra).

A *Noetherian space* is a space where every descending sequence of closed sets stabilizes.

Definition 19.0.3 (Noetherian rings and spaces)
$X \in \mathsf{Sch}$ is **locally Noetherian** iff there exists an affine open cover with $U_i = \operatorname{Spec} A_i$ for A_i Noetherian. X is **(globally) Noetherian** if X is additionally *quasicompact*, i.e. every open cover has a finite subcover.

Example 19.0.4*(Non-Noetherian rings can produce Noetherian spaces):* The hypothesis of being a Noetherian space isn't enough in general. Consider the ring of *puiseux series* studied by Newton, $R = \bigcup_{n \geq 1} k[[t^{\frac{1}{n}}]]$. Then $\operatorname{Spec} R$ has 2 points:

$$\operatorname{Spec} R = \left\{ \mathfrak{p} := \langle t^r \rangle_{r \in \mathbf{Q}_{\geq 0}}, \langle 0 \rangle \right\}.$$

Here $\langle 0 \rangle$ has closure containing $\mathfrak{p}$, so $\mathfrak{p}$ is a generic point. This is a valuation ring, just not a DVR, and is a Noetherian topological space since there are only two closed sets. However, R is not Noetherian, since there is an infinite chain of ideals:

$$\{I_j\}_{j \geq 1} : \quad \langle t \rangle \subsetneq \left\langle t^{\frac{1}{2}} \right\rangle \subsetneq \left\langle t^{\frac{1}{3}} \right\rangle \subsetneq \left\langle t^{\frac{1}{3}} \right\rangle \subsetneq \cdots.$$

However, $V(I_j) = V(I_k)$ for all j, k, and all equal to $V(\langle p \rangle)$, so $\operatorname{Spec} R$ is a Noetherian space!

Fun fact: $\mathrm{ff}(R) = \bigcup_{n \geq 1} k\left(\left(t^{\frac{1}{n}}\right)\right) = \overline{k((t))}$ when $k = \bar{k}$.

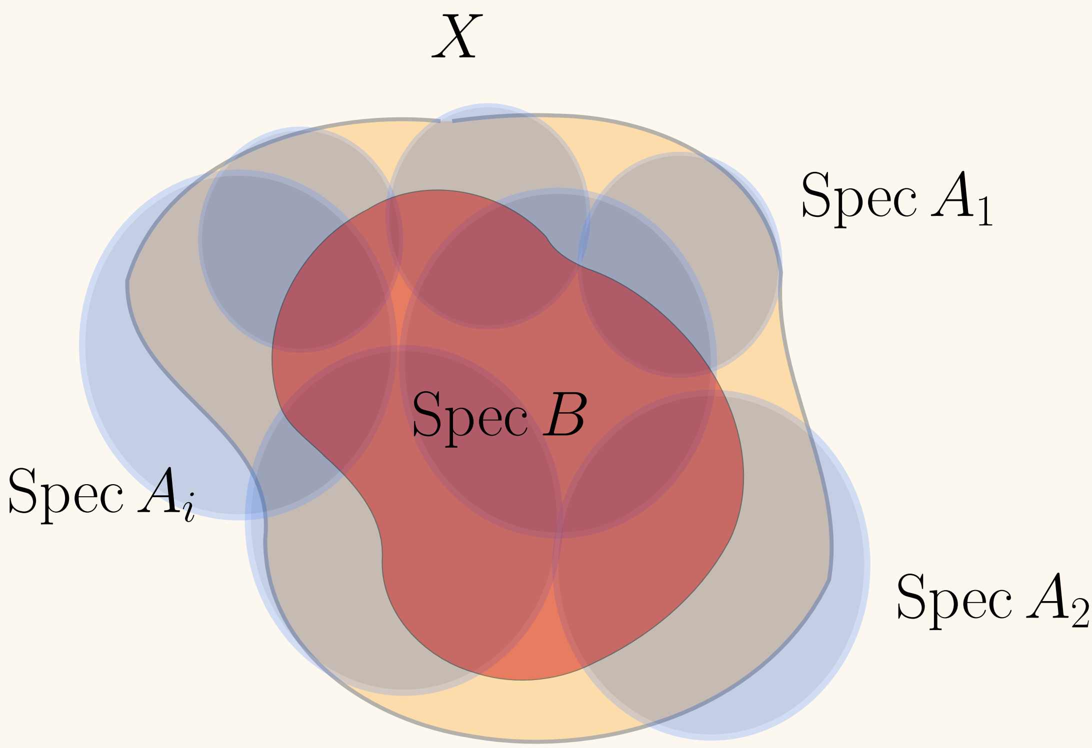
Remark 19.0.5: There are many theorems of the form "a scheme is locally *something*". Here we required an open affine cover by $\operatorname{Spec} R$ for R Noetherian rings. The following two conditions will thus be equivalent:

- A property P holds on every affine open $U \subseteq X$,
- There exists some affine cover $\mathcal{U} \rightrightarrows X$ satisfying property P.

Theorem 19.0.6 ***(?).***
$X \in \mathsf{Sch}$ is locally Noetherian iff for any affine open $U = \operatorname{Spec} A \subseteq X$, A is a Noetherian ring.

Proof (of theorem).
$\Longleftarrow$: Definitional, just apply the hypothesis to some affine open cover.
$\Longrightarrow$: The more nontrivial direction.
Fact ***(from ring theory)***
The localization of any Noetherian ring is again Noetherian

Let $U \subseteq X$ be an affine open, so $U = \operatorname{Spec} B$, and let $\mathcal{U} \rightrightarrows X$ be an affine cover:



Consider $U \cap \mathcal{U}_i \subseteq \mathcal{U}_i$ open, which can be covered by distinguished open sets. So write $U \cap \mathcal{U}_i = \bigcup_j V_{ij}$ with $V_{ij} = D(f_{ij}) \subseteq \operatorname{Spec} A_i$. Then U is covered by $\operatorname{Spec} A_i\left[f_{ij}{}^{-1}\right]$, i.e. spectra of local Noetherian rings. Can we conclude that B is Noetherian from this? This will follow from the fact that we can further decompose $V_{ij} = \bigcup W_{ijk}$ where $W_{ijk} = D_B(f_{ijk})$.
So we want to show the following ring-theoretic statement: let $B \in \mathsf{Ring}$ and $\{g_i\} \subseteq B$

be a collection such that $\operatorname{Spec} B = \bigcup \operatorname{Spec} B\left[g_i^{-1}\right]$ with each $B\left[g_i^{-1}\right]$ Noetherian, then B is necessarily Noetherian. Equivalently, we need $\langle g_i \rangle = \langle 1 \rangle$, which corresponds to $\cap V(g_i) = \emptyset$. ■

20 Locally Noetherian Schemes vs Noetherian Covers (Friday, October 08)

20.1 Proof of Theorem

Recall that we were proving the following:

Theorem 20.1.1 *(?).*
$X \in \mathsf{Sch}$ is locally Noetherian iff for any affine open $U = \operatorname{Spec} A \subseteq X$, A is a Noetherian ring.

Remark 20.1.2: Recall that we covered X by U_i, had some affine open U isomorphic to Spec of a ring, and then covered each intersection $U \cap U_i$ by distinguished opens which were $\operatorname{Spec} R\left[f_i^{-1}\right] = D(f_i) = \{\mathfrak{p} \not\ni f_i\}$. Then $R\left[f_i^{-1}\right]$ is Noetherian iff $\bigcup_{i\in I} D(f_i) = \operatorname{Spec} R$, which implies $\bigcap_{i\in I} V(f_i) = \emptyset$, and thus $\nexists p \in \operatorname{Spec} R$ prime with $p \ni f_i$ for all $i \in I$. Then $\left\langle f_i \;\middle|\; i \in I \right\rangle = \langle 1 \rangle$.

Proposition 20.1.3 *(?).*
$\operatorname{Spec} R$ is quasicompact.

Proof (of proposition).
Let $\mathcal{U} \rightrightarrows \operatorname{Spec} R$, so $\operatorname{Spec} R = \bigcup_{i\in I} U_i$, then we want to find a finite subcover. Take $\{D(f_{ij})\} \rightrightarrows U_i$; it suffices to find a finite subcover of the refined cover by distinguished opens, so reduce to $U_i = D(f_i)$ for each i. Using the argument from the above remark, $\left\langle f_i \;\middle|\; i \in I \right\rangle = \langle 1 \rangle$ since this is a cover. But then there exists a finite sum $\sum_{j=1}^N a_j f_{ij} = 1$ for some $a_j \in R$, so $\{f_{ij}\}_{j=1}^N = \langle 1 \rangle$ which implies that $\bigcup_{j=1}^N D(f_{ij}) = \operatorname{Spec} R$. ■

Remark 20.1.4: Applying the proposition above, we can find a finite set $\{f_i\}$ such that $\left\langle f_i \;\middle|\; i \in I \right\rangle = \langle 1 \rangle$ with each $R\left[f_i^{-1}\right]$ Noetherian. We'll use the following:

Lemma 20.1.5 *(?).*
Let $J \trianglelefteq R$ and $\varphi_i : R \to R\left[f_i^{-1}\right]$ by the canonical localization morphism. Setting $\langle f_1, \cdots, f_s \rangle =$

$\langle 1 \rangle$,

$$J = \cap_{i=1}^{s} \varphi_i \left(\varphi_i^{-1}(J) R\left[f_i^{-1}\right] \right).$$

Proof (?).
Note that $\varphi_i\left(\varphi_i^{-1}(J)R\left[f^{-1}\right]\right) \neq J$ generally, e.g. if $f \in J$. So that J is contained in the right-hand side is clear. For the reverse containment, let $b \in \bigcap_i \varphi_i^{-1}(\varphi_i(J)R\left[f_i^{-1}\right])$. Then $\varphi_i(b) \in \varphi_i(J)R\left[f_i^{-1}\right]$ for all i, so $b \sim a_i/f_i^{n_i}$ in the localization for some $a_i \in J$.
Since $\{f_i\}$ is finite, assume that that $n_i = n$ for some uniform n, e.g. $n = \max\{n_i\}$. Then $b \sim a_i/f_i^N$, so there exist m_i such that $f_i^{m_i}(f_i^N b - a_i) = 0$ in the original ring R. So now pick $M := \max\{m_i\}$ to obtain $f_i^M(f_i^N - a_i) = 0$.
Now a trick: use that $f_i^{M+N} b \in J$ for all i, and the claim is that $\left\langle f_i^{M+N} \right\rangle_{i\in I} = \langle 1 \rangle$. More generally, raising all generators of a unit ideal to a fixed power still generates the unit ideal. This follows from writing $1 = \sum_{i=1}^r c_i f_i \implies 1 = 1^M = \left(\sum c_i f_i\right)^M$, so choose M large enough so that some f_i occurs with an exponent of at least $m+n$, e.g. choosing $M = r(m+n)$.
Example 20.1.6*(?)*: If $1 = \langle x, y\rangle$, then $\langle x^2, y^2\rangle = 1$ by taking $ax+by = 1$ and (say) expanding $(ax+by)^4 = 1$ and noting that any term is divisible by either x^2 or y^2.
Now writing $\sum_{i=1}^r c_i f_i^{m+n} = 1$, we get $\sum_{i=1}^r c_i f_i^{m+n} \in J$, and thus $b \in J$.

■

Remark 20.1.7: We now know that the $R\left[f_i^{-1}\right]$ are Noetherian. Let $J_1 \subseteq J_2 \subseteq \cdots$ be an ascending chain of ideals in R, we'll show it stabilizes. Since the $R\left[f_i^{-1}\right]$ are Noetherian, there is an ascending chain $J_1 R\left[f_i^{-1}\right] \subseteq J_2 R\left[f_i^{-1}\right] \subseteq \cdots$ in $R\left[f_i^{-1}\right]$, which is Noetherian and thus stabilizes. So for some $N = N(i) \gg 0$, $J_k R\left[f_i^{-1}\right] = J_{k+1} R\left[=^{-1}\right] \cdots$ for all $k \geq N$. But there are only finitely many f_i, so we can choose some uniformly large $\tilde{N} \gg 0$ not depending on i where $J_k = J_{k+1} = \cdots$ for all $k \geq \tilde{N}$ by applying the lemma.

Remark 20.1.8: On applying the lemma: use that

$$\begin{aligned}
\varphi_i^{-1}(J_k R\left[f_i^{-1}\right]) = \varphi_i^{-1}(J_{k+1} R\left[f_i^{-1}\right]) \quad \forall k \geq N \\
&\implies \cap_i \varphi_i^{-1}(J_k R\left[f_i^{-1}\right]) = \cap_i \varphi_i^{-1}(J_{k+1} R\left[f_i^{-1}\right]) \quad \forall k \geq N \\
&\implies J_k = Jk+1 \quad \forall k \neq N.
\end{aligned}$$

20.2 Other Properties

Example 20.2.1*(A scheme that is not quasicompact)*: Let $X = \mathbf{Z}$ with the discrete topology (so every set is open) and set $\mathcal{O}_X(U) = \mathsf{Set}(U, k)$ to be the local ring of arbitrary set functions. Then for $p \in X$, the stalks are $\mathcal{O}_{X,p}\mathcal{O}_X(\{P\}) = \mathsf{Set}(p, k) = k$, which is local. This is a locally ringed space, since it's locally isomorphic to $\operatorname{Spec} R$: we can take an open cover of such, or find a neighborhood where it holds, but $p \in \{p\}$ which is open and letting $\mathcal{F} := \mathcal{O}_X|_{\{p\}}$, we have $(\{p\}, \mathcal{F}) \cong \operatorname{Spec} k$. Then $X = \bigcup_{p\in X} \{p\}$ is an open cover with no finite subcover.

So $\mathbf{Z}$ with the discrete topology is not $\operatorname{Spec} R$ with the Zariski topology for any ring.

Exercise 20.2.2 (?)
$X := \operatorname{Spec} \overline{\mathbf{Q}}[t] = \{\langle 0 \rangle\} \cup \left\{ \langle t - a_i \rangle \;\middle|\; i \in I \right\}$ where I is a countable enumeration of $\overline{\mathbf{Q}}$. Is this quasicompact?

Exercise 20.2.3 (?)
Consider $R := \prod_{n \in \mathbf{Z}} \mathbf{C}$, then there is a set map $\{I \trianglelefteq R\} \xrightarrow{\sim} \mathcal{P}(\mathbf{Z})$, given by sending any subset to the ideal $\mathbf{C} \oplus \mathbf{C} \oplus \cdots$ which are zeroed out at entries according to the complement of S. What is $\operatorname{Spec} R$, and what is the topology? Is $\operatorname{Spec} R$ quasicompact?

Consider $I := \left\{ (a_i) \;\middle|\; a_i = 0 \quad i \geq N \gg 0 \right\}$, which forms an ideal. Is I prime? Are there prime ideals not containing I?

21 | Monday, October 11

Remark 21.0.1: Last time: Noetherian isn't a purely topological property. Today: another guiding principle in AG is that one can put properties on schemes, or alternatively on morphisms, usually to $\operatorname{Spec} k$.

Definition 21.0.2 (Finite type morphisms)
A morphism $X \xrightarrow{f} Y$ of schemes is **locally of finite type** if there exists an affine open cover $\mathcal{V} \rightrightarrows Y$ with $V_i = \operatorname{Spec} B_i$ such that there exists an affine open cover $\mathcal{U} \rightrightarrows f^{-1}(\mathcal{V})$ so $U_{ij} \supseteq f^{-1}(V_i)$ where $U_{ij} = \operatorname{Spec} A_{ij}$ such that the restricted function $f : U_{ij} \to V_i$ is induced by a ring morphism $B_i \to A_{ij}$ finitely generated over B_i.

Remark 21.0.3: This globalizes the notion of being a finitely generated ring, essentially by covering the scheme morphisms by ring morphisms with the desired property. As a special case, let $X \in \mathsf{Sch}_{/k}$, so there is a morphism $X \to \operatorname{Spec} k$. Let $X = \cup_j U_j$ with $U_j = \operatorname{Spec} A_j$, then we want that the map $\operatorname{Spec} A_j \to \operatorname{Spec} k$ is induced by a finitely-generated ring morphism $k \to A_j$, so A_j is a finitely-generated k-algebra. So this is like having a sheaf of k-algebras. As an abuse of notation/terminology, we say that X is **finite type** over k (since the target doesn't need to be covered).

⚠ **Warning 21.0.4**
Note that a subring of a finitely generated ring need not be finitely generated!

Example 21.0.5 *(?)*: Let $A \in {}_k\mathsf{Alg}^{\mathrm{fg}}$, then $\operatorname{Spec} A$ is finite type over k. One can change the definition from "there exists an open cover" to "for all open covers" – this amounts to checking localizations of ring maps.

Example 21.0.6 *(?)*: Consider $X := \operatorname{Proj} k[x, y] = \mathbf{P}^1_{/k}$, then recall that $(D(f), \mathcal{O}_X|_{D(f)}) \cong$

Spec $R\left[f^{-1}\right]$. Then

$$\mathbf{P}^1_{/k} = D(x)\coprod_f D(y) \cong \mathbf{A}^1_{/k}\coprod_f \mathbf{A}^1_{/k},$$

glued along inversion. Then $k[x,y]\left[x^{-1}\right] \cong k\left[\frac{y}{x}\right]$ and $k[x,y]\left[y^{-1}\right] \cong k\left[\frac{x}{y}\right]$. One can check that $\mathbf{P}^1_{/k} \to \operatorname{Spec} k$ is finite type, and this works for $\mathbf{P}^n$ as well.

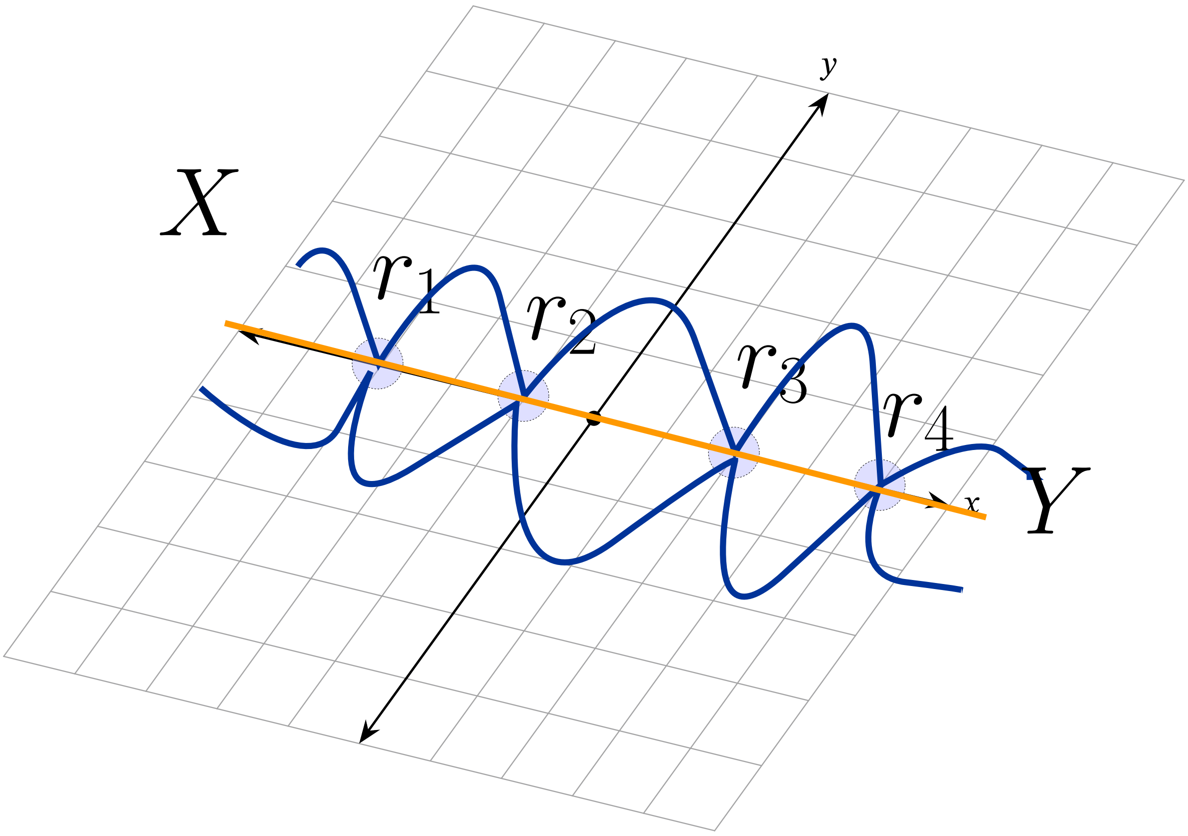
Note that if $S := k[x_1, \cdots, x_n]/I$ for I a homogeneous ideal, then $\operatorname{Proj} S$ is also finite type over k. We can write $\operatorname{Proj} S = \bigcup_{i=0}^n D(\bar{x}_i)$, since taking complements yields $\emptyset = \bigcap_{i=0}^n V(\bar{x}_i)$, which is the set of homogeneous prime ideals $p \in \operatorname{Spec} S$ with $p \supseteq \bar{x}_i$ for all i and $p \not\supseteq S^+$ the irrelevant ideal, which is empty. So $S\left[x_i^{-1}\right]$ is a finitely generated ring, with generators of the form x_j/x_i.

Example 21.0.7 ***(a non-example):*** $\operatorname{Spec} k[\![t]\!] \to \operatorname{Spec} k$ is not a finite type morphism, i.e. $k[\![t]\!]$ is not a finitely generated k-algebra. Toward a contradiction suppose $k[\![t]\!] = \langle f_1, \cdots, f_r \rangle$, so there is a ring map $k[f_1, \cdots, f_r] \twoheadrightarrow k[\![t]\!]$. Take $k = \mathbf{Q}$, or any countable field, then the LHS is countable but the right-hand side is not.

Definition 21.0.8 (Finite morphisms)
Recall that a morphism $\varphi : B \to A \in \mathsf{CRing}$ is a **finite morphism** if A is finitely-generated as a B-module. A morphism $X \xrightarrow{f} Y \in \mathsf{Sch}$ is **finite** iff there exists an affine open cover $\mathcal{V} \rightrightarrows Y$ with $V_i = \operatorname{Spec} B_i$ and $f^{-1}(V_i) = \operatorname{Spec} A_i$, and the induced ring maps $B_i \to A_i$ are finite.

Remark 21.0.9: Here the module structure is $b \cdot a := \varphi(b)a$. Note that finite type required finite generation as rings, so $B(g_1, \cdots, g_r) \twoheadrightarrow A$, but here we require that any $a \in A$ is of the form $a = \sum_{i=1}^r \varphi(b_i)a_i$ for some b_i.

Example 21.0.10 ***(?):*** Consider $X := \operatorname{Spec} R := \operatorname{Spec} \mathbf{C}[x,y]/\left\langle y^2 - f(x)\right\rangle$ where f has no repeated roots, which yields a hyperelliptic curve. This is a reduced ring, so X is the scheme associated to a variety. Letting r_i be the roots of f, we have the following:

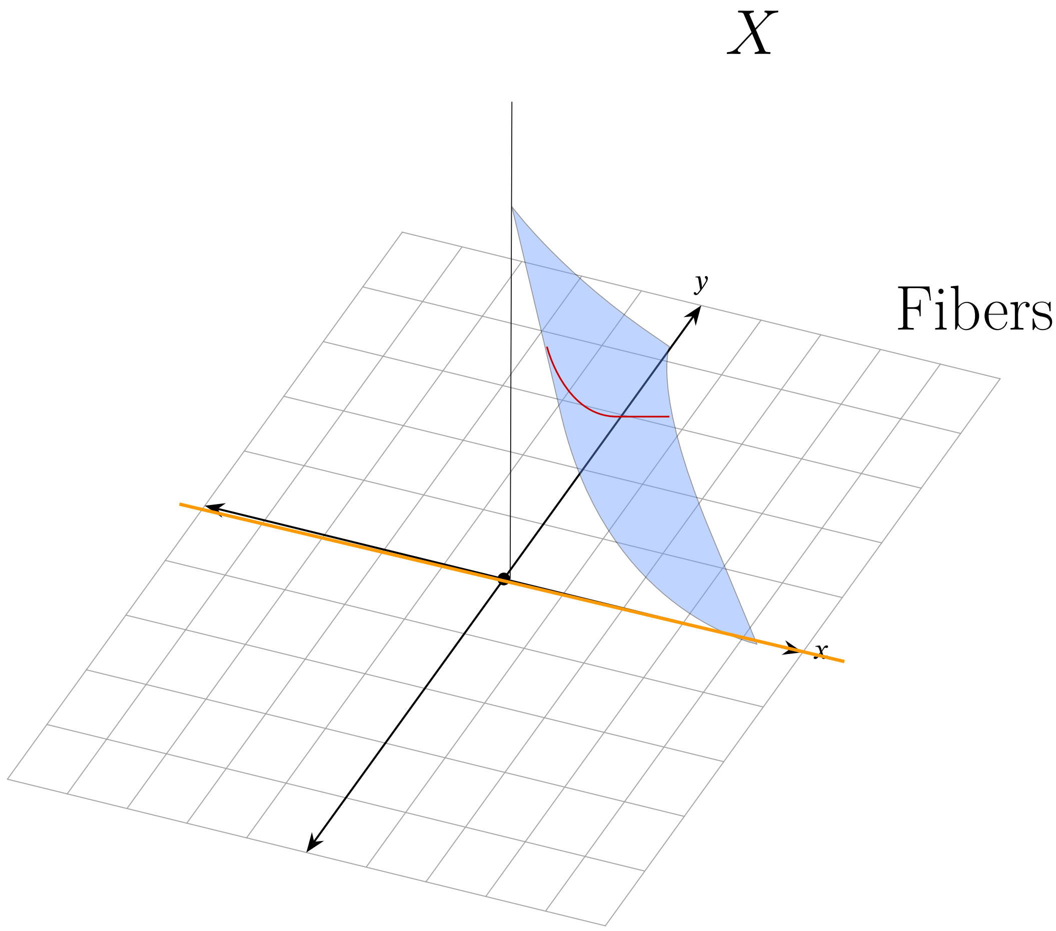


Consider the function $f : X \to Y$ induced by the following ring map:

$$\begin{aligned} \mathbf{C}[x] &\to R \\ x &\mapsto x, \end{aligned}$$

which is projection onto the axis. Note that $R \cong \mathbf{C}[x] \langle 1 \rangle \oplus \mathbf{C}[x] \langle y \rangle$ as a $\mathbf{C}[x]$-module.

Example 21.0.11 ***(?):*** Consider $\operatorname{Spec} \mathbf{C}[x, y, z]/ \langle xyz - 1 \rangle \to \operatorname{Spec} \mathbf{C}[x]$, whose real locus yields a hyperboloid:



Note that finite type should approximately be spec of finite type k-algebras, i.e. essentially varieties, where for finiteness the fibers should be finite.

Example 21.0.12*(?)*: Consider $\operatorname{Spec} \mathbf{C}[x, x^{-1}] \to \operatorname{Spec} \mathbf{C}[x]$, i.e. $\mathbf{G}_m \hookrightarrow \mathbf{A}^1$. However, $\mathbf{C}[x, x^{-1}]$ is not finitely-generated as a $\mathbf{C}[x]$ modules, even though it has finite fibers. Given any finite set of generators, one can take $\mathbf{C}[x]\left\langle \frac{f_i}{x^{k_i}} \right\rangle$ which doesn't contain $1/x^{\max k_i + 1}$.

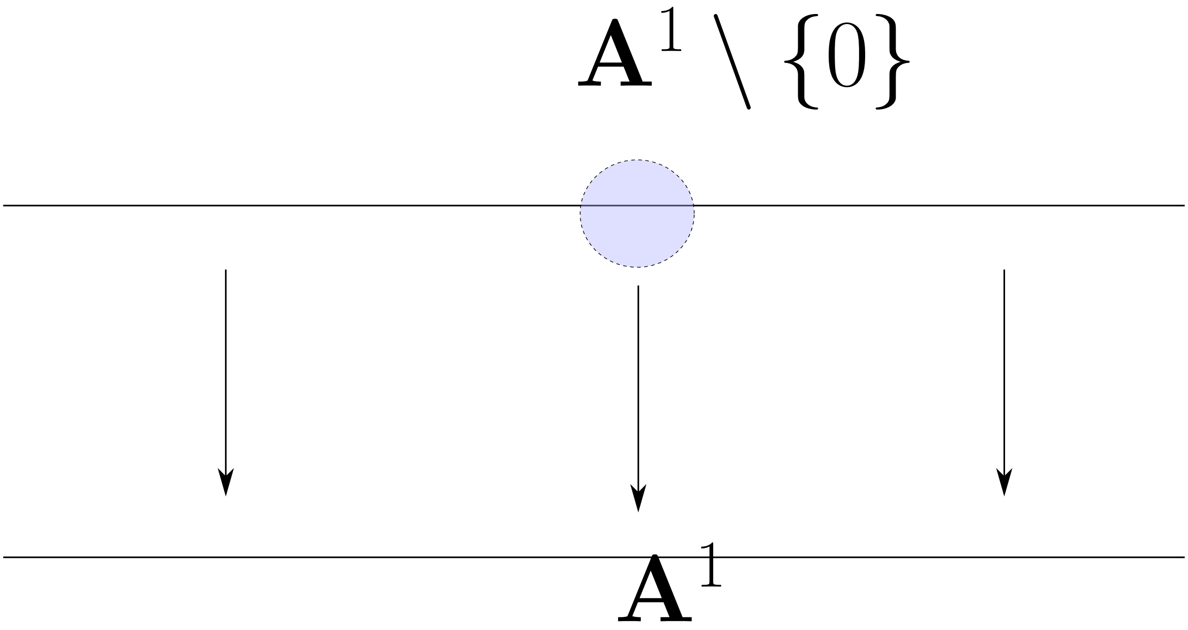
Remark 21.0.13: We'll define subschemes soon.

22 | Wednesday, October 13

Remark 22.0.1: Result from last time: there doesn't exist a surjection $k[f_1, \cdots, f_m] \twoheadrightarrow k[\![x]\!]$ for any finite collection $\{f_i\}_{i=1}^m$ of polynomials. This can be checked by just considering their dimension as a k-modules, where $\dim_k LHS = \#\mathbb{N}$ and $\dim_k RHS = \#\mathbf{R}$.

We said that a morphism $X \xrightarrow{f} Y$ is **locally finite type** if it is locally modeled on $\operatorname{Spec} A \to \operatorname{Spec} B$ with $B \to A$ a finitely-generated ring morphism, and is **finite** if locally modeled on $B \to A$ module-finite. Note that in the first case, we require $f^{-1}(U) \supseteq U \to V$, but in the latter $U = f^{-1}(V)$.

Example 22.0.2 ***(?):*** As an example, the map $D(x) \to \mathbf{A}^1_{/k}$ was not finite since $k[x] \to k[x, x^{-1}]$ is not module-finite, despite the fact that this geometrically corresponds to $\mathbf{A}^1 \setminus \{0\} \hookrightarrow \mathbf{A}^1$:



Example 22.0.3 ***(Not finite type:*** Spec ***of a local ring of a variety):*** Let $p \in V$ be a k-variety for k an infinite field, which we can assume to be affine (so $k[x_1, \cdots, x_n]/I$ for I reduced). Then $\mathcal{O}_{V,p}$ is not not finitely-generated as a k-algebra, and $\operatorname{Spec} \mathcal{O}_{V,p} \to \operatorname{Spec} k$ is not of finite type. Consider the local ring of $\mathbf{A}^1_{/k}$ at the prime ideal $p \coloneqq \langle x \rangle$, then $k[x]\left[(p^c)^{-1}\right] = \left\{ f/g \;\middle|\; g \notin p \right\}$, so $g(0) \neq 0$ for such g. Note that this is not $k[x]\left[x^{-1}\right]$!

Idea: there are only finitely many denominators: if

$$\varphi : k[f_1/g_1, \cdots, f_r/g_r] \twoheadrightarrow k[x]\left[(p^c)^{-1}\right],$$

then $\operatorname{im} \varphi$ contains contains those f/g such that $V(g) \subseteq \cup V(g_i)$, so such a φ can not exist. Note that this is still true for k a finite field, just not by this proof.

22.1 Open/Closed Subschemes

Definition 22.1.1 (Open subschemes)
Given $X \in \mathsf{Sch}$, an **open subscheme** of X is an open subset $U \subseteq |X|$ with structure sheaf $\mathcal{O}_U = \mathcal{O}_X|_U$.

Remark 22.1.2: Why does $(U, \mathcal{O}_U) \in \mathsf{LocRingSp}$ form a scheme? One needs to check that it's locally isomorphic to the spectrum of a ring: let $\{X_i\} \rightrightarrows X$ be an open affine cover, then $U_i \coloneqq U \cap X_i$ is open in U and in X_i, so can be covered by distinguished opens V_{ij}. But then $\{V_{ij}\} \rightrightarrows U$ is a cover by affine schemes. The inclusion $(U, \mathcal{O}_U) \hookrightarrow (X, \mathcal{O}_X)$ is clearly a morphism in $\mathsf{LocRingSp}$.

Definition 22.1.3 (Open Immersion)
The inclusion above is an **open immersion**.

Remark 22.1.4: A small dictionary

AG	Rest of Math
Immersion	Embedding
Nothing!	Immersion

Remark 22.1.5: Here I write $|X| \coloneqq \operatorname{sp}(X)$ as an alternative for Hartshorne's notation.

Definition 22.1.6 (Closed immersion)
A **closed immersion** is a morphism $X \xrightarrow{f} Y \in \mathsf{Sch}$ such that

1. The induced map $|X| \to |Y| \in \mathsf{Top}$ is a homeomorphism onto a closed subset of $|Y|$,
2. $f^\# : \mathcal{O}_Y \twoheadrightarrow f_* \mathcal{O}_X \in \mathsf{Sh}(Y)$ is surjective.

Remark 22.1.7: Set $U = D(f) \subseteq \operatorname{Spec} A$ defines an open immersion $\operatorname{Spec} A\left[f^{-1}\right] \to \operatorname{Spec} A$. So this corresponds to the ring map $A \hookrightarrow A\left[f^{-1}\right]$ since $\operatorname{Spec} A\left[f^{-1}\right] \subseteq \operatorname{Spec} A$ are those ideals not containing f.

Example 22.1.8 *(?)***:** Consider $U \coloneqq \mathbf{A}^2_{/k} \setminus \{[x, y]\} \hookrightarrow \mathbf{A}^2_{/k}$. Then $\{[x, y]\} = V(x, y)$, and this is a subscheme of an affine scheme which is not itself affine. One can use that $\dim D(f)^c \geq 1$

Exercise 22.1.9 (?)
Prove that this is not affine. Hint: check $\mathcal{O}_U = k[x, y]$,[a] and use that for any $X \in \mathsf{AffSch}$ we have $\mathcal{O}_X(X) = R$. However, $\operatorname{Spec} k[x, y] = \mathbf{A}^2 \neq U$.

[a] This says that any regular function on U actually extends to all of $\mathbf{A}^2$

Check "affinization"? It fills in holes of at codimension at most 2, and satisfies a universal property. Consider $X = \mathbf{A}^1 \times \mathbf{P}^1$.

Remark 22.1.10: All examples are locally of this form: $F : X \hookrightarrow Y = \operatorname{Spec} A$ where $|X| \to U \subseteq |Y|$ maps homeomorphically onto a closed subset. Recall that the closed subsets are of the form $U = V(I)$, and here we need $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ surjective. Let $X = \operatorname{Spec} A/I$, and recall that every surjective ring map is of the form $A \to A/I$. Here $q : A \twoheadrightarrow A/I$ where $\mathfrak{p} \hookleftarrow q^{-1}(\mathfrak{p})$, so we get some map $\operatorname{Spec} A/I \to \operatorname{Spec} A$, and this is homeomorphism onto $V(I) \subseteq \operatorname{Spec} A$:

$$
\begin{aligned}
\operatorname{Spec} A/I &\to V(I) \\
\mathfrak{p} &\mapsto q^{-1}(\mathfrak{p}) \supseteq I \\
q(\mathfrak{q}) &\hookleftarrow \mathfrak{q}.
\end{aligned}
$$

We also get an induced map $A\left[g^{-1}\right] \to (A/I)\left[g^{-1}\right]$, which is precisely

$$
f^\#(D(g)) : \mathcal{O}_Y(D(g)) \to \mathcal{O}_X(f^{-1}(D(g)))
$$

and is thus surjective. Since it's surjective on a basis, by gluing it'll be surjective on the entire space.

23 | Friday, October 15

Remark 23.0.1: Last time: open and closed subschemes, where openness was easy since we required $f : X \to Y$ to be a homeomorphism onto a closed subset of Y with $f^\#$ surjective. Any example of a closed subscheme is locally of the following form: $V(I) := \operatorname{Spec} A/I \to \operatorname{Spec} A$ induced by some $A \twoheadrightarrow A/I$ in rings. Here $A\left[g^{-1}\right] \twoheadrightarrow (A/I)\left[g^{-1}\right]$ implies that $f^(D(g))$ surjective for every distinguished open, so $f^\#$ is a surjective sheaf map. However, this need not be surjective on global sections.

Example 23.0.2 ***(?):*** Recall that $\mathbf{P}^1_{/k} = \mathbf{A}^1 \coprod_{x \mapsto x^{-1}} \mathbf{A}^1$ and $\mathcal{O}_{\mathbf{P}^1}(\mathbf{P}^1) = k$ where we glued $k[t] \cap k[s] = k$ along $s = 1/t$. Consider the closed subscheme of $\mathbf{A}^1$ given by $X := \operatorname{Spec} \mathbf{C}[t]/t^2$ and the global restriction map

$$
\begin{aligned}
f^\#(\mathbf{P}^1) : \mathcal{O}_{\mathbf{P}^1}(\mathbf{P}^1) &\to \mathcal{O}_X(X) \\
\mathbf{C} &\to \mathbf{C}[t]/t^2,
\end{aligned}
$$

which is not surjective.

Example 23.0.3 ***(?):*** Consider $\mathbf{A}^2_{/k}$ for $k = \bar{k}$, how many closed subschemes are homeomorphic onto the origin 0 corresponding to $\langle x, y \rangle \in \operatorname{Spec} k[x, y]$. Since they're all locally of the form $V(I)$, these correspond to ideals I where $V(I) = 0$. These are ideals I where $\langle x, y \rangle$ is the only ideal containing I, so we can write this as $\left\{ I \,\middle|\, \sqrt{I} = \langle x, y \rangle \right\}$, i.e. the primary decomposition of I has only 1 prime, namely $\langle x, y \rangle$. Some ideals of this form:

- $\langle x,y\rangle^k$ for any $k \geq 0$,
- $\left\langle x^a y^b, x^c y^d \right\rangle$ where $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$, e.g. $\langle x^2, y\rangle$.
- $\langle (x+y)^2, y\rangle$
- $\langle f,g\rangle$ where $V(f) \cap V(g) = 0$ as a set, e.g. two curves only intersecting at the origin.

Remark 23.0.4: What kinds of schemes are these? For example, considering $V(x-y^2)$ and $V(y)$, we have $\langle y-x^2, y\rangle = \langle x^2, y\rangle$, yielding a non-reduced scheme. We have $k[x,y]/\langle x^2,y\rangle = k \bigoplus kx \in {}_k\mathsf{Mod}$, thought of as functions as the tangent vector at 0 pointing horizontally. Similarly, $k[x,y] = \langle x^2, xy, y^2\rangle = k \oplus kx \oplus ky$, which can be thought of as functions on $\mathbf{T}_p\mathbf{A}^2$ for $p=0$. The rough idea: we want $\mathbf{T}_0\mathbf{A}^2 \cong \operatorname{Spec} k[x,y]/\langle x,y\rangle^2$.

Definition 23.0.5 (Reduced subscheme structures)
Let $Z \subseteq |Y|$ be closed, then there exists a unique scheme structure X on Z such that $|X| = Z$, the **reduced subscheme structure on** Z. Affine locally, for $Z \subseteq |\operatorname{Spec} A|$ given by $V(I)$ for some ideal I, and we define this as $\operatorname{Spec} A/\sqrt{I}$. This will glue because passing to reduction will commute with localization, i.e. $(A_{\mathrm{red}})\left[f_{\mathrm{red}}{}^{-1}\right] = (A\left[f^{-1}\right])_{\mathrm{red}}$ where $A_{\mathrm{red}} = A/\sqrt{0}$.

Example 23.0.6 *(?):* Take $0 \in \left|\mathbf{A}^2_{/k}\right|$, then its reduced subscheme structure is $\operatorname{Spec} k[x,y]/\langle x,y\rangle$.

Remark 23.0.7: Any closed subscheme structure along Z is locally given by $\operatorname{Spec} A/I$ with $V(I) = Z$, and there's always a map $\operatorname{Spec} A/\sqrt{I} \to \operatorname{Spec} A/I$ dual to the reduction map $A/I \to (A/I)_{\mathrm{red}}$. For any closed subscheme $X \subseteq Y$, we define X_{red} as the reduced subscheme associated to $|X|$, and there is a morphism $X_{\mathrm{red}} \to X$.

Idea: this is a space such that all of its functions kill nilpotents.

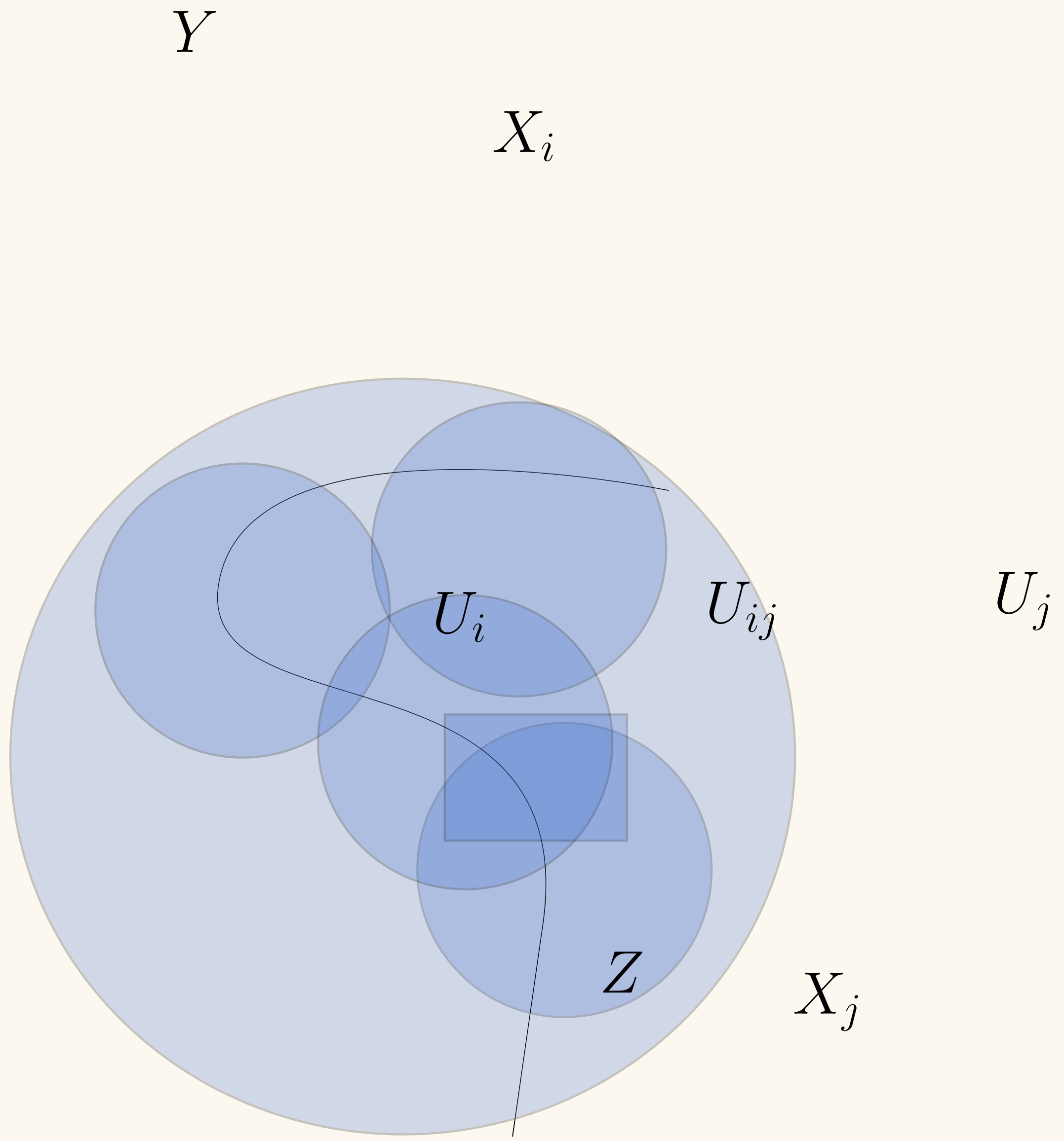
Proposition 23.0.8 *(?).*
X_{red} is well-defined

Proof (?).
Let $Y \in \mathsf{Sch}$ and $Z \subset |Y| \in \mathsf{Top}$ closed. Take a cover $\mathcal{U} \to Y$ with $U_i = \operatorname{Spec} A_i$, then $Z \cap |U_i|$ is closed and thus equal to some $V(I_i)$. Define a reduced scheme $X_i := \operatorname{Spec}(A_i/\sqrt{I_i})$, which we'll try to glue to define X_{red}. Note that we can write

$$\sqrt{I_i} = \bigcap_{\mathfrak{p} \in Z \cap |U_i|} \mathfrak{p},$$

which generalizes $\sqrt{0_R} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} R} \mathfrak{p}$.
To give a gluing amounts to defining isomorphisms:

$$f_{ij} : X_i \cap U_j \to X_j \cap U_i.$$



So pass to an open affine cover. We'll have $(A_i)[f^{-1}] = (A_j)[g^{-1}]$ for some f, g, and this will induce isomorphisms

$$(A_i)[f^{-1}]/\sqrt{I_i} \xrightarrow{\sim} \mathcal{O}_{X_j \cap U_i}(Z \cap D(f)).$$

■

24 | Monday, October 18

24.1 Dimension

Question 24.1.1
If $X = \operatorname{Spec} A$ is affine and $U \subset |X|$ is open, is the inclusion $U \hookrightarrow X$, represented say by $\operatorname{Spec} A' \hookrightarrow \operatorname{Spec} A$, represented by a ring map $A \to A'$?

Definition 24.1.2 (Dimension)
For $X \in \mathsf{Sch}$, write $\dim X := \dim_{\mathsf{Top}} |X|$ as the topological dimension of the underlying space, which is the length of the longest chain of irreducible closed subsets

$$\emptyset \subsetneq Z_0 \subseteq Z_1 \subsetneq \cdots \subsetneq Z_n \subseteq |X|,$$

where equality at the end is possible if $|X|$ is irreducible.

Example 24.1.3 *(?):*

- $\dim \operatorname{Spec} k = 0$
- $\dim \operatorname{Spec} \mathbf{Z}_{\widehat{p}}$: consider $\emptyset \subsetneq \mathrm{pt} \subseteq \operatorname{Spec} \mathbf{Z}_{\widehat{p}}$, where pt is a generic point, so $\dim \operatorname{Spec} \mathbf{Z}_{\widehat{p}} = 1$.
- $\dim \mathbf{P}^n_{/k} = \dim \mathbf{A}^n_{/k} = n$.

Example 24.1.4 *(?):* If $X = \operatorname{Spec} A$ is affine for A then $\dim X = \operatorname{krulldim} A$ is the Krull dimension of the ring A. This follows because irreducible closed subsets of $\operatorname{Spec} A$ biject with prime ideals of A. Why is this true?

$\Longleftarrow$:

Suppose $p \subseteq A$ is prime, then note that $V(p) = \left\{ q \in \operatorname{Spec} A \;\middle|\; q \supseteq p \right\}$. If $V(p) = V(I) \cup V(J) = V(IJ)$, then $p \supseteq IJ$ so p contains one of I, J. But then $V(p) = V(I)$ wlog, so $V(p)$ is an irreducible closed subset.

$\Longrightarrow$: We can reverse almost all of these implications:

- $V(p) = V(IJ)$
- $\iff p \supseteq IJ$
- $\iff p \subseteq I$ or $p \subseteq J$
- $\iff V(p) = V(I)$ or $V(J)$.

Note that bijections preserve strict containments, so we have correspondences on chains:

$$\begin{aligned} \emptyset \subsetneq Z_0 \subsetneq \cdots \subsetneq Z_n \subset X = \operatorname{Spec} A \\ \iff \\ \langle 1 \rangle \supsetneq p_0 \supsetneq \cdots \supsetneq p_n. \end{aligned}$$

Remark 24.1.5: So we can use that $\operatorname{krulldim} k[x_1, \cdots, x_n] = n$ to show $\dim \mathbf{A}^n_{/k} = n$. For $\mathbf{P}^n_{/k}$, use that any maximal chain contains a point $\{z_0\}$, so choosing such a point and intersecting z_i with the embedded copy of $\mathbf{A}^n_{/k} \hookrightarrow \mathbf{P}^n_{/k}$. Then use that there is a chain $\langle 0 \rangle \subsetneq \langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \cdots \subsetneq \langle x_1, \cdots, x_n \rangle$, so $\dim X \geq n$. For the reverse inequality: this is hard! See Atiyah-MacDonald's discussion of regular systems of parameters.

Definition 24.1.6 (Codimension)
The **codimension** $\operatorname{codim}(Z, X)$ for $Z \subseteq X$ a closed irreducible subset is the length of the longest chain starting at Z:

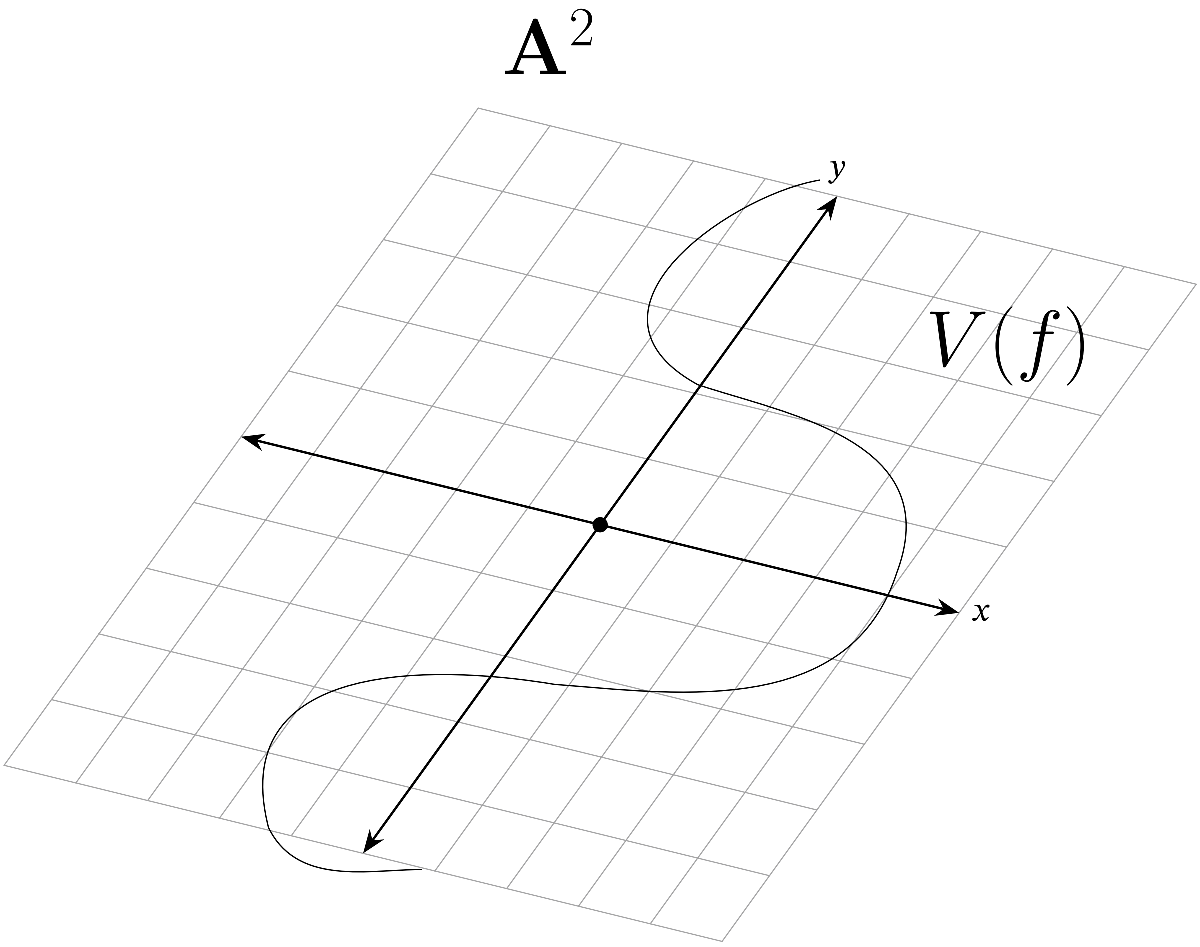
$$Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subset X.$$

Fact 24.1.7
For $X = \operatorname{Spec} A$ and $A \in {}_k\mathsf{Alg}^{\mathrm{fg}}$, there is a formula

$$\dim(Z) + \operatorname{codim}(X, Z) = \dim(X).$$

Remark 24.1.8: This is not true in general, even for Noetherian rings – see **catenary rings**, where any chain of prime ideals can be extended to a chain of fixed maximal length n. Without this, one can extend chains to maximal chains of differing lengths.

Example 24.1.9 ***(?):*** $\dim \operatorname{Spec} \mathbf{Z} = 1$, instead of having dimension zero! This is because there's always a chain $0 \to \langle p \rangle \to \mathbf{Z}$ for any prime. An analogy here is a curve $\operatorname{Spec} k[x, y] / \langle f(x, y) \rangle$:



One can similarly do this for $\mathcal{O}_K$ the ring of integers in a number field K and get $\dim \operatorname{Spec} \mathcal{O}_K = 1$. This leads to a good theory of divisors (free modules on codimension 1 subvarieties) and the Picard group, so a useful geometrization of number theory.

24.2 Fiber Products

Remark 24.2.1: Perhaps the most important construction in schemes! Picks up intersection multiplicities.

Definition 24.2.2 (Fiber products)
Let $X, Y \in \mathsf{Sch}_{/S}$ then $X \underset{S}{\times} Y \in \mathsf{Sch}_{/S}$ is an S-scheme equipped with morphisms of S-schemes onto X, Y satisfying a universal property. For any Z with maps to X and Y, there is a unique θ making the following diagram commute:

$$\begin{tikzcd}
 & & Z \arrow[ddll, bend right] \arrow[d, dashed, "\exists!\theta"] \arrow[ddrr, bend left] & & \\
 & & X \underset{S}{\times} Y \arrow[dl] \arrow[dr] & & \\
Y \arrow[dr] & & & & X \arrow[dl] \\
 & S & & &
\end{tikzcd}$$

Link to Diagram

Remark 24.2.3: Note that on the ring side, this yields a tensor product over S.

25 | Wednesday, October 20

Remark 25.0.1: Today: only the most important property of schemes, the existence of fiber products! Let $X, Y \in \mathsf{Sch}_{/S}$, then the fiber product $X \underset{S}{\times} Y \in \mathsf{Sch}_{/S}$ is an object satisfying a universal property:

$$\begin{tikzcd}
\forall Z \arrow[dr, dashed, "\exists!\theta"] \arrow[drr, "\beta"] \arrow[ddr, "\alpha"'] & & \\
 & X \underset{S}{\times} Y \arrow[r, "q"] \arrow[d, "p"] & X \arrow[d] \\
 & Y \arrow[r] & Z
\end{tikzcd}$$

Link to Diagram

Note that needing the square involving Z and $X \underset{S}{\times} Y$ to commute is automatic, since we're working in $\mathsf{Sch}_{/S}$ instead of just Sch. Note that $X \times Y = X \underset{\operatorname{Spec} \mathbf{Z}}{\times} Y$ recovers the product.

Question 25.0.2
Do fiber products exist? They're unique up to isomorphism if they do, so we just need to construct it.

Theorem 25.0.3 *(Existence of fiber products).*
Fiber products exist and are unique up to isomorphism.

25.1 The 7-Step Proof

Proof (Step 1: Prove for X, Y, S are affine.).
Write $X = \operatorname{Spec} A, Y = \operatorname{Spec} B, S = \operatorname{Spec} R$. We start in Ring, and noting contravariance of $\operatorname{Spec}(-)$, the claim is that $\operatorname{Spec}(A \otimes_R B) \cong X \underset{S}{\times} Y$. Use that $\operatorname{Spec} : \mathsf{Ring} \to \mathsf{Sch}$ is fully faithful, which almost allows just reversing arrows if some care is taken. A map $Z \to \operatorname{Spec} A$ is the same data as a map $A \to \mathcal{O}_Z(Z)$. Let $\mathcal{U} \rightrightarrows Z$ with $U_i = \operatorname{Spec} C_i$, then a morphism $Z \to \operatorname{Spec} A \in \mathsf{Sch}_{/S}$ is equivalently a collection of compatible morphisms $A \to C_i \in \mathsf{Ring}$ by the sheaf condition, so the restrictions to $\mathcal{O}_{Z_i \cap Z_j}$ are compatible. So we can interchange any two diagrams of the following form:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd} \Longrightarrow \begin{tikzcd} \mathcal{O}_Z(Z) & B \arrow[l] \\ A \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd}$$

Link to Diagram

Now the universal property of $A \otimes_R B \in \mathsf{Ring}$ yields a unique map $A \otimes_R B \xrightarrow{\theta*} \mathcal{O}_Z(Z)$, so equivalently $Z \xrightarrow{\theta} \operatorname{Spec}(A \otimes_R B)$. ■

For step 2, the universal property will imply uniqueness if it exists, which we'll need for gluing.

Proof (Step 3: Gluing morphisms on covers).
A morphism $X \to Y \in \mathsf{Sch}$ is equivalently the data of $\mathcal{U} \rightrightarrows X$ and morphism $U_i \xrightarrow{f_i} Y \in \mathsf{Sch}$ with $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. This is true more generally for any $X \in \mathsf{Top}$ and $F \in \mathsf{Sh}_{/X}(\mathsf{C})$ with values in any category. ■

Proof (Step 4: Passing to open subsets of a factor).
Let $X, Y \in \mathsf{Sch}_{/S}$ be arbitrary and $U \subseteq X$ open. If $X \underset{S}{\times} Y$ exists, it is equipped with morphisms q to X and p to Y. Note that every open subset has a canonical open subscheme structure.

Claim: $U \underset{S}{\times} Y \cong p^{-1}(U)$, noting that we don't yet know that fibers are schemes.

Proof (?).
Let $Z \in \mathsf{Sch}_{/S}$ such that we have the following:

$$\begin{tikzcd}
Z \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[rr, bend left, "\alpha'"] & U \arrow[r, hook, "\iota"] & X \\
Y & &
\end{tikzcd}$$

Link to Diagram

If $X \underset{S}{\times} Y$ exists, then $\exists! \theta : Z \to X \underset{S}{\times} Y$. Then $\Theta(Z) \subseteq p^{-1}(U)$ since $p \circ \Theta = \alpha' := \iota \circ \alpha$, so $\operatorname{im}(p \circ \Theta) \subseteq U$. So $\theta : Z \to p^{-1}(U)$ is unique, making $p^{-1}(U) \cong U \underset{S}{\times} Y$. ∎

So if $X \underset{S}{\times} Y$ exists then $U \underset{S}{\times} Y$ exists. ∎

Proof (Step 5: Gluing fiber products from an open cover).
Let $X, Y \in \mathsf{Sch}_{/S}$ and suppose $\mathcal{X} \rightrightarrows X$ where $X_i \underset{S}{\times} Y$ exists, then the claim is that $X \underset{S}{\times} Y$ exists. Define $X_{ij} := X_i \cap X_j$, then by step 4 $p_i^{-1}(X_{ij})$ is a fiber product $X_{ij} \underset{S}{\times} Y$, and similarly $p_j^{-1}(X_{ij}) = X_{ij} \underset{S}{\times} Y$. By uniqueness in step 2, there is a unique isomorphism $\varphi_{ij} : p_i^{-1}(X_{ij}) \to p_j^{-1}(X_{ij})$ of fiber products. Furthermore, the cocycle condition is satisfied since φ_{ik} is unique, so $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$. These are schemes (or more generally any ringed space), so we can glue to get *some* scheme which we'll suggestively write $X \underset{S}{\times} Y$. The claim is that this satisfies the correct universal property. First: there are morphisms $X \underset{S}{\times} Y \xrightarrow{p} X$ and $X \underset{S}{\times} Y \xrightarrow{q} Y$. Note that the $X \underset{S}{\times} Y$ cover $X_i \underset{S}{\times} Y$ and $X = \coprod X_i / \sim$ Define the following maps:

$$\begin{tikzcd}
 & Z_i := \alpha^{-1}(X_i) \arrow[r, "\alpha|_{Z_i}"] \arrow[d, hook] \arrow[dl, bend right, "\beta|_{Z_i}"'] & X_i \arrow[d, hook] \\
Y & Z \arrow[l, "\beta"'] \arrow[r, "\alpha"] & X
\end{tikzcd}$$

Link to Diagram

Then $\exists! \Theta_i : Z_i \to X_i \underset{S}{\times} Y$ where the Θ_i agree on overlaps Z_{ij} as morphisms $Z_{ij} \to X_{ij} \underset{S}{\times} Y$. By step 3, these glue to a unique $\Theta : Z \to X \underset{S}{\times} Y$, since the gluing is defined as $X \underset{S}{\times} Y = \coprod_i (X_i \underset{S}{\times} Y) / \varphi_{ij}(p) \sim p$. Why does it make the above diagram commute?

$$\begin{tikzcd}
Z \arrow[dr, "\Theta"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& X \underset{S}{\times} Y \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

Link to Diagram

This commutes because such a map is determined on an open cover, and we have commutativity in the following:

$$\begin{tikzcd}
Z_i \arrow[dr, "\Theta_i"] \arrow[drr, "\alpha_i"] & & \\
& X_i \underset{S}{\times} Y \arrow[r, "p_i"] & X
\end{tikzcd}$$

Link to Diagram

So $X \underset{S}{\times} Y$ exists if $X_i \underset{S}{\times} Y$ exists for $\mathcal{X} \rightrightarrows X$.

■

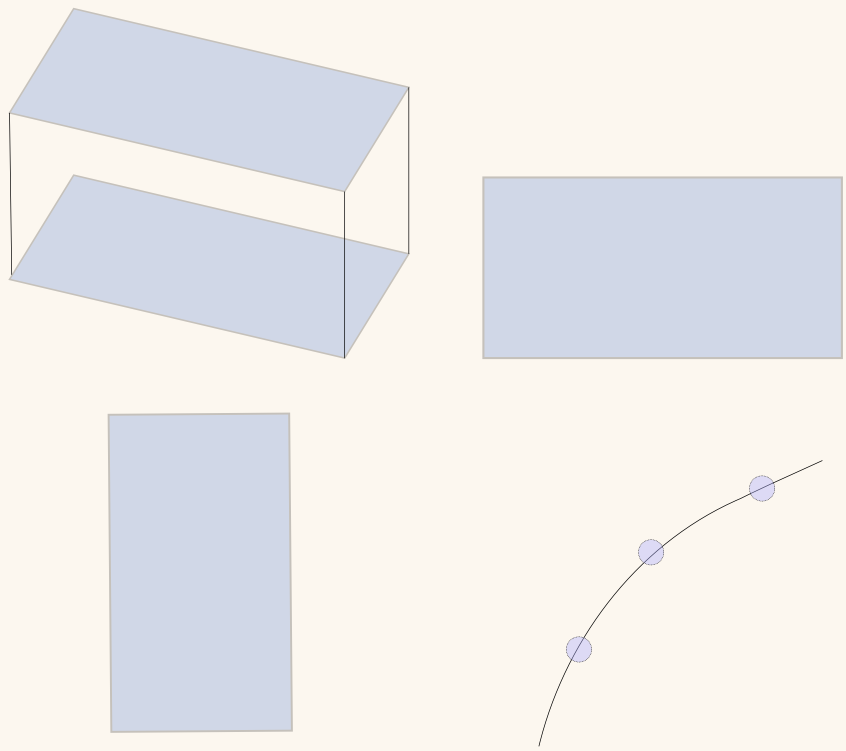
Proof (Step 6: Affine base).
Let $S \in \mathsf{AffSch}$, then by step 1 $X_i \underset{S}{\times} Y_j$ exists, to $X_i \underset{S}{\times} Y$ exists by step 5, which implies $X \underset{S}{\times} Y$ exists

■

Proof (Step 7: Arbitrary).
Let $X, Y, S \in \mathsf{Sch}_{/S}$ be arbitrary. Take $\mathcal{S} \rightrightarrows S$, and set $X_i = p^{-1}(S_i)$ and $Y_i = q^{-1}(S_i)$. Then $X_i \underset{S_i}{\times} Y_i$ exists, and the claim is that there is an isomorphism

$$X_i \underset{S_i}{\times} Y_i \cong X \underset{S}{\times} Y_i \in \mathsf{Sch}_{/S}.$$

Then there exist $Z \to Y_i$, and $\mathrm{im}(Z \to S)$ must lie in S.



■

26 | Fiber Products (Friday, October 22)

Remark 26.0.1: Last time: we defined and proved the existence of fiber products in $\mathsf{Sch}_{/S}$, and for $X, Y, S \in \mathsf{AffSch}$ equal to $\operatorname{Spec} A, \operatorname{Spec} B, \operatorname{Spec} R$ respectively,

$$X\mathrm{fp}SY = \operatorname{Spec}(A \otimes_R B).$$

Definition 26.0.2 (Residue field)
For $X \xrightarrow{f} Y \in \mathsf{Sch}$ and $p \in Y$, define the **residue field**

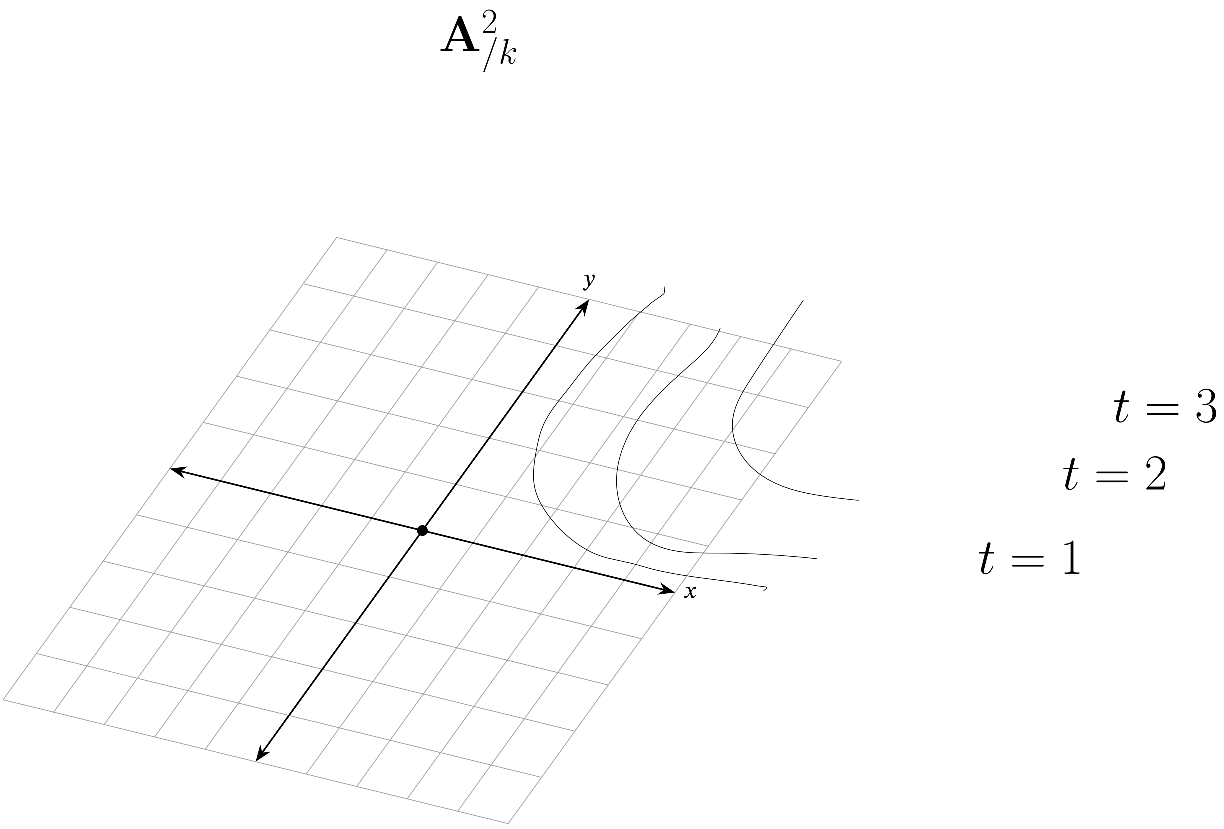
$$k(p) := \mathcal{O}_{Y,p}/\mathfrak{m}_{Y,p}.$$

Remark 26.0.3: There is a closed immersion $\operatorname{Spec} k(p) \hookrightarrow Y$ if p is a closed point (since it came from a quotient map), and we can take a fiber product

$$\begin{tikzcd}
\operatorname{Spec} k(p) \underset{Y}{\times} \operatorname{Spec} k(p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\
\operatorname{Spec} k(p) \arrow[r] & Y
\end{tikzcd}$$

Link to Diagram

Example 26.0.4*(?)*: Consider $\operatorname{Spec} k[x, y, t] / \langle xy - t \rangle \to \operatorname{Spec} k[t]$:



What is the fiber over $p \coloneqq \langle t - 1 \rangle$ or $q = \langle t \rangle$?

- $k(p) = k[t] / \langle t - 1 \rangle \cong k$,
- $k(q) = k[t] / \langle t \rangle \cong k$,

so they are abstractly isomorphic. We have the following tensor product in rings:

$$\begin{tikzcd}
k[x,y,t]/\langle xy=t\rangle \otimes_{k[t]} k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & k[x,y,t]/\langle xy-t\rangle \\
k \cong k[t]/\langle t-1\rangle & k[t] \arrow[l] \arrow[u]
\end{tikzcd}$$

Link to Diagram

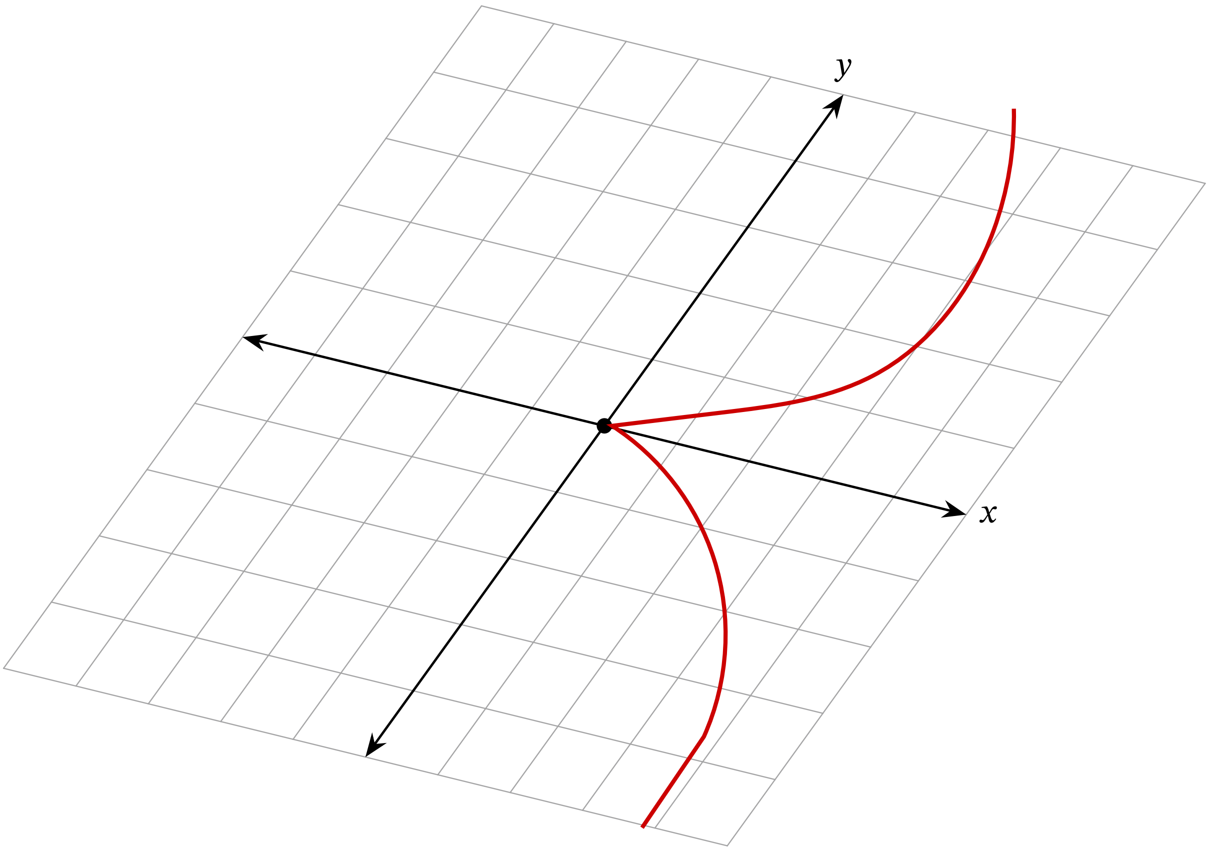
Generally, pulling back over $k[t]/\langle t-c\rangle$ has the effect of setting $t=c$ in the tensor product, and thus the fiber products are given by

- $k(p) \underset{\operatorname{Spec} k[t]}{\times} X = \operatorname{Spec} k[x,y]/\langle xy-1\rangle$
- $k(q) \underset{\operatorname{Spec} k[t]}{\times} X = \operatorname{Spec} k[x,y]/\langle xy\rangle$

Example 26.0.5 ***(Fiber products aren't quite set products):*** Consider

$$X := \mathbf{A}^1_{/k} \underset{\operatorname{Spec} k}{\times} \mathbf{A}^1_{/k} \cong \operatorname{Spec}(k[s] \otimes_k k[t]) \cong \operatorname{Spec}(k[s,t]) \cong \mathbf{A}^2_{/k}.$$

However, X is not the set-theoretic product of the two constituent sets, although it does contain the product. Why? Consider $p := \langle y^2 - x^3\rangle \in \operatorname{Spec} \mathbf{A}^2_{/k}$, which is prime (check irreducibility in each variable!) and thus yields a point which is not the product of any two points in $\mathbf{A}^1_{/k}$.



Example 26.0.6 ***(Reduction mod*** p***):*** Let $X \in \mathsf{Sch} \xrightarrow{\sim} \mathsf{Sch}_{/\operatorname{Spec}\mathbf{Z}}$ with structure map $X \to \operatorname{Spec}\mathbf{Z}$, and let $p = \langle P \rangle \in \operatorname{Spec}\mathbf{Z}$. Then $k(p) = \mathbf{Z}/p = \mathbf{F}_p$, so consider the fiber over p:

$$\begin{tikzcd}
X \times \operatorname{Spec}\mathbf{F}_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
\operatorname{Spec}\mathbf{F}_p \arrow[r] & \operatorname{Spec}\mathbf{Z}
\end{tikzcd}$$

Link to Diagram

Call this the **reduction mod** p, denoted $X_{\mathbf{F}_p}$. If $X = \operatorname{Spec}R$, then $X_{\mathbf{F}_p} = \operatorname{Spec}(R \otimes_{\mathbf{Z}} \mathbf{Z}/p) = \operatorname{Spec}(R/\langle p \rangle)$.

Example 26.0.7 ***(?):*** Take $X := \operatorname{Spec}\mathbf{Z}[x,y,z]/\langle x^5 + y^5 = z^5 \rangle$, so nontrivial $\mathbf{Z}$-points yield counterexamples to Fermat. Then $X_{\mathbf{F}_p} = \mathbf{F}_p[x,y,z]/\langle x^5 + y^5 + z^5 \rangle$, which reduces the coefficients of the equations.

How are these related to models of a scheme?

Example 26.0.8 *(?)*: Take $X = \operatorname{Spec} \mathbf{C}$ to get $\mathbf{C} \otimes_{\mathbf{F}_p} \mathbf{Z}$ – what is this ring? One has to regard $\mathbf{C}$ as a ring over $\mathbf{Z}$ first, so write $\mathbf{C} = \overline{\mathbf{Q}}(T)$ where T is an uncountable basis of transcendental elements. So this yields $\overline{\mathbf{F}_p}(T)$.

Remark 26.0.9: Consider $X \in \mathsf{Var}_{/\mathbf{C}}$, e.g. $X = \operatorname{Spec} \mathbf{C}[x,y,z]/\left\langle x = \sqrt{2}y, y^2 = \pi z^3 \right\rangle$. Then consider the (much smaller) subring generated by the coefficients of the defining equations, so $R \coloneqq \mathbf{Z}[\sqrt{2}, \pi] \cong \mathbf{Z}[\sqrt{2}][t]$ and consider $\operatorname{Spec} R[x,y,z]/\left\langle x = \sqrt{y}, y^2 = \pi z^3 \right\rangle$. This has the exact same equations but is now defined of $\operatorname{Spec} R$. Note that having finitely many equations yields a finitely generated as a $\mathbf{Z}$-module.

Since $R \hookrightarrow \mathbf{C}$ we get a morphism $\operatorname{Spec} \mathbf{C} \to \operatorname{Spec} R$, and we get a diagram

$$\begin{tikzcd}
Y \coloneqq R[x,y,z]/\left\langle x = \sqrt{2}y, y^2 = \pi z^3 \right\rangle \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\
\operatorname{Spec} R \arrow[r] & \operatorname{Spec} \mathbf{C}
\end{tikzcd}$$

Link to Diagram

Warning 26.0.10
So in the literature, *reduction of X mod p* generally means $Y_{\mathbf{F}_p}$ and not $X_{\mathbf{F}_p}$.

Definition 26.0.11 (Base Change)
Given $X, Y \in \mathsf{Sch}_{/S}$ with structure maps f, g respectively, the **base change of f along g** is defined as the fiber product $X \underset{S}{\times} Y \in \mathsf{Sch}_{/Y}$. So there is a functor

$$- \underset{S}{\times} Y : \mathsf{Sch}_{/S} \to \mathsf{Sch}_{/Y}.$$

What is the adjoint? Probably the forgetful functor given by composing along g.

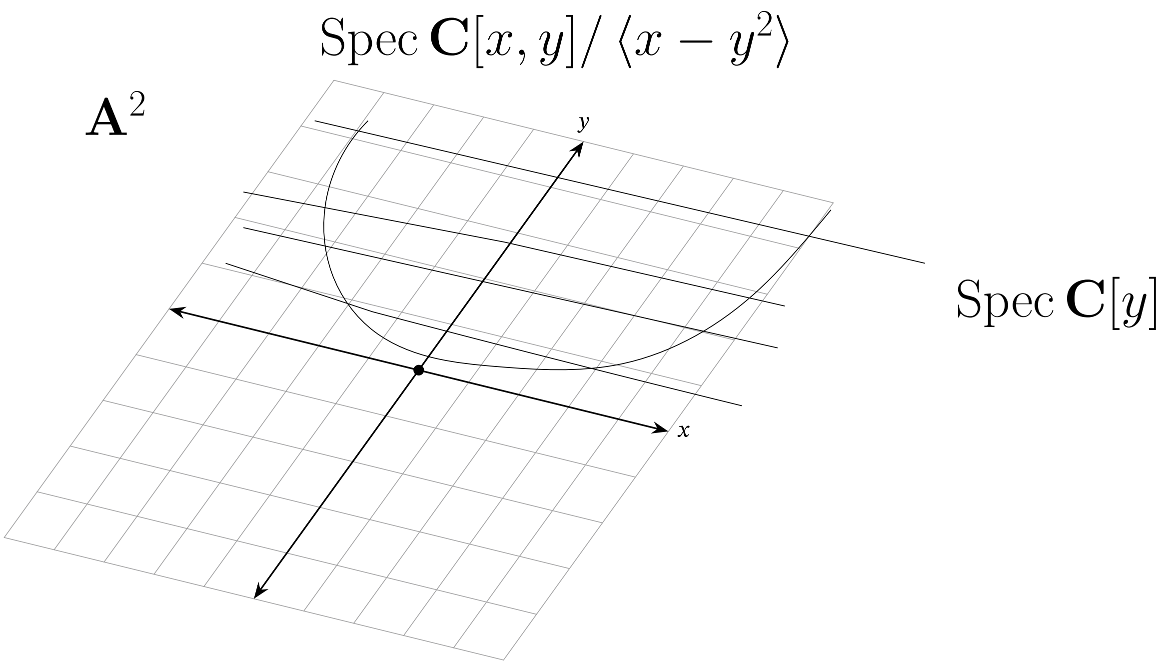
27 | Monday, October 25

27.1 Length

Remark 27.1.1: A correction from last time: we said $\mathbf{C} = \overline{\mathbf{Q}}(t_j \mid j \in J)$ for some uncountable set of generators J. Noting that $R \otimes_{\mathbf{Z}} \mathbf{F}_p = R/pR$, which is zero if $\frac{1}{p} \in R$, so $\mathbf{C} \otimes_{\mathbf{Z}} \mathbf{F}_p = 0$. However, this doesn't happen for $\overline{\mathbf{Z}}(t_j \mid j \in J)$, so passing to a ring given by adjoining coefficients of equations is still a reasonable thing to do.

Last time: for $X \xrightarrow{f} Y$, the fiber over $p \in Y$ was $X \underset{Y}{\times} \operatorname{Spec} k(p)$ where $k(p) := \mathcal{O}_{Y,p}/\mathfrak{m}_p$, sometime denoted $f^{-1}(p)$, the *scheme-theoretic fiber.*

Example 27.1.2 ***(?):*** Consider intersecting a parabola with a family of lines:



Then there is a map $\operatorname{Spec} \mathbf{C}[x,y]/\langle y - x^2\rangle \to \operatorname{Spec} \mathbf{C}[y]$ corresponding to a ring map $\mathbf{C}[y] \to \mathbf{C}[x,y]/\langle y - x^2\rangle$. We showed the scheme theoretic fiber over $y = c_0$ is precisely $\operatorname{Spec} \mathbf{C}[x]/\langle c_0 - x^2\rangle \cong \mathbf{C}^{\oplus^2}$ if $c_0 \neq 0$, and $\operatorname{Spec} \mathbf{C}[x]/\langle x^2\rangle$ if $c_0 = 0$. The former has no nilpotents while the latter does, so the fibers are reduced away from $c_0 = 0$.

Definition 27.1.3 (Length)
If $R \in {}_k\mathsf{Alg}^{\mathrm{fg}}$ with $\operatorname{krulldim}(R) = 0$, then $\operatorname{length}(R) := \dim_k R < \infty$. $\operatorname{Spec} R$ is called a length l scheme over k.

Remark 27.1.4: Note that $\dim_{\mathbf{C}} \mathbf{C}[x,y] = \infty$, but has Krull dimension 2. Most k-algebras will have infinite k-dimension in this setting.

Remark 27.1.5: If R is reduced and $k = \bar{k}$, one can prove that $R = k^{\oplus^\ell}$ and $\operatorname{Spec} R = \coprod_{i \leq \ell} \operatorname{Spec} k$ where ℓ is the number of reduced points.

Example 27.1.6 *(?)*: Take

$$\operatorname{Spec} \mathbf{C} \oplus \mathbf{C}[x]/\left\langle x^3 \right\rangle \oplus \mathbf{C}[x,y]/\left\langle x^2, xy, y^3 \right\rangle.$$

The terms have dimension $1, 3, 4$ respectively, yielding a 0-dimensional length 8 scheme. Note that $\operatorname{Spec} R = \operatorname{Spec} R_{\text{red}}$, and reducing this ring yields $\mathbf{C}^{\oplus^3}$.

Remark 27.1.7: Let $Y \in \mathsf{Sch}^{\text{irr}}_{/k}$ and $X \xrightarrow{f} Y$, then Y has a generic point $\sqrt{0} \in \operatorname{Spec} A_i$ for some cover $\{\operatorname{Spec} A_i\} \rightrightarrows Y$. This corresponds to the irreducible closed subset Y itself, and yields a unique open generic point Y_{gen} One can then take the fiber $f^{-1}(Y_{\text{gen}})$ – what fiber product is this? Check that $Y_{\text{gen}} = A_i\left[\sqrt{0}^{-1}\right]/\sqrt{0} = \operatorname{ff}(A_i)$ is exactly the fraction field. Note that if $\operatorname{Spec} B_{ij} \subseteq \operatorname{Spec} A_i$ a distinguished open, we have $\operatorname{ff}(B_{ij}) \subseteq \operatorname{ff}(A_i)$, so this doesn't depend on the choice of the affine open $\operatorname{Spec} A_i$.

Example 27.1.8 *(?)*: Consider $\operatorname{Spec} \mathbf{Z}[x,y]/\left\langle y^2 = x^3 - 1 \right\rangle \in \mathsf{Sch}_{/\mathbf{Z}}$, which comes equipped with a map to $\operatorname{Spec} \mathbf{Z}$. The generic fiber is the base change to $\operatorname{Spec} \mathbf{Q}$:

$$\begin{tikzcd}
\operatorname{Spec} \mathbf{Q}[x,y]/\left\langle y^2 = x^3 - 1 \right\rangle \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\
\operatorname{Spec} \mathbf{Q} \arrow[r] & \operatorname{Spec} \mathbf{Z}
\end{tikzcd}$$

Link to Diagram

This can be done more generally to base change from a number field to its fraction field. Consider a degree 2 field extension $K \to K[x]/\left\langle x^2 - y \right\rangle$, then for example if $K = \mathbf{C}(y)$ we can construct the following:

$$\begin{tikzcd}
\operatorname{Spec} \mathbf{C}[x,y]/\left\langle y - x^2 \right\rangle \arrow[r] & \operatorname{Spec} \mathbf{C}[y] \\
\operatorname{Spec} \mathbf{C}(y)[x]/\left\langle y - x^2 \right\rangle \arrow[r] \arrow[u] & \operatorname{Spec} \mathbf{C}(y) = \{\text{pt}\} \arrow[u]
\end{tikzcd}$$

Link to Diagram

Note that $\operatorname{Spec} \mathbf{C}(y)$ is just a single point! So this doesn't quite pick up that any specific choice of generic point splits into two components, since $x^2 - y$ doesn't split unless $\sqrt{y} \in K$. One can remedy this by passing to $\overline{\mathbf{C}(y)}$ in this case. For f a finite morphism to an irreducible Y, one can define the **degree** of f as the degree of the extension associated to a generic point.

Example 27.1.9 *(?)*: Consider two lines projecting onto the y axis, say $(x-1)(x+1) = 0$, then this splits/factors over the generic point.

27.2 Separated/Proper Morphisms

Definition 27.2.1 (Diagonal)

Let $X \in \mathsf{Sch}_{/S}$ with structure map $X \xrightarrow{f} S$, then the **diagonal** $\Delta : X \to X^{\times_S^2}$ is the following induced map:

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X^{\times_S^2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & S
\end{tikzcd}$$

Link to Diagram

Definition 27.2.2 (Separated)

A structure map $X \xrightarrow{f} S$ is **separated** if the diagonal $\Delta : X \to X^{\times_S^2}$ is a closed embedding. X itself is separated if $\Delta : X \to X^{\times_{\operatorname{Spec} \mathbf{Z}}^2}$ is separated.

Warning 27.2.3

The usual "Hausdorff iff diagonal is closed" depends on a separation axiom! Which will often not hold in AG: for example, $\operatorname{Spec} R$ is separated for any ring but never Hausdorff.

Proposition 27.2.4 *(?).*
Any morphism in AffSch is separated.

Proposition 27.2.5 *(?).*
Consider:

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[dr, "\Delta"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \operatorname{Spec}(B^{\otimes_A^2}) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
& \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

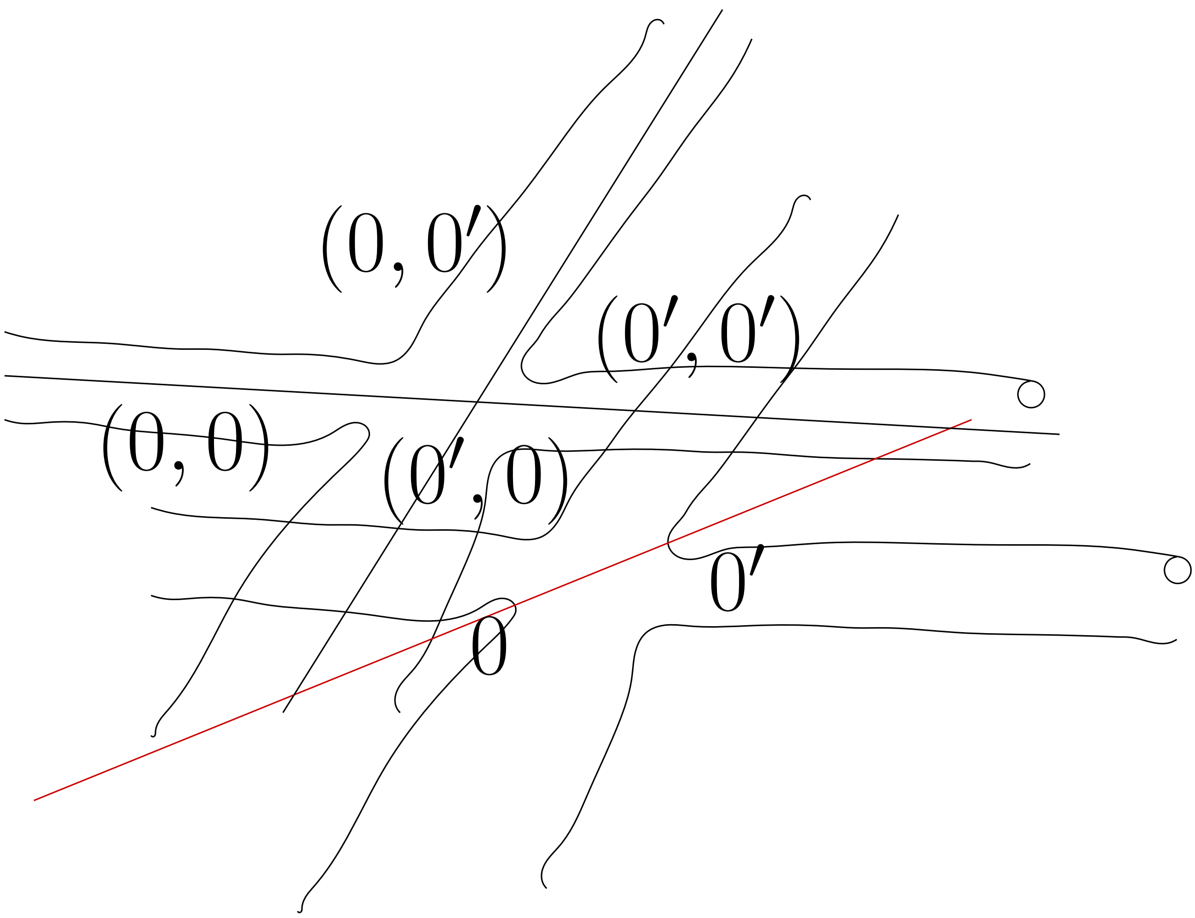
Link to Diagram

Then there is a ring morphism

$$\begin{aligned} \Delta^* B^{\otimes_A^2} &\to B && \in \mathsf{CRing} \\ b_1 \otimes b_2 &\mapsto b_1 b_2. \end{aligned}$$

Since this is surjective, Δ is a closed immersion.

Example 27.2.6 ***(A classic non-example):*** Let X be $\mathbf{A}^1$ with the doubled origin, so $X = \mathbf{A}^1 \coprod_f \mathbf{A}^1$ where for $U := \mathbf{A}^1 \setminus \{0\}$, we glue by id_U. Taking the product $X \underset{\operatorname{Spec} k}{\times} X$ yields the following:



Note that $(0,0') \notin \Delta(X)$ is not closed, but $(0,0') \in \overline{\Delta(X)}$ is in its closure.

28 | Wednesday, October 27

Remark 28.0.1: Recall that if $X \in \mathsf{Sch}_{/S}$ with $f : X \to S$, there is an induced diagonal map

$$\Delta : X \to X^{\times^2_S}$$

,

which is induced by $(\mathrm{id}_X, \mathrm{id}_X) : X \to X^{\times^2}$. We said f is **separated** if Δ is a closed immersion, which in particular is a homeomorphism onto a closed subset.

Example 28.0.2 ***(?):*** An example: any morphism of affine schemes $f \in \mathsf{Sch}(\operatorname{Spec} B, \operatorname{Spec} A)$.

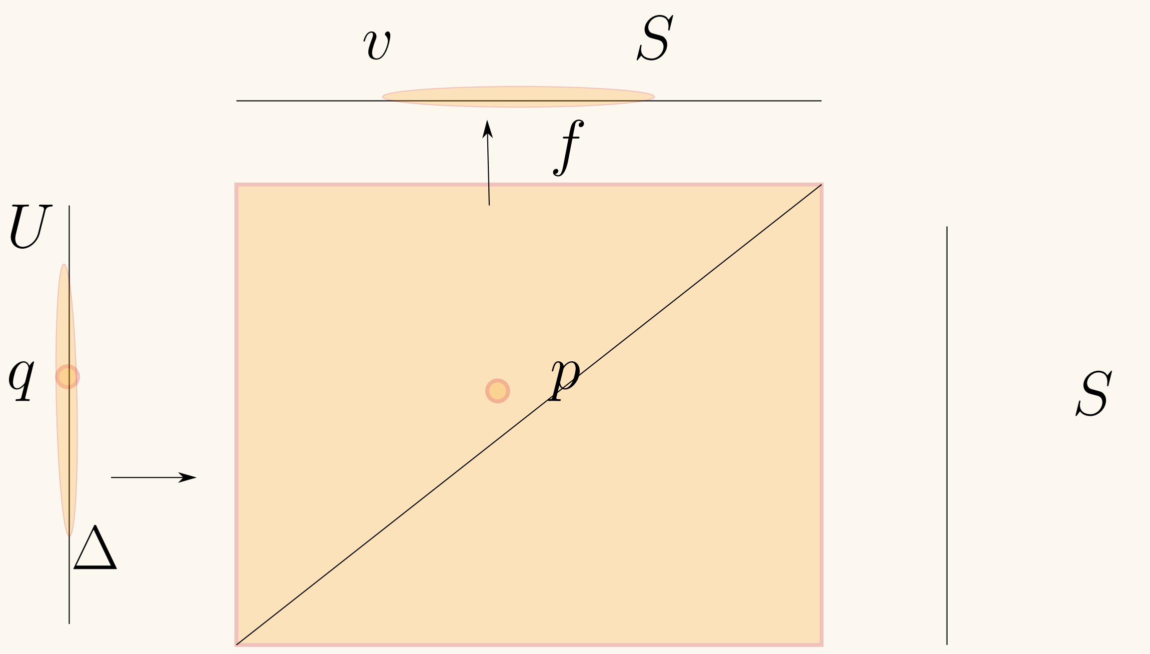
A non-example: $\mathbf{A}^1_{/k} \coprod_{\mathbf{A}^1_{/k} \setminus \{0\}} \mathbf{A}^1_{/k}$, the line with the doubled origin. We saw $(0, 0') \in \partial\Delta(X) = \overline{\Delta(X)} \setminus \Delta(X)$, and in fact $X^{\times^2_{\operatorname{Spec} k}} = \cup_{i \leq 4} \mathbf{A}^2_{/k}$.

Proposition 28.0.3 ***(?).***
$f \in \mathsf{Sch}(X, S)$ is separated $\iff$ $\operatorname{im}(\Delta)$ is closed.

Proof (?).
$\Longrightarrow$: This is definitional.
$\Longleftarrow$: First show Δ is a homeomorphism onto its image. Use the universal property to get $p_1 \circ \delta = \mathrm{id}_X$ where $p_i : X^{\times^2_S} \to X$ are the projections. Since both are continuous, Δ is a homeomorphism onto its image, which is closed.
It then suffices to show $\Delta^\# : \mathcal{O}_{X^{\times^2_S}} \to \Delta_* \mathcal{O}_X$ is a surjective map in $\mathsf{Sh}(X^{\times^2_S})$ to get a closed immersion. For any $p \in X^{\times^2_S}$, we need to show that there exists an open $N \ni p$ such that there is a surjection on sections

$$\Delta^\#(N) : \mathcal{O}_{X^{\times^2_S}}(N) \twoheadrightarrow \Delta_* \mathcal{O}_X(N) \in \mathsf{CRing}.$$

Observe that if $p \notin \operatorname{im} \Delta$ and $N = (\operatorname{im} \Delta)^c$ is open, then $\Delta_* \mathcal{O}_X(N) = \mathcal{O}_X(\Delta^{-1}(N)) = \mathcal{O}_X(\emptyset) = 0$ and anything surjects onto the zero ring. So if $p \notin \operatorname{supp} \Delta_* \mathcal{O}_X$, this is surjective. For $p \in \operatorname{im} \Delta$, write $p = \Delta(q)$ using the Δ is a homeomorphism onto its image and thus q is unique, and choose $U \ni q$ an affine open in X. Then $f(U) \subseteq V$ is contained in an affine open in S. The gluing construction of fiber products yields that $U^{\times^2_V} \subseteq X^{\times^2_S}$ is again an affine open, and $N \ni p$.



Then $\Delta^*(N)$ is surjective since U, V are affine and thus $f|_U : U \to V$ is separated. ∎

Example 28.0.4 ***(?):*** A morphism of schemes can be a homeomorphism onto a closed subset but *not* a closed immersion. Consider $k \hookrightarrow k[x]/\left\langle x^2 \right\rangle$, inducing $\operatorname{Spec} k[x]/\left\langle x^2 \right\rangle \to \operatorname{Spec} k$. This is not a surjective map of rings, and on affines this is equivalent to being a closed immersion.

Note that even though f is a closed immersion here, Δ is not: the fiber product is given by

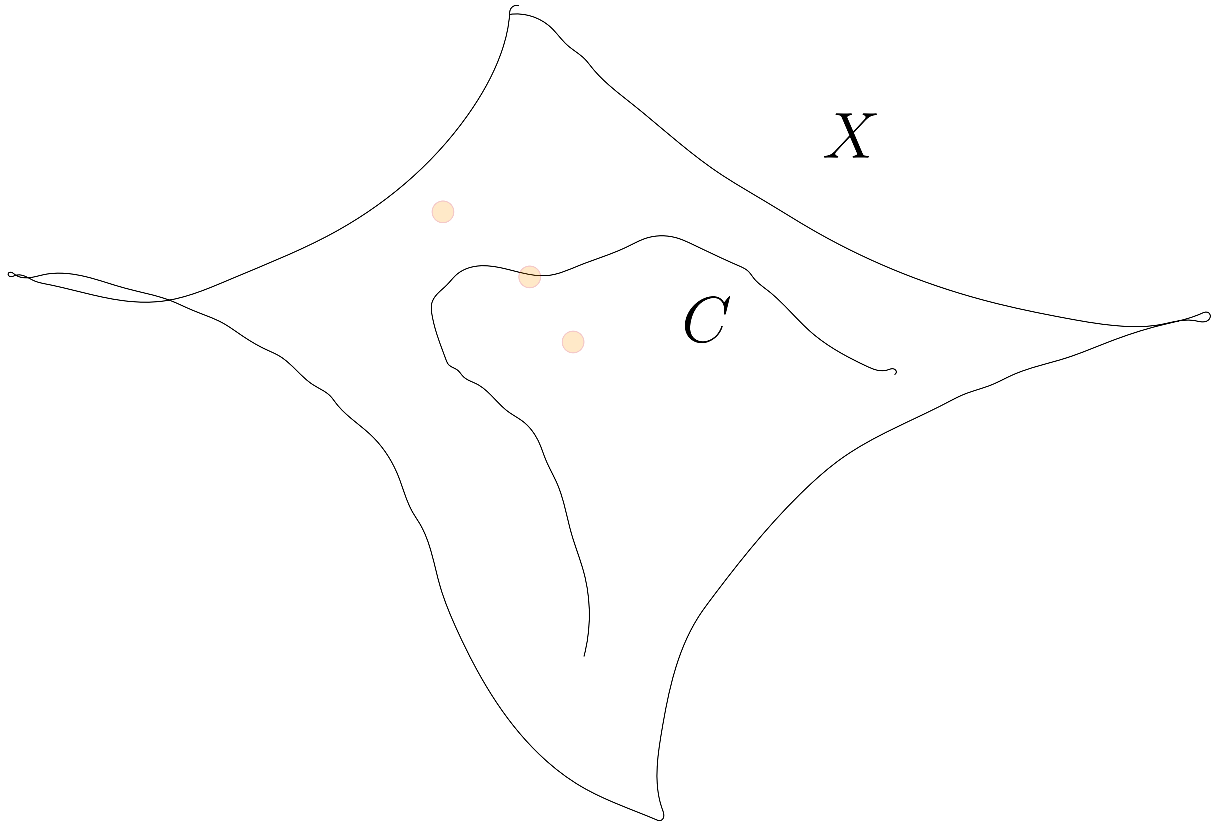
$$\operatorname{Spec} k[x]/x^2 \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} k[y]/y^2 = \operatorname{Spec} k[x,y]/\left\langle x^2, y^2 \right\rangle.$$

Note that taking $k[x]/x^2 \to k$ where $x \to 0$ *is* a closed immersion.

To google: if f is a homeomorphism onto its image and satisfies some condition for the induced map on Zariski tangent spaces, is it necessarily a closed immersion?

28.1 Valuative Criterion of Separatedness

Remark 28.1.1: A map from a punctured curve should extend uniquely!



Theorem 28.1.2 ***(?).***
Let $f \in \mathsf{Sch}(X, Y)$ with X Noetherian, then f is separated iff for any $R \in \mathsf{DVR}$ and $K := \mathrm{ff}(R)$, if the following lift exists, it is unique:

$$\begin{tikzcd} K \\ R \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "!\theta"] & Y \end{tikzcd}$$

LinktoDiagram

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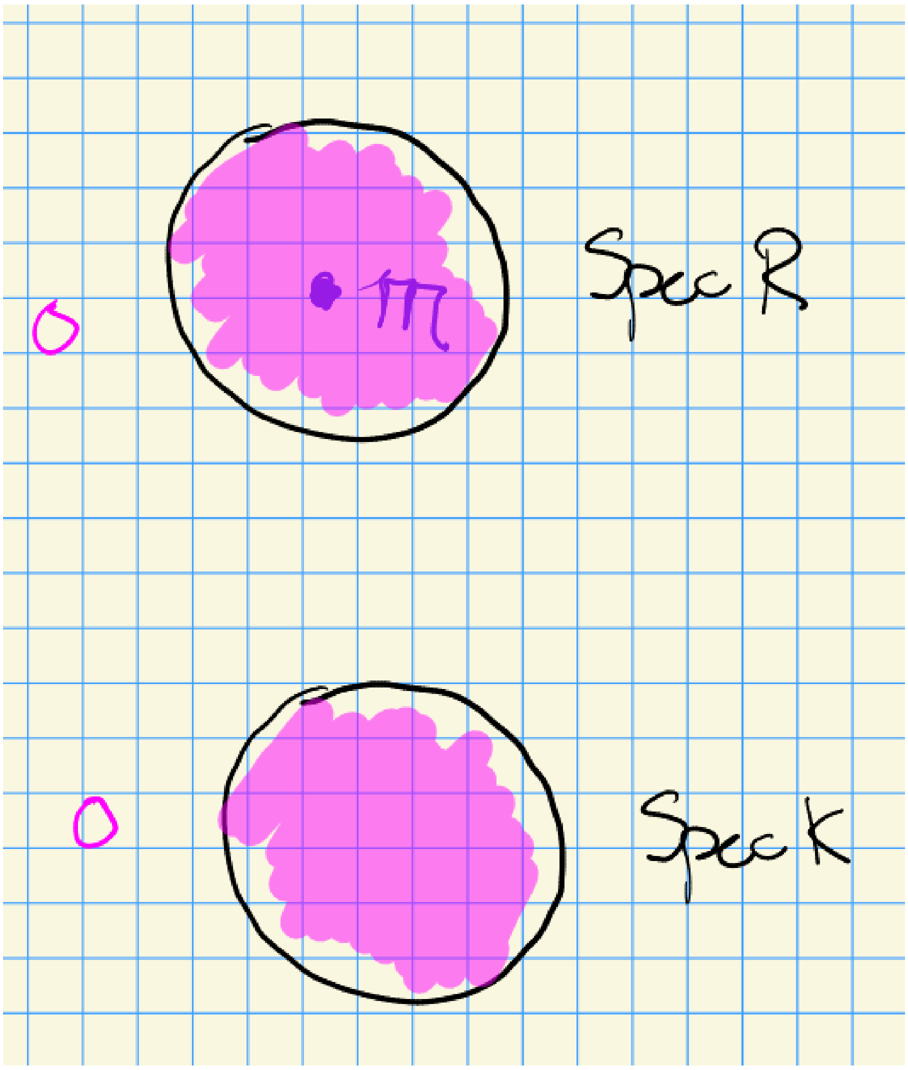
Example 28.1.3 ***(?):*** For $\operatorname{Spec} R = \{\mathfrak{m}, 0\}$, $\operatorname{Spec} K = \{0\}$ is just the generic point:



And moreover the generic points must be mapped to each other.

Example 28.1.4 ***(?):*** Consider $R \coloneqq \mathbf{C}[t]\left[t^{-1}\right]$, which is the stalk $\mathcal{O}_{\mathbf{A}^1_{/\mathbf{C}},0}$, and $K = \mathbf{C}(t)$.

Slogan 28.1.5
Spec R for $R \in \mathsf{DVR}$ is like a small piece of a curve.

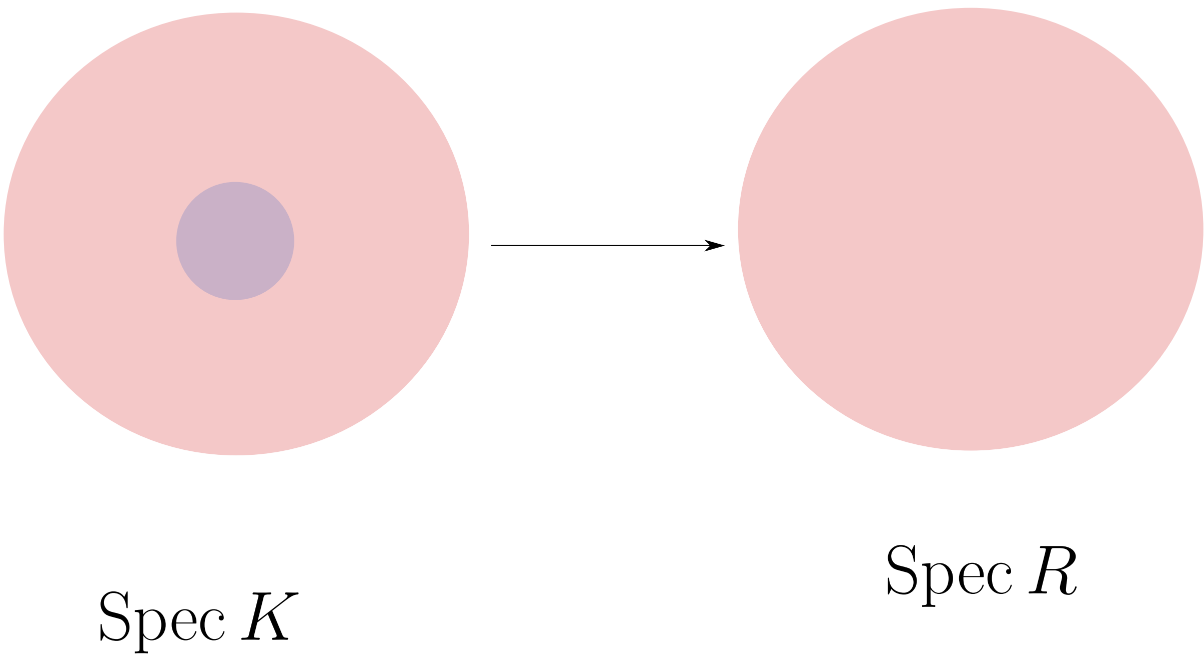
29 | Monday, November 01

Remark 29.0.1: Recall that $f : X \to Y$ is **separated** if $\Delta X \to X^{\times_Y^2}$ is a closed immersion, or equivalently $\Delta(X) \subseteq X^{\times_Y^2}$ is closed. We discussed the valuative criterion of separatedness, which is slightly more useful when proving things, but only holds for Noetherian (quasicompact, admits a a finite cover of affines) schemes: X is separated iff any diagram admitting a lift Θ of the following form admits a *unique* lift:

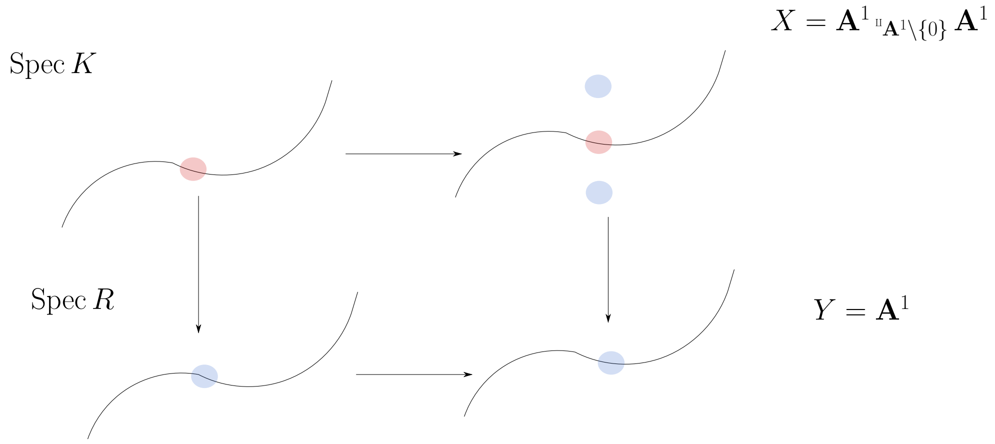
$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} R, R \in \mathsf{DVR} \arrow[r] \arrow[ur, dashed, "\exists ! \Theta"] & Y
\end{tikzcd}$$

Link to Diagram

Here $R \in \mathsf{DVR}$ and $K \in \operatorname{ff}(R)$.



Example 29.0.2*(?):* Consider mapping to the line with two origins:



Then given $f : X \to Y$ and $\operatorname{Spec} R \to Y$ there is an induced map $k[t]\left[t^{-1}\right] \leftarrow k[t]$. But note that there are two distinct extensions $\operatorname{Spec} R \to X$, say Θ_1, Θ_2, and there is an extension of the following form:

$$\begin{tikzcd}
\operatorname{Spec} k(t) \arrow[r] \arrow[d] & \operatorname{Spec} k[t] \\
\operatorname{Spec} k[t]\left[t^{-1}\right] \arrow[ur, shift left, "\exists\Theta_1"] \arrow[ur, shift right, "\exists\Theta_2"'] &
\end{tikzcd}$$

Link to Diagram

Remark 29.0.3: Taking fraction fields corresponds to throwing out everything but the generic point.

Proof (of valuative criterion).
Omitted, see Hartshorne. We'll discuss one key idea: **specialization**. ■

29.1 Specialization

Remark 29.1.1: Consider the data of a morphism from $\operatorname{Spec} K$:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\varphi"] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r, "\psi"] & Y
\end{tikzcd}$$

Link to Diagram

This is the data of a point $p \in X$, so $\varphi(\langle 0 \rangle) = p \in |X|$, and (it suffices to have) a pushforward $\varphi_p^\sharp$ which is a morphism of local rings inducing a diagram:

$$\begin{tikzcd}
\mathcal{O}_{X,p} \arrow[r, "\varphi_p^\sharp"] \arrow[d, "\operatorname{mod} \mathfrak{m}_p"'] & K \\
\kappa(p) \arrow[ur] &
\end{tikzcd}$$

Link to Diagram

Here $\varphi_p^\sharp(\mathfrak{m}_p) = \langle 0 \rangle$ and $\kappa(p)$ is the residue field at p.

Remark 29.1.2: What data is needed to specify $\psi : \operatorname{Spec} R \to Y$? We need two points $p_0, p_1 \in |X|$ with $p_1 = \psi(\langle 0 \rangle) \in \psi(\operatorname{Spec} K)$ and $p_0 = \psi(\mathfrak{m})$. Since $\psi^{-1}\left(\overline{\{p_1\}}\right)$ is closed, we also need $p_0 \in \overline{\{p_1\}}$, so $\kappa(p_1) \subseteq K$.

Consider $Z \coloneqq \overline{\{p_1\}} \ni p_0$ with its structure as a reduced closed subscheme of X. This yields a map $\operatorname{Spec} R \to Z$, and we need an injective (dominant) ring map $\mathcal{O}_{Z,p_0} \to R$. Why does this produce a map $\operatorname{Spec} R \to Y$? We have a closed immersion $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} \mathcal{O}_{Z,p_0} \to Z \hookrightarrow X$

Definition 29.1.3 (Specialization)
A point p_0 is a **specialization** of p_1 relative to $R \in \mathsf{DVR}$ and $K = \operatorname{ff}(R)$ if p_1 is a K-point, so $\kappa(p_1) \subseteq K$ such that $p_0 \in \overline{p_1}$ and $\mathcal{O}_{\bar{p}_1,p} \twoheadrightarrow R$

Example 29.1.4*(?)*: Take $R \coloneqq k[\![t]\!]$ and $\operatorname{ff}(R) = k((t))$ for $k \in \mathsf{Field}$. Consider $\operatorname{Spec} k \to \mathbf{A}^1_{/k}$ corresponding to $k[t] \hookrightarrow k((t))$. Setting $p_1 = \operatorname{im} \langle 0 \rangle = \langle 0 \rangle \in k[t]$ to be the generic point of $\mathbf{A}^1_{/k}$, we have $\overline{p_1} = \mathbf{A}^1_{/k}$. Set $p_0 = \langle t \rangle$, and note that $p_0 \in \overline{p_1}$, we then want a ring map $\mathcal{O}_{\overline{p_1},p_0} \to R = k[[t]]$. Note that $\mathcal{O}_{\overline{p_1},p_0} = k[t]\,[t^{-1}]$, and there is a ring map $\left\{ f(t)/g(t) \;\middle|\; g(0) \neq 0 \right\} \to k[[t]]$. This is injective, yielding a domination of rings.

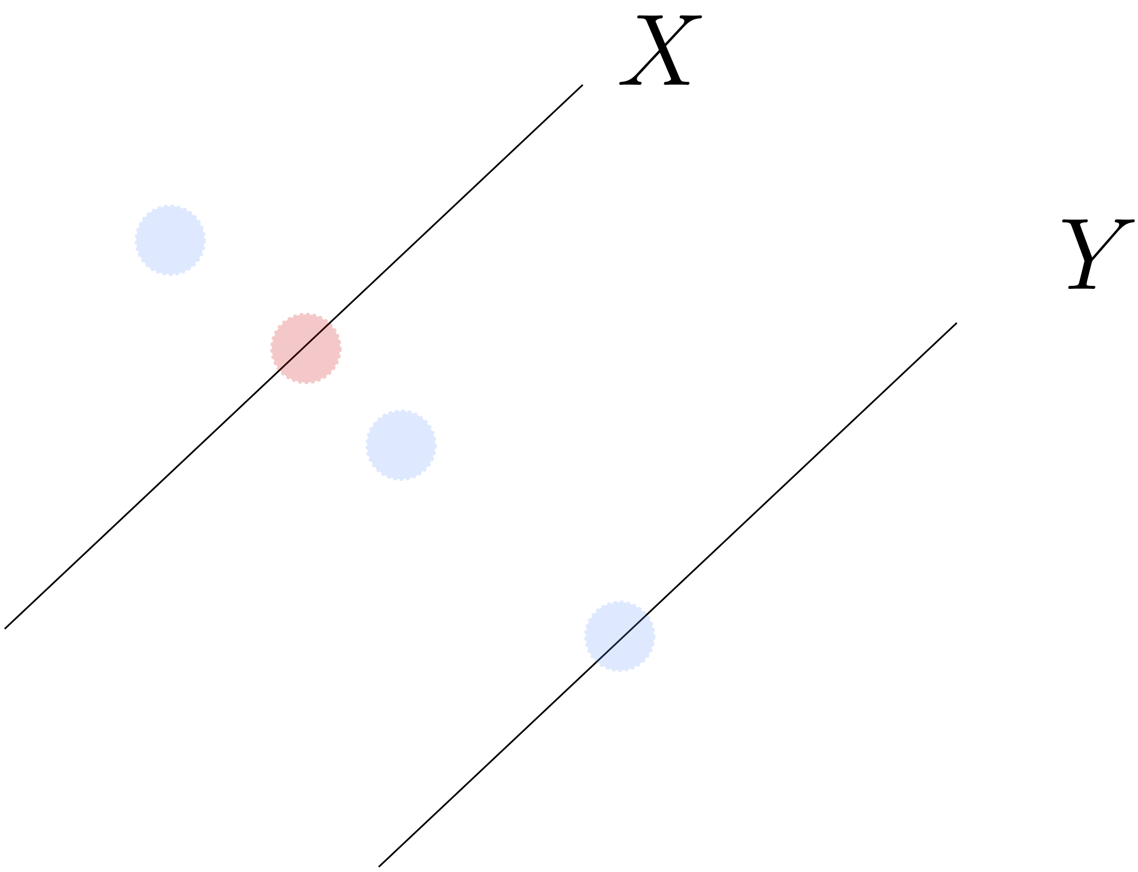
30 | Wednesday, November 03

Remark 30.0.1: Let $R \in \mathsf{DVR}, k = \mathrm{ff}(R)$, then a k-point of X is a morphism $\operatorname{Spec} k \to X$ and is given by the data of an inclusion $p_1 \in |X|$ and an inclusion $\kappa(p_1) \hookrightarrow k$.

Example 30.0.2 *(?):* Why these are called k-points? Given $\operatorname{Spec} \mathbf{Q} \to \operatorname{Spec} S := \operatorname{Spec} \mathbf{Z}[x, y, z]/\left\langle x^5 + y^5 - z^5 \right\rangle$, then there is a ring map $S \to \mathbf{Q}$ where $x, y, z \mapsto x_0, y_0, z_0$ satisfying $x_0^2 + y_0^2 = z_0^2$. So these are rational solutions to the defining equations.

Remark 30.0.3: Lifting a k-point to an R-point $\operatorname{Spec} R \to X$ requires $p_0 \in \mathrm{cl}(\{p_1\})$ and a domination $\mathcal{O}_{\mathrm{cl}(\{p_1\}), p_0} \to R$ inducing the k-point in the sense that the generic point of $\operatorname{Spec} R$ maps to the generic point of $\operatorname{Spec} \mathcal{O}_{\mathrm{cl}\{p_1\}, p_0}$ corresponding to an inclusion of fields. So we get a morphism of local rings.

Remark 30.0.4: We saw that $f : X \to Y$ is separated iff $\Delta(X) \hookrightarrow X^{\times_Y^2}$ is closed iff any k-point of X has at most one specialization over a given R-point of Y. Idea: rules out two lifts.



Note that this needs X to be Noetherian.

Is separatedness local? Perhaps X only locally Noetherian would suffice.

Remark 30.0.5: Review the difference between "of finite type" and "*locally* of finite type".

Definition 30.0.6 (Closed and universally closed)
A morphism $f : X \to Y$ is **closed** if the underlying map $|f| : |X| \to |Y| \in \mathsf{Top}$ is a closed continuous map (so images of closed sets are closed), and f is **universally closed** if for all $Y' \to Y$ the change $f' : X \underset{Y}{\times} Y' \to Y'$ is closed.

Example 30.0.7 ***(?):*** Identity maps $\mathrm{id}_X : X \to X$ are closed, using that $Y \underset{X}{\times} X \cong Y$ and pulling back id_X yields id_Y.

Example 30.0.8 ***(A non-example):*** Consider $\mathbf{A}^1_{/k} \to \operatorname{Spec} k$, which is closed since $\operatorname{Spec} k = \mathrm{pt}$ and has the discrete topology. This is not universally closed, since we have

$$\begin{tikzcd} \mathbf{A}^2_{/k} \arrow[r] \arrow[d, "f'"'] & \mathbf{A}^1_{/k} \arrow[d, "f"] \\ \mathbf{A}^1_{/k} \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

Link to Diagram

Consider $Z \coloneqq V(xy-1) \subseteq \mathbf{A}^2_{/k}$, then $f'(Z) = \mathbf{A}^2_{/k} \setminus \{0\}$ is projection onto the x-axis and is not closed in $\mathbf{A}^2_{/k}$. What this is a projection onto the x-axis: this comes from the map $f : k[x] \hookrightarrow k[x, y] \cong k[x] \otimes_k k[y]$ where $f^{-1}(\langle x - x_0, y - y_0 \rangle) = \langle x - x_0 \rangle$, so geometrically this yields the $(x_0, y_0) \to x_0$.

Example 30.0.9 ***(?):*** Consider $\mathbf{P}^1_{/k} \coloneqq \operatorname{proj} k[x, y]$ and consider $\mathbf{P}^1_{/k} \to \operatorname{Spec} k$. Here $\mathbf{P}^1_{/k}$ is supposed to be "compact" in the sense that graphs of all functions are closed.

Exercise 30.0.10 (?)
Are compact spaces universally closed in Top?

Definition 30.0.11 (Proper)
A morphism $f : X \to Y$ is **proper** if

1. f is of finite type,
2. f is separated,
3. f is *universally closed*

Remark 30.0.12: This ranges over all possible base changes, so it's quite hard to actually check! The following result gives an easier way:

Theorem 30.0.13 ***(Valuative criterion of properness).***
Let $f : X \to Y$ be a finite type morphism with X Noetherian. Then f is proper $\iff$ there exists unique lifts Θ of the following form:

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "!\Theta" description] & Y
\end{tikzcd}$$

Link to Diagram

Remark 30.0.14: Most spaces in practice are separated and of finite type, unless you're working with moduli of K3 surfaces!

Proof ($\implies$).
Suppose f is proper, then f is separated and we have uniqueness for any lifts by the valuative criterion for separatedness. This uses that X is Noetherian. It then suffices to show existence of Θ, using that f is universally closed. Consider the base change $X_R := X \underset{Y}{\times} \operatorname{Spec} R$, then using commutativity we get a morphism $s : \operatorname{Spec} k \to X_R$. Let $p_1 = s(\langle 0 \rangle) \subseteq X_R$, we'll then try to specialize p_1. Let $Z := \operatorname{cl}\{p_1\} \subseteq X_R$, then since f is proper and Z is closed in X_R, $f_R(Z)$ is closed:

$$\begin{tikzcd}
X_R \arrow[r] \arrow[d, "f_R"'] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r] & Y
\end{tikzcd}$$

Link to Diagram

We can compose $\operatorname{Spec} k \to Z \xrightarrow{f_R} \operatorname{Spec} R$ to get $\tilde{f}$, which is an inclusion of the generic point:

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[dr] \arrow[drr] \arrow[ddr, "\tilde{f}"'] & & \\
& X_R \arrow[r] \arrow[d, "f_R" description] & X \arrow[d, "f"] \\
& \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "!\Theta" description] & Y
\end{tikzcd}$$

Link to Diagram

Then $f_R(Z) = \operatorname{Spec} R$ and so there exists $p_0 \in Z$ with $f_R(p_0) = \mathfrak{m}$, the closed point in $\operatorname{Spec} R$. So we get

$$\begin{aligned} g : Z &\to \operatorname{Spec} R \\ \operatorname{cl}\{p_1\} \ni p_0 &\mapsto \mathfrak{m} \\ p_1 &\mapsto \langle 0 \rangle . \end{aligned}$$

Taking stalks yields a local ring morphism $g^\sharp_{p_0} : R \to \mathcal{O}_{Z,p_0}$, and this completes to a diagram:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \mathcal{O}_{Z,p_0} \arrow[d] \\ k \arrow[r] & \operatorname{ff} \mathcal{O}_{Z,p_0} = \kappa(p_1) \end{tikzcd}$$

Link to Diagram

But R is final with respect to domination for local rings R' in k with $\operatorname{ff} R' = k$, and if final objects admit morphisms to other objects, those objects must also be final, so $R = \mathcal{O}_{Z,p_0}$. This yields a domination $\mathcal{O}_{Z,p_0} \to R$, which corresponds to a lift $\operatorname{Spec} R \to X$.

■

31 | Friday, November 05

Remark 31.0.1: Last time: valuative criterion for properness. A morphism $f : X \to Y \in \mathsf{Sch}$ is proper $\iff$

- f is separated,
- f is of finite type,
- f is universally closed (closed to closed, and preserved under base change)

If X is Noetherian and f is of finite type, then f is proper $\iff$ for $R \in \mathsf{DVR}, K = \operatorname{ff}(R)$, we have lifts:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, "\exists ! \Theta"] & Y \end{tikzcd}$$

Link to Diagram

We proved that f proper implies $\exists\Theta$. Erratum: we said $R \subseteq K$ is final with respect to local rings contained in K with fraction field K, but rather it's maximal. As an example, $\mathbf{Z}_{[p^{-1}]}2, \mathbf{Z}_{[p^{-1}]}3 \hookrightarrow \mathbf{Q}$ but there's no common ring they map to. Proof of $\Longleftarrow$: see Hartshorne.

Corollary 31.0.2 *(?).*
Some applications/corollaries of the valuative criterion for properness:

Separated	Proper
Open or closed immersions	Closed immersions[a]
Compositions	Compositions
Stable under base change	Stable under base change
Products	Products
Local on base	Local on base
$X \xrightarrow{f} Y \xrightarrow{g} Z$ with $g \circ f$ separated $\implies f$ separated	$X \xrightarrow{f} Y \xrightarrow{g} Z$ with $g \circ f$ proper and g separated $\implies f$ separated

- "Stable under base change" means that whenever $X \xrightarrow{f} Y$ has a property $P(f)$, any fiber product along $Y' \to Y$ yields the same property $P(f')$:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

Link to Diagram

- A *product* of morphisms in $\mathsf{Sch}_{/S}$ is the product in $\mathsf{Sch}_{/S}$, or equivalently the fiber product over S. So given $f : X \to Y$ and $f' : X' \to Y'$, the product is $(f, f') : X \underset{S}{\times} X' \to Y \underset{S}{\times} Y'$. So here "Products" means that if $P(f), P(f')$ holds, then $P(f, f')$ holds.

- P is *local on the base* if whenever P holds for $X \xrightarrow{f} Y$ then for all open $U \subseteq Y$, the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ also satisfies P

[a] Very rarely proper! Only if inclusion of a connected component.

Proof (stability under base change).
Diagram chases involving the valuative criteria and universal properties of the fiber product. For example, we'll do stability under base change: let $X \xrightarrow{f} Y$ be separated and $Y' \to Y$, we'll show $X' \xrightarrow{f'} Y'$ is separated where $X' := X \underset{Y}{\times} Y'$. We need to show an extension Θ of the following form is unique if it exists:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[r, "\beta"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r, "\alpha"'] \arrow[ur, dashed, shift left, "\Theta_1"] \arrow[ur, dashed, shift right, "\Theta_2"'] & Y' \arrow[r] & Y
\end{tikzcd}$$

Link to Diagram

Note that $\beta \circ \theta_1 = \beta \circ \Theta_1$, since f is separated, using the valuative criterion for separatedness. Since X' is a fiber product, by the universal property there exists a unique product morphism $(\beta \circ \Theta_1, \alpha) = (\beta \circ \Theta_2, \alpha)$. So $\Theta_1 = \Theta_2$ and f' is separated by the valuative criterion of separatedness.

■

Proof (of products).
We want to show that if $f : X \to Y, f' : X' \to Y'$ are proper then $(f, f') : X \underset{S}{\times} X' \to Y \underset{S}{\times} Y'$ is proper. We can produce a diagram:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[rr] \arrow[dr] \arrow[ddrr, dotted, bend right] & & X \underset{S}{\times} X' \arrow[rr] \arrow[dr] \arrow[dd] & & X' \arrow[dr] \arrow[dd] & \\
& \operatorname{Spec} R \arrow[rr] \arrow[ur, dashed, "\text{WTS: } \exists!\Theta"] \arrow[urrr, squiggly, "\exists!"] \arrow[dr, squiggly, "\exists!"] \arrow[ddrr, dotted, bend right] & & Y \underset{S}{\times} Y' \arrow[rr] \arrow[dd] & & Y' \arrow[dd] \\
& & X \arrow[rr] \arrow[dr] & & S \arrow[dr] & \\
& & & Y \arrow[rr] & & S
\end{tikzcd}$$

Link to Diagram

Here we get existence of unique maps $\operatorname{Spec} R \to X, \operatorname{Spec} R \to X'$, which thus yields a unique map $\operatorname{Spec} R \to X \underset{S}{\times} X'$.

■

Proof (locality on base).
Suppose $X \xrightarrow{f} Y$ is proper, we'll show that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is proper for $U \subseteq Y$. We use that $f^{-1}(U)$ is a fiber product and apply the universal property:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & f^{-1}(U) \arrow[r] \arrow[d, "f|_{f^{-1}(U)}"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\therefore \exists!"] \arrow[urr, "\exists!"] & U \arrow[r] & Y
\end{tikzcd}$$

Link to Diagram

For the converse, it suffices to check properness on an open cover $\mathcal{U} \rightrightarrows Y$. Why? It follows if for any diagram of the following form, there exists an open $U_i \subseteq Y$ such that $\operatorname{im} \operatorname{Spec} R \subseteq U_i$:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} R \arrow[r] & Y
\end{tikzcd}$$

Link to Diagram

Note that $\operatorname{Spec} R$ has two points, so this is not completely trivial. Consider the closed point $\mathfrak{m} \in \operatorname{Spec} R$ and let $p_0 = \operatorname{im}(\mathfrak{m})$.

■

32 | Monday, November 08

Remark 32.0.1: Let $f : X \to Y \in \mathsf{Sch}$, then f is **proper** iff

- f is separated,
- f is of finite type,
- f is universally closed, so for all $Y' \to Y$, the base change morphisms $X \underset{Y}{\times} Y' \to Y'$ is closed.

The valuative criterion of properness stated that if R is a valuation ring and $K := \operatorname{ff}(R)$, so $\operatorname{Spec} K \to \operatorname{Spec} R$, there are unique lifts of the following form:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\exists"] & Y
\end{tikzcd}$$

Link to Diagram

Definition 32.0.2 (Projective space over a scheme)
Let $Y \in \mathsf{Sch}$, then define **projective space over** Y as $\mathbf{P}^n_{/Y} := \mathbf{P}^n_{/\mathbf{Z}} \times Y := \mathbf{P}^n_{/\mathbf{Z}} \underset{\operatorname{Spec} \mathbf{Z}}{\times} Y$.

Remark 32.0.3: This is analogous to $\mathbf{P}^n_{/R} := \operatorname{Proj} R[x_0, \cdots, x_n]$ for $R \in \mathsf{CRing}$, and these two constructions turn out to be the same. Note that $R[x_0, \cdots, x_n] \cong \mathbf{Z}[x_0, \cdots, x_n] \otimes_{\mathbf{Z}} R$.

Definition 32.0.4 (Projective morphisms)
A morphism $X \xrightarrow{f} Y \in \mathsf{Sch}$ is **projective** iff f factors as $X \hookrightarrow \mathbf{P}^n_{/Y} \twoheadrightarrow Y$, a closed immersion into projective space followed by projection onto Y.

Example 32.0.5 *(?)*: Let $S \in \mathsf{gr}_{\mathbf{Z}}\mathsf{CRing}$ be a graded ring, then $S_0 \leq S$ is a subring. Suppose S is finitely generated over S_0 by S_1, so there exists a finite generating set $\{x_i\}_{i \leq n}$ and a surjective map $S_0[x_0, \cdots, x_n] \twoheadrightarrow S$ sending x_i to elements of S_1. Note that this preserves the grading, since S_0 elements have degree zero and x_i have degree 1 on both sides, so we get a map ψ : $\operatorname{Proj} S \to \operatorname{Proj} S_0[x_0, \cdots, x_n] \coloneqq \mathbf{P}^n_{/S_0}$. So the ring maps on affine opens will be surjective, since they are localizations of ψ, so this yields a closed immersion and thus a projective morphism $\operatorname{Proj} S \to \operatorname{Spec} S_0$.

Example 32.0.6 *(?)*: If $S = k \in \mathsf{Field}$ with $k = \bar{k}$, then if $R \in \mathsf{gr}_{\mathbf{Z}}\mathsf{CRing}$ is finitely generated in degree 1, then $\operatorname{proj} R \to \operatorname{Spec} k$ is projective, since these are exactly quotients of $k[x_1, \cdots, x_n]$ by a homogeneous ideal. If this homogeneous ideal is radical, then these correspond to projective varieties over k.

Remark 32.0.7: We'll show that projective implies proper, which will furnish many examples of proper maps.

Theorem 32.0.8 *(?)*.
Any projective morphism $f : X \to Y$ is proper.

Exercise 32.0.9 (Hartshorne 3.13, checking when a morphism is finite type)
See Hartshorne, try it!

Proof (?).
We know that base changes of proper morphisms are proper, using the valuative criterion. With the above example and exercise, it suffices to show $\mathbf{P}^n_{/\mathbf{Z}}$ is proper. Why? We have a closed immersion $X \to \mathbf{P}^n_{/Y} \coloneqq \mathbf{P}^n_{/\mathbf{Z}} \times Y$ by the definition of $f : X \to Y$ being proper. Then the projection $\mathbf{P}^n_{/\mathbf{Z}} \times Y \to Y$ is proper, and is a base change of $\mathbf{P}^n_{/\mathbf{Z}} \to \operatorname{Spec} \mathbf{Z}$. So if we know the latter is proper, compositions of proper maps are proper and thus f is proper.

todo, try to form a diagram here.

Idea: clear denominators in a minimal way. We can cover $\mathbf{P}^n_{/\mathbf{Z}}$ by affine opens:

$$D(x_i) \coloneqq \left\{ \mathfrak{p} \in \mathbf{Z}[x_0, \cdots, x_n]_{\text{homog}} \;\middle|\; z_i \not\in \mathfrak{p} \right\} \cong \operatorname{Spec} \mathbf{Z}\left[\tfrac{x_0}{x_1}, \cdots, \tfrac{x_n}{x_1}\right].$$

Then $\{D(x_i)\}_{1 \leq i \leq n+1} \rightrightarrows \mathbf{P}^n_{/\mathbf{Z}}$, making $\mathbf{P}^n_{/\mathbf{Z}}$ finite type since it admits a finite cover by $\operatorname{Spec} R_i$ for $R_i \in \mathsf{Alg}^{\text{fg}}_{/\mathbf{Z}}$. It thus suffices to verify the valuative criterion of properness, since this will imply separatedness. So we'll show any morphism $g : \operatorname{Spec} K \to \mathbf{P}^n_{/\mathbf{Z}}$ lifts uniquely to a morphism $\operatorname{Spec} K \to \mathbf{P}^n_{/\mathbf{Z}}$. Given g, $g(0)$ is a single point, and by a linear change of coordinates we can ensure $g(0) \in \bigcap_i D(x_i)$. So we have

$$g(0) \in D(x_0, \cdots, x_n) \cong \operatorname{Spec} T := \operatorname{Spec} \mathbf{Z}\left[\left\{ \frac{x_i}{x_j} \right\}_{0 \leq j \leq n}\right] \qquad \forall i.$$

So g factors through the open immersion $\operatorname{Spec} T \hookrightarrow \mathbf{P}^n_{/\mathbf{Z}}$, and is thus a map of affine schemes and equivalently the data of a ring map $\varphi : \operatorname{Spec} T \to K \in \mathsf{CRing}$. Let $\varphi_{ij} = \varphi\left(\frac{x_i}{x_j}\right)$, and note that $\varphi_{ij} \in K^\times$ for every i, j. These satisfy a cocycle condition $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$, so letting $v_i := v(\varphi_{i,0})$ for v the valuation, there is some minimal v_i.

Continued next time.

■

33 | Wednesday, November 10

Remark 33.0.1: We defined projective space $\mathbf{P}^n_{/Y} := \mathbf{P}^n_{/\mathbf{Z}} \underset{\operatorname{Spec} \mathbf{Z}}{\times} Y$, and a projective morphism as one that factors as a closed immersion into $\mathbf{P}^n_{/Y}$ for some Y followed by projection onto Y. Continuing the proof from last time: we reduced to $\mathbf{P}^n_{/\mathbf{Z}}$ and produced a map $\operatorname{Spec} R \to \mathbf{P}^n_{/\mathbf{Z}}$.

Proof (continued).
We noted that $\varphi_{ij} \in k^\times$ since $\varphi_{ij}\varphi ji = 1$, and more generally $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$. We chose $v_i = \operatorname{val}(\varphi_{i,0}) \in \mathbf{Z}$ minimally, so assume without loss of generality by relabeling that it is v_1, so $v_i \geq v_1$ for all i. Then use that $\varphi_{ij} = \varphi_{i1}/\varphi_{j1} \implies v(\varphi_{i1}) = v(\varphi_{i0}) - v(\varphi_{10}) = v_2 - v_1 \geq 0$. Writing $R = \left\{\varphi \in k \;\middle|\; v(\varphi) \geq 0\right\}$, we have $\varphi_{i1} \in R \subseteq k$. Consider the ring map

$$\begin{aligned} \mathbf{Z}\left[\frac{x_0}{x_1}, \cdots, \frac{x_n}{x_1}\right] &\to R \\ \frac{x_i}{x_1} &\mapsto \varphi_{i1}. \end{aligned}$$

This yields a map $\operatorname{Spec} R \to \mathbf{A}^n_{/\mathbf{Z}} \cong D(x_1)$, which restricts to a map $\operatorname{Spec} K \to D(x_0, \cdots, x_n)$, a smaller open set.

Fire alarm! Class canceled.

■

34 | Friday, November 12

Remark 34.0.1: Continuing from last time: this is equivalent to $\Delta(\mathbf{P}^n_{/\mathbf{Z}})$ being closed. Since every affine scheme is separated, $\Delta(\mathbf{A}^n_{/\mathbf{Z}})$ is closed for every $D(x_i)$. Suppose a closed point (P, P') lies in the closure of $\Delta(\mathbf{P}^n_{/\mathbf{Z}})$, then if $P, P' \in D(x_i)$ for some i then $P = P'$ since $D(x_i)$ is separated. We can ensure this is possible by potentially taking a linear change of coordinates. Introducing new variables $x_i' = \sum n_i x_i$ with $N := (n_i)_{i \in I} \in \mathrm{GL}_{n+1}(\mathbf{Z})$. Then there is an isomorphism of graded rings $\mathbf{Z}[x_0, \cdots, x_n] \to \mathbf{Z}[x_0', \cdots, x_n']$ inducing an isomorphism $\mathbf{P}^n_{/\mathbf{Z}} \circlearrowleft$. By doing this we can replace x_0 with any linear combination of x_is, and we need to show that there exists a linear map L such that $L(x) = \sum n_i x_i$ for which $L(x) \neq P, L(x) \neq P'$, so $P, P' \in D(L(x))$. Consider the image of $L(x)$ in $\mathbf{Z}[x_0, \cdots, x_n]/P$ and similarly for P'. These are (finite) fields since P, P' are maximal. Any map $\mathbf{Z}^{\times^{n+1}} \to \mathbf{F}_q$ is a group morphism, and the kernel is a finite index sublattice. One can always find an element of $\mathbf{Z}^{\times^{n+1}}$ which isn't on the union of two strict sublattices, i.e. $1/a + 1/b - 1/ab < 1$.

Example 34.0.2 ***(?):*** A projective variety over k is proper over $\operatorname{Spec} k$. These are of the form $\operatorname{Proj} k[x_1, \cdots, x_n]/I$ for I a homogeneous ideal, and thus come with a closed immersion into $\mathbf{P}^n_{/k}$.

Example 34.0.3 ***(The main class of examples):*** If $X \xrightarrow{f} Y \in \operatorname{Proj} \mathsf{Var}$ or $\mathsf{Sch}_{/k}$. Then the maps $X \to \operatorname{Spec} k$ and $Y \to \operatorname{Spec} k$ are proper, and the second is separated. Peeling off the compositions shows f is proper.

Example 34.0.4 ***(?):*** Let $X \xrightarrow{f} Y$ be any morphism from a projective scheme to a separated scheme of finite type over k. This is also proper, and thus universally closed, and its image in Y is also proper using that closed subschemes of separated schemes are separated and of finite type, and morphisms factor through their images.

Corollary 34.0.5 ***(?).***
Any regular function on a projective (or even proper) variety is locally constant.

Proof (?).
A regular function on projective X is a morphism $X \xrightarrow{f} \mathbf{A}^1$, so consider the open immersion $\mathbf{A}^1 \hookrightarrow \mathbf{P}^1$. The composition $x \circ f : X \to \mathbf{P}^1$ is projective, thus proper, so $(i \circ f)(X) \subseteq \mathbf{A}^1 \subseteq \mathbf{P}^1$ is closed, but the only such closed sets are finite. Thus $i \circ f$ and thus f is constant on a connected component of X.

■

Corollary 34.0.6 ***(?).***
Any morphism from a proper variety to an affine variety is locally constant.

Proof (?).
If $X \xrightarrow{f} Y$ with X proper and Y affine, then there is an open immersion $\iota : Y \hookrightarrow \mathbf{A}^n$. The composition of $\iota \circ f$ is locally constant on each coordinate by the previous corollary, making f locally constant.

■

Example 34.0.7 ***(?):*** For X any variety and Y proper, $X \times Y \to X$ is proper because it is the base change along $Y \to \operatorname{Spec} k$. So e.g. $\mathbf{P}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$ is proper. Another example: blow up a projective variety at an ideal.

Slogan 34.0.8
Proper means compact fibers.

35 | Monday, November 22

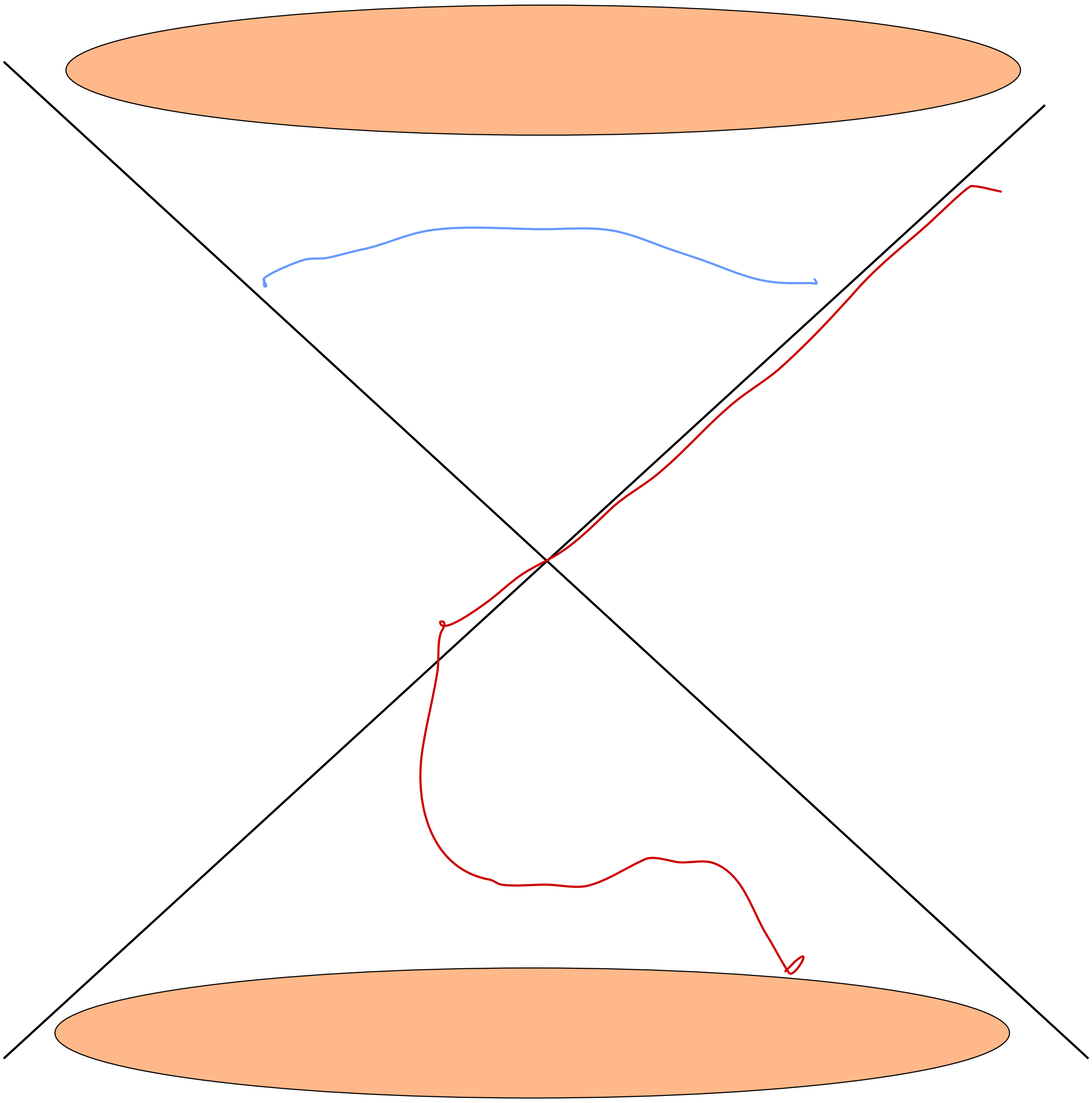
Remark 35.0.1: Last time: a local ring R is **regular** iff the number of generators of $\mathfrak{m}_R$ is equal to the Krull dimension of R. There is a slightly weaker notion: a scheme is regular in codimension 1 iff every local ring $\mathcal{O}_{X,x}$ of dimension 1 is regular. Note that this is locally the generic point associated to a height 1 prime ideal.

Remark 35.0.2: Note that not every local ring is a domain, e.g. $\mathcal{O}_{V(xy),\mathbf{0}}$. Algebra fact: a 1-dimensional regular local ring is a DVR, since this forces $\mathfrak{m}$ to be generated by 1 element and thus principal.

Remark 35.0.3: A new standing assumption: $X \in \mathsf{Sch}$ is

- Noetherian,
- Integral (covered by spec of integral domains, equivalently reduced and irreducible)
- Separated,
- Regular in codimension 1.

Example 35.0.4*(?)*: Examples are smooth projective varieties, but may include singular varieties, e.g. $V(xy - z^2) \subseteq \mathbf{A}^3_{/k}$. Note that the partials vanish at $\mathbf{0}$, an singularity is picked up by the fact that $k[x, y]/\langle f \rangle\left[\langle x,y \rangle^{-1}\right]$ and this is conical hyperboloid:



Note that some curves may be singular, namely those passing through 0, but generically they are non-singular. An equation of a line on X might be $V(x,z)$, so consider $\mathfrak{p} = \langle x, y \rangle \in k[x,y,z]/\langle xy - z^2 \rangle$.

Exercise 35.0.5 (A good one)
Check that $R[\mathfrak{p}^{-1}]$ is regular, and $\mathfrak{p}R[\mathfrak{p}^{-1}]$ is principal.

Definition 35.0.6 (Prime divisors)
A **prime divisor** on X is an integrable subscheme of codimension 1.

Example 35.0.7 ***(?):*** Take $V(y^2 - x^3) \subseteq \mathbf{A}^2_{/k}$, or $V(x,z) \subseteq \operatorname{Spec} R$. Generally, if $f \in R$ is irreducible then $V(f) \subseteq \operatorname{Spec} R$ is a prime divisor. Note that the second example is codimension 1 in $\mathbf{A}^2_{/k}$.

Remark 35.0.8: If $X = \operatorname{Spec} R$, then the prime divisors are in 1-to-1 correspondence with height 1 prime ideals in R. Check that $\langle 0 \rangle \subsetneq \mathfrak{p}$ since R is a domain, and no prime ideal can fit between these. Note that in the above example, $\langle 0 \rangle \subsetneq \langle x, z \rangle \subseteq k[x,y,z]/\langle xy - z^2 \rangle$, where e.g. $\langle x \rangle$ isn't prime because $xy \in \langle x \rangle \implies z^2 \in \langle x \rangle \implies z \in \langle x \rangle$.

Example 35.0.9 *(?)***:** Some examples of prime divisors:

- For X a nice variety: the irreducible subvarieties of codimension 1.
- For $X = \operatorname{Spec} \mathbf{Z}$: closed points, i.e. any maximal ideal.
- For $X = \operatorname{Spec} \mathbf{Z}[\sqrt{-5}]$: an example might be $\langle 2, 1 + \sqrt{-5} \rangle$.
- For $X = \mathbf{F}_3[t]$, consider $\langle t - a_i \rangle$ for $a_i = 0, 1, 2$ and $\langle t^2 - 2 \rangle$. Note that being a prime ideal is not preserved under base change, e.g.

$$\begin{tikzcd}
X' = \operatorname{Spec} \mathbf{F}_3[t] \otimes \mathbf{F}_q = \operatorname{Spec} \mathbf{F}_q[t] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X = \operatorname{Spec} \mathbf{F}_3[t] \arrow[d] \\
\operatorname{Spec} \mathbf{F}_q \arrow[r] & \operatorname{Spec} \mathbf{F}_3
\end{tikzcd}$$

Link to Diagram

Definition 35.0.10 (Weil Divisor)
The **Weil divisors** on X is the free $\mathbf{Z}$-module on the prime divisors, and is denoted $\operatorname{Div}(X)$.

Example 35.0.11 *(?)***:**

- $1[\langle x, y \rangle] \in \operatorname{Div}(\operatorname{Spec} k[x,y,z]/\langle xy - z^2 \rangle)$.
- $2[\langle z \rangle] - [\langle 3 \rangle] + 8[\langle 7 \rangle] \in \operatorname{Div}(\operatorname{Spec} \mathbf{Z})$.
- $[V(y^2 - x^3)] + 2[V(y)] \in \operatorname{Div}(\mathbf{A}^2_{/k})$
- $[0] - [\infty] \in \operatorname{Div}(\mathbf{P}^1_{/k})$.
- For C an irreducible reduced curve, any linear combination of closed points.

Note that *Cartier* divisors are those locally cut out by a single equation.

Remark 35.0.12: Since X is integral, it has a generic point η, so define a **rational function** as a nonzero element of $\mathcal{O}_{X,\eta}$. Equivalently, if $\operatorname{Spec} R \subseteq X$ is an affine chart, an element of $k^\times$ where $k = \operatorname{ff}(R)$. Note that this is independent of further localizing R! Any rational function φ on X gives an element $\varphi \in \operatorname{ff} \mathcal{O}_{X,x} \cong k$ for any point $x \in X$. In particular, the standing assumptions (specifically being regular in codimension 1) implies that $\mathcal{O}_{X,x}$ is a DVR when x is the generic point of a prime divisor. Let $Y \subseteq X$ be a prime divisor, then define $v_Y(\varphi)$ to be the valuation $v(\varphi)$ in $\mathcal{O}_{X,Y}$.

Example 35.0.13 *(?)***:** The element $4/7$ is a rational function on $\operatorname{Spec} \mathbf{Z}$, which is exactly $\mathbf{Q}^\times$. Moreover $4/7 \in \mathcal{O}_{\operatorname{Spec} \mathbf{Z}, 2}$ and $\operatorname{val}_{\mathfrak{m}}(4/7) = 2$ for $\mathfrak{m} = \langle 2 \rangle$.

Definition 35.0.14 (Divisors of functions, principal divisors, class groups)

$$\mathrm{Div}(\varphi) := \sum_{Y \subseteq X \text{prime}} v_Y(\varphi)[Y].$$

These are called **principal divisors**, and form a group $\mathrm{cl}(X)$ the **class group**.

Example 35.0.15 *(?):* $\mathrm{Div}(4/7) = 2[2] - 1[7]$.

36 | Monday, November 29

Remark 36.0.1: Standing assumption: $X \in \mathsf{Sch}$ is

- Integral: covered by $\operatorname{Spec} R$ for R integral domains
- Noetherian: covered by Noetherian rings.
- Separated: Δ is closed.
- Regular in codimension 1: $\dim \mathcal{O}_{X,x} = 1 \implies \mathcal{O}_{X,x}$ is regular and thus a DVR.

Definition 36.0.2 (Prime and Weil divisors)
A **prime divisor** $Y \subseteq X$ is a closed integral subscheme of codimension 1, and a **Weil divisor** is a formal $\mathbf{Z}$-linear combination $\sum_{1 \leq i \leq k} n_i Y_i$. The divisor is **effective** if $n_i \geq 0$ for all i.

Example 36.0.3 *(?):* If $\eta \in Y$ is the generic point, $\mathcal{O}_{X,\eta} := \mathcal{O}_{X,Y}$ is a local ring of dimension 1, and thus a DVR. This yields a valuation:

$$v_Y : \mathrm{ff}(\mathcal{O}_{X,Y})^\times = k^\times \to \mathbf{Z},$$

where k is the residue field of the generic point of X, also called the *rational functions on* X.

Definition 36.0.4 (Principal divisors)
If $f \in k^\times$ then there is an associated divisor:

$$\mathrm{Div}(f) := \sum_{Y \text{prime divisors}} v_Y(f)[Y].$$

Any divisor of a rational function is **principal**.

Example 36.0.5 *(?):* For $X := \operatorname{Spec} \mathbf{Z}$, the generic point is $\langle 0 \rangle$ and the prime divisors are prime ideals of height 1, so here just prime ideals $\langle p \rangle$. So the integral closed codimension subschemes correspond to primes $p \in \mathbf{Z}$, and there are valuations $v_p : \mathbf{Q}^\times \to \mathbf{Z}$. Write $k = \mathrm{ff}(\mathcal{O}_{X,\langle 0 \rangle}) = \mathbf{Q}$, then e.g. $\mathrm{Div}(4/7) = 2[2] - 1[7]$.

Example 36.0.6 *(?)*: Set $X := \mathbf{A}^2_{/\mathbf{C}}$ and $f(x,y) = x/y$. What is $v_{[V(x)]}(f)$? Then $\operatorname{Div}(f) = [V(x)] - [V(y)]$, and

$$v_{[V(x)]} : \mathbf{C}[x,y]\left[\langle x\rangle^{-1}\right] \to ?.$$

So the answer is 1.

Proposition 36.0.7 *(?)*.
$\operatorname{Div}(f)$ is well-defined, i.e. $v_f(Y) = 0$ for all but finitely many Y.

Proof (?).
Let $f \in k^\times = \operatorname{ff}(A)$ for $\operatorname{Spec} A \subseteq X$ an affine open. Write $f = a/b$ for some $b \in A \setminus \{0\}$, noting that A is a domain since we assumed X integral. Passing to $D(b)$, we can assume f is a regular function on some affine open $U \subseteq X$. Since $X \setminus U$ is a proper closed subset and X is Noetherian, it contains only finitely many prime divisors – each irreducible component has $\operatorname{codim}_X \geq 1$, and conversely any prime divisor must be an irreducible component, and Noetherian spaces have finitely many irreducible components. So it suffices to show $\operatorname{Div}(f)$ is well-defined for $f \in A$ when $X = \operatorname{Spec} A$. Just use that $V(f) \subseteq \operatorname{Spec} A$ is a proper closed subset, the same argument shows $V(f)$ contains finitely many prime divisors. Since $f \in A$, we have $f \in \mathcal{O}_{X,Y}$ and thus $v_Y(f) \geq 0$. Moreover if $v_Y(f) = 0$ then $Y \subseteq V(f)$ – use that $f \in pA\left[p^{-1}\right]$ and $pA\left[p^{-1}\right] \cap A = p$ to get $f \in p$.

■

Definition 36.0.8 (Divisor class groups)
The **divisor class group** of X is defined as

$$\operatorname{Cl}(X) = \operatorname{Div}(X) / \operatorname{Prin}\operatorname{Div}(X),$$

where $\operatorname{Prin}\operatorname{Div}(X) = \left\{\operatorname{Div} f \;\middle|\; f \in k^\times\right\}$. Since $v_Y(fg) = v_Y(f) = v_Y(g)$, $\operatorname{Prin}\operatorname{Div}(X)$ forms a subgroup of $\operatorname{Div}(X)$.

Example 36.0.9 *(?)*: Consider $X := \operatorname{Spec}\mathbf{Z}$, then

$$\operatorname{Div}(X) = \bigoplus_{p \text{ prime}} \mathbf{Z}[p].$$

Then $\operatorname{Prin}\operatorname{Div}(\operatorname{Spec}\mathbf{Z}) = \operatorname{Div}(X)$ by sending $\sum n_p[p] \to \operatorname{Div}(\prod_p p^{n_p})$, so $\operatorname{Cl}(\operatorname{Spec}\mathbf{Z}) = 0$.

Example 36.0.10 *(?)*: For $K \in \mathsf{NumberField}$ and $\mathcal{O}_K$ its ring of integers, we can consider $\operatorname{Cl}(\mathcal{O}_K)$. For example, $\operatorname{Cl}(\operatorname{Spec}\mathbf{Z}[\sqrt{-5}]) = C_2 = \langle 2, 1+\sqrt{-5}\rangle$, using the Dedekind domains admit unique factorization into prime ideals.

Proposition 36.0.11 *(?)*.
Let A be a Noetherian domain, then A is a UFD iff $\operatorname{Cl}(\operatorname{Spec} A) = 1$ is trivial.

Proof (?).
Use the lemma that A is a UFD $\iff$ every height 1 prime ideal is principal. Note that $\langle 2, 1+\sqrt{-5}\rangle$ is height 1 but not principal!
$\implies$: Let $Y \subseteq X = \operatorname{Spec} A$ be a prime divisor, so $Y = V(p)$ for p a height 1 prime ideal, so we can write $V(p) = V(f)$ for some f. Then $\operatorname{Div}(f) = [Y]$, and any prime divisor is principal, and now just use that $[Y]$ generate $\operatorname{Div}(X)$.
$\impliedby$: Suppose $\operatorname{Cl}(X) = 0$ and let $Y \subseteq X$ be a prime divisor with $\operatorname{Div}(f) = Y$ for some $f \in k^\times$. We want to show $V(f) = Y$. If $\operatorname{Div}(f) = [Y]$, then for all $Y' \subseteq X$ prime divisors we have $v_{Y'}(f) \geq 0$. By ring theory, $f \in A$. If $V(f) = 1$ then $f \in pA\left[p^{-1}\right]$ for $p = V(y)$, so $f \in pA\left[p^{-1}\right] \cap A$ and thus $f \in p$. The claim is that $p = \langle f\rangle$ – suppose $g \in p$, then $v_{Y'}(g) \geq 0$ and $v_Y(g) \geq 1$ implies $v_{Y'}(g/f) \geq 0$ for all Y'. So $g/f \in A$, making $g \in \langle f\rangle$. ■

Remark 36.0.12: Use that valuations are non-negative on prime divisors and that the valuations are either 0 or 1.

37 | Wednesday, December 01

Remark 37.0.1: Recall:

- $X \in \mathsf{Sch}$ is Noetherian, integral, separated, regular in codimension 1,
- $Y \subseteq X$ a prime divisor is an integral closed codimension 1 subscheme,
- $\operatorname{Div}(X) = \mathbf{Z}[\{\text{Prime divisors}\}]$,
- $\operatorname{Prin}\operatorname{Div}(X) = \left\{\operatorname{Div}(f) \mathrel{\Big|} f \in k^\times, \text{ the rational functions on } X\right\}$ where $\operatorname{Div}(f) \coloneqq \sum_Y v_Y(f)[Y]$,
- $\operatorname{Cl}(X) \coloneqq \operatorname{Div}(X)/\operatorname{Prin}\operatorname{Div}(X)$.

We proved that if $X = \operatorname{Spec} A$,

$$\operatorname{Cl}(X) = 1 \iff A \text{ is a UFD.}$$

This shows that any height 1 prime ideal contained in A is principal. The key commutative algebra fact was

$$\bigcap_{\operatorname{ht}(p)=1} A\left[p^{-1}\right] = A.$$

.

Remark 37.0.2: A quick review of why $\operatorname{Cl}(X) = 1 \implies$ every $p \in \operatorname{Spec} A$ with $\operatorname{ht}(p) = 1$ is principal. Let $Y = V(p)$, then $[Y] \in \operatorname{Cl}(X) = 1$ means that $[Y] = \operatorname{Div}(\varphi)$ for some $\varphi \in k^\times$. Since $v_{Y'} \geq 0$ for all Y' (since they're just zero for $Y \neq Y'$) implies that $f \in A$, and the claim is that

$\langle\varphi\rangle = p$. Taking $f \in p$, then $\operatorname{Div}(f) = [Y] + \varepsilon$ where $\varepsilon \geq 0$ is effective. Then $\operatorname{Div}(f/\varphi) \geq 0$ is effective, i.e. $f/\varphi \in A\left[p'^{-1}\right]$ for all p'. But then $f/\varphi \in \bigcap_{\operatorname{ht}(p)=1} A\left[p^{-1}\right] = A$, so $f \in \langle\varphi\rangle$. So $p = \langle\varphi\rangle$.

Proof (of the commutative algebra fact).
To show $\bigcap_{\operatorname{ht}(p)=1} A\left[p^{-1}\right] = A$, let $a/b \in k^\times$, then $\dim A/\langle b\rangle = \dim A - 1$. Why? Let $\bar{p}_0 \subset \cdots \subset A/\langle b\rangle$, then $q^{-1}(\bar{p}_0) = p \ni b$.

■

Remark 37.0.3: The geometric analog: if $X \coloneqq \operatorname{Spec} A$ and $V(p)$ is a variety, intersecting with a hyperplane yields a codimension 1 locus (in nice cases).

Example 37.0.4 ***(Affine space):*** For $k \in \mathsf{Field}$ not necessarily algebraically closed,

$$\operatorname{Cl}(\mathbf{A}^n_{/k}) = 1.$$

Proof: k is a UFD, so $k[x_1, \cdots, x_n]$ is a UFD, so apply the proposition.

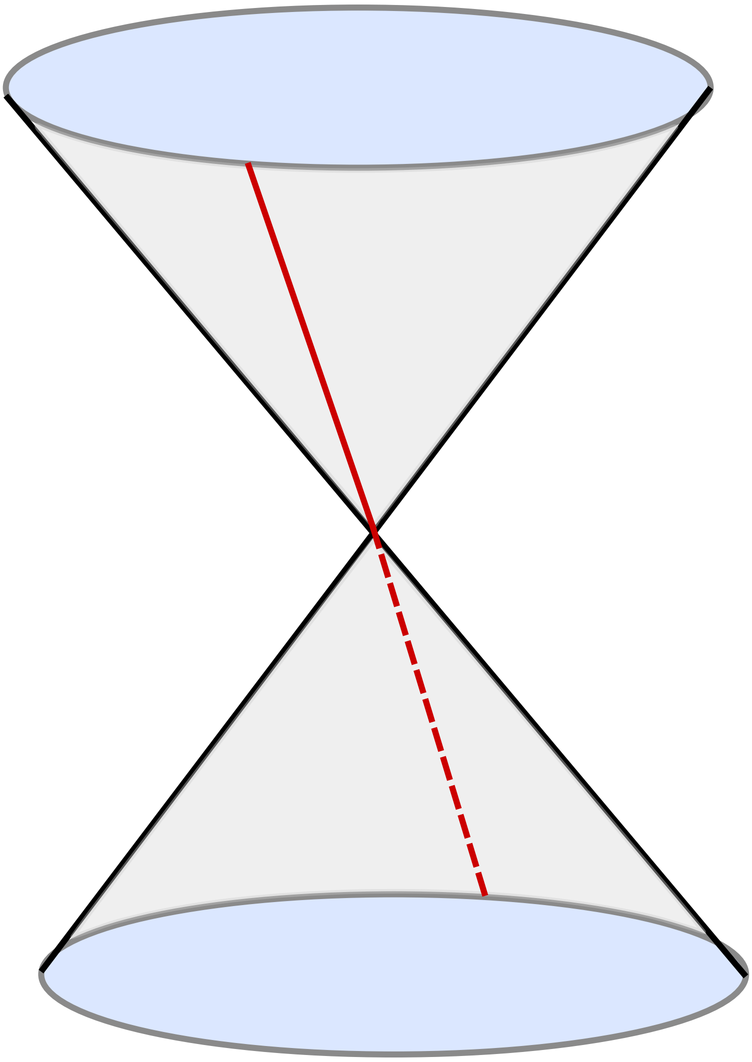
Example 37.0.5 ***(Number fields):*** For $K \in \mathsf{NumberField}$ and $\mathcal{O}_K$ its ring of integers,

$$\operatorname{Cl}(\operatorname{Spec} \mathcal{O}_K) = 1 \iff \mathcal{O}_K \text{ is a UFD},$$

and this coincides with $\operatorname{Cl}(\mathcal{O}_K)$ from number theory.

Example 37.0.6 ***(A geometric non-example):*** If $X \coloneqq V(xy - z^2) \subseteq \mathbf{A}^3_{/k}$, then $\operatorname{Cl}(X) \neq 1$ since $xy = zz$ in A, exhibiting failure of unique factorization. How to find an irreducible subscheme:

$$X = V(xy - z^2)$$



$V(x,z)$

We use that $\langle x, z\rangle \in k[x,y,z]/\langle xy - z^2\rangle$ is not principal. Note that if $Y = V(p)$ then $2[Y] = 0$ in $\mathrm{Cl}(X)$: show that $x \in A[p^{-1}]$, then $v_p(x) = 2$.

38 | Friday, December 03

Theorem 38.0.1 *(?).*
For $X := \mathbf{P}^n_{/k}$ and $D \in \mathrm{Div}(X)$, define

$$\deg D := \sum n_i \deg Y_i \qquad \text{where } D = \sum n_i [Y_i].$$

Let $H := \{x_0 = 0\}$ by a hyperplane, then

- $D \sim \deg(D)H$
- $f \in k^\times \implies \deg(f) = 0$,
- $\deg : \mathrm{Cl}(X) \to \mathbf{Z}$ is an isomorphism.

Proof (?).
Missed, see Hartshorne.

■

Proposition 38.0.2 *(?).*
For $Z \subseteq X$ proper and closed with $U := X \setminus Z$, if $\operatorname{codim} Z = 2$, then $\mathrm{Cl}(X) \cong \mathrm{Cl}(U)$. If Z is irreducible and $\operatorname{codim} Z = 1$, there is an exact sequence $\mathbf{Z} \xrightarrow{f} \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0$ where $f(1) = [Z]$.

Remark 38.0.3: Note that this $\mathbf{Z} \to \mathrm{Cl}(X)$ isn't injective in general: take $X := \mathbf{A}^n$ so $\mathrm{Cl}(X) = 1$.

Proof (?).
Define a map

$$\begin{aligned} \varphi : \mathrm{Div}(X) &\to \mathrm{Div}(U) \\ Y &\mapsto \begin{cases} Y \cap U & Y \cap U \neq \emptyset \\ 0 & Y \cap U = \emptyset. \end{cases} \end{aligned}$$

Then φ is $\mathbf{Z}$-linear, and $k^\times(X) \cong k^\times(U)$, and descends to a map $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$. Moreover $\ker \varphi$ is generated by prime divisors contained in Z, so if $\operatorname{codim} Z \geq 2$ this is empty and we have an isomorphism. Otherwise if $\operatorname{codim} Z = 1$ with Z irreducible, then the only prime divisors in Z is Z itself, so Z generates $\ker \varphi$.

■

Example 38.0.4 *(?):* Let $Z \subseteq \mathbf{P}^2$ be an irreducible degree d curve and let $U = \mathbf{P}^2_{/k} \setminus \{0\}$. Then

$\mathrm{Cl}(U) = C_d$ is cyclic of order d. We have

$$\begin{aligned} \mathbf{Z} &\to \mathrm{Cl}(\mathbf{P}^2_{/k}) \cong \mathbf{Z} \to \mathrm{Cl}(U) \to 0 \\ 1 &\mapsto [Z] \cong \deg d. \end{aligned}$$

38.1 Divisors on Curves

Definition 38.1.1 (Curve)
Let $k = \bar{k}$, then a **curve** $X \in \mathsf{Sch}_{/k}$ is an integral (so reduced) separated of finite type of dimension 1. We say X is **complete** if $X \to \operatorname{Spec} k$ is proper, and **smooth** if X is regular (equivalently regular in codimension 1).

Example 38.1.2 *(?)*: $X := V(f) \subseteq \mathbf{P}^2_{/k}$ where f is irreducible. This is complete since it is closed in $\mathbf{P}^2_{/k}$, which is proper.

Proposition 38.1.3 *(?)*.
If $f : X \to Y$ is a morphism of curves and X is complete and nonsingular, then $\operatorname{im} f$ is either a point or all of Y. If $\operatorname{im} = Y$, then f is finite.

Proof (?).
X proper implies $f(X)$ is closed in Y, and X irreducible implies $f(X)$ is irreducible. Since Y is irreducible, this forces $f(X) = \mathrm{pt}$ or Y. Let $V \subseteq Y$ be an affine open and $U := f^{-1}(V)$, the claim is that U is affine and the pullback $f^* : \mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is a module-finite extension. We have a map on function fields $f^* : k(X) \to k(Y)$, and since $\dim X, \dim Y = 1$, these are fields of transcendence degree 1 over k. Therefore f^* is a finite extension of fields (use Noether normalization), and $\mathcal{O}_Y(V) \subseteq k(Y)$. We can write $\mathcal{O}_Y(V) = \cap_{p \in V} \mathcal{O}_{Y,p}$. The following gives module-finiteness:

Claim: The integral closure of $\mathcal{O}_Y(V)$ in $k(X)$ is $\mathcal{O}_X(U)$.

To see that U is affine: exercise! ■

39 Curves and Divisors: Ramification and Degree (Monday, December 06)

Remark 39.0.1: Recall that we defined a curve as a 1-dimensional integral separated scheme of finite type over an algebraically closed field. Here nonsingular corresponds to regular, and complete corresponds to proper. We were proving the following:

Proposition 39.0.2 *(?).*
If $f : X \to Y$ is a morphism of curves with X complete and nonsingular, then

- $f(X) = \text{pt}$ or all of Y
- If $f(X) = Y$, then f is finite and $f^* : K(Y) \to K(X)$ is a finite extension of fields.

Remark 39.0.3: If $\operatorname{Spec} B \hookrightarrow Y$ is an affine open, then defining A as the integral closure of B in $K(X)$ we get $\operatorname{Spec} A \hookrightarrow X$ and $\operatorname{Spec} A = f^{-1}(\operatorname{Spec} B)$. This relies on X being complete and nonsingular – prove this as an exercise.

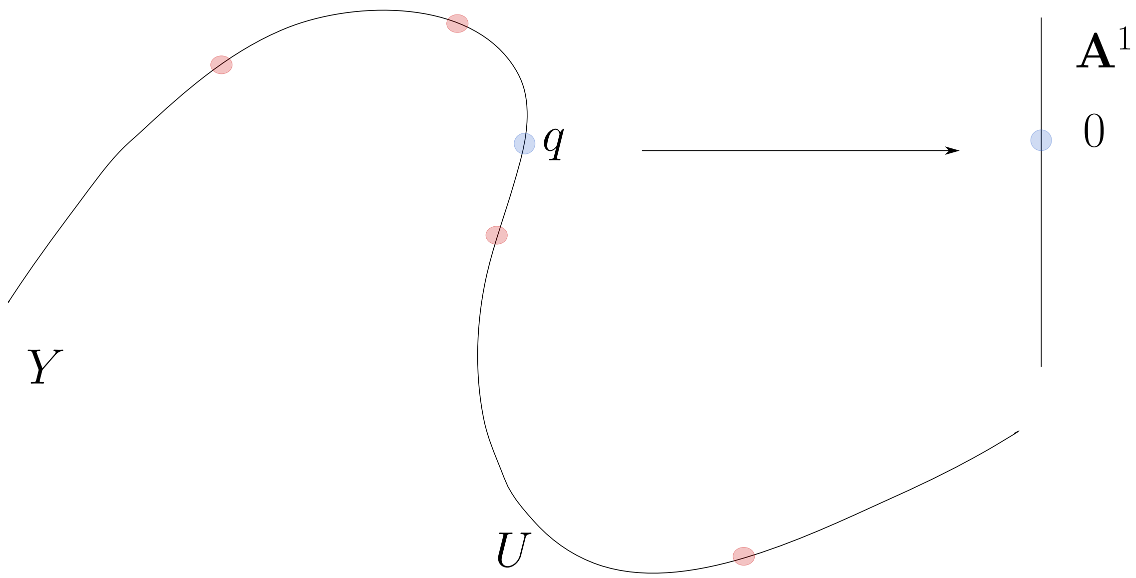
Definition 39.0.4 (Degree of a surjective morphism of curves)
Let $f : X \to Y$ as before and suppose f is surjective. The **degree** of f is defined as

$$\deg f := [K(X) : K(Y)].$$

Remark 39.0.5: Define a pullback of divisors

$$f^* : \operatorname{Div}(Y) \to \operatorname{Div}(X),$$

defined on closed points (and extended $\mathbf{Z}$-linearly) as follows: let $q \in Y$ be a closed point, then since Y regular in codimension 1 there exists a generator $t \in K(Y)$ such that $t \in \mathcal{O}_{Y,q}$ and $\mathfrak{m}_q = \langle t \rangle$. We'll call t a **local parameter** at q. Take an open $U \ni q$ where t is regular and $V(t) = \{q\}$.



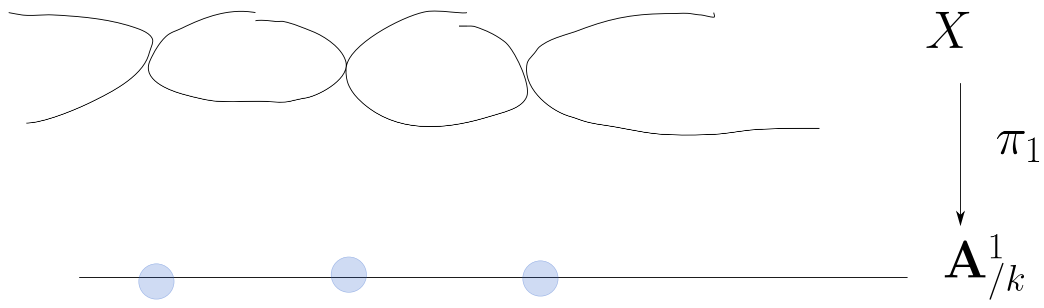
Write $f^*t \in \mathcal{O}_X(f^{-1}(U))$, then

$$\operatorname{Div}_{f^{-1}(U)} f^*(t) := f^*[q] = \sum_{f(p)=q} v_p(f^*t)[p].$$

Example 39.0.6 *(?)*: Consider $C = \{y^2 = x^3 + ax + b\}$ and $X \coloneqq V(C)$, and take the projection

$$\begin{aligned} X &\to \mathbf{A}^1_{/k} \\ (x, y) &\mapsto x. \end{aligned}$$

Assume the discriminant $\Delta(a, b) \neq 0$ so X is nonsingular and the roots of $f(x) \coloneqq x^3 + ax + b$ are distinct:



Roots of $f(x)$

Consider $(a, b) = (0, -1)$ so $y^2 = x^3 - 1$ and work over $\mathbf{C}$. Check that $(x, y) = (0, \pm i)$ are solutions, and we can write

$$f^*[0] = \sum_{f(p)=0} v_p(f^*x) = 1 \cdot [(0, i)] + 1 \cdot [(0, -i)],$$

since $\mathfrak{m}_p = \langle x, y \mp i \rangle = \langle x \rangle$. Similarly,

$$f^*[1] = v_{(1,0)} \langle x - 1 \rangle = 2[(1, 0)],$$

so the function x is not a local coordinate at 1, but y is. Consider $\left(\frac{k[x,y]}{\langle y^2 - x^3 - +1 \rangle} \right) [\langle x-1, y \rangle^{-1}]$; then $\mathfrak{m} = \langle x - 1, y \rangle$ and we can factor $y^2 = x^3 - 1 = (x - 1)(x^2 + x + 1)$, and we can invert to write $x - 1 = \frac{y^2}{x^2+x+1} \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$.

Remark 39.0.7: The punchline: even though the size of the set-theoretic fibers changed, in both cases we pulled back degree 1 divisors and got degree 2 divisors, and this is evidently a 2-to-1 cover. Note that this example wasn't complete, but we can take the projective closure by homogenizing to get $V(y^2z =^3 + axz^2 + bz^3)$, and we can extend our map $\pi : X \to \mathbf{A}^1$ to $\tilde{\pi} : \tilde{X} \to \mathbf{P}^1$ by mapping the new point to ∞.

Definition 39.0.8 (Ramification, branching)
Let $f : X \to Y$ be a morphism of smooth complete curves with $f(X) = Y$. A **ramification point** of f is a point $p \in X$ where $e_p(f) \coloneqq v_p(f^*t) > 1$ for t a local parameter at $q = f(p)$.

Such a point q is said to be a **branch point**. The **ramification divisor** of f is defined as

$$R_f := \sum_{p \in X} (e_p(f) - 1)\,[p].$$

Remark 39.0.9: This is a finite sum: show that for all but finitely many points (i.e. a Zariski open), the pullback of a local parameter will again be a local parameter on the cover, potentially after subtracting a constant to shift the image to 0. More precisely, for any $f \in K(X)$, $f - f(p)$ will be a local parameter at p for a Zariski open.

Proposition 39.0.10 *(?).*
Let $f : X \to Y$ be a nonconstant morphism of smooth complete curves, then

$$\deg f^* D = \deg f \cdot \deg D,$$

where $\deg\left(\sum n_p[p]\right) = \sum n_p$.

Proof (?).
The 30s version: write $\operatorname{Spec} V \subset Y$ with A defined as the integral closure of B in $K(X)$. Then $B \to A$ is module-finite of dimension $\deg f = [K(X) : K(Y)]$. Taking there is an induced map on the local ring

$$\begin{aligned} B\left[q^{-1}\right] &\to \bigoplus_{f(p)=q} A\left[p^{-1}\right] \\ t &\mapsto \bigoplus f^* t. \end{aligned}$$

Then

$$\dim\left(\bigoplus A\left[p^{-1}\right]/t \bigoplus A\left[p^{-1}\right]/B\left[q^{-1}\right]/tB\left[q^{-1}\right]\right) = \dim\left(\bigoplus_{f(p)=q} k[t]/t^{e_p(f)}/k\right) = \deg f.$$

Note that this uses CRT: $A/tA \cong \bigoplus_{f(p)=q} A\left[p^{-1}\right]/tA\left[p^{-1}\right]$. ■

40 | Tuesday, December 07

Proposition 40.0.1 *(?).*
For $f \in K(C)^\times$ for C a smooth complete curve, $\deg(\operatorname{Div} f) = 0$.

Proof (?).
If f is constant this is trivial, so assume not. Define

$$U_1 := \left\{ p \in C \,\middle|\, v_p(F) \geq 0 \right\}$$
$$U_2 := \left\{ p \in C \,\middle|\, v_p(F) \leq 0 \right\}.$$

Then $U_1 \cap U_2$ is precisely the set of closed points of C. Suppose f is regular on U_1, so $1/f$ is regular on U_2. Define a map $\widehat{f} : C \to \mathbf{P}^1$ by writing $\mathbf{P}^1 = \mathbf{A}_1 \cup \mathbf{A}_1$ and defining $f|_{U_i} : U_i \to \mathbf{A}^1$ to map into the ith factor. Note that $\widehat{f}\Big|_{U_2} = 1/f$. Then

$$\operatorname{Div}(f) = \sum_{p \in C} v_p(f)[p] = f^* \left([0] - [\infty]\right) := f^*(D).$$

Then $\deg(\operatorname{Div}(f)) = \deg f \cdot \deg(D) = 0$ since $\deg(D) = 0$, noting that $\deg f$ is the degree of the corresponding field extension.

■

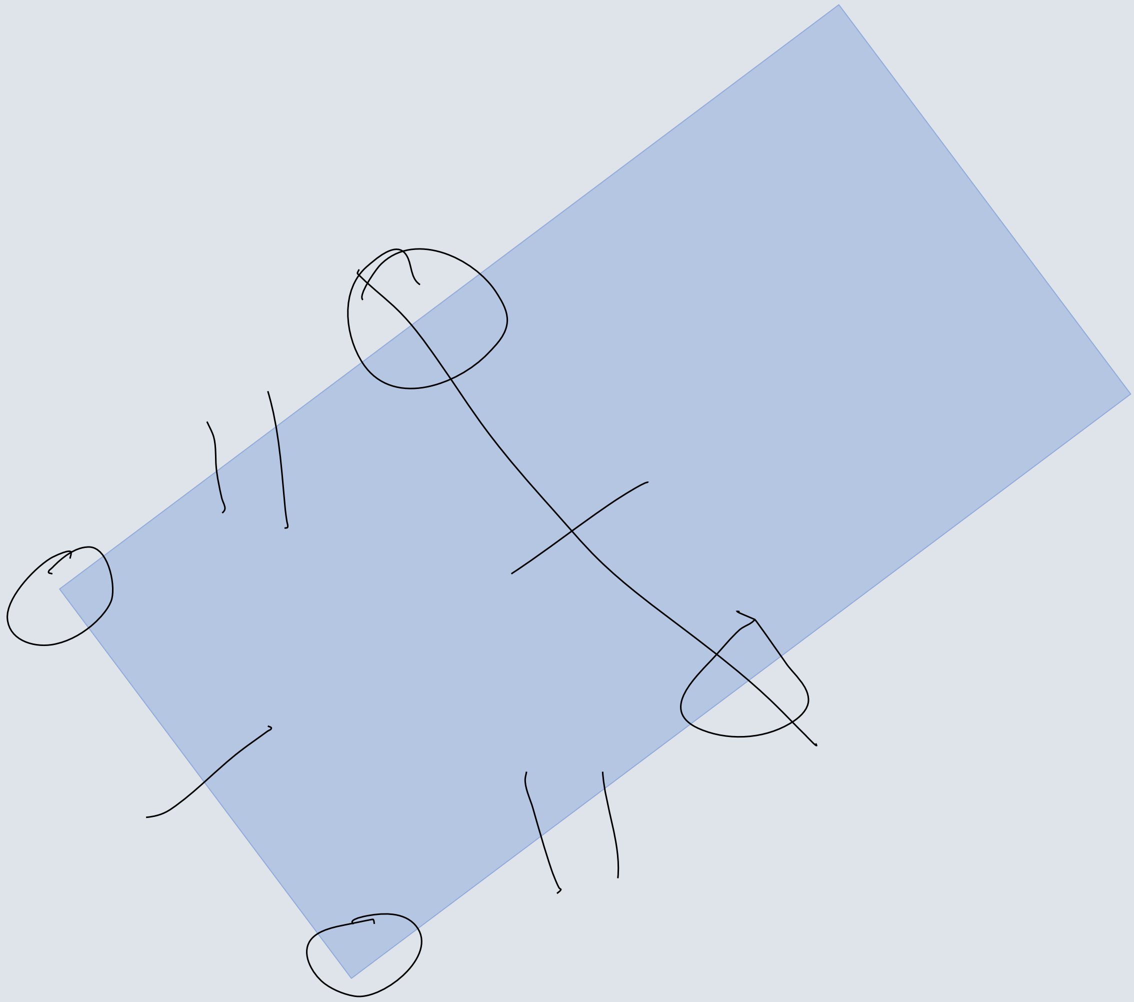
Corollary 40.0.2 *(?).*
For a smooth complete curve, the degree map descends to a well-defined map:

$$\begin{aligned} \deg : \operatorname{Cl}(C) &\to \mathbf{Z} \\ \sum n_p p &\mapsto \sum n_p. \end{aligned}$$

Definition 40.0.3 (Cl_0)
Define

$$\operatorname{Cl}_0(C) := \ker\left(\operatorname{Cl}(C) \xrightarrow{\deg} \mathbf{Z}\right).$$

Definition 40.0.4 (Elliptic curve)
An **elliptic curve** E is a smooth complete genus 1 curve over k with a distinguished closed point $0 \in E$ called the origin. Complex analytically, $E = \mathbf{C}/\Lambda$ where $\Lambda \subset \mathbf{C}$ is an integral lattice with $\Lambda \otimes_{\mathbf{Z}} \mathbf{R} \cong \mathbf{C}$, so the basis vectors are independent.



Example 40.0.5 ***(?):*** The cubic in $\mathbf{P}^3_{/k}$ defined by

$$C \coloneqq V(zy^2 = x^3 + axz^2 + bz^3), \quad 0 \coloneqq [0 : 1 : 0].$$

Definition 40.0.6 (Weierstrass $\wp$ function)
Define a complex analytic function

$$\begin{aligned} \wp : \mathbf{C} &\to \mathbf{P}^1 \\ z &\mapsto \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} = z^{-2} + \mathsf{O}(z^2). \end{aligned}$$

Remark 40.0.7: Note that

- $\wp(z+\lambda) = \wp(z)$ for all $\lambda \in \Lambda$
- $\wp : E \to \mathbf{P}^1$, so $\wp \in \mathbf{C}(E)^\times$

- $\wp'(z) = \sum_{\lambda \in \Lambda} \frac{-2}{(z-\lambda)^3} \in \mathbf{C}(E)^\times = -\frac{2}{z^3} + \mathsf{O}(z)$.
- $\wp'(z)^2 = \frac{4}{z^6} + \frac{c_1}{z^2} + \mathsf{O}(1)$
- $\wp(z)^3 = z^{-6} + c_2 z^{-2} + \mathsf{O}(1)$

So there is a relation

$$F: \quad \wp'(z)^2 = 4\wp(z)^3 + G_2(\Lambda)\wp(z) + G_3(\Lambda) + \mathsf{O}(z).$$

Note that this cancels the poles at the lattice points, making it a bounded holomorphic function and thus constant. Since it's $\mathsf{O}(z)$, this forces it to be zero. So define a map

$$\begin{aligned} E &\to \mathbf{P}^2_{/\mathbf{C}} \\ z &\mapsto [\wp(z) : \wp'(z) : 1], \end{aligned}$$

and note that $0 \mapsto [0:1:0]$ so this factors through $V(zy^2 = 4x^3 + G_2(\Lambda)xz^2 + G_3(\Lambda)z^3)$ biholomorphically, using that $\deg \wp, \wp' = 2, 3$ to get injectivity. This makes E an algebraic variety.

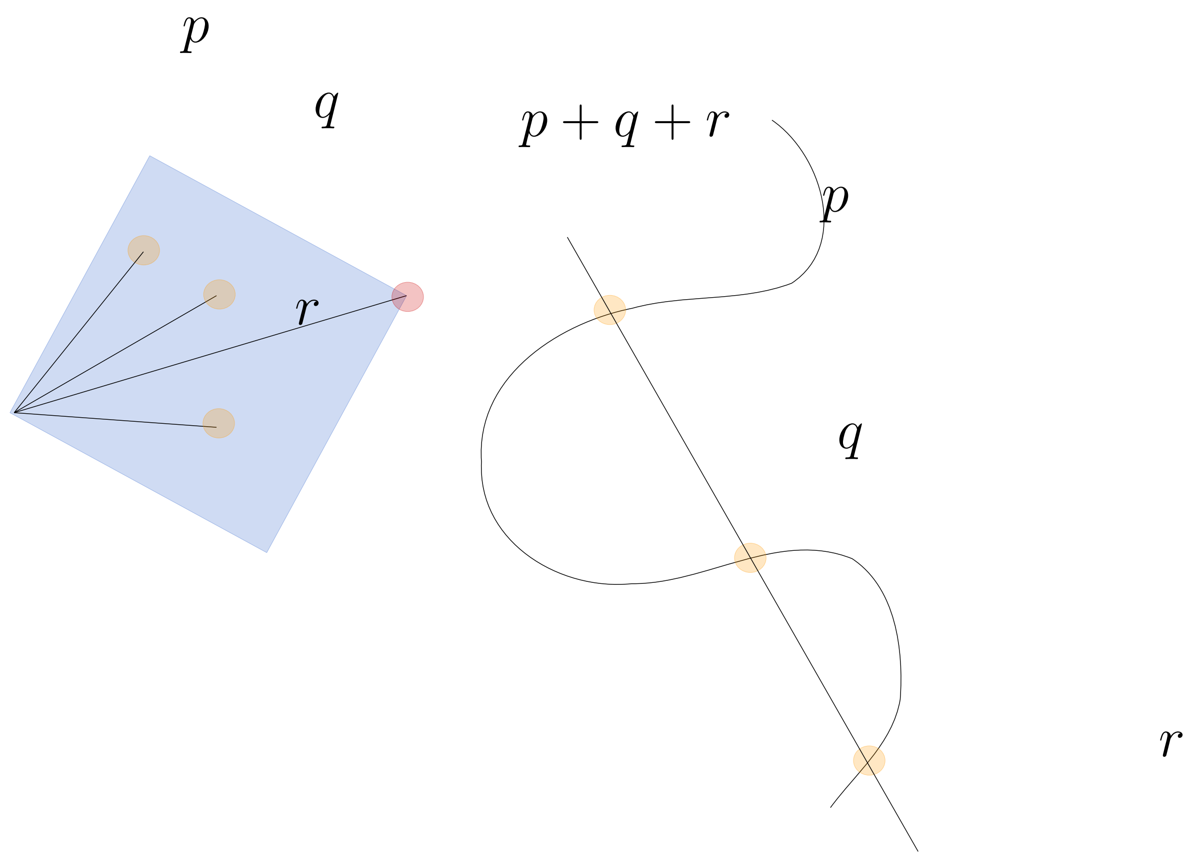
Remark 40.0.8: An aside: suppose $f : C_1 \to C_2$ is a degree 1 holomorphic map of compact complex curves. Then $f' = 0$ at only finitely many points, so f is invertible on an open set and f^{-1} extends continuously to C_2. Now use the Riemann removable singularity theorem: extending a holomorphic function continuously over a puncture implies that the new function is holomorphic.

Remark 40.0.9: Why is this algebraic structure unique? Use an overpowered theorem: Serre's GAGA, i.e. there is a unique variety structure on a compact complex manifold over $\mathbf{C}$. In our case, it suffices to show $\wp(z-c), \wp'(z-c) \in \mathbf{C}(\wp, \wp')$. In fact, the rational functions are given by $K(X) = \operatorname{ff}\left(\mathbf{C}[\wp, \wp'] / \langle \text{a cubic} \rangle\right)$.

Remark 40.0.10: Write E for the vanishing locus of the cubic F above, and consider a map

$$\mathbf{C}/\Lambda \xrightarrow{\sim} E \subseteq \mathbf{P}^2_{/\mathbf{C}}.$$

Consider a line $L \subseteq \mathbf{P}^2_{/\mathbf{C}}$, then $L \cap C = p + q + r = \operatorname{Div}(L)$ generically. We claim $p + q + r \equiv 0 \bmod \Lambda$.



To prove this: write $L_0 \coloneqq V(z)$, so $\operatorname{Div}(L_0) = 3[0]$, and consider $L_0 \cap V(F)$. We have $\operatorname{Div}(L/L_0) = [p] + [q] + [r] - 3[0]$.

Claim: If $f \in K(E)^\times$ and $\operatorname{Div} f = \sum n_p[p]$ then $\sum n_p p \equiv 0 \bmod \Lambda$ after taking these as honest points in $\mathbf{C}$.

To prove this, do some kind of contour integral over the fundamental domain and use lattice periodicity of f. This yields $p + q + r - 3 \cdot 0 \in \Lambda$, so given any two points we can solve for the third.

Remark 40.0.11: This can be used as a reduction algorithm:

$$\begin{aligned} [p_1] + [p_2] + 2[p_3] - 4[p_4] &= [p_1 + p_2] + [0] + 2[p_3] - 4[p_4] \in \operatorname{Div}^0(E) \\ &= 2[p_3 - p_4] - 2[p_4] - 2[0] \\ \cdots &= [p] - [0] \implies, \end{aligned}$$

so there is an isomorphism

$$\begin{aligned} E &\to \operatorname{Cl}^0(E) \\ p &\mapsto [p] - [0]. \end{aligned}$$

41 | Appendix

41.1 Notation

- $\mathrm{ff}(R)$ denotes the fraction field (or field of quotients) of R.
- $R[S^{-1}]$ is the ring R localized at the multiplicative set $S \subseteq R$, i.e. the subset of the fraction field $\mathrm{ff}(R)$ with denominators only in S. This differs from the usual notation $S^{-1}R$.
- $\mathbf{Z}_{\widehat{p}}$ is the p-adic integers, i.e. the ring $R = \mathbf{Z}$ completed at the ideal $\langle p \rangle$. This differs from the usual notation $\mathbf{Z}_p$.
- $R[\![t]\!]$ is the topological ring of formal power series in t, i.e. infinite sums $\sum_{i \geq 0} r_i t^i$ with the t-adic topology.
- $R(\!(t)\!)$ is the topological ring of formal Laurent series, i.e. half-infinite sums $\sum_{i \geq -N} r_i t^i$.
  - Note that $R(\!(t)\!) = R[\![t]\!][S^{-1}]$ where $S = \{1, x, x^2, \cdots\}$. If R is a field, $R(\!(t)\!) = \mathrm{ff}(R[\![t]\!])$.

41.2 Facts

Remark 41.2.1: Some useful facts:

- The equalizer diagram for a sheaf $\mathcal{F}$:

$$\emptyset \longrightarrow \mathcal{F}(U) \rightrightarrows \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i<j \in I} \mathcal{F}(U_{ij}) \cdots$$

- The inverse image / pushforward ("direct image") adjunction:

$$\mathsf{Sh}(X) \underset{f^{-1}}{\overset{f_*}{\underset{\perp}{\rightleftarrows}}} \mathsf{Sh}(Y) \implies \mathsf{Sh}(X)(f^{-1}\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \mathsf{Sh}(Y)(\mathcal{G}, f_*\mathcal{F}).$$

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