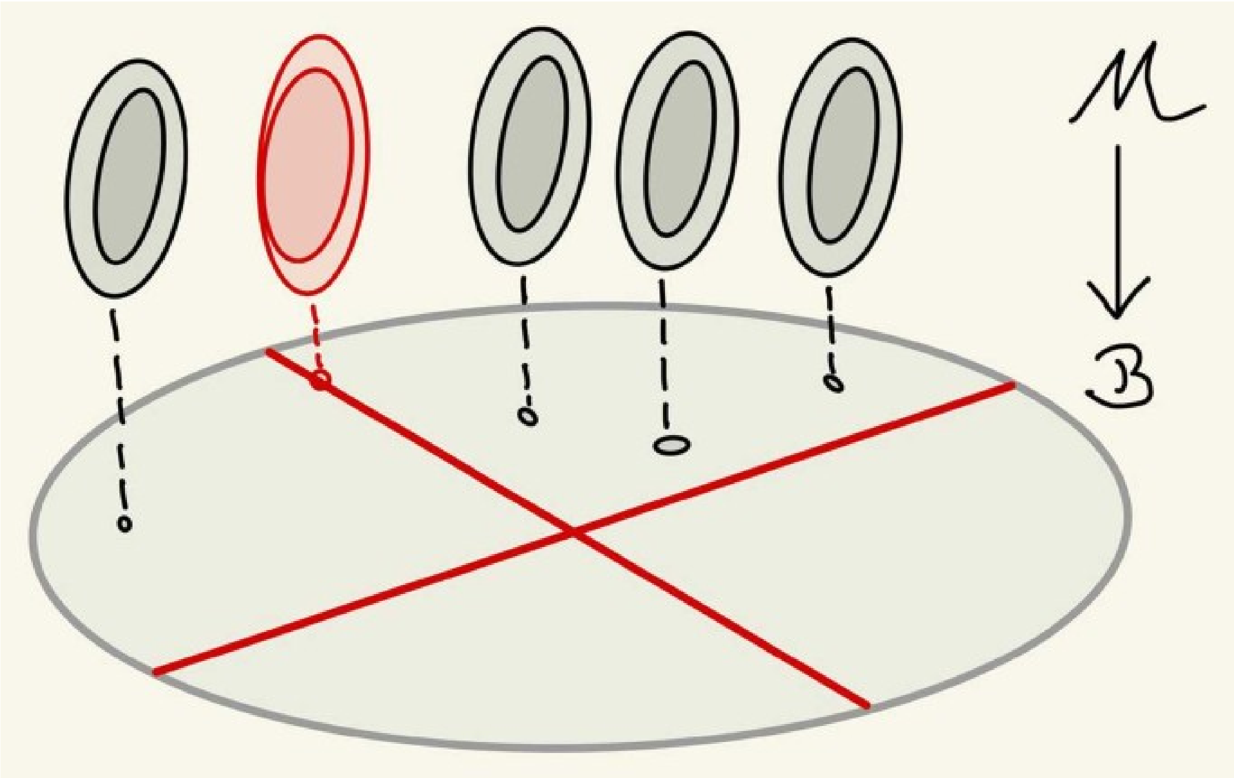


Notes: These are notes on an online graduate course in stacks by Jarod Alper in Spring 2021. As such, any errors or inaccuracies are almost certainly my own.

Introduction to Stacks and Moduli

Lectures by Jarod Alper. University of Washington, Spring 2021

D. Zack Garza
University of Georgia
dzackgarza@gmail.com

Last updated: 2021-09-06

Table of Contents

Contents

1 Friday, July 30

References:

- Course website: https://sites.math.washington.edu/~jarod/math582C.html
- Gómez 99: Expository article on algebraic stacks

Remark 1.0.1: Stated goal of the course: prove that the moduli space $\overline{\mathcal{M}_g}$ of stable curves (for $g \geq 2$) is a smooth, proper, irreducible Deligne-Mumford stack of dimension $3g - 3$. Moreover, it admits a projective coarse moduli space.

In the process we'll define **algebraic spaces** and **stacks**.

Prerequisites:

- Schemes
- Existence of Hilbert schemes
- Artin approximation
- Resolution of singularities for surfaces
- Deformation theory

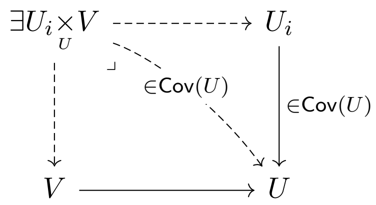
2 Lecture 3: Groupoids and Prestacks (Monday, September 06)

2.1 Groupoids

Remark 2.1.1: Last time: functors, sheaves on sites, descent, and Artin approximation. Today: groupoids and stacks.

Recall that a **site** S is a category such that for all $U \in \mathrm{Ob}(\mathsf{S})$, there exists a set $\mathsf{Cov}(U) := \{U_i \to U\}_{i \in I}$ (a *covering family*) such that

- $\mathrm{id}_U \in \mathsf{Cov}(U)$,
- $\mathsf{Cov}(U)$ is closed under composition.
- $\mathsf{Cov}(U)$ is closed under pullbacks:



Link to Diagram

Example 2.1.2 ***(The big étale site):*** Take $\mathsf{S} \coloneqq \mathsf{Sch}_{\text{Ét}}$ to be the big étale site: the category of all schemes, with covering families given by étale morphisms $\{U_i \to U\}_{i \in I}$ such that $\coprod_i U_i \twoheadrightarrow U$. Note that there is a special covering family given by *surjective* etale morphisms.

Reducing to case of single surjective etale cover somehow?

Definition 2.1.3 (Sheaves on sites)
Let C be a category (e.g. $\mathsf{C} \coloneqq \mathsf{Set}$) and recall that a *presheaf* on a category S is a contravariant functor $\mathsf{S} \to \mathsf{C}$.
A C-valued **sheaf** on a site S is a presheaf

$$\mathcal{F} : \mathsf{S} \to \mathsf{C}$$

such that for all $U_i, U_j \in \mathsf{Cov}(U)$, the following equalizer diagram is exact in C

$$0 \longrightarrow F(U) \Longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \underset{U}{\times} U_j)$$

Exercise 2.1.4 (Criterion for sheaves on the big etale site)
Show that a presheaf F is a sheaf on $\mathsf{Sch}_{\text{Ét}}$ iff

- F is a sheaf on $\mathsf{Sch}_{\text{Zar}}$ and
- For all etale surjections $U' \twoheadrightarrow_{\text{ét}} U$ of affines, the equalizer diagram is exact.

Proposition 2.1.5 ***(Yoneda).***
For $X \in \mathsf{Sch}$, the presheaf

$$h_X \coloneqq \operatorname{Mor}(-, X) : \mathsf{Sch} \to \mathsf{Set}$$

is a sheaf on $\mathsf{Sch}_{\text{Ét}}$.

Remark 2.1.6: We'll often consider *moduli functors*: functors $F : \mathsf{Sch} \to \mathsf{Set}$ where $F(S)$ is a family of objects over S. Then F will be a sheaf iff families glue uniquely in the étale topology, and representability of such functors will imply they are sheaves.

Example 2.1.7 ***(A non-sheaf):*** Consider the following moduli functor:

$$F_{\mathsf{Alg}} : \mathsf{Sch} \to \mathsf{Set}$$

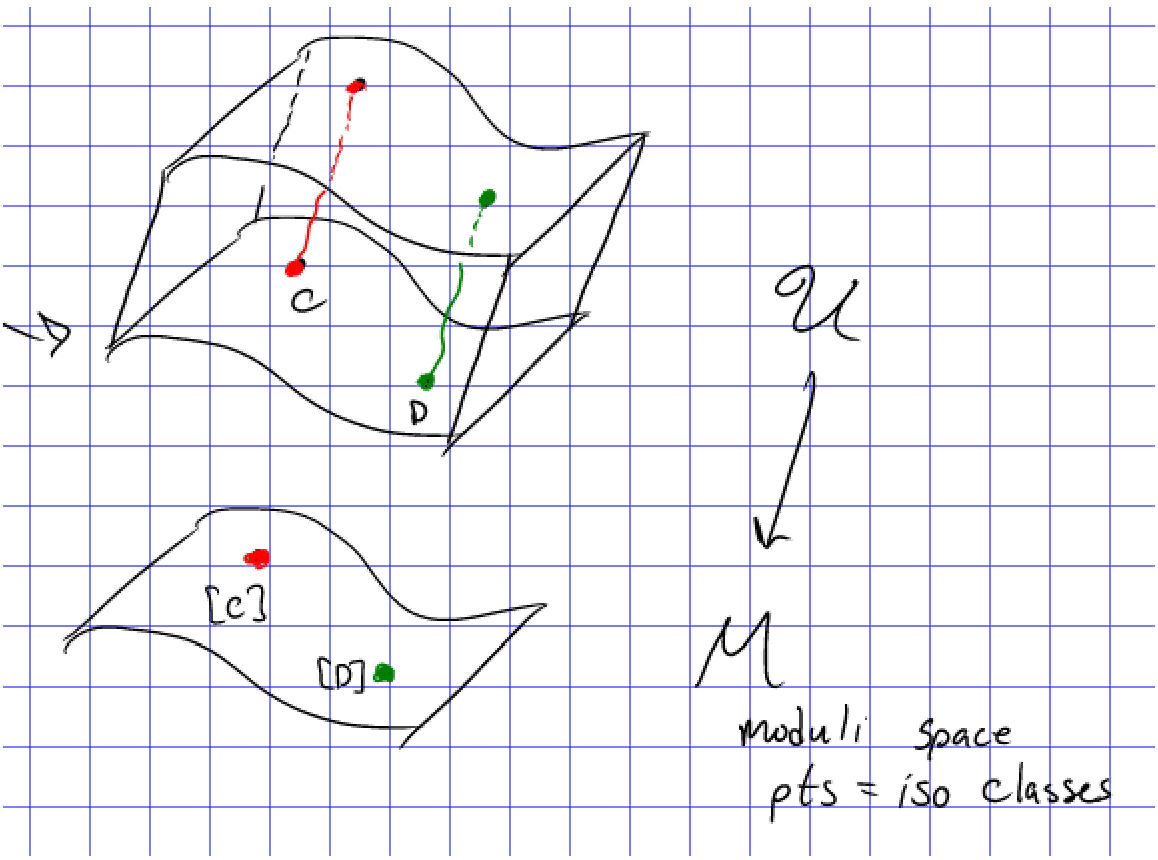
$$S \mapsto \left\{ \begin{array}{c} \mathcal{C} \\ \downarrow \\ S \end{array} \quad \text{Smooth families of genus } g \text{ curves.} \right.$$

This is *not* representable by a scheme and not a sheaf.

Remark 2.1.8: Why care about representability? Suppose there were a scheme M, so

$$F_{\mathsf{Alg}}(S) \simeq \operatorname{Mor}(S, M).$$

Then taking $\mathrm{id}_M \in \operatorname{Mor}(M, M)$ should yield a universal family $\mathcal{U} \to M$:



Then the points of M would correspond to isomorphism classes of curves, and every family of curves would be a pullback of this.

For any $S \in \mathsf{Sch}$ and a family $\mathcal{C} \xrightarrow{f} S$, the fiber $f^{-1}(s) \in \mathcal{C}$ is a curve for any $s \in S$, so one could define a map

$$\begin{aligned} g : S &\to M \\ s &\mapsto [s], \end{aligned}$$

where we send a curve to its isomorphism class. Then $\mathcal{C}$ would fit into a pullback diagram:

$$\begin{array}{ccc} \mathcal{C} & \dashrightarrow & \mathcal{U} \\ \downarrow & \lrcorner & \downarrow \\ S & \longrightarrow & M \end{array}$$

Link to Diagram

If S was itself a curve, then $g : S \to M$ would be a path in M deforming a base curve.

2.2 Groupoids

Remark 2.2.1: Recall that a **groupoid** is a category where every morphism is an isomorphism. Morphisms of groupoids are functors, and isomorphisms of groupoids are equivalences of categories.

Example 2.2.2 ***(Groupoid of a set):*** A basic example is the category of sets where

$$\operatorname{Mor}(A, B) := \begin{cases} \mathrm{id}_A & A = B \\ \emptyset & \text{else.} \end{cases}$$

A similar construction: for any set Σ, one can form a groupoid $\mathcal{C}_\Sigma$:

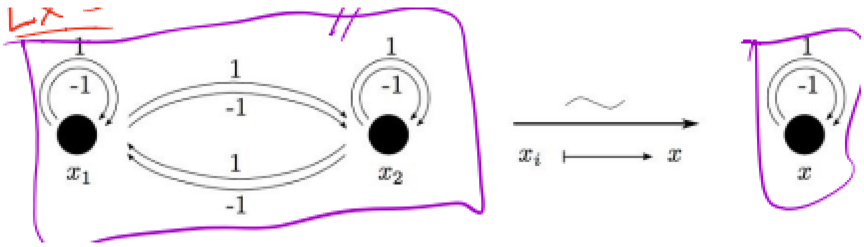
- Object: Elements $x \in \Sigma$.
- Morphisms: id_x

Example 2.2.3 ***(Moduli of curves):*** Define a category $\mathcal{M}_g(\mathbb{C})$:

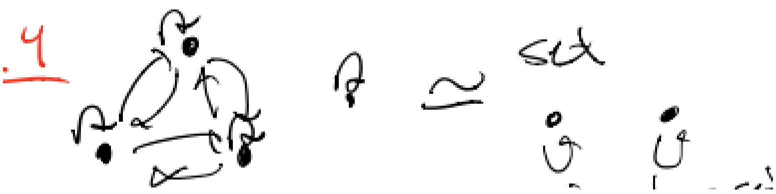
- Objects: smooth projective curves over $\mathbb{C}$ of genus g.
- Morphisms:

$$\operatorname{Mor}(C, C') = \operatorname*{Isom}_{\mathsf{Sch}_{/\mathbb{C}}}(C, C') \subseteq \operatorname{Mor}_{\mathsf{Sch}_{/\mathbb{C}}}(C, C').$$

Example 2.2.4 ***(Equivalence of groupoids):*** Groupoids are equivalent iff they are equivalent as categories. The following is an example of mapping the quotient groupoid $[C_2/C_4]$ to $\mathsf{B}C_2$:



Example 2.2.5 ***(Groupoids equivalent to sets):*** If a groupoid $\mathfrak{X}$ is equivalent to C_Σ for any $\Sigma \in \mathsf{Set}$, we say $\mathfrak{X}$ is **equivalent to a set**. For example, the following groupoid is equivalent to a 2-element set:



Example 2.2.6 ***(Quotient groupoids):*** For $G \curvearrowright \Sigma$ a group acting on any set, define the **quotient groupoid** $[\Sigma/G]$ in the following way:

- Objects: $x \in \Sigma$, i.e. one object for each element of the set Σ.
- Morphisms: $\mathrm{Mor}(x, x') = \left\{ g \in G \;\middle|\; gx' = x \right\}$.

> **Exercise 2.2.7** (Groupoids equivalent to sets)
> Show that $[\Sigma/G]$ is equivalent to a set iff $G \curvearrowright \Sigma$ is a free action.

Example 2.2.8 ***(Classifying stacks):*** For $\Sigma = \{\mathrm{pt}\}$, we obtain

$$\mathsf{B}G := [\mathrm{pt}/G],$$

where there is one object pt and $\mathrm{Mor}(\mathrm{pt}, \mathrm{pt}) = G$.

Example 2.2.9 ***(from representation stability):*** Define FinSet to be the category of finite sets where the morphisms are set bijections. Then $\mathsf{FinSet} = \coprod_{n \in \mathbb{Z}_{\geq 0}} \mathsf{B}S_n$ for S_n the symmetric group.

> **Definition 2.2.10** (Fiber products of groupoids)
> For $C, D' \to D$ morphisms of groupoids, we can construct their **fiber product** as the cartesian diagram:

$$\begin{array}{ccc} C\underset{D}{\times}D' & \xrightarrow{\mathrm{pr}_2} & D' \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[f]{} & D \end{array}$$

Link to Diagram

It can be constructed as the following category:

$$\mathrm{Ob}(C\underset{D}{\times}D') := \left\{ (c, d', \alpha) \;\middle|\; \begin{array}{l} c \in C, d' \in D', \\ \alpha : f(c) \xrightarrow{\sim} g(d') \end{array} \right\}$$

$$\mathrm{Mor}((c_1, d_1', \alpha_1), (c_2, d_2', \alpha_2)) := \left\{ \begin{array}{c} c_1 \xrightarrow{\beta} c_2 \\ \\ d_1' \xrightarrow{\gamma} d_2' \end{array} \;\middle|\; \begin{array}{ccc} f(c_1) & \xrightarrow{f(\beta)} & f(c_2) \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ g(d_1') & \xrightarrow[g(\gamma)]{} & g(d_2') \end{array} \right\}$$

Exercise 2.2.11 (Universal property of pullbacks in Groupoids)
Describe the universal property of the pullback in the 2-category of groupoids.

Example 2.2.12 ***(G is a pullback of $\mathsf{B}G$):*** G regarded as a groupoid is the pullback over inclusions of points into $\mathsf{B}G$:

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{pt} \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathsf{B}G \end{array}$$

Link to Diagram

Example 2.2.13 ***(Orbit/Stabilizer):*** Let $G \curvearrowright \Sigma$ and $x \in \Sigma$, and let Gx be the orbit and G_x be the stabilizer. Then there is a morphism of groupoids $f \in \mathrm{Mor}(\mathsf{B}G_x, [\Sigma/G])$ inducing a pullback:

$$\begin{array}{ccc} G_x & \longrightarrow & \Sigma \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{B}G_x & \xrightarrow{\exists f} & [\Sigma/G] \end{array}$$

$$\mathrm{pt} \longmapsto x$$

Link to Diagram

2.3 Prestacks

Remark 2.3.1: Motivation: to specify a moduli functor, we'll need the data of

- Families over S,
- How to pull back families under morphisms, and
- *How* objects are isomorphic.

As a first attempt, we might try to define a 2-functor $F : \mathsf{Sch} \to \mathsf{Grpd}$ between 2-categories, where the latter is the category of groupoids. For this, we need the following data:

- For all $S \in \mathsf{Sch}$, an assignment of a groupoid $F(S)$,
- For all morphisms $f \in \mathrm{Mor}_{\mathsf{Sch}}(S, T)$, an assignment of morphisms of groupoids
$$f^* \in \mathrm{Mor}_{\mathsf{Grpd}}(F(T), F(S)).$$
- For compositions of morphisms of schemes $S \xrightarrow{f} T \xrightarrow{g} U$, an isomorphism of functors
$$\psi_{fg} : g^* \circ f^* \xrightarrow{\sim} (g \circ f)^*.$$
- Compatibility of these isomorphisms on chains of compositions $S \to T \to U \to V \to \cdots$. [1]

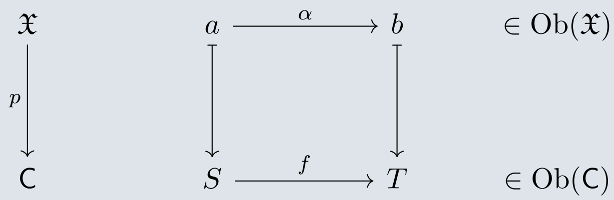
This is a lot of data to track, so instead we'll construct a large category $\mathfrak{X}$ that encodes all of this, along with a fibration

$$\begin{array}{ccc} \mathfrak{X} := \coprod_{S \in \mathsf{Sch}} F(S) & & (S, \alpha \in F(S)) \\ \downarrow p & & \downarrow \\ \mathsf{Sch} & & S \end{array}$$

Here $S \in \mathsf{Sch}$ and $F(S) \in \mathsf{Grpd}$, so the "fibers" above S are groupoids.

[1] This leads to the notion of **lax** or **pseudofunctors**.

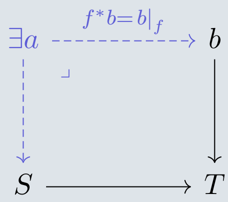
Definition 2.3.2 (Prestack)
Let $p : \mathfrak{X} \to \mathsf{C}$ be a functor between two 1-categories, so we have the following data:



Link to Diagram

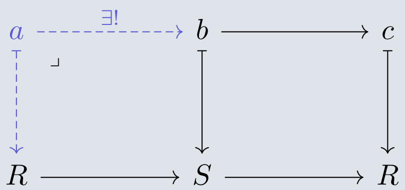
Then $\mathfrak{X}, p$ define a **prestack** over C iff

- Pullbacks exist: for $S \xrightarrow{f} T$, there exists a (not necessarily unique) map f^*b, sometimes denoted $b|_f$, yielding a cartesian square:



Link to Diagram

- A universal property making $\mathfrak{X}$ a *fibered category*: every arrow in $\mathfrak{X}$ is a pullback, so there are always lifts of the following form:



Link to Diagram

Slogan 2.3.3
An alternative definition: a prestack is a category *fibered in groupoids*.

Warning 2.3.4
We often conflate $\mathfrak{X}$ and the functor $\mathfrak{X} \xrightarrow{p} S$, and don't spell out the composition law in $\mathfrak{X}$. Moreover, we write f^*b or $b|_f$ for a *choice* of a pullback.

Definition 2.3.5 (Fiber Categories)
For $p : \mathfrak{X} \to \mathsf{C}$ a functor and $S \in \mathrm{Ob}(\mathsf{C})$ any fixed object, the associated **fiber category over** S, denoted $\mathfrak{X}(S)$, is the subcategory of $\mathfrak{X}$ defined by:

- Objects: $a \in \mathrm{Ob}(\mathfrak{X})$ such that $a \xrightarrow{p} S$,
- Morphisms: $\mathrm{Mor}(a, a')$ are morphisms $f \in \mathrm{Mor}_{\mathfrak{X}}(a, a')$ over id_S:

$$\begin{array}{ccc} a & \xrightarrow{f} & a' \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

Remark 2.3.6: We can now equivalently define presheaves as categories fibered in sets.

Exercise 2.3.7 (Justifying 'category fibered in groupoids')
Show that if $\mathfrak{X} \to \mathsf{C}$ is a prestack, then for all $S \in \mathsf{C}$, all maps in $\mathfrak{X}(S)$ are invertible. Conclude that the fiber categories $\mathfrak{X}(S)$ are all groupoids.

Example 2.3.8 ***(Presheaves):*** Every presheaf forms a prestack. Let $F \in \mathsf{Sh}_{\mathsf{pre}}(\mathsf{Sch}, \mathsf{Set})$ be a presheaf of sets, and define $\mathfrak{X}_F$ as the following category:

- Objects: Pairs $(S, a \in F(S))$ where $S \in \mathsf{Sch}$ and $F(s) \in \mathsf{Set}$.
- Morphisms:

$$\mathrm{Mor}((S, a), (T, b)) := \left\{ S \xrightarrow{f} T \;\middle|\; a = f^* b \right\}.$$

Note that we'll often conflate F and $\mathfrak{X}_F$. This yields the fibration

$$\begin{array}{ccc} \mathfrak{X}_F & & (S, a) \\ \downarrow p & & \downarrow \\ \mathsf{Sch} & & S \end{array}$$

Link to Diagram

Example 2.3.9 ***(Schemes):*** For $X \in \mathsf{Sch}$, take its Yoneda functor $h_X : \mathsf{Sch} \to \mathsf{Set}$. Then define the category $\mathfrak{X}_X$:

- Objects: Morphisms $S \to X$ of schemes.
- Morphisms: $\mathrm{Mor}(S \to X, T \to X)$ are morphisms over X:

$$\begin{array}{ccc} S & \longrightarrow & T \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

This yields the fibration

$$\begin{array}{ccc} \mathfrak{X}_X & & (S \to X) \\ \downarrow{\scriptstyle p} & & \big\downarrow \\ \mathsf{Sch} & & S \end{array}$$

Link to Diagram

Example 2.3.10 ***(Moduli of curves):*** Define $\mathcal{M}_g$ as the following category:

- Objects: families $\mathcal{C} \to S$ of smooth genus g curves,
- Morphisms: $\mathrm{Mor}(\mathcal{C} \to S, \mathcal{C}' \to S')$: cartesian squares

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}' \\ \big\downarrow & \lrcorner & \big\downarrow \\ S & \longrightarrow & S' \end{array}$$

Link to Diagram

This yields a fibration

$$\begin{array}{ccc} \mathcal{M}_g & & (\mathcal{C} \to S) \\ \big\downarrow & & \big\downarrow \\ \mathsf{Sch} & & S \end{array}$$

Example 2.3.11 ***(Bundles):*** For C a smooth connected projective curve over k a field, define $\mathsf{Bun}(C)$ as the following category:

- Objects: pairs (S, F) where F is a vector bundle over $C \times S$.
- Morphisms:

$$\mathrm{Mor}((S,F),(S',F')) = \left\{ \begin{array}{l} f \in \mathrm{Mor}_{\mathsf{Sch}}(S,S') \\ \text{and a chosen isomorphism} \\ \alpha : (f \times \mathrm{id})^* \circ F' \xrightarrow{\sim} F \end{array} \right\}.$$

Remark 2.3.12: A technical point: the choice of pushforward here is not necessarily canonical. However, as part of the data, one can take morphisms $F' \to (f \times \mathrm{id})_* \circ F$ such that the adjunction yields an isomorphism.

Example 2.3.13 ***(Quotient prestack):*** Let $X_{/S} \in \mathsf{GrpSch}$ where $G \curvearrowright X$. Then define a category $[X/G]^{\mathsf{pre}}$:

- Objects: Morphisms over id_S:

$$\begin{array}{ccc} T & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

- Morphisms:

$$\mathrm{Mor}(T \to X, T' \to X) := \left\{ T \to T' \;\middle|\; \begin{array}{l} (T \to T' \to X) = g(T \to X) \\ g \in G(T) \\ G(T) \curvearrowright X(T) \end{array} \right\}.$$

Remark 2.3.14: A group scheme can alternatively be thought of as a functor with a factorization through Grp.

Exercise 2.3.15 (Quotient prestacks and quotient groupoids)
Show that for $T \in \mathsf{Sch}$, there is an equivalence

$$[X/G]^{\mathsf{pre}}(T) \xrightarrow{\sim} [X(T)/G(T)],$$

where the left-hand side is a fibered category over T and the right-hand side is a quotient groupoid.

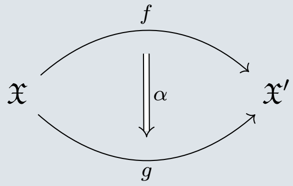
2.3.1 Morphisms of Prestacks

Definition 2.3.16 (Morphisms of prestacks)
A **morphism of prestacks** is a functor $\mathfrak{X} \xrightarrow{f} \mathfrak{X}'$ such that there is a (strictly) commutative triangle

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{f} & \mathfrak{X}' \\ & {\scriptstyle p_X}\searrow \quad \swarrow{\scriptstyle p_Y} & \\ & \mathsf{C} & \end{array}$$

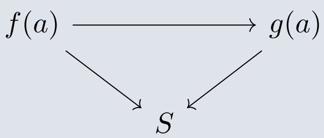
Link to Diagram

Here we require a strict equality $p_X(a) = p_Y(f(a))$ for any $a \in \mathfrak{X}$
A **2-morphism** α between morphisms f, g is a natural transformation:



Link to Diagram

such that for all $a \in \mathfrak{X}$, the following triangle $\alpha_a \in \mathrm{Mor}_{\mathfrak{X}'}(f(a), g(a))$ is a morphisms over id_S for any $S \in \mathsf{C}$:

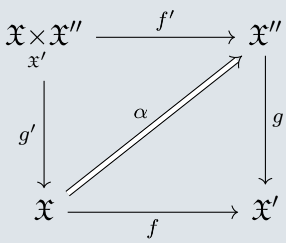


We define a category $\mathrm{Mor}(\mathfrak{X}, \mathfrak{X}')$ by:

- Objects: morphisms of prestacks.
- Morphisms: 2-morphisms of prestacks.

Exercise 2.3.17 (?)
Show that $\mathrm{Mor}(\mathfrak{X}, \mathfrak{X}')$ is a groupoid.

Definition 2.3.18 (2-commutativity)
A diagram is **2-commutative** iff there exists a 2-morphism $\alpha : g \circ f' \xrightarrow{\sim} f \circ g'$ which is an isomorphism:



Link to Diagram

Definition 2.3.19 (Isomorphisms of prestacks)
An **isomorphism** of prestacks is a 1-isomorphism of prestacks $f : \mathfrak{X} \to \mathfrak{X}'$ along with 2-isomorphisms $g \circ f \xrightarrow{\sim} \mathrm{id}_{\mathfrak{X}}$ and $f \circ g \xrightarrow{\sim} \mathrm{id}_{\mathfrak{X}'}$.

Exercise 2.3.20 (Isomorphisms of prestacks can be checked on fibers)
Show that $\mathfrak{X} \to \mathfrak{X}'$ is an isomorphism iff $\mathfrak{X}(S) \xrightarrow{\sim} \mathfrak{X}'(S)$ is an isomorphism on all fibers.

Proposition 2.3.21 ***(2-Yoneda).***
If $\mathfrak{X} \in \mathsf{St}_{\mathsf{pre}}/\mathsf{C}$ is a prestack over C, then for any $S \in \mathrm{Ob}(\mathsf{C})$, there is an equivalence of categories induced by the following functor:

$$\begin{aligned} \mathrm{Mor}(S, \mathfrak{X}) &\xrightarrow{\sim} \mathfrak{X}(S) \\ f &\mapsto f_S(\mathrm{id}_S). \end{aligned}$$

Remark 2.3.22: For $S \in \mathsf{Sch}$, view S as a prestack and consider a morphism $f : S \to \mathfrak{X}$. How is this specified? For all $T \in \mathsf{Sch}$, the objects of $S_{/T}$ are morphisms

$$f_T : \mathrm{Mor}(T, S) \to \mathfrak{X}(T)$$

and if $T = S$ this sends id_S to $f_S(\mathrm{id}_S) \in \mathfrak{X}(S)$.

What is the inverse? For $a \in \mathfrak{X}(S)$ and for each $T \xrightarrow{g} S$, **choose** a pullback g^*a. Then define $f : S \to \mathfrak{X}$ by

$$\begin{aligned} f_T : \mathrm{Mor}(T, S) &\to \mathfrak{X}(T) \\ g &\mapsto g^*a. \end{aligned}$$

Exercise 2.3.23 (?)
Define what this equivalence should do on morphisms.

Remark 2.3.24: Next time: fiber products of prestacks.

ToDos

List of Todos

2

Definitions

2 Theorems

2 Exercises

Figures

List of Figures