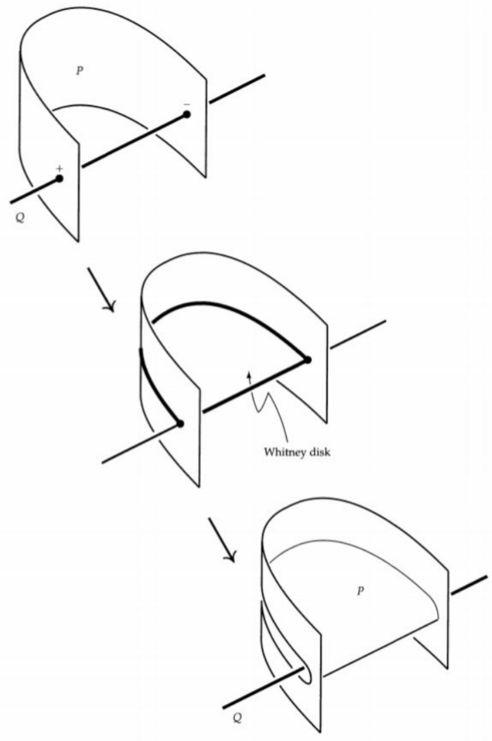

1.26. The Whitney trick

Notes: These are notes live-tex'd from a graduate course in 4-Manifolds taught by Philip Engel at the University of Georgia in Spring 2021. As such, any errors or inaccuracies are almost certainly my own.

4-Manifolds

Lectures by Philip Engel. University of Georgia, Spring 2021

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1 | Tuesday, January 12

1.1 Background

From Phil's email:

Personally, I found the following online references particularly useful:

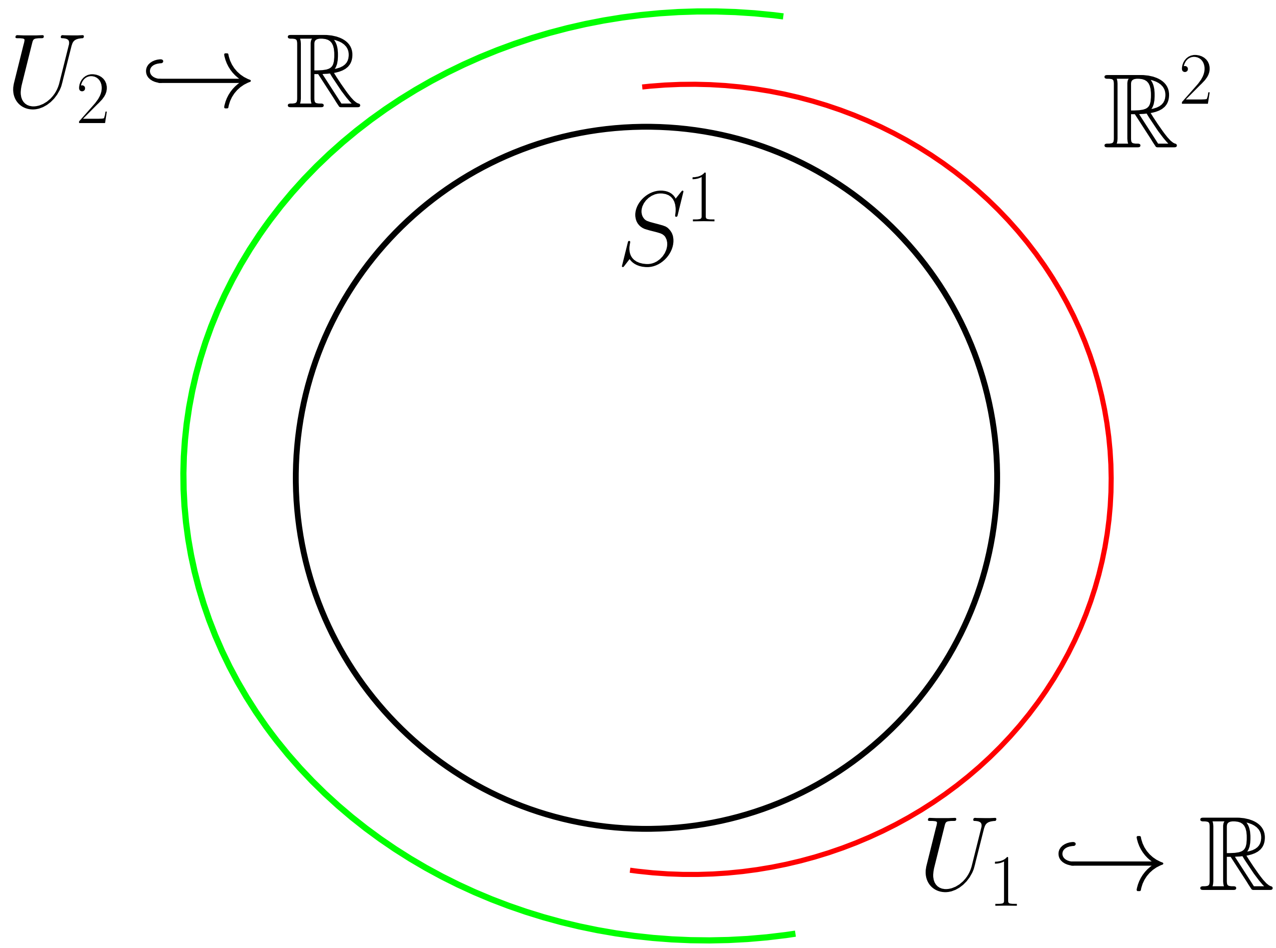
- Dietmar Salamon: Spin Geometry and Seiberg-Witten Invariants [5]
- Richard Mandelbaum: Four-dimensional Topology: An Introduction [2]
  - This book has a nice introduction to surgery aspects of four-manifolds, but as a warning: It was published right before Freedman's famous theorem. For instance, the existence of an exotic R^4 was not known. This actually makes it quite useful, as a summary of what was known before, and provides the historical context in which Freedman's theorem was proven.
- Danny Calegari: Notes on 4-Manifolds [1]
- Yuli Rudyak: Piecewise Linear Structures on Topological Manifolds [4]
- Akhil Mathew: The Dirac Operator [3]
- Tom Weston: An Introduction to Cobordism Theory [6]

  A wide variety of lecture notes on the Atiyah-Singer index theorem, which are available online.

1.2 Introduction

> **Definition 1.2.1** (Topological Manifold)
> Recall that a **topological manifold** (or C^0 manifold) X is a Hausdorff topological space *locally homeomorphic* to $\mathbb{R}^n$ with a countable topological base, so we have charts $\varphi_u : U \to \mathbb{R}^n$ which are homeomorphisms from open sets covering X.

Example 1.2.2 ***(The circle):*** S^1 is covered by two charts homeomorphic to intervals:



Remark 1.2.3: Maps that are merely continuous are poorly behaved, so we may want to impose extra structure. This can be done by imposing restrictions on the transition functions, defined as

$$t_{uv} := \varphi_V \to \varphi_U^{-1} : \varphi_U(U \cap V) \to \varphi_V(U \cap V).$$

Definition 1.2.4 (Restricted Structures on Manifolds)

- We say X is a **PL manifold** if and only if t_{UV} are piecewise-linear. Note that an invertible PL map has a PL inverse.
- We say X is a C^k **manifold** if they are k times continuously differentiable, and **smooth** if infinitely differentiable.
- We say X is **real-analytic** if they are locally given by convergent power series.
- We say X is **complex-analytic** if under the identification $\mathbb{R}^n \cong \mathbb{C}^{n/2}$ if they are holomorphic, i.e. the differential of t_{UV} is complex linear.
- We say X is a **projective variety** if it is the vanishing locus of homogeneous polynomials on $\mathbb{CP}^N$.

Remark 1.2.5: Is this a strictly increasing hierarchy? It's not clear e.g. that every C^k manifold is PL.

Question 1.2.6
Consider $\mathbb{R}^n$ as a topological manifold: are any two smooth structures on $\mathbb{R}^n$ diffeomorphic?

Remark 1.2.7: Fix a copy of $\mathbb{R}$ and form a single chart $\mathbb{R} \xrightarrow{\text{id}} \mathbb{R}$. There is only a single transition function, the identity, which is smooth. But consider

$$\begin{aligned} X &\to \mathbb{R} \\ t &\mapsto t^3. \end{aligned}$$

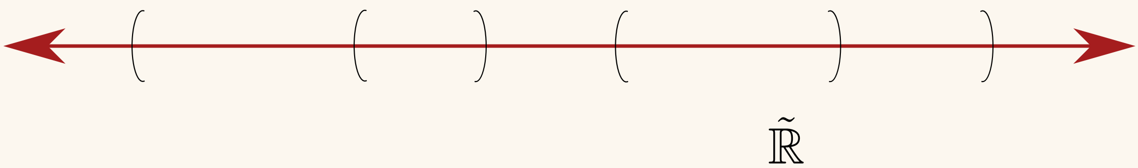
This is also a smooth structure on X, since the transition function is the identity. This yields a different smooth structure, since these two charts don't like in the same maximal atlas. Otherwise there would be a transition function of the form $t_{VU} : t \mapsto t^{1/3}$, which is not smooth at zero. However, the map

$$\begin{aligned} X &\to X \\ t &\mapsto t^3. \end{aligned}$$

defines a diffeomorphism between the two smooth structures.

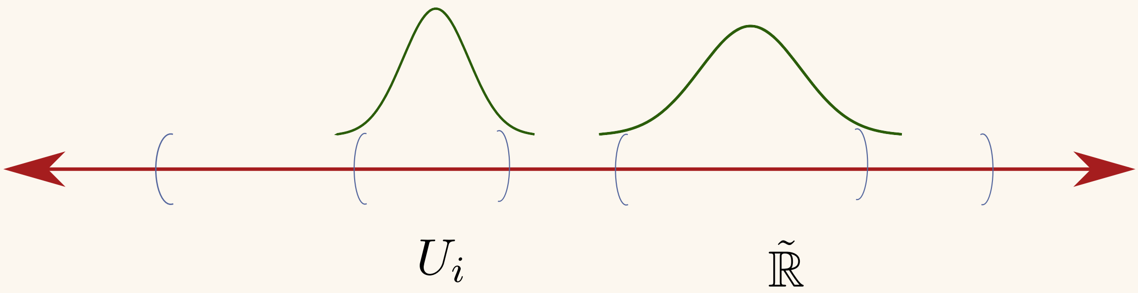
Claim: $\mathbb{R}$ admits a unique smooth structure.

Proof (sketch).
Let $\tilde{\mathbb{R}}$ be some exotic $\mathbb{R}$, i.e. a smooth manifold homeomorphic to $\mathbb{R}$. Cover this by coordinate charts to the standard $\mathbb{R}$:



Fact
There exists a cover which is *locally finite* and supports a *partition of unity*: a collection of smooth functions $f_i : U_i \to \mathbb{R}$ with $f_i \geq 0$ and supp$f \subseteq U_i$ such that $\sum f_i = 1$ (*i.e., bump functions*). It is also a purely topological fact that $\tilde{\mathbb{R}}$ is orientable.

So we have bump functions:



Take a smooth vector field V_i on U_i everywhere aligning with the orientation. Then $\sum f_i V_i$ is a smooth nowhere vector field on X that is nowhere zero in the direction of the orientation. Taking the associated flow

$$\begin{aligned} \mathbb{R} &\to \tilde{\mathbb{R}} \\ t &\mapsto \varphi(t). \end{aligned}$$

such that $\varphi'(t) = V(\varphi(t))$. Then φ is a smooth map that defines a diffeomorphism. This follows from the fact that the vector field is everywhere positive.

■

Slogan 1.2.9
To understand smooth structures on X, we should try to solve differential equations on X.

Remark 1.2.10: Note that here we used the existence of a global frame, i.e. a trivialization of the tangent bundle, so this doesn't quite work for e.g. S^2.

Question 1.2.11
What is the difference between all of the above structures? Are there obstructions to admitting any particular one?

Answer 1.2.12

1. (Munkres) Every C^1 structure gives a unique C^k and C^∞ structure.[1]

2. (Grauert) Every C^∞ structure gives a unique real-analytic structure.

3. Every PL manifold admits a smooth structure in $\dim X \leq 7$, and it's unique in $\dim X \leq 6$, and above these dimensions there exists PL manifolds with no smooth structure.

4. (Kirby–Siebenmann) Let X be a topological manifold of $\dim X \geq 5$, then there exists a

[1] Note that this doesn't start at C^0, so topological manifolds are genuinely different! There exist topological manifolds with no smooth structure.

cohomology class $\mathrm{ks}(X) \in H^4(X; \mathbb{Z}/2\mathbb{Z})$ which is 0 if and only if X admits a PL structure. Moreover, if $\mathrm{ks}(X) = 0$, then (up to concordance) the set of PL structures is given by $H^3(X; \mathbb{Z}/2\mathbb{Z})$.

5. (Moise) Every topological manifold in $\dim X \leq 3$ admits a unique smooth structure.

6. (Smale et al.): In $\dim X \geq 5$, the number of smooth structures on a topological manifold X is finite. In particular, $\mathbb{R}^n$ for $n \neq 4$ has a unique smooth structure. So dimension 4 is interesting!

7. (Taubes) $\mathbb{R}^4$ admits uncountably many non-diffeomorphic smooth structures.

8. A compact oriented smooth surface Σ, the space of complex-analytic structures is a complex orbifold [2] of dimension $3g - 2$ where g is the genus of Σ, up to biholomorphism (i.e. *moduli*).

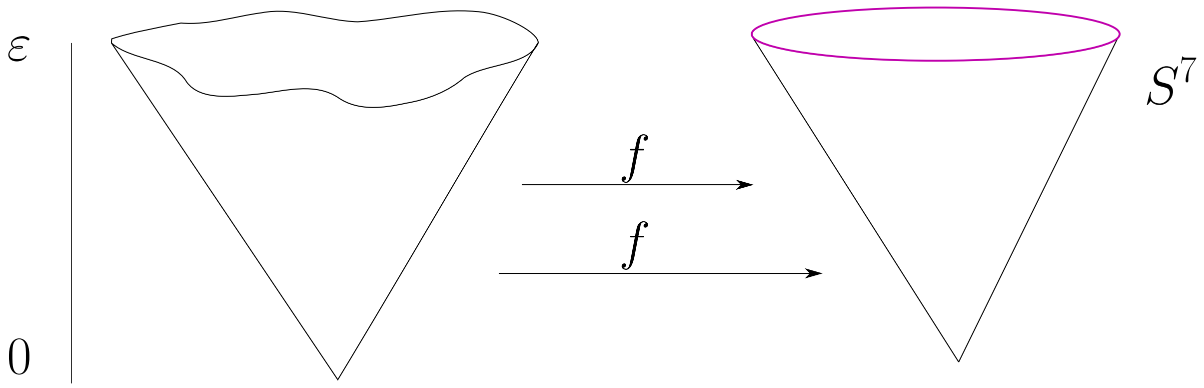
Remark 1.2.13: Kervaire-Milnor: S^7 admits 28 smooth structures, which form a group.

2 | Friday, January 15

Remark 2.0.1: Let

$$V := \left\{ a^2 + b^2 + c^2 + d^3 + e^{6k-1} = 0 \right\} \subseteq \mathbb{C}^5$$
$$S_\varepsilon := \left\{ |a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2 = 1 \right\}.$$

Then $V_k \cap S_\varepsilon \cong S^7$ is a homeomorphism, and taking $k = 1, 2, \cdots, 28$ yields the 28 smooth structures on S^7. Note that V_k is the cone over $V_k \cap S_\varepsilon$.

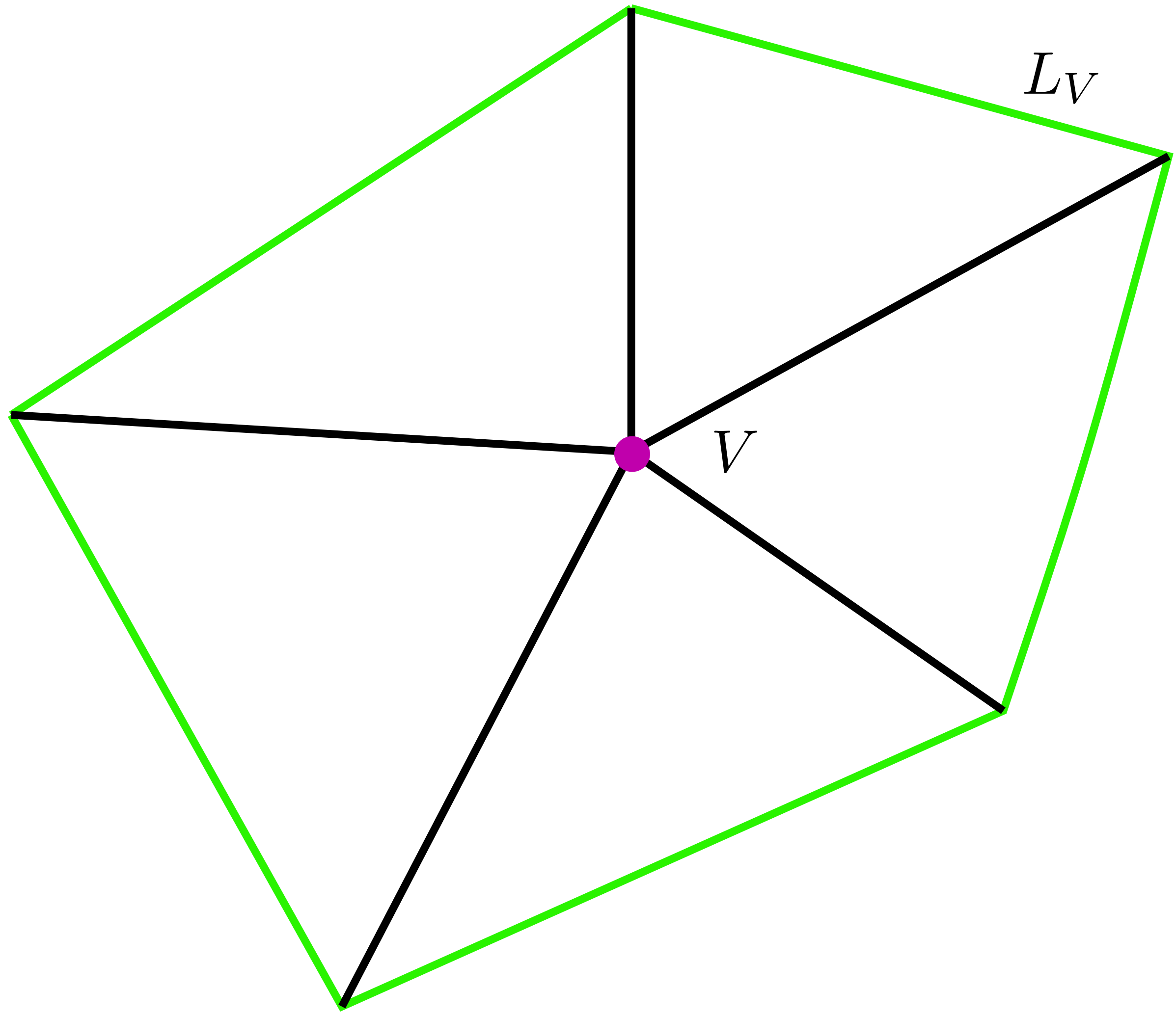


? Admits a smooth structure, and $\overline{V}_k \subseteq \mathbb{CP}^5$ admits no smooth structure.

[2] Locally admits a chart to $\mathbb{C}^n/\Gamma$ for Γ a finite group.

Question 2.0.2
Is every triangulable manifold PL, i.e. homeomorphic to a simplicial complex?

Answer 2.0.3
No! Given a simplicial complex, there is a notion of the **combinatorial link** L_V of a vertex V:



It turns out that there exist simplicial manifolds such that the link is not homeomorphic to a sphere, whereas every PL manifold admits a "PL triangulation" where the links are spheres.

Remark 2.0.4: What's special in dimension 4? Recall the **Kirby-Siebenmann** invariant $\mathrm{ks}(x) \in H^4(X;\mathbb{Z}_2)$ for X a topological manifold where $\mathrm{ks}(X) = 0 \iff X$ admits a PL structure, with the caveat that $\dim X \geq 5$. We can use this to cook up an invariant of 4-manifolds.

Definition 2.0.5 (Kirby-Siebenmann Invariant of a 4-manifold)
Let X be a topological 4-manifold, then

$$\operatorname{ks}(X) := \operatorname{ks}(X \times \mathbb{R}).$$

Remark 2.0.6: Recall that in $\dim X \geq 7$, every PL manifold admits a smooth structure, and we can note that

$$H^4(X; \mathbb{Z}_2) = H^4(X \times \mathbb{R}; \mathbb{Z}_2) = \mathbb{Z}_2, .$$

since every oriented 4-manifold admits a fundamental class. Thus

$$\operatorname{ks}(X) = \begin{cases} 0 & X \times \mathbb{R} \text{ admits a PL and smooth structure} \\ 1 & X \times \mathbb{R} \text{ admits no PL or smooth structures} \end{cases}.$$

Remark 2.0.7: $\operatorname{ks}(X) \neq 0$ implies that X has no smooth structure, since $X \times \mathbb{R}$ doesn't. Note that it was not known if this invariant was ever nonzero for a while!

Remark 2.0.8: Note that $H^2(X; \mathbb{Z})$ admits a symmetric bilinear form Q_X defined by

$$\begin{aligned} Q_X : H^2(X; \mathbb{Z})^{\otimes 2} &\to \mathbb{Z} \\ \alpha \otimes \beta &\mapsto \int_X \alpha \wedge \beta := (\alpha \smile \beta)([X]). \end{aligned}$$

where $[X]$ is the fundamental class.

3 | Main Theorems for the Course

Remark 3.0.1: Proving the following theorems is the main goal of this course:

Theorem 3.0.2 ***(Freedman).***
If X, Y are compact oriented topological 4-manifolds, then $X \cong Y$ are homeomorphic if and only if $\operatorname{ks}(X) = \operatorname{ks}(Y)$ and $Q_X \cong Q_Y$ are isometric, i.e. there exists an isometry

$$\varphi : H^2(X; \mathbb{Z}) \to H^2(Y; \mathbb{Z}).$$

that preserves the two bilinear forms in the sense that $\langle \varphi\alpha,\ \varphi\beta \rangle = \langle \alpha,\ \beta \rangle$.
Conversely, every **unimodular** bilinear form appears as $H^2(X; \mathbb{Z})$ for some X, i.e. the pairing induces a map

$$\begin{aligned} H^2(X; \mathbb{Z}) &\to H^2(X; \mathbb{Z})^\vee \\ \alpha &\mapsto \langle \alpha,\ - \rangle. \end{aligned}$$

which is an isomorphism. This is essentially a classification of simply-connected 4-manifolds.

Remark 3.0.3: Note that preservation of a bilinear form is a stand-in for "being an element of the orthogonal group", where we only have a lattice instead of a full vector space.

Remark 3.0.4: There is a map $H^2(X;\mathbb{Z}) \xrightarrow{PD} H_2(X;\mathbb{Z})$ from Poincaré , where we can think of elements in the latter as closed surfaces $[\Sigma]$, and

$$\langle \Sigma_1,\ \Sigma_2 \rangle = \text{signed number of intersections points of } \Sigma_1 \pitchfork \Sigma_2.$$

Note that Freedman's theorem is only about homeomorphism, and is not true smoothly. This gives a way to show that two 4-manifolds are homeomorphic, but this is hard to prove! So we'll black-box this, and focus on ways to show that two *smooth* 4-manifolds are *not* diffeomorphic, since we want homeomorphic but non-diffeomorphic manifolds.

Definition 3.0.5 (Signature)
The **signature** of a topological 4- manifold is the signature of Q_X, where we note that Q_X is a symmetric nondegenerate bilinear form on $H^2(X;\mathbb{R})$ and for some a, b

$$(H^2(X;\mathbb{R}), Q_x) \xrightarrow{\text{isometric}} \mathbb{R}^{a,b}.$$

where a is the number of $+1$s appearing in the matrix and b is the number of -1s. This is $\mathbb{R}^{ab}$ where $e_i^2 = 1, i = 1 \cdots a$ and $e_i^2 = -1, i = a+1, \cdots b$, and is thus equipped with a specific bilinear form corresponding to the Gram matrix of this basis.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} = I_{a \times a} \oplus -I_{b \times b}.$$

Then the signature is $a - b$, the dimension of the positive-definite space minus the dimension of the negative-definite space.

Theorem 3.0.6 ***(Rokhlin's Theorem).***
Suppose $\langle \alpha,\ \alpha \rangle \in 2\mathbb{Z}$ and $\alpha \in H^2(X;\mathbb{Z})$ and X a simply connected **smooth** 4-manifold. Then 16 divides $\operatorname{sig}(X)$.

Remark 3.0.7: Note that Freedman's theorem implies that there exists topological 4-manifolds with no smooth structure.

Theorem 3.0.8 ***(Donaldson).***
Let X be a smooth simply-connected 4-manifold. If $a = 0$ or $b = 0$, then Q_X is diagonalizable and there exists an orthonormal basis of $H^2(X;\mathbb{Z})$.

Remark 3.0.9: This comes from Gram-Schmidt, and restricts what types of intersection forms can occur.

3.1 Warm Up: $\mathbb{R}^2$ Has a Unique Smooth Structure

Remark 3.1.1: Last time we showed $\mathbb{R}^1$ had a unique smooth structure, so now we'll do this for $\mathbb{R}^2$. The strategy of solving a differential equation, we'll now sketch the proof.

Definition 3.1.2 (Riemannian Metrics)
A **Riemannian metric** $g \in \Gamma(\mathrm{Sym}^2 T^\vee X)$ for X a smooth manifold is a metric on every T_pX, so $g_p \in (T_pX^{\otimes 2})^\vee$, such that

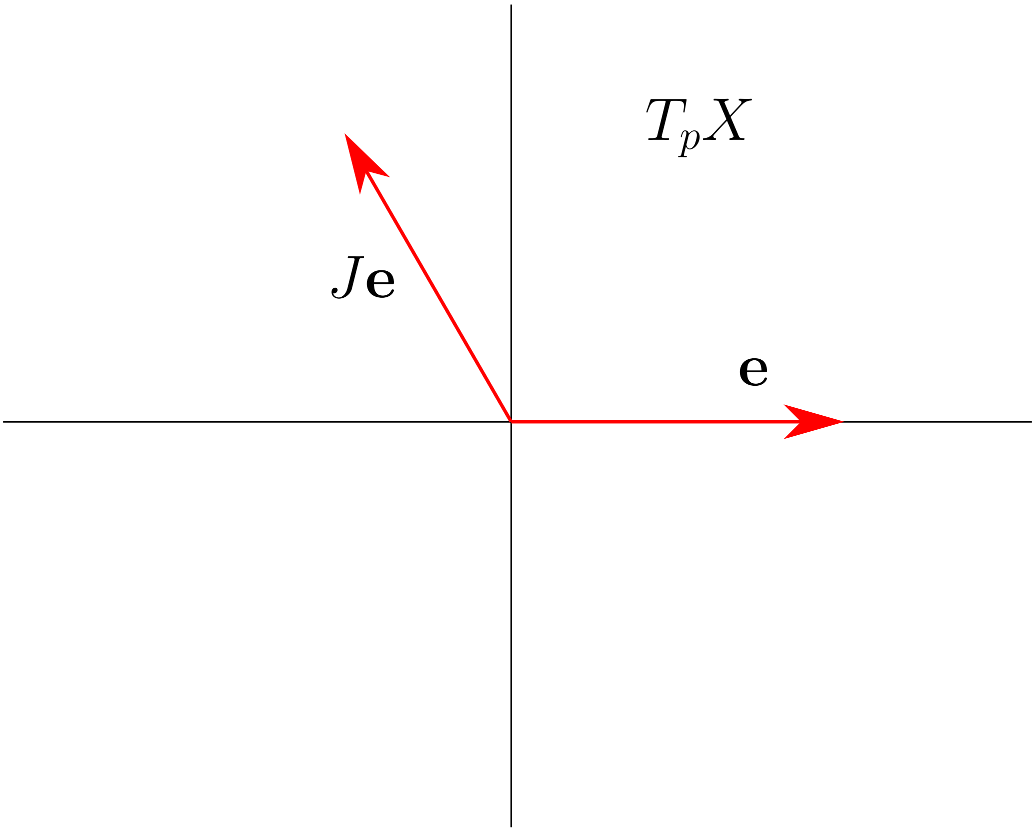
$$g_p : T_pX \otimes T_pX \to \mathbb{R} \qquad\qquad g(v,v) \geq 0, \quad g(v,v) = 0 \iff v = 0.$$

Definition 3.1.3 (Almost complex structure)
An **almost complex structure** is a morphism $J \in \operatorname*{End}_{\mathsf{Vect}(X)}(TX)$ of vector bundles over X such that $J^2 = -\mathrm{id}_{TX}$.

Definition 3.1.4 (Integrable)
An almost-complex structure is **integrable** J if it comes from a complex structure in the following sense: for a complex manifold $M \in \mathsf{Mfd}(\mathbb{C})$, take holomorphic coordinates $z = x + iy$ and set $J\frac{\partial}{\partial x} := \frac{\partial}{\partial y}$ and $J\frac{\partial}{\partial y} := -\frac{\partial}{\partial x}$.

Remark 3.1.5: A manifold $M \in \mathsf{smMfd}(\mathbb{R})$ admits an almost-complex structure iff TM admits a reduction of structure group $\mathrm{GL}_{2n}(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{C})$.

Remark 3.1.6: Let $e \in T_pX$ and $e \neq 0$, then if X is a surface then $\{e, J_pe\}$ is a basis of T_pX, where J_p is the restriction of J to T_pX:



Exercise 3.1.7 (?)
Show that $\{e, J_pe\}$ are linearly independent in T_pX. In particular, J_p is determined by a point in $\mathbb{R}^2 \setminus \{\text{the } x\text{-axis}\}$

Proof (That R2 admits a unique smooth structure (sketch)).
Let $\tilde{\mathbb{R}}^2$ be an exotic $\mathbb{R}^2$.

3.1.1 Step 1

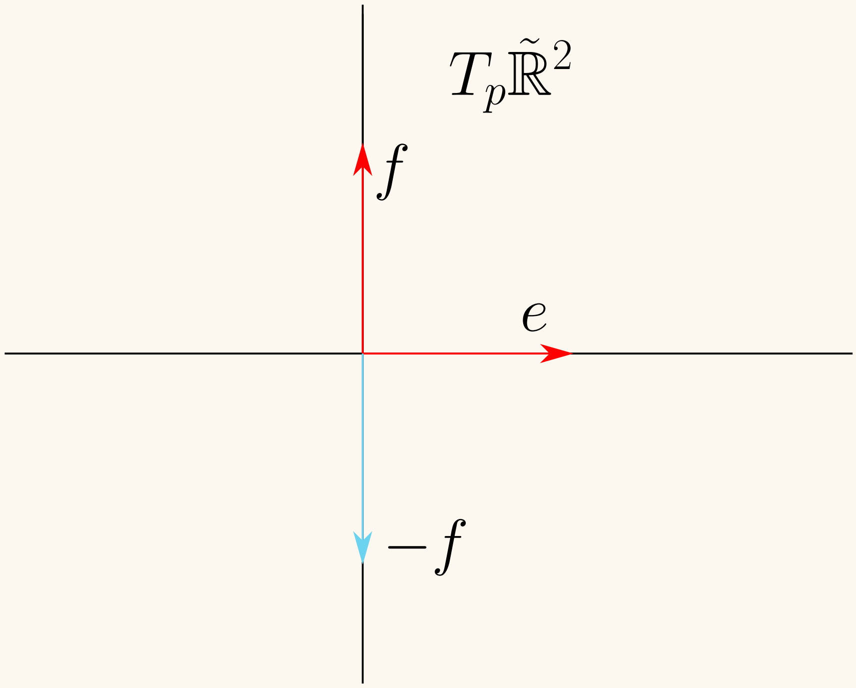
Choose a metric on $\tilde{\mathbb{R}}^2$, say $g := \sum f_I g_i$ with g_i metrics on coordinate charts U_i and f_i a partition of unity.

3.1.2 Step 2

Find an almost complex structure on $\tilde{\mathbb{R}}^2$. Choosing an orientation of $\tilde{\mathbb{R}}^2$, the metric g defines a unique almost complex structure $J_pe := f \in T_p\tilde{\mathbb{R}}^2$ such that

- $g(e, e) = g(f, f)$
- $g(e, f) = 0$.
- $\{e, f\}$ is an oriented basis of $T_p\tilde{\mathbb{R}}^2$

This is because after choosing e, there are two orthogonal vectors, but only one choice yields an *oriented* basis.



3.1.3 Step 3

We then apply a theorem:

> **Theorem 3.1.8** ***(Almost-complex structures on surfaces come from complex structures).***
> Any almost complex structure on a surface comes from a complex structure, in the sense that there exist charts $\varphi_i : U_i \to \mathbb{C}$ such that J is multiplication by i.

So

$$d\varphi(J \cdot e) = i \cdot d\varphi_i(e),$$

and $(\tilde{\mathbb{R}}^2, J)$ is a complex manifold. Since it's simply connected, the Riemann Mapping Theorem shows that it's biholomorphic to $\mathbb{D}$ or $\mathbb{C}$, both of which are diffeomorphic to $\mathbb{R}^2$.

■

Remark 3.1.9: See the Newlander-Nirenberg theorem, a result in complex geometry.

4 Sheaves, Bundles, Connections (Lecture 3, Wednesday, January 20)

4.1 Sheaves

Definition 4.1.1 (Presheaves and Sheaves)
Recall that if X is a topological space, a **presheaf** of abelian groups $\mathcal{F}$ is an assignment $U \to \mathcal{F}(U)$ of an abelian group to every open set $U \subseteq X$ together with a restriction map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for any inclusion $V \subseteq U$ of open sets. This data has to satisfying certain conditions:

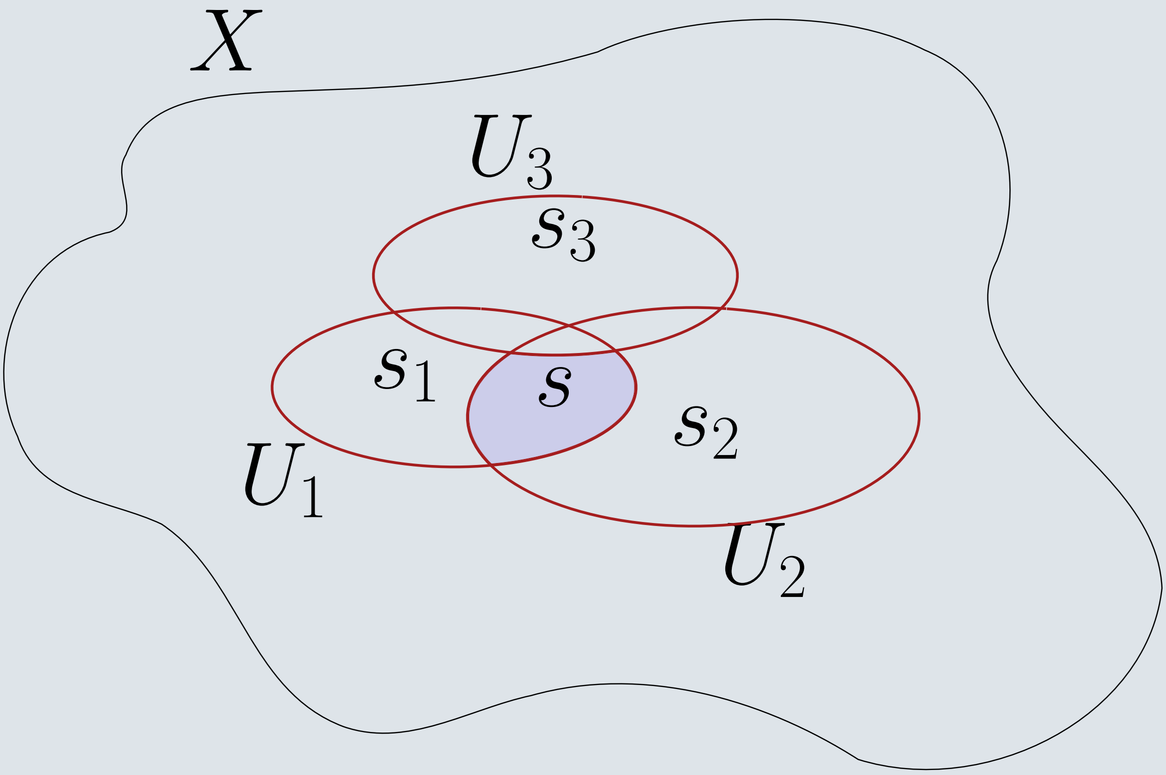
a. $\mathcal{F}(\emptyset) = 0$, the trivial abelian group.

b. $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U) = \mathrm{id}_{\mathcal{F}(U)}$

c. Compatibility if restriction is taken in steps: $U \subseteq V \subseteq W \implies \rho_{VW} \circ \rho_{UV} = \rho_{UW}$.

We say $\mathcal{F}$ is a **sheaf** if additionally:

d. Given $s_i \in \mathcal{F}(U_i)$ such that $\rho_{U_i \cap U_j}(s_i) = \rho_{U_i \cap U_j}(s_j)$ implies that there exists a unique $s \in \mathcal{F}\left(\bigcup_i U_i\right)$ such that $\rho_{U_i}(s) = s_i$.



Example 4.1.2 *(?)*: Let X be a topological manifold, then $\mathcal{F} := C^0(-, \mathbb{R})$ the set of continuous functionals form a sheaf. We have a diagram

$$\begin{tikzcd}
U \arrow[r, "\mathcal{F}"] & C^0(U;\mathbb{R}) \arrow[d, dashed, hook, "\text{restrict cts. functions}"] \\
V \arrow[u, hook] \arrow[r, "\mathcal{F}"'] & C^0(V;\mathbb{R})
\end{tikzcd}$$

Link to diagram

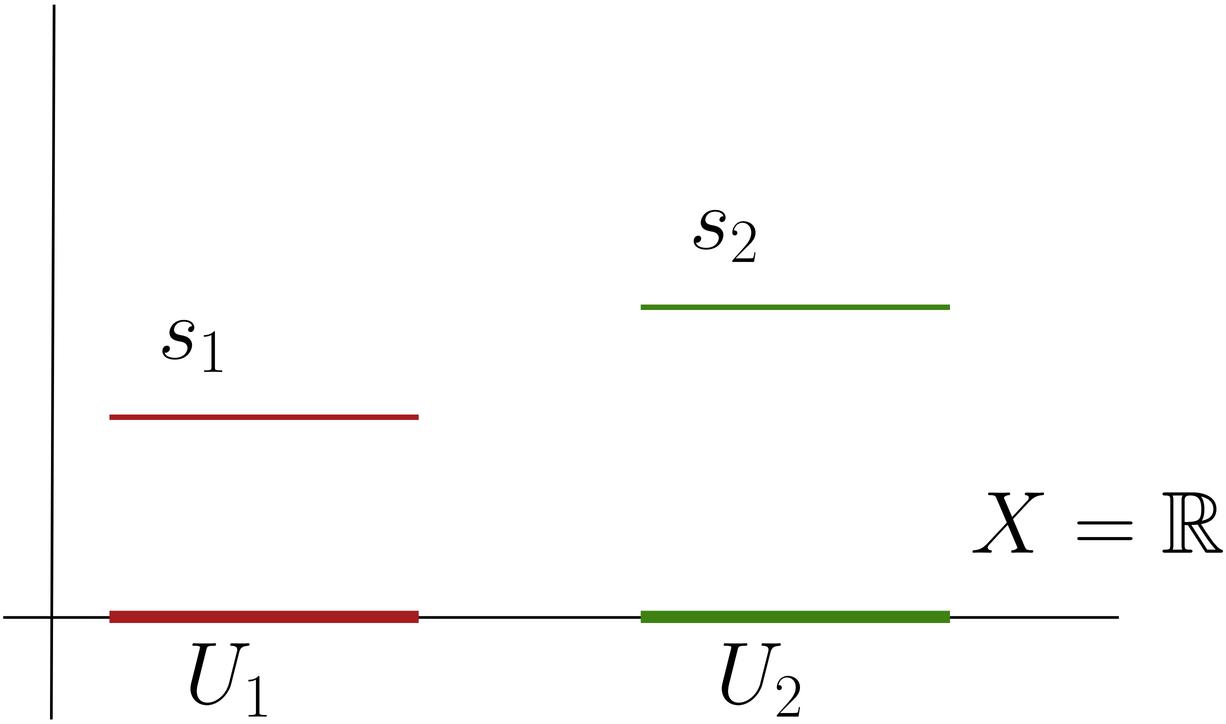
Property (d) holds because given sections $s_i \in C^0(U_i;\mathbb{R})$ agreeing on overlaps, so $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there exists a unique $s \in C^0\left(\bigcup_i U_i;\mathbb{R}\right)$ such that $s|_{U_i} = s_i$ for all i – i.e. continuous functions glue.

Remark 4.1.3: Recall that we discussed various structures on manifolds: PL, continuous, smooth, complex-analytic, etc. We can characterize these by their sheaves of functions, which we'll denote $\mathcal{O}$. For example, $\mathcal{O} := C^0(-;\mathbb{R})$ for topological manifolds, and $\mathcal{O} := C^\infty(-;\mathbb{R})$ is the sheaf for smooth manifolds. Note that this also works for PL functions, since pullbacks of PL functions are again PL. For complex manifolds, we set $\mathcal{O}$ to be the sheaf of holomorphic functions.

Example 4.1.4*(Locally Constant Sheaves):* Let $A \in \mathsf{Ab}$ be an abelian group, then $\underline{A}$ is the sheaf defined by setting $\underline{A}(U)$ to be the locally constant functions $U \to A$. E.g. let $X \in \mathsf{Mfd}_{\mathsf{Top}}$ be a topological manifold, then $\underline{\mathbb{R}}(U) = \mathbb{R}$ if U is connected since locally constant $\implies$ globally constant in this case.

⚠Warning 4.1.5

Note that the presheaf of constant functions doesn't satisfy (d)! Take $\mathbb{R}$ and a function with two different values on disjoint intervals:



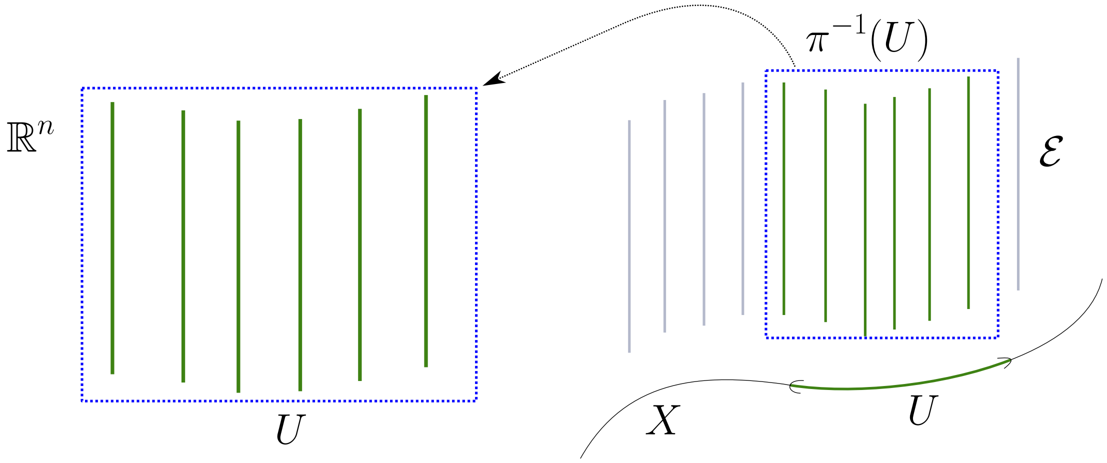
Note that $s_1|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2}$ since the intersection is empty, but there is no constant function that restricts to the two different values.

4.2 Bundles

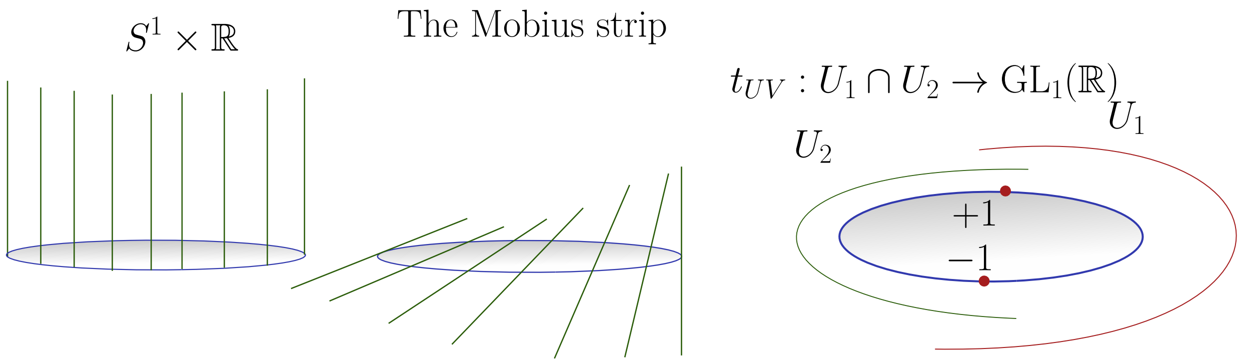
Remark 4.2.1: Let $\pi : \mathcal{E} \to X$ be a **vector bundle**, so we have local trivializations $\pi^{-1}(U) \xrightarrow{h_u} Y^d \times U$ where we take either $Y = \mathbb{R}, \mathbb{C}$, such that $h_v \circ h_u^{-1}$ preserves the fibers of π and acts linearly on each fiber of $Y \times (U \cap V)$. Define

$$t_{UV} : U \cap V \to \mathrm{GL}_d(Y)$$

where we require that t_{UV} is continuous, smooth, complex-analytic, etc depending on the context.



Example 4.2.2 ***(Bundles over*** S^1***):*** There are two $\mathbb{R}^1$ bundles over S^1:



Note that the Mobius bundle is not trivial, but can be locally trivialized.

Remark 4.2.3: We abuse notation: $\mathcal{E}$ is also a sheaf, and we write $\mathcal{E}(U)$ to be the set of sections $s : U \to \mathcal{E}$ where s is continuous, smooth, holomorphic, etc where $\pi \circ s = \mathrm{id}_U$. I.e. a bundle is a sheaf in the sense that its sections *form* a sheaf.

Example 4.2.4 ***(?):*** The trivial line bundle gives the sheaf $\mathcal{O}$: maps $U \xrightarrow{s} U \times Y$ for $Y = \mathbb{R}, \mathbb{C}$ such that $\pi \circ s = \mathrm{id}$ are the same as maps $U \to Y$.

Definition 4.2.5 ($\mathcal{O}$-modules)
An $\mathcal{O}$**-module** is a sheaf $\mathcal{F}$ such that $\mathcal{F}(U)$ has an action of $\mathcal{O}(U)$ compatible with restriction.

Example 4.2.6 ***(?):*** If $\mathcal{E}$ is a vector bundle, then $\mathcal{E}(U)$ has a natural action of $\mathcal{O}(U)$ given by $f \curvearrowright s := fs$, i.e. just multiplying functions.

Example 4.2.7 ***(Non-example):*** The locally constant sheaf $\underline{\mathbb{R}}$ is not an $\mathcal{O}$-module: there isn't natural action since the sections of $\mathcal{O}$ are generally non-constant functions, and multiplying a constant function by a non-constant function doesn't generally give back a constant function.

Remark 4.2.8: We'd like a notion of maps between sheaves:

Definition 4.2.9 (Morphisms of Sheaves)
A **morphism** of sheaves $\mathcal{F} \to \mathcal{G}$ is a group morphism $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for all opens $U \subseteq X$ such that the diagram involving restrictions commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{UV}"] & \mathcal{G}(U) \arrow[d, "\rho_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{F}(V) \end{tikzcd}$$

Example 4.2.10 ***(An $\mathcal{O}$-module that is not a vector bundle.):*** Let $X = \mathbb{R}$ and define the **skyscraper sheaf** at $p \in \mathbb{R}$ as

$$\mathbb{R}_p(U) := \begin{cases} \mathbb{R} & p \in U \\ 0 & p \notin U. \end{cases}.$$

The $\mathcal{O}(U)$-module structure is given by

$$\begin{aligned} \mathcal{O}(U) \times \mathcal{O}(U) &\to \mathbb{R}_p(U) \\ (f, s) &\mapsto f(p)s. \end{aligned}$$

This is not a vector bundle since $\mathbb{R}_p(U)$ is not an infinite dimensional vector space, whereas the space of sections of a vector bundle is generally infinite dimensional (?). Alternatively, there are arbitrarily small punctured open neighborhoods of p for which the sheaf makes trivial assignments.

Example 4.2.11 ***(of morphisms):*** Let $X = \mathbb{R} \in \mathsf{smMfd}$ viewed as a smooth manifold, then multiplication by x induces a morphism of structure sheaves:

$$\begin{aligned} (x\cdot) : \mathcal{O} &\to \mathcal{O} \\ s &\mapsto x \cdot s \end{aligned}$$

for any $x \in \mathcal{O}(U)$, noting that $x \cdot s \in \mathcal{O}(U)$ again.

Exercise 4.2.12 ***(The kernel of a sheaf morphism is a sheaf)***
Check that $\ker \varphi$ is naturally a sheaf and $\ker(\varphi)(U) = \ker(\varphi(U)) : \mathcal{F}(U) \to \mathcal{G}(U)$

Here the kernel is trivial, i.e. on any open U we have $(x\cdot) : \mathcal{O}(U) \hookrightarrow \mathcal{O}(U)$ is injective. Taking the cokernel $\operatorname{coker}(x\cdot)$ as a presheaf, this assigns to U the quotient presheaf $\mathcal{O}(U)/x\mathcal{O}(U)$, which turns out to be equal to $\mathbb{R}_0$. So $\mathcal{O} \to \mathbb{R}_0$ by restricting to the value at 0, and there is an exact sequence

$$0 \to \mathcal{O} \xrightarrow{(x\cdot)} \mathcal{O} \to \mathbb{R}_0 \to 0.$$

This is one reason sheaves are better than vector bundles: the category is closed under taking quotients, whereas quotients of vector bundles may not be vector bundles.

5 | Lecture 4 (Friday, January 22)

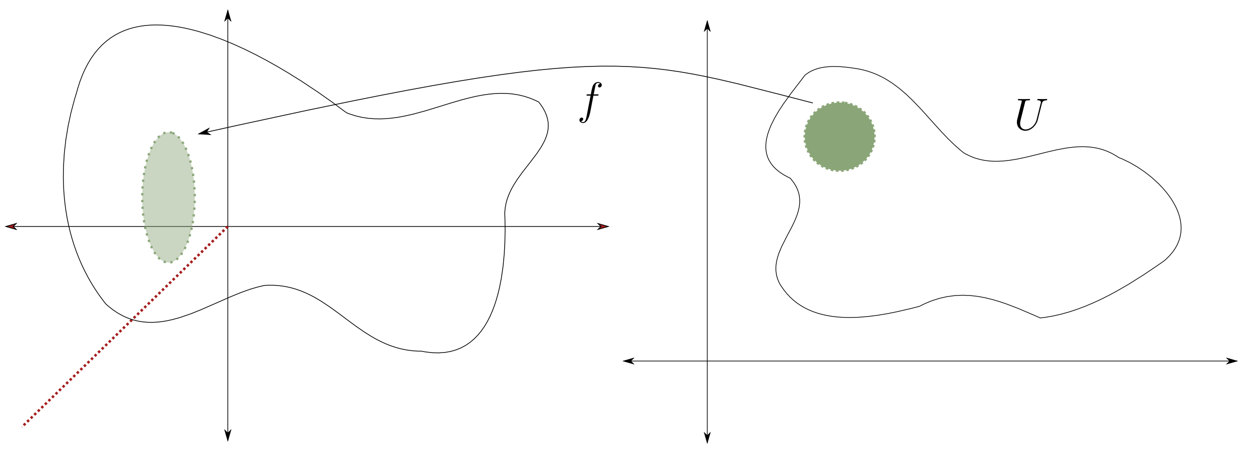
5.1 The Exponential Exact Sequence

Remark 5.1.1: Let $X = \mathbb{C}$ and consider $\mathcal{O}$ the sheaf of holomorphic functions and $\mathcal{O}^\times$ the sheaf of *nonvanishing* holomorphic functions. The former is a vector bundle and the latter is a sheaf of abelian groups. There is a map $\exp : \mathcal{O} \to \mathcal{O}^\times$, the **exponential map**, which is the data $\exp(U) : \mathcal{O}(U) \to \mathcal{O}^\times(U)$ on every open U given by $f \mapsto e^f$. There is a kernel sheaf $2\pi i\underline{\mathbb{Z}}$, and we get an exact sequence

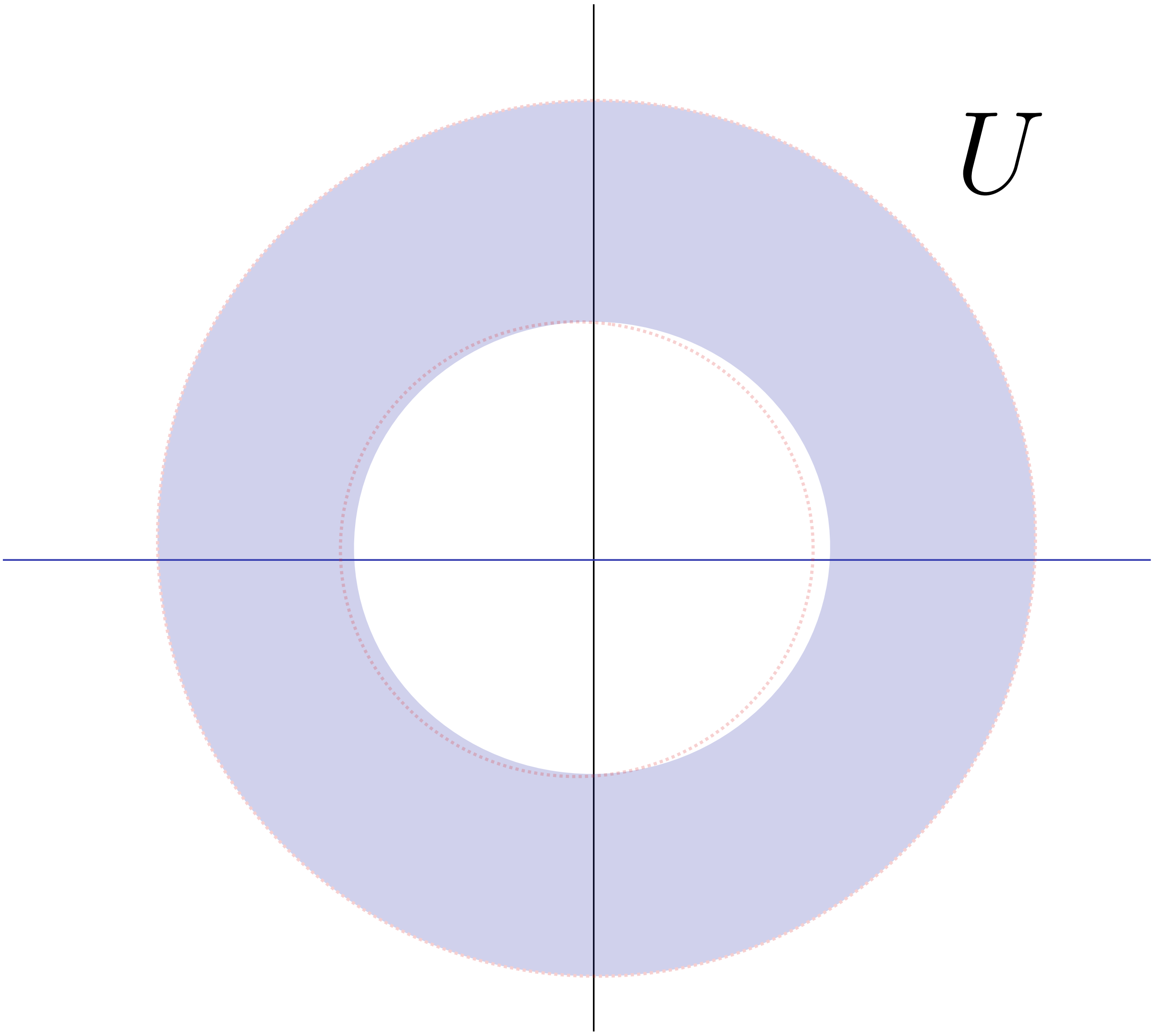
$$0 \to 2\pi i \underline{\mathbb{Z}} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to \operatorname{coker}(\exp) \to 0.$$

Question 5.1.2
What is the cokernel sheaf here?

Remark 5.1.3: Let U be a contractible open set, then we can identify $\mathcal{O}^\times(U)/\exp(\mathcal{O}^\times(U)) = 1$.



Any $f \in \mathcal{O}^\times(U)$ has a logarithm, say by taking a branch cut, since $\pi_1(U) = 0 \implies \log f$ has an analytic continuation. Consider the annulus U and the function $z \in \mathcal{O}^\times(U)$, then $z \notin \exp(\mathcal{O}(U))$ – if $z = e^f$ then $f = \log(z)$, but $\log(z)$ has monodromy on U:



Thus on any sufficiently small open set, coker(exp) = 1. This is only a presheaf: there exists an open cover of the annulus for which $z|_{U_i}$, and so the naive cokernel doesn't define a sheaf. This is because we have a locally trivial section which glues to z, which is nontrivial.

Exercise 5.1.4 (Fixing the sheaf cokernel)
Redefine the cokernel so that it is a sheaf. Hint: look at sheafification, which has the defining property

$$\operatorname*{Hom}_{\substack{\mathsf{Sh} \\ \mathsf{pre}}}(\mathcal{G}, \mathcal{F}^{\mathsf{pre}}_{\mathsf{Sh}}) = \operatorname*{Hom}_{\mathsf{Sh}}(\mathcal{G}, \mathcal{F}^{\mathsf{Sh}})$$

for any sheaf $\mathcal{G}$.

5.2 Global Sections

Definition 5.2.1 (Global Sections Sheaf)
The **global sections** sheaf of $\mathcal{F}$ on X is given by $H^0(X;\mathcal{F}) = \mathcal{F}(X)$.

Example 5.2.2 ***(?):***

- $C^\infty(X) = H^0(X, C^\infty)$ are the smooth functions on X
- $VF(X) = H^0(X;T)$ are the smooth vector fields on X for T the tangent bundle
- If X is a complex manifold then $\mathcal{O}(X) = H^0(X;\mathcal{O})$ are the globally holomorphic functions on X.
- $H^0(X;\mathbb{Z}) = \underline{\mathbb{Z}}(X)$ are ??

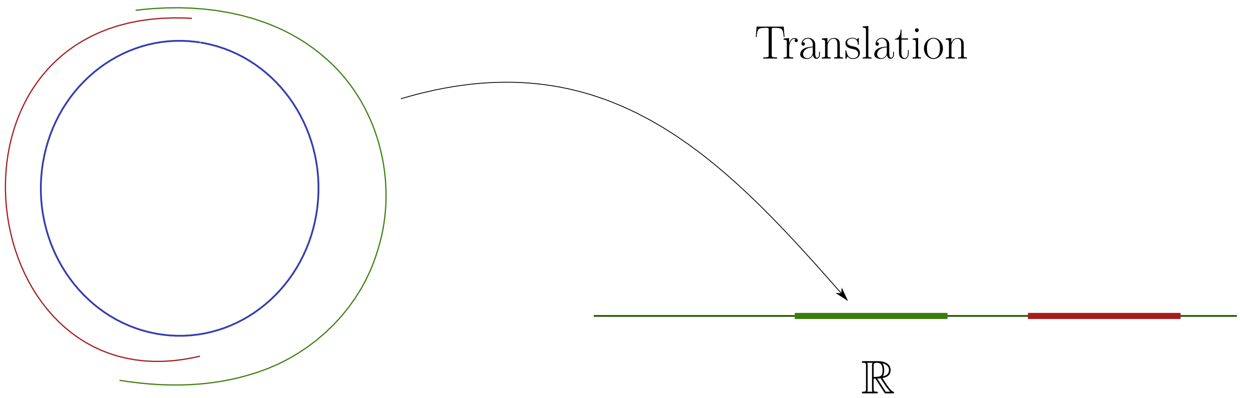
Remark 5.2.3: Given vector bundles V, W, we have constructions $V \oplus W, V \otimes W, V^\vee, \operatorname{Hom}(V,W) = V^\vee \otimes W, \operatorname{Sym}^n V, \bigwedge^p V$, and so on. Some of these work directly for sheaves:

- $\mathcal{F} \oplus \mathcal{G}(U) := \mathcal{F}(U) \oplus \mathcal{G}(U)$
- For tensors, duals, and homs $\mathscr{H}\mathrm{om}(V,W)$ we only get presheaves, so we need to sheafify.

Warning 5.2.4
$\operatorname{Hom}(V,W)$ will denote the *global* homomorphisms $\mathscr{H}\mathrm{om}(V,W)(X)$, which is a sheaf.

Example 5.2.5 ***(?):*** Let $X^n \in \mathsf{Mfd}_{\mathsf{sm}}$ and let Ω^p be the sheaf of smooth p-forms, i.e $\bigwedge^p T^\vee$, i.e. $\Omega^p(U)$ are the smooth p forms on U, which are locally of the form $\sum f_{i_1,\cdots,i_p}(x_1,\cdots,x_n)dx_{i_1}\wedge dx_{i_2}\wedge\cdots dx_{i_p}$ where the $f_{i_1,\cdots,i_p}$ are smooth functions.

Example 5.2.6 ***(Sub-example):*** Take $X = S^1$, writing this as $\mathbb{R}/\mathbb{Z}$, we have $\Omega^1(X) \ni dx$. There are two coordinate charts which differ by a translation on their overlaps, and $dx(x+c) = dx$ for c a constant:

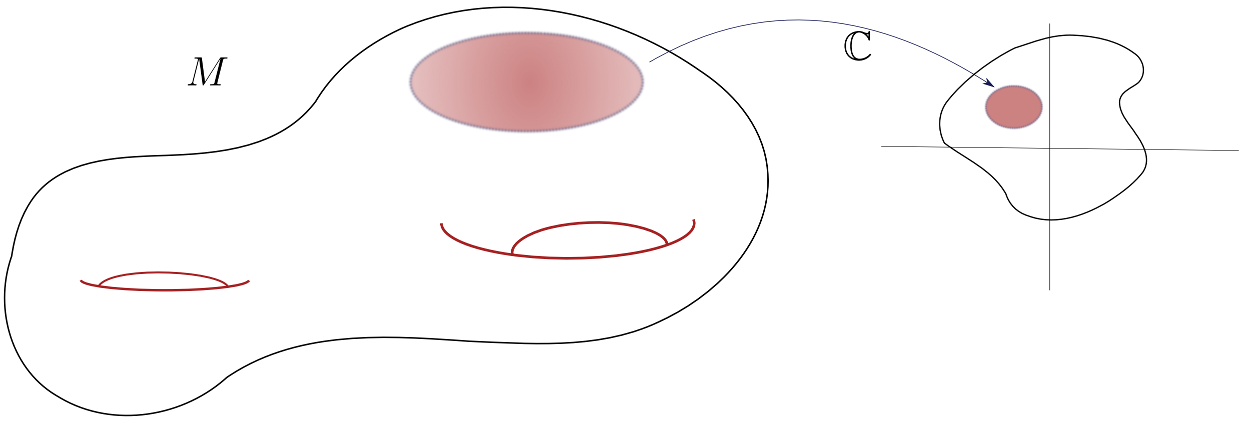


Exercise 5.2.7 ***(?)***
Check that on a torus, dx_i is a well-defined 1-form.

Remark 5.2.8: Note that there is a map $d : \Omega^p \to \Omega^{p+1}$ where $\omega \mapsto d\omega$.

⚠ Warning 5.2.9
d is **not** a map of $\mathcal{O}$-modules: $d(f \cdot \omega) = f \cdot \omega + df \wedge \omega$, where the latter is a correction term. In particular, it is not a map of vector bundles, but is a map of sheaves of abelian groups since $d(\omega_1 + \omega_2) = d(\omega_1) + d(\omega_2)$, making d a sheaf morphism.

Remark 5.2.10: Let $X \in \mathsf{Mfd}_{\mathbb{C}}$, we'll use the fact that TX is complex-linear and thus a $\mathbb{C}$-vector bundle.



Remark 5.2.11 ***(Subtlety 1):*** Note that Ω^p for complex manifolds is $\bigwedge^p T^\vee$, and so if we want to view $X \in \mathsf{Mfd}_{\mathbb{R}}$ we'll write $X_{\mathbb{R}}$. $TX_{\mathbb{R}}$ is then a real vector bundle of rank $2n$.

Remark 5.2.12 ***(Subtlety 2):*** Ω^p will denote *holomorphic* p-forms, i.e. local expressions of the form

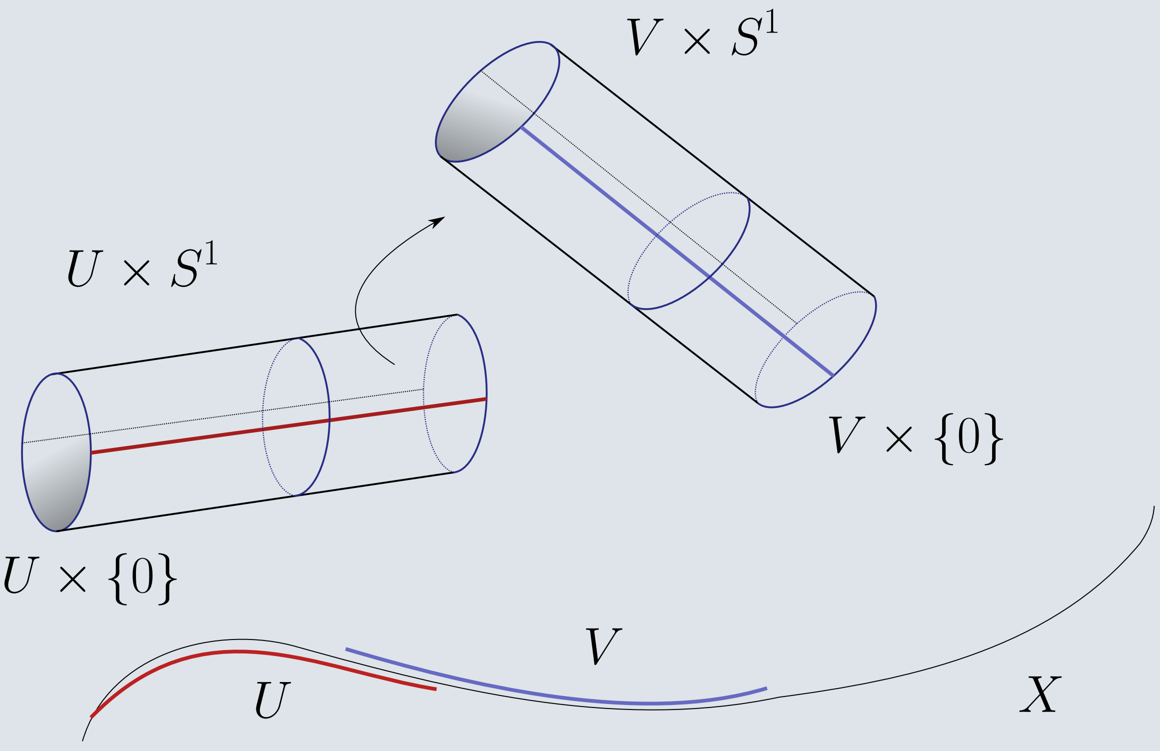
$$\sum f_I(z_1, \cdots, z_n) \bigwedge dz_I.$$

For example, $e^z dz \in \Omega^1(\mathbb{C})$ but $z\bar{z}dz$ is not, where $dz = dx + idy$. We'll use a different notation when we allow the f_I to just be smooth: $A^{p,0}$, the sheaf of $(p,0)$-forms. Then $z\bar{z}dz \in A^{1,0}$.

Remark 5.2.13: Note that $T^\vee X_{\mathbb{R}} \otimes_{\mathbb{C}} = A^{1,0} \oplus A^{0,1}$ since there is a unique decomposition $\omega = fdz + gd\bar{z}$ where f, g are smooth. Then $\Omega^d X_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=d} A^{p,q}$. Note that $\Omega^p_{\mathsf{sm}} \neq A^{p,q}$ and these are really quite different: the former are more like holomorphic bundles, and the latter smooth. Moreover $\dim \Omega^p(X) < \infty$, whereas Ω^1_{sm} is infinite-dimensional.

6 Principal G-Bundles and Connections (Monday, January 25)

Definition 6.0.1 (Principal Bundles)
Let G be a (possibly disconnected) Lie group. Then a **principal G-bundle** $\pi : P \to X$ is a space admitting local trivializations $h_u : \pi^{-1}(U) \to G \times U$ such that the transition functions are given by left multiplication by a continuous function $t_{UV} : U \cap V \to G$.



Remark 6.0.2: Setup: we'll consider TX for $X \in \mathsf{Mfd}_{\mathrm{Sm}}$, and let g be a metric on the tangent bundle given by

$$g_p : T_p X^{\otimes 2} \to \mathbb{R},$$

a symmetric bilinear form with $g_p(u, v) \geq 0$ with equality if and only if $v = 0$.

Definition 6.0.3 (The Frame Bundle)
Define $\underset{p}{\mathrm{Frame}}(X) := \{\text{bases of } T_p X\}$, and $\mathrm{Frame}(X) := \bigcup_{p \in X} \underset{p}{\mathrm{Frame}}(X)$.

Remark 6.0.4: More generally, $\mathrm{Frame}(\mathcal{E})$ can be defined for any vector bundle $\mathcal{E}$, so $\mathrm{Frame}(X) := \mathrm{Frame}(TX)$. Note that $\mathrm{Frame}(X)$ is a principal $\mathrm{GL}_n(\mathbb{R})$-bundle where $n := \mathrm{rank}(\mathcal{E})$. This follows from the fact that the transition functions are fiberwise in $\mathrm{GL}_n(\mathbb{R})$, so the transition functions are given by left-multiplication by matrices.

Remark 6.0.5 ***(Important):*** A principal G-bundle admits a G-action where G acts by *right* multiplication:

$$\begin{aligned} P \times G &\to P \\ ((g,x),h) &\mapsto (gh,x). \end{aligned}$$

This is necessary for compatibility on overlaps. **Key point**: the actions of left and right multiplication commute.

Definition 6.0.6 (Orthogonal Frame Bundle)
The **orthogonal frame bundle** of a vector bundle $\mathcal{E}$ equipped with a metric g is defined as $\underset{p}{\mathrm{OFrame}}(\mathcal{E}) := \{\text{orthonormal bases of } \mathcal{E}_p\}$, also written $O_r(\mathbb{R})$ where $r := \mathrm{rank}(\mathcal{E})$.

Remark 6.0.7: The fibers $P_x \to \{x\}$ of a principal G-bundle are naturally **torsors** over G, i.e. a set with a free transitive G-action.

Definition 6.0.8 (Hermitian metric)
Let $\mathcal{E} \to X$ be a complex vector bundle. Then a **Hermitian metric** is a hermitian form on every fiber, i.e.

$$h_p : \mathcal{E}_p \times \overline{\mathcal{E}_p} \to \mathbb{C}.$$

where $h_p(v, \bar{v}) \geq 0$ with equality if and only if $v = 0$. Here we define $\overline{\mathcal{E}_p}$ as the fiber of the complex vector bundle $\overline{\mathcal{E}}$ whose transition functions are given by the complex conjugates of those from $\mathcal{E}$.

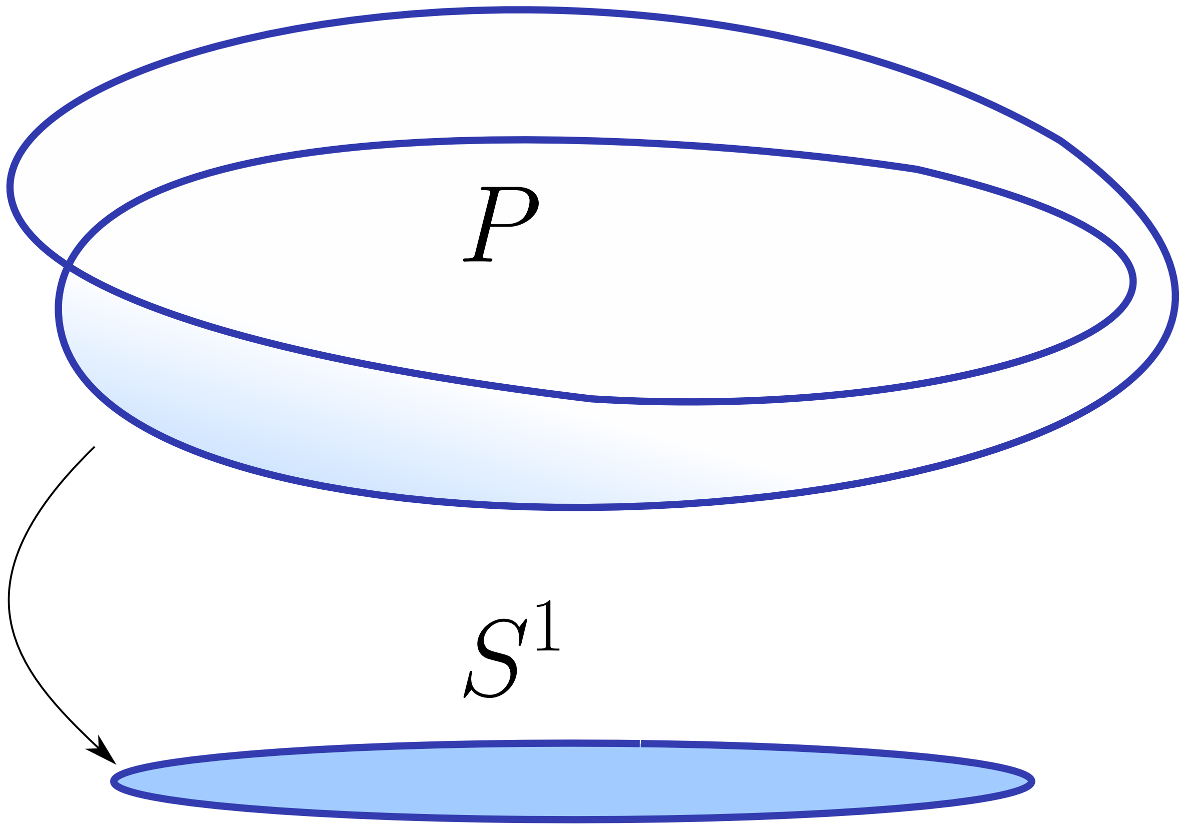
Remark 6.0.9: Note that $\mathcal{E}, \overline{\mathcal{E}}$ are genuinely different as complex bundles. There is a *conjugate-linear* map given by conjugation, i.e. $L(cv) = \bar{c}L(v)$, where the canonical example is

$$\begin{aligned} \mathbb{C}^n &\to \mathbb{C}^n \\ (z_1, \cdots, z_n) &\mapsto (\overline{z_1}, \cdots, \overline{z_n}). \end{aligned}$$

Definition 6.0.10 (Unitary Frame Bundle)
We define the **unitary frame bundle** $\mathrm{UFrame}(\mathcal{E}) := \bigcup_p \mathrm{UFrame}(\mathcal{E})_p$, where at each point this is given by the set of orthogonal frames of $\mathcal{E}_p$ given by $(e_1, \cdots, e_n)$ where $h(e_i, \overline{e_j}) = \delta_{ij}$.

Remark 6.0.11: This is a principal G-bundle for $G = U_r(\mathbb{C})$, the invertible matrices $A_{/\mathbb{C}}$ satisfy $A\overline{A}^t = \mathrm{id}$.

Example 6.0.12 ***(of more principal bundles):*** For $G = \mathbb{Z}/2\mathbb{Z}$ and $X = S^1$, the Möbius band is a principal G-bundle:



Example 6.0.13 ***(more principal bundles):*** For $G = \mathbb{Z}/2\mathbb{Z}$, for any (possibly non-oriented) manifold X there is an **orientation principal bundle** P which is locally a set of orientations on U, i.e.

$$P := \left\{ (x, O) \;\middle|\; x \in X,\ O \text{ is an orientation of } T_p X \right\}.$$

Note that P is an oriented manifold, $P \to X$ is a local isomorphism, and has a canonical orientation. (?) This can also be written as $P = \operatorname{Frame}(X)/\operatorname{GL}_n^+(\mathbb{R})$, since an orientation can be specified by a choice of n linearly independent vectors where we identify any two sets that differ by a matrix of positive determinant.

Definition 6.0.14 (Associated Bundles)
Let $P \to X$ be a principal G-bundle and let $G \to \operatorname{GL}(V)$ be a continuous representation. The **associated bundle** is defined as

$$P \times_G V = \left\{ (p, v) \;\middle|\; p \in P,\ v \in V \right\} / \sim \qquad \text{where } (p, v) \sim (pg, g^{-1}v),$$

which is well-defined since there is a right action on the first component and a left action on the second.

Example 6.0.15 ***(?):*** Note that $\operatorname{Frame}(\mathcal{E})$ is a $\operatorname{GL}_r(\mathbb{R})$-bundle and the map $\operatorname{GL}_r(\mathbb{R}) \xrightarrow{\text{id}} \operatorname{GL}(\mathbb{R}^r)$ is a representation. At every fiber, we have $G \times_G V = (p, v)/\sim$ where there is a unique representative of this equivalence class given by (e, pv). So $P \times_G V_p \to \{p\} \cong V_x$.

Exercise 6.0.16 *(?)*
Show that $\mathrm{Frame}(\mathcal{E}) \times_{\mathrm{GL}_r(\mathbb{R})} \mathbb{R}^r \cong \mathcal{E}$. This follows from the fact that the transition functions of $P \times_G V$ are given by left multiplication of $t_{UV} : U \cap V \to G$, and so by the equivalence relation, $\operatorname{im} t_{UV} \in \mathrm{GL}(V)$.

Remark 6.0.17: Suppose that M^3 is an oriented Riemannian 3-manifold. Them $TM \to \mathrm{Frame}(M)$ which is a principal $\mathrm{SO}(3)$-bundle. The universal cover is the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$, so can the transition functions be lifted? This shows up for spin structures, and we can get a $\mathbb{C}^2$ bundle out of this.

7 | Wednesday, January 27

7.1 Bundles and Connections

Definition 7.1.1 (Connections)
Let $\mathcal{E} \to X$ be a vector bundle, then a **connection** on $\mathcal{E}$ is a map of sheaves of abelian groups

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$$

satisfying the *Leibniz rule*:

$$\nabla(fs) = f\nabla s + s \otimes ds$$

for all opens U with $f \in \mathcal{O}(U)$ and $s \in \mathcal{E}(U)$. Note that this works in the category of complex manifolds, in which case ∇ is referred to as a **holomorphic connection**.

Remark 7.1.2: A connection ∇ induces a map

$$\begin{aligned}\tilde{\nabla} : \mathcal{E} \otimes \Omega^p &\to \mathcal{E} \otimes \Omega^{p+1} \\ s \otimes \omega &\mapsto \nabla s \wedge w + s \otimes d\omega.\end{aligned}$$

where $\wedge : \Omega^p \otimes \Omega^1 \to \Omega^{p+1}$. The standard example is

$$\begin{aligned}d : \mathcal{O} &\to \Omega^1 \\ f &\mapsto df.\end{aligned}$$

where the induced map is the usual de Rham differential.

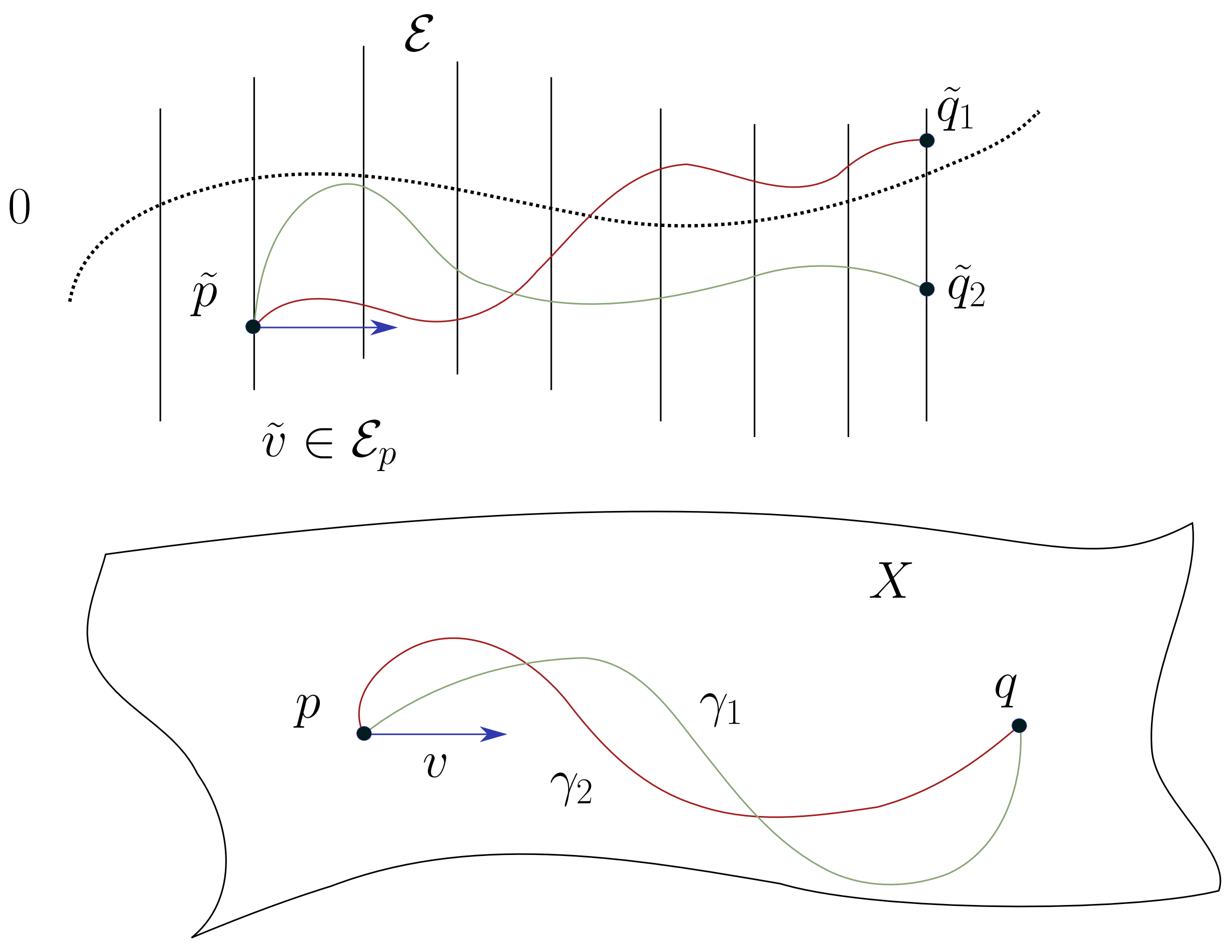
Exercise 7.1.3 (?)

Prove that the *curvature* of ∇, i.e. the map

$$F_\nabla := \nabla \circ \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^2$$

is $\mathcal{O}$-linear, so $F_\nabla(fs) = f\nabla \circ \nabla(s)$. Use the fact that $\nabla s \in \mathcal{E} \otimes \Omega^1$ and $\omega \in \Omega^p$ and so $\nabla s \otimes \omega \in \mathcal{E}\Omega^1 \otimes \Omega^p$ and thus reassociating the tensor product yields $\nabla s \wedge \omega \in \mathcal{E} \otimes \Omega^{p+1}$.

Remark 7.1.4: Why is this called a connection?

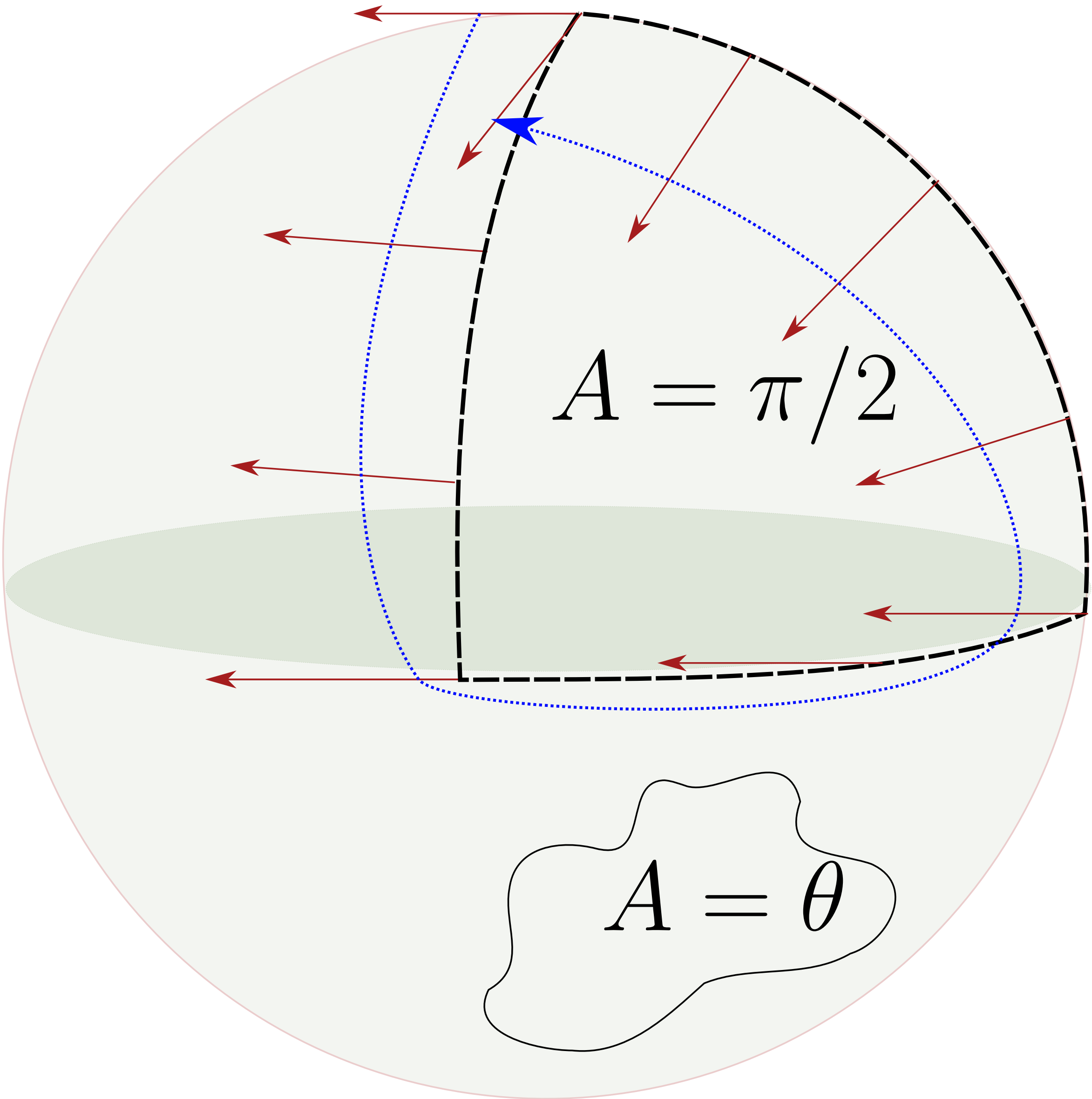


This gives us a way to transport $v \in \mathcal{E}_p$ over a path γ in the base, and ∇ provides a differential equation (a flow equation) to solve that lifts this path. Solving this is referred to as **parallel transport**. This works by pairing $\gamma'(t) \in T_{\gamma(t)}X$ with Ω^1, yielding $\nabla s = (\gamma'(t)) = s(\gamma(t))$ which are sections of γ.

Note that taking a different path yields an endpoint in the same fiber but potentially at a different point, and $F_\nabla = 0$ if and only if the parallel transport from p to q depends only on the homotopy class of γ.

Note: this works for any bundle, so can become confusing in Riemannian geometry when all of the bundles taken are tangent bundles!

Example 7.1.5 ***(A classic example):*** The Levi-Cevita connection ∇^{LC} on TX, which depends on a metric g. Taking $X = S^2$ and g is the round metric, there is nonzero curvature:



In general, every such transport will be rotation by some vector, and the angle is given by the area of the enclosed region.

Definition 7.1.6 (Flat Connection and Flat Sections)
A connection is **flat** if $F_\nabla = 0$. A section $s \in \mathcal{E}(U)$ is **flat** if it is given by

$$L(U) \coloneqq \left\{ s \in \mathcal{E}(U) \;\middle|\; \nabla s = 0 \right\}.$$

Exercise 7.1.7 (?)
Show that if ∇ is flat then L is a *local system*: a sheaf that assigns to any sufficiently small open set a vector space of fixed dimension. An example is the constant sheaf $\underline{\mathbb{C}^d}$. Furthermore $\operatorname{rank}(L) = \operatorname{rank}(\mathcal{E})$.

Remark 7.1.8: Given a local system, we can construct a vector bundle whose transition functions are the same as those of the local system, e.g. for vector bundles this is a fixed matrix, and in general these will be constant transition functions. Equivalently, we can take $L \otimes_{\mathbb{R}} \mathcal{O}$, and $L \otimes 1$ form flat sections of a connection.

7.2 Sheaf Cohomology

Definition 7.2.1 (Čech complex)
Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X, and let $\mathfrak{U} \coloneqq \{U_i\} \rightrightarrows X$ be an open cover of X. Let $U_{i_1, \cdots, i_p} \coloneqq U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_p}$. Then the **Čech Complex** is defined as

$$C^p_{\mathfrak{U}}(X, \mathcal{F}) \coloneqq \prod_{i_1 < \cdots < i_p} \mathcal{F}(U_{i_1, \cdots, i_p})$$

with a differential

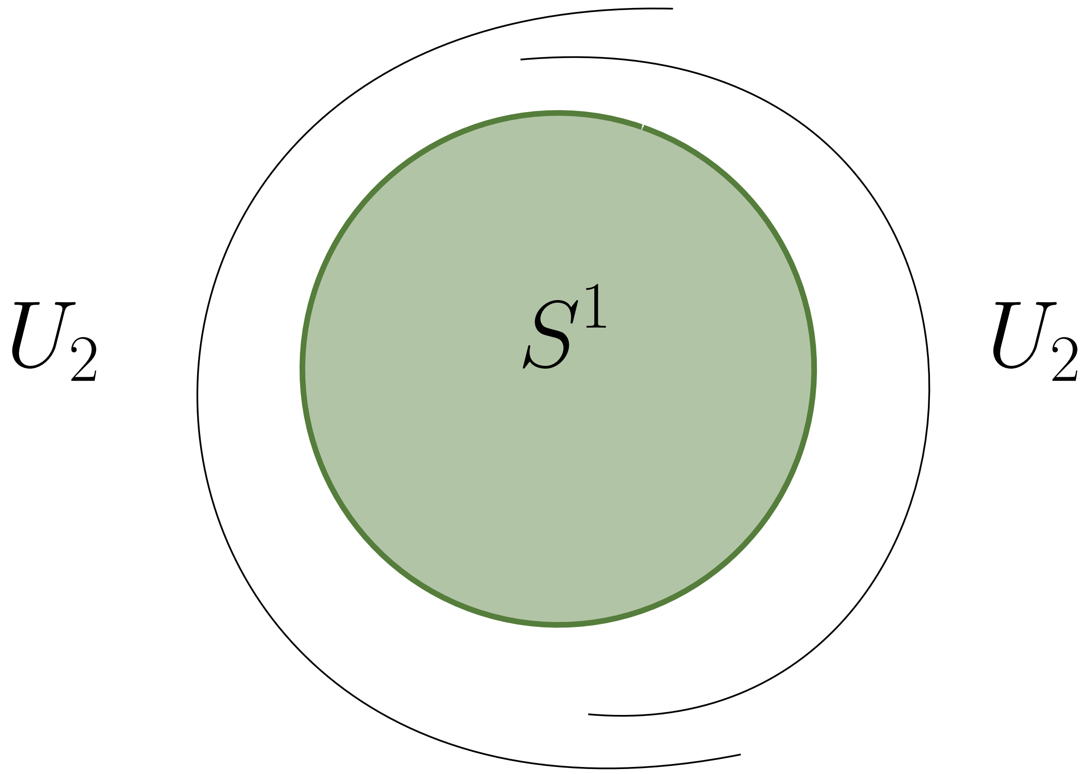
$$\begin{aligned} \partial^p : C^p_{\mathfrak{U}}(X, \mathcal{F}) &\to C^{p+1}_{\mathfrak{U}}(X\mathcal{F}) \\ \sigma &\mapsto (\partial\sigma)_{i_0, \cdots, i_p} \coloneqq \prod_j (-1)^j \sigma_{i_0, \cdots, \widehat{i_j}, \cdots, i_p}\Big|_{U_{i_0, \cdots, i_p}} \end{aligned}$$

where we've defined this just on one given term in the product, i.e. a p-fold intersection.

Exercise 7.2.2 (?)
Check that $\partial^2 = 0$.

Remark 7.2.3: The Čech cohomology $H^p_{\mathfrak{U}}(X, \mathcal{F})$ with respect to the cover $\mathfrak{U}$ is defined as $\ker \partial^p / \operatorname{im} \partial^{p-1}$. It is a difficult theorem, but we write $H^p(X, \mathcal{F})$ for the Čech cohomology for any sufficiently refined open cover when X is assumed paracompact.

Example 7.2.4*(?):* Consider S^1 and the constant sheaf $\underline{\mathbb{Z}}$:



ere we have

$$C^0(S^1, \underline{\mathbb{Z}}) = \underline{\mathbb{Z}}(U_1) \oplus \underline{\mathbb{Z}}(U_2) = \underline{\mathbb{Z}} \oplus \underline{\mathbb{Z}},$$

and

$$C^1(S^1, \mathbb{Z}) = \bigoplus_{\substack{\text{double} \\ \text{intersections}}} \underline{\mathbb{Z}}(U_{ij})\underline{\mathbb{Z}}(U_{12}) = \underline{\mathbb{Z}}(U_1 \cap U_2) = \underline{\mathbb{Z}} \oplus \underline{\mathbb{Z}}.$$

We then get

$$\begin{gathered} C^0(S^1, \underline{\mathbb{Z}}) \xrightarrow{\partial} C^1(S^1, \underline{\mathbb{Z}}) \\ \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \\ (a, b) \mapsto (a - b, a - b), \end{gathered}$$

Which yields $H^*(S^1, \underline{\mathbb{Z}}) = [\mathbb{Z}, \mathbb{Z}, 0, \cdots]$.

8 | Sheaf Cohomology (Friday, January 29)

Last time: we defined the Čech complex $C^p_{\mathfrak{U}}(X, \mathcal{F}) := \prod_{i_1, \cdots, i_p} \mathcal{F}(U_{i_1} \cap \cdots \cap U_{i_p})$ for $\mathfrak{U} := \{U_i\}$ is an open cover of X and F is a sheaf of abelian groups.

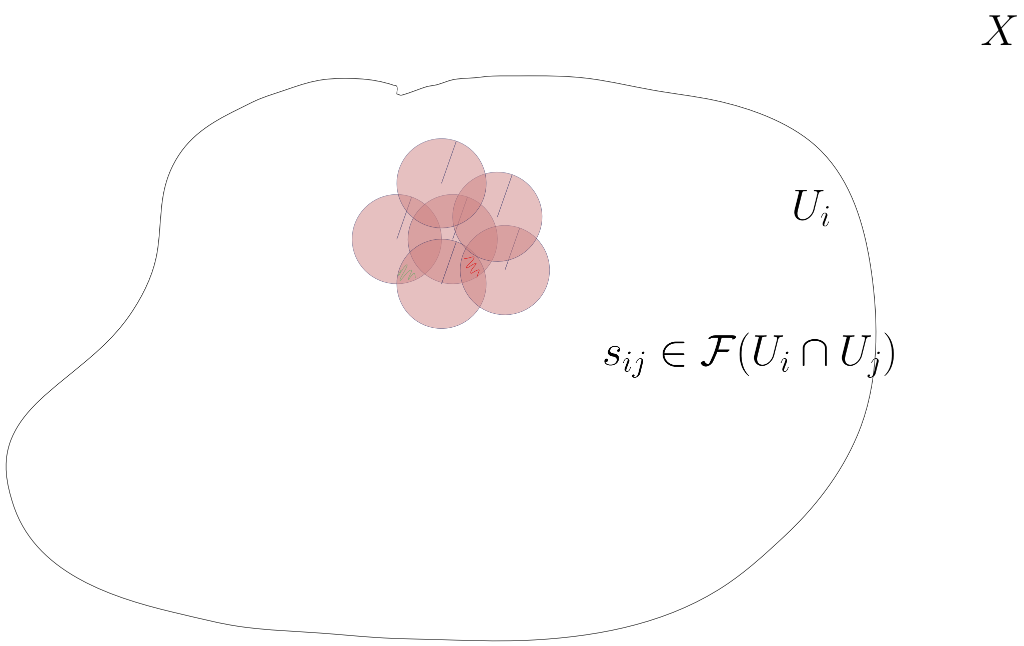
Fact 8.0.1
If $\mathfrak{U}$ is a sufficiently fine cover then $H^p_{\mathfrak{U}}(X, \mathcal{F})$ is independent of $\mathfrak{U}$, and we call this $H^p(X; \mathcal{F})$.

Remark 8.0.2: Recall that we computed $H^p(S^1, \underline{\mathbb{Z}} = [\mathbb{Z}, \mathbb{Z}, 0, \cdots]$.

Theorem 8.0.3 ***(When sheaf cohomology is isomorphic to singular cohomology).*** Let X be a paracompact and locally contractible topological space. Then $H^p(X, \underline{\mathbb{Z}}) \cong H^p_{\text{Sing}}(X, \mathbb{Z})$. This will also hold more generally with $\underline{\mathbb{Z}}$ replaced by $\underline{A}$ for any $A \in \mathsf{Ab}$.

Definition 8.0.4 (Acyclic Sheaves)
We say $\mathcal{F}$ is *acyclic* on X if $H^{>0}(X; \mathcal{F}) = 0$.

Remark 8.0.5: How to visualize when $H^1(X; \mathcal{F}) = 0$:



On the intersections, we have $\operatorname{im} \partial^0 = \left\{ (s_i - s_j)_{ij} \;\middle|\; s_i \in \mathcal{F}(U_i) \right\}$, which are *cocycles*. We have $C^1(X; \mathcal{F})$ are collections of sections of $\mathcal{F}$ on every double overlap. We can check that $\ker \partial^1 = \left\{ (s_{ij}) \;\middle|\; s_{ij} - s_{ik} + s_{jk} = 0 \right\}$, which is the cocycle condition. From the exercise from last class, $\partial^2 = 0$.

Theorem 8.0.6 ***((Important!)).***

Let X be a paracompact Hausdorff space and let

$$0 \to \mathcal{F}_1 \xrightarrow{\varphi} \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a SES of sheaves of abelian groups, i.e. $\mathcal{F}_3 = \operatorname{coker}(\varphi)$ and φ is injective. Then there is a LES in cohomology:

$$\begin{tikzcd}
0 \arrow[r] & H^0(X;\mathcal{F}_1) \arrow[r] & H^0(X;\mathcal{F}_2) \arrow[r] & H^0(X;\mathcal{F}_3) \arrow[dll] \\
& H^1(X;\mathcal{F}_1) \arrow[r] & H^1(X;\mathcal{F}_2) \arrow[r] & H^1(X;\mathcal{F}_3) \arrow[dll] \\
& \dots & &
\end{tikzcd}$$

Example 8.0.7 ***(?):*** For X a manifold, we can define a map and its cokernel sheaf:

$$0 \to \underline{\mathbb{Z}} \xrightarrow{\cdot 2} \underline{\mathbb{Z}} \to \underline{\mathbb{Z}/2\mathbb{Z}} \to 0.$$

Using that cohomology of constant sheaves reduces to singular cohomology, we obtain a LES in homology:

$$\begin{tikzcd}
0 \arrow[r] & H^0(X;\mathbb{Z}) \arrow[r] & H^0(X;\mathbb{Z}) \arrow[r] & H^0(X;\mathbb{Z}/2\mathbb{Z}) \arrow[dll] \\
& H^1(X;\mathbb{Z}) \arrow[r] & H^1(X;\mathbb{Z}) \arrow[r] & H^1(X;\mathbb{Z}/2\mathbb{Z}) \arrow[dll] \\
& \dots & &
\end{tikzcd}$$

Corollary 8.0.8 ***(of theorem).***
Suppose $0 \to \mathcal{F} \to I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \xrightarrow{d_2} \cdots$ is an exact sequence of sheaves, so on any sufficiently small set kernels equal images., and suppose I_n is acyclic for all $n \geq 0$. This is referred to as an **acyclic resolution**. Then the homology can be computed at $H^p(X;\mathcal{F}) = \ker(I_p(X) \to I_{p+1}(X)) / \operatorname{im}(I_{p-1}(X) \to I_p(X))$.

Note that locally having kernels equal images is different than satisfying this globally!

Proof (of corollary).
This is a formal consequence of the existence of the LES. We can split the LES into a collection of SESs of sheaves:

$$\begin{aligned} 0 \to \mathcal{F} \to I_0 \xrightarrow{d_0} \mathrm{im}(d_0) \to 0 &\qquad \mathrm{im}(d_0) = \ker(d_1) \\ 0 \to \ker(d_1) \hookrightarrow I_1 \to I_1/\ker(d_1) = \mathrm{im}(d_1) &\qquad \mathrm{im}(d_1) = \ker(d_2) \\ . & \end{aligned}$$

Note that these are all exact sheaves, and thus only true on small sets. So take the associated LESs. For the SES involving I_0, we obtain:

$$\begin{tikzcd} & & \cdots \arrow[dll] \\ H^{p-1}(\mathcal{F}) \arrow[r] & H^{p-1}(\mathcal{I}_{\prime}) = 0 \arrow[r] & H^{p-1}(\mathrm{im}(\lceil_{\prime})) \arrow[dll, "\cong"] \\ H^p(\mathcal{F}) \arrow[r] & \cdots = 0 & \end{tikzcd}$$

The middle entries vanish since I_* was assumed acyclic, and so we obtain $H^p(\mathcal{F}) \cong H^{p-1}(\mathrm{im}\, d_0) \cong H^{p-1}(\ker d_1)$. Now taking the LES associated to I_1, we get $H^{p-1}(\ker d_1) \cong H^{p-2}(\mathrm{im}\, d_1)$. Continuing this inductively, these are all isomorphic to $H^p(\mathcal{F}) \cong H^0(\ker d_p)/d_{p-1}(H^0(I_{p-1}))$ after the pth step.

■

Corollary 8.0.9 ***(of the previous corollary).***
Suppose $\mathfrak{U} \rightrightarrows X$, then if $\mathcal{F}$ is acyclic on each $U_{i_1, \cdots, i_p}$, then $\mathfrak{U}$ is sufficiently fine to compute Čech cohomology, and $H^p_{\mathfrak{U}}(X; \mathcal{F}) \cong H^p(X; \mathcal{F})$.

Proof (?).
See notes.

■

Corollary 8.0.10 ***(of corollary).***
Let $X \in \mathsf{Mfd}_{\mathsf{sm}}$, then $H^p(X, \underline{\mathbb{R}}) = H^p_{\mathrm{dR}}(X;\ RR)$.

Proof (?).
Idea: construct an acyclic resolution of the sheaf $\underline{\mathbb{R}}$ on M. The following exact sequence works:

$$0 \to \underline{\mathbb{R}} \to \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to \cdots .$$

So we start with locally constant functions, then smooth functions, then smooth 1-forms, and so on. This is an exact sequence of sheaves, but importantly, not exact on the total space. To check this, it suffices to show that $\ker d^p = \operatorname{im} d^{p-1}$ on any contractible coordinate chart. In other words, we want to show that if $d\omega = 0$ for $\omega \in \Omega^p(\mathbb{R}^n)$ then $\omega = d\alpha$ for some $\alpha \in \Omega^{p-1}(\mathbb{R}^n)$. This is true by integration! Using the previous corollary, $H^p(X; \underline{\mathbb{R}}) = \ker(\Omega^p(X) \xrightarrow{d} \Omega^{p+1}(X)) / \operatorname{im}(\Omega^{p-1}(X) \xrightarrow{d} \Omega^p(X))$.

■

Check Hartshorne to see how injective resolutions line up with derived functors!

9 | Monday, February 01

Remark 9.0.1: Last time $\underline{\mathbb{R}}$ on a manifold M has a resolution by vector bundles:

$$0 \to \underline{\mathbb{R}} \hookrightarrow \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots .$$

This is an exact sequence of sheaves of any smooth manifold, since locally $d\omega = 0 \implies \omega = d\alpha$ (by the *Poincaré d-lemma*). We also want to know that Ω^k is an acyclic sheaf on a smooth manifold.

Exercise 9.0.2 (?)
Let $X \in \mathit{Top}$ and $\mathcal{F} \in \mathsf{Sh}(\mathsf{Ab})_{/X}$. We say $\mathcal{F}$ is **flasque** if and only if for all $U \supseteq V$ the map $\mathcal{F}(U) \xrightarrow{\rho_{UV}} \mathcal{F}(V)$ is surjective. Show that $\mathcal{F}$ is acyclic, i.e. $H^i(X; \mathcal{F}) = 0$. This can also be generalized with a POU.

Example 9.0.3 ***(?):*** The function $1/x \in \mathcal{O}(\mathbb{R} \setminus \{0\})$, but doesn't extend to a continuous map on $\mathbb{R}$. So the restriction map is not surjective.

Remark 9.0.4: Any vector bundle on a smooth manifold is acyclic. Using the fact that Ω^k is acyclic and the above resolution of $\underline{\mathbb{R}}$, we can write $H^k(X; \mathbb{R}) = \ker(d_k) / \operatorname{im} d_{k-1} := H^k_{dR}(X; \mathbb{R})$.

Remark 9.0.5: Now letting $X \in \mathsf{Mfd}_{\mathbb{C}}$, recalling that Ω^p was the sheaf of holomorphic p-forms. Locally these are of the form $\sum_{|I|=p} f_I(\mathbf{z}) dz^I$ where $f_I(\mathbf{z})$ is holomorphic. There is a resolution

$$0 \to \Omega^p \to A^{p,0},$$

where in $A^{p,0}$ we allowed also f_I are *smooth*. These are the same as bundles, but we view sections differently. The first allows only holomorphic sections, whereas the latter allows smooth sections. What can you apply to a smooth $(p, 0)$ form to check if it's holomorphic?

Example 9.0.6 ***(?):*** For $p = 0$, we have

$$0 \to \mathcal{O} \to A^{0,0}.$$

where we have the sheaf of holomorphic functions mapping to the sheaf of smooth functions. We essentially want a version of checking the Cauchy-Riemann equations.

Definition 9.0.7 (∂ and $\bar{\partial}$ operators)
Let $\omega \in A^{p,q}(X)$ where

$$d\omega = \sum \frac{\partial f_I}{\partial z_j} dz^j \wedge dz^I \wedge d\bar{z}^J + \sum_j \frac{\partial f_I}{\partial \bar{z}_j} d\bar{z}^j \wedge dz^I d\bar{z}^J := \partial + \bar{\partial}$$

with $|I| = p, |J| = q$.

Example 9.0.8 ***(?):*** The function $f(z) = z\bar{z} \in A^{0,0}(\mathbb{C})$ is smooth, and $df = \bar{z}dz + zd\bar{z}$. This can be checked by writing $z^j = x^j + iy^j$ and $\bar{z}^j = x^j - iy_j$, and $\dfrac{\partial}{\partial \bar{z}} g = 0$ if and only if g is holomorphic. Here we get $\partial\omega \in A^{p+1,q}(X)$ and $\bar{\partial} \in A^{p,q+1}(X)$, and we can write $d(z\bar{z}) = \partial(z\bar{z}) + \bar{\partial}(z\bar{z})$.

Definition 9.0.9 (Cauchy-Riemann Equations)
Recall the Cauchy-Riemann equations: ω is a holomorphic $(p, 0)$-form on $\mathbb{C}^n$ if and only if $\bar{\partial}\omega = 0$.

Remark 9.0.10: Thus to extend the previous resolution, we should take

$$0 \to \Omega^p \hookrightarrow A^{p,0} \xrightarrow{\bar{\partial}} A^{p,1} \xrightarrow{\bar{\partial}} A^{p,2} \to \cdots .$$

The fact that this is exact is called the *Poincaré $\bar{\partial}$-lemma.*

Remark 9.0.11: There are no bump functions in the holomorphic world, and since Ω^p is a holomorphic bundle, it may not be acyclic. However, the $A^{p,q}$ *are* acyclic (since they are smooth vector bundles and thus admit POUs), and we obtain

$$H^q(X; \Omega^p) = \ker(\bar{\partial}_q) / \operatorname{im}(\bar{\partial}_{q-1}).$$

Note the similarity to H_{dR}, using $\bar{\partial}$ instead of d. This is called **Dolbeault cohomology**, and yields invariants of complex manifolds: the **Hodge numbers** $h^{p,q}(X) := \dim_{\mathbb{C}} H^q(X; \Omega^p)$. These are analogies:

Smooth	Complex
$\mathbb{R}$	Ω^p
Ω^k	$A^{p,q}$
Betti numbers β_k	Hodge numbers $h^{p,q}$

Note the slight overloading of terminology here!

Theorem 9.0.12 ***(Properties of Singular Cohomology).***
Let $X \in \mathsf{Top}$, then $H^i_{\text{Sing}}(X;\mathbb{Z})$ satisfies the following properties:

- Functoriality: given $f \in \underset{\mathsf{Top}}{\text{Hom}}(X,Y)$, there is a pullback $f^* : H^i(Y;\mathbb{Z}) \to H^i(X;\mathbb{Z})$.
- The cap product: a pairing

$$H^i(X;\mathbb{Z}) \otimes_{\mathbb{Z}} H_j(X;\mathbb{Z}) \to H_{j-i}(X;\mathbb{Z})$$
$$\varphi \otimes \sigma \mapsto \varphi\left(\sigma|_{\Delta_{0,\cdots,j}}\right) \sigma|_{\Delta_{i,\cdots,j}}.$$

  This makes H_* a module over H^*.

- There is a ring structure induced by the cup product:

$$H^i(X;\mathbb{R}) \times H^j(X;\mathbb{R}) \to H^{i+j}(X;\mathbb{R}) \qquad \alpha \cup \beta = (-1)^{ij} \beta \cup \alpha.$$

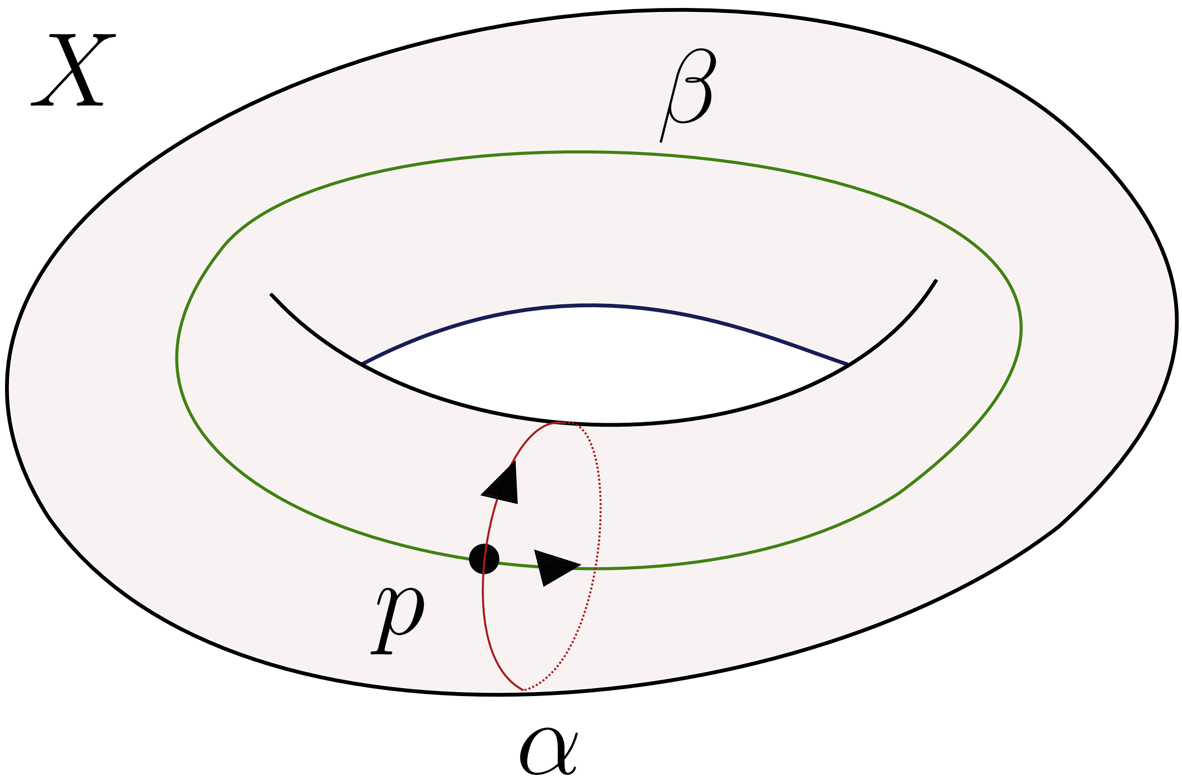
- Poincaré Duality: If X is an oriented manifold, there exists a fundamental class $[X] \in H_n(X;\mathbb{Z}) \cong \mathbb{Z}$ and $(-)\cap X : H^i \to H_{n-i}$ is an isomorphism.

Remark 9.0.13: Let $M \subset X$ be a submanifold where X is a smooth oriented n-manifold. Then $M \hookrightarrow X$ induces a pushforward $H_n(M;\mathbb{Z}) \xrightarrow{\iota_*} H_n(X;\mathbb{Z})$ where $\sigma \mapsto \iota \circ \sigma$. Using Poincaré duality, we'll identify $H_{\dim M}(X;\mathbb{Z}) \to H^{\operatorname{codim} M}(X;\mathbb{Z})$ and identify $[M] = PD(\iota_*([M]))$. In this case, if $M \pitchfork N$ then $[M] \cap [N] = [M \cap N]$, i.e. the cap product is given by intersecting submanifolds.

⚠ **Warning 9.0.14**
This can't always be done! There are counterexamples where homology classes can't be represented by submanifolds.

10 | Wednesday, February 03

Consider an oriented surface, and take two oriented submanifolds

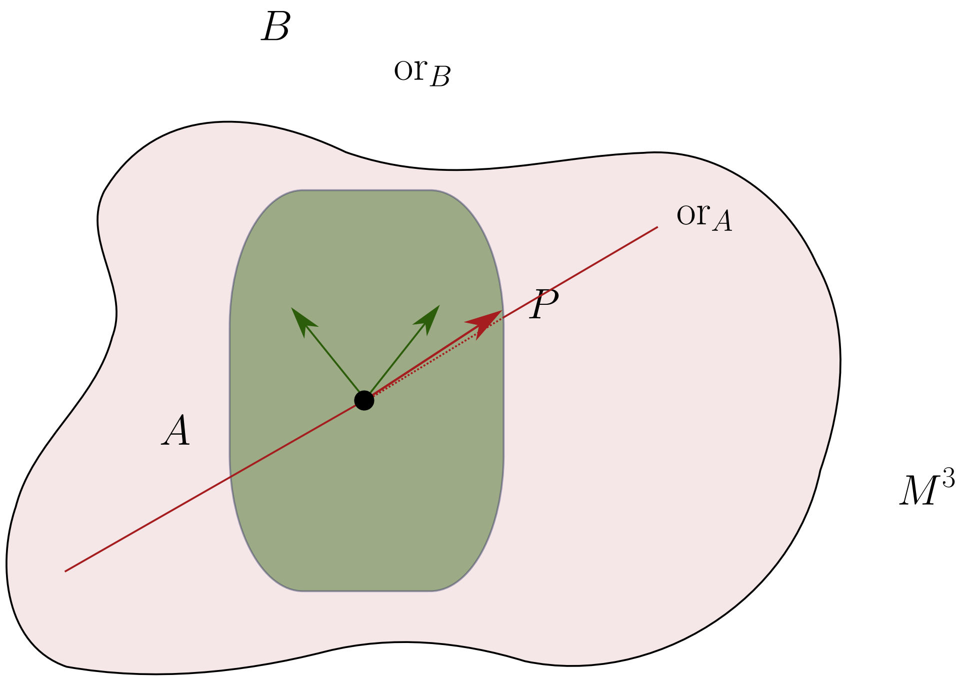


We can then take the fundamental classes of the submanifolds, say $[\alpha], [\beta] \in H^1(X;\mathbb{Z}) \xrightarrow{PD} H^1(X,\mathbb{Z})$. Here $T_p\alpha \oplus T_p\beta = T_pX$, since the intersections are transverse. Since α, β are oriented, let $\{e\}$ be a basis of $T_p\alpha$ (up to $\mathbb{R}^+$) and similarly $\{f\}$ a basis of $T_p\beta$. We can then ask if $\{e, f\}$ constitutes an *oriented* basis of T_pX. If so, we write $\alpha \cdot_p \beta := +1$ and otherwise $\alpha \cdot_p \beta = -1$. We thus have

$$[\alpha] \smile [\beta] \in H^2(X;\mathbb{Z}) \xrightarrow{PD} H_0(X;\mathbb{Z}) = \mathbb{Z}$$

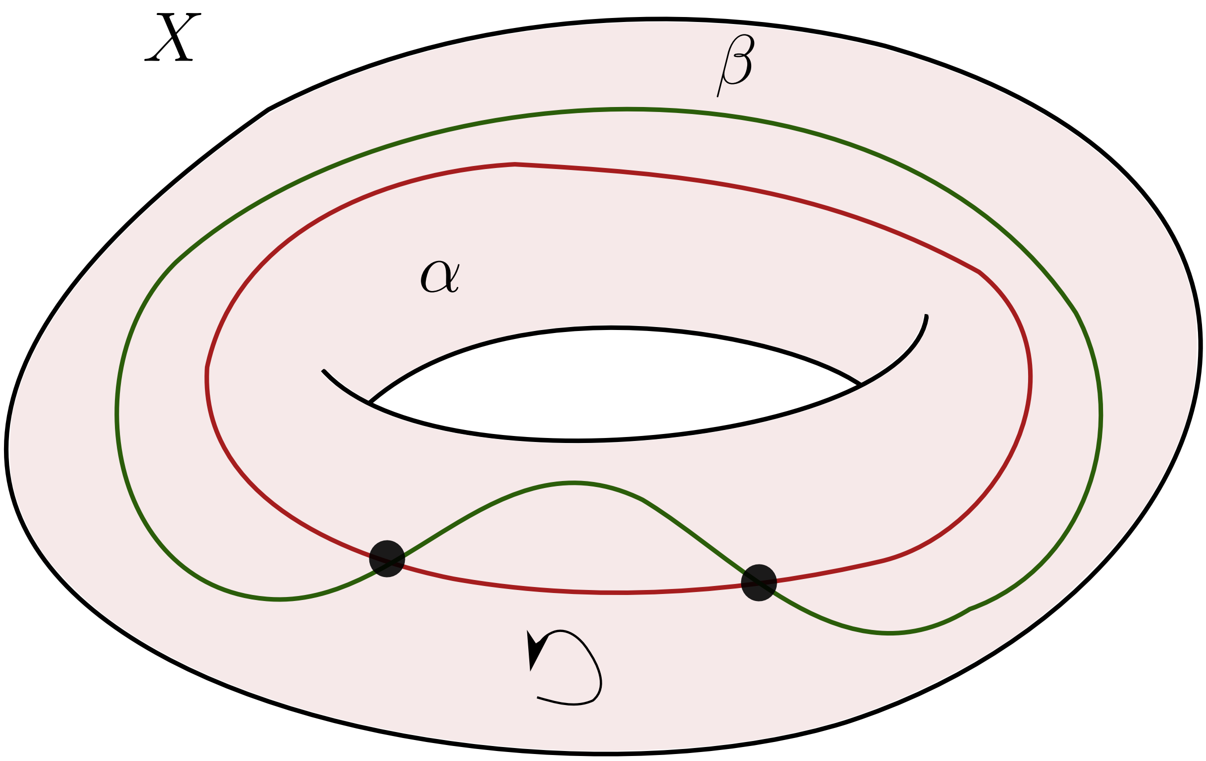
since X is connected. We can thus define the **intersection form** $\alpha \cdot \beta := [\alpha] \smile [\beta]$. In general if A, B are oriented transverse submanifolds of M which are themselves oriented, we'll have $[A] \smile [B] = [A \cap B]$. We need to be careful: how do we orient the intersection? This is given by comparing the orientations on A and B as before.

Example 10.0.1 ***(?):*** If $\dim M = \dim A + \dim B$, then any $p \in A \cap B$ is oriented by comparing $\{\mathrm{or}_A, \mathrm{or}_B\}$ to or_M.



Here it suffices to check that $\{e, f_1, f_2\}$ is an oriented basis of T_pM.

Example 10.0.2*(?):* In this case, $[\alpha] \smile [\beta] = 0$ and so $\alpha \cdot \beta = 0$:



Remark 10.0.3: Note that cohomology with $\mathbb{Z}$ coefficients can be defined for any topological space, and Poincaré duality still holds.

Remark 10.0.4: We'll be considering $M = M^4$, smooth 4-manifolds. How to visualize: take a 3-manifold and cross it with time!

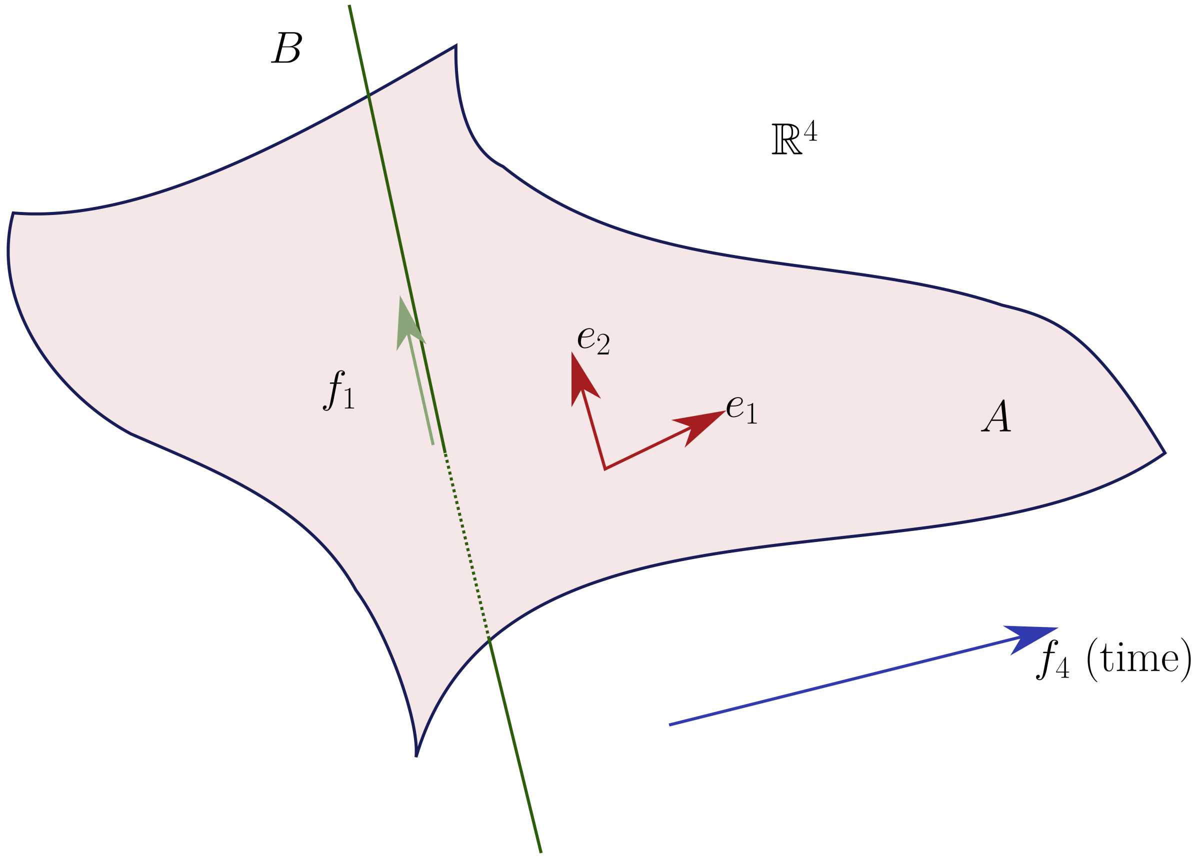


Figure 1: Picking one basis element in the time direction

Here ? is oriented in the "forward time" direction, and this is a surface at time $t = 0$. Where $A \cdot B = +1$, since $\{e_1, e_2, f_1, f_2\} = \{e_x, e_y, e_z, e_t\}$ is a oriented basis for $\mathbb{R}^4$. For $?^2$, switching the order of α, β no longer yields an oriented basis, but in this case it is ? and $A \cdot B = B \cdot A$. This is because

$$A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies \det(A) = -1 \qquad\qquad \det \begin{bmatrix} A & \\ & A \end{bmatrix} = 1.$$

Remark 10.0.5: Let M^{2n} be an oriented manifold, then the cup product yields a bilinear map $H^n(M; \mathbb{Z}) \otimes H^n(M; \mathbb{Z}) \to \mathbb{Z}$ which is symmetric when n is odd and antisymmetric (or symplectic) when n is even. This is a **perfect** (or **unimodular**) pairing (potentially after modding out by torsion) which realizes an isomorphism:

$$\begin{aligned} (H^n(M; \mathbb{Z})/\text{tors})^\vee &\xrightarrow{\sim} H^n(M; \mathbb{Z})/\text{tors} \\ \alpha \smile - &\hookleftarrow \alpha, \end{aligned}$$

where the LHS are linear functionals on cohomology.

Remark 10.0.6: Recall the universal coefficients theorem:

$$H^i(X;\mathbb{Z})/\text{tors} \cong (H_i(X;\mathbb{Z})/\text{tors})^\vee.$$

The general theorem shows that $H^i(X;\mathbb{Z})_{\text{tors}} = H_{i-1}(X;\mathbb{Z})_{\text{tors}}$.

Remark 10.0.7: Note that if M is an oriented 4-manifold, then

$$\begin{array}{ccc ccccc}
 & \text{tors} & \text{torsionfree} & & & \text{tors} & \text{torsionfree} \\
H^0 & 0 & \mathbb{Z} & & H_0 & 0 & \mathbb{Z} \\
H^1 & 0 & \mathbb{Z}^{\beta_1} & & H_1 & A & \mathbb{Z}^{\beta_1} \\
H^2 & A & \mathbb{Z}^{\beta_2} & \xrightarrow{PD} & H_2 & A & \mathbb{Z}^{\beta_2} \\
H^3 & A & \mathbb{Z}^{\beta_1} & & H_3 & 0 & \mathbb{Z}^{\beta_1} \\
H^4 & 0 & \mathbb{Z} & & H_4 & 0 & \mathbb{Z}
\end{array}$$

In particular, if M is simply connected, then $H_1(M) = \mathsf{Ab}(\pi_1(M)) = 0$, which forces $A = 0$ and $\beta_1 = 0$.

Definition 10.0.8 (Lattice)
A **lattice** is a finite-dimensional free $\mathbb{Z}$-module L together with a symmetric bilinear form

$$\begin{aligned} \cdot : L^{\otimes 2} &\to \mathbb{Z} \\ \ell \otimes m &\mapsto \ell \cdot m. \end{aligned}$$

The lattice $(L, \cdot)$ is **unimodular** if and only if the following map is an isomorphism:

$$\begin{aligned} L &\to L^\vee \\ \ell &\mapsto \ell \cdot (-). \end{aligned}$$

Remark 10.0.9: How to determine if a lattice is unimodular: take a basis $\{e_1, \cdots, e_n\}$ of L and form the *Gram matrix* $M_{ij} := (e_i \cdot e_j) \in \text{Mat}(n \times n, \mathbb{Z})^{\text{Sym}}$. Then $(L, \cdot)$ is unimodular if and only if $\det(M) = \pm 1$ if and only if M^{-1} is integral. In this case, the rows of M^{-1} will form a basis of the dual basis.

Definition 10.0.10 (Index of a lattice)
The **index** of a lattice is $|\det M|$.

Exercise 10.0.11 (?)
Prove that $|\det M| = |L^\vee / L|$.

Remark 10.0.12: In general, for M^{4k}, the H^{2k}/tors is unimodular. For M^{4k+2}, the H^{2k+1}/tors is a unimodular *symplectic* lattice, which is obtained by replacing the word "symmetric" with "antisymmetric" everywhere above.

Example 10.0.13*(?):* For the torus, since the dimension is $2 \pmod 4$, you get the skew-symmetric matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Check!

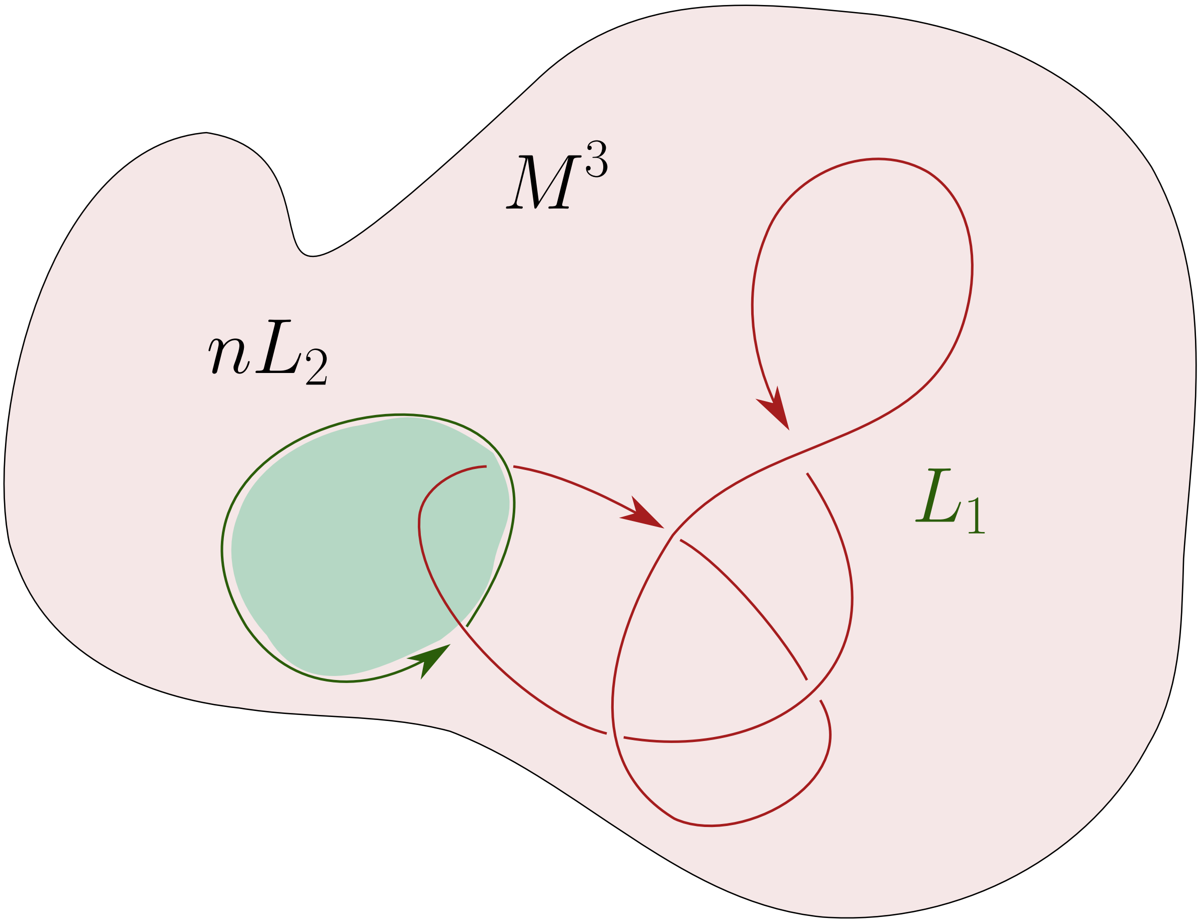
Definition 10.0.14 (Nondegenerate lattices)
A lattice is **nondegenerate** if $\det M \neq 0$.

Definition 10.0.15 (Base change of lattices)
The tensor product $L \otimes_{\mathbb{Z}} \mathbb{R}$ is a vector space with an $\mathbb{R}$-valued symmetric bilinear form. This allows extending the lattice from $\mathbb{Z}^n$ to $\mathbb{R}^n$.

Remark 10.0.16: If $(L, \cdot)$ is nondegenerate, then Gram-Schmidt will yield an orthonormal basis $\{v_i\}$. The number of positive norm vectors is an invariant, so we obtain $\mathbb{R}^{p,q}$ where p is the number of $+1$s in the Gram matrix and q is the number of -1s. The **signature** of $(L, -)$ is (p, q), or by abuse of notation $p - q$. This is an invariant of the 4-manifold, as is the lattice itself $H^2(X; \mathbb{Z})/\mathrm{tors}$ equipped with the intersection form.

Remark 10.0.17: There is a perfect pairing called the **linking pairing**:

$$H^i(X; \mathbb{Q}/\mathbb{Z}) \otimes H^{n-i-1}(X; \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}.$$



Remark 10.0.18: $A \cdot B \coloneqq \sum_{p \in A \cap B} \operatorname{sgn}_p(A, B)$, where $A \pitchfork B$ and this turns out to be equal to the cup product. This works for topological manifolds – but there are no tangent spaces there, so taking oriented bases doesn't work so well! You can also view

$$[A] \smile [\omega] = \int_A \omega.$$

11 | Friday, February 05

Remark 11.0.1: Recall that a lattice is **unimodular** if the map $L \to L^\vee \coloneqq \operatorname{Hom}(L, \mathbb{Z})$ is an isomorphism, where $\ell \mapsto \ell \cdot (-)$. To check this, it suffices to check if the Gram matrix M of a basis $\{e_i\}$ satisfies $|\det M| = 1$.

Example 11.0.2 ***(Determinant 1 Integer Matrices):*** The matrices $[1]$ and $[-1]$ correspond to the lattice $\mathbb{Z}e$ where either $e^2 \coloneqq e \cdot e = 1$ or $e^2 = -1$. If M_1, M_2 both have absolute determinant 1, then so does

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}.$$

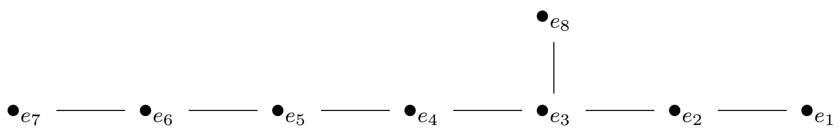
So if L_1, L_2 are unimodular, then taking an orthogonal sum $L_1 \oplus L_2$ also yields a unimodular lattice. So this yields diagonal matrices with p copies of $+1$ and q copies of -1. This is referred to as $rm1_{p,q}$, and is an *odd* unimodular lattice of signature (p,q) (after passing to $\mathbb{R}$). Here *odd* means that there exists a $v \in L$ such that v^2 is odd.

Example 11.0.3 ***(Even unimodular lattices):*** An even lattice must have no vectors of odd norm, so all of the diagonal elements are in $2\mathbb{Z}$. This is because $(\sum n_i e_i)^2 = \sum_i n_i^2 e_i^2 + \sum_{i<j} 2n_i, n_j e_i \cdot e_j$. Note that the matrix must be symmetric, and one example that works is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We'll refer to this lattice as H, sometimes referred to as the *hyperbolic cell* or *hyperbolic plane.*

Example 11.0.4 ***(A harder even unimodular lattice):*** This is built from the E_8 Dynkin diagram:



The rule here is

$$e_i \cdot e_j = \begin{cases} -2 & i = j \\ 1 & e_i \to e_j \\ 0 & \text{if not connected.} \end{cases}$$

So for example, $e_2 \cdot e_6 = 0, e_1 \cdot e_3 = 1, e_2^2 = -2$. You can check that $\det(e_i \cdot e_j) = 1$, and this is referred to as the E_8 lattice. This is of signature $(0,8)$, and it's negative definite if and only if $v^2 < 0$ for all $v \neq 0$. One can also negate the intersection form to define $-E_8$. Note that any simply-laced Dynkin diagram yields some lattice. For example, E_{10} is unimodular of signature $(1,9)$, and it turns out that $E_{10} \cong E_8 \oplus H$.

Definition 11.0.5 (Unimodular lattice II)
Take

$$\mathbf{II}_{a,a+8b} := \bigoplus_{i=1}^{a} H \oplus \bigoplus_{j=1}^{b} E_8,$$

which is an even unimodular lattice since the diagonal entries are all -2, and using the fact that the signature is additive, is of signature $(a, a + 8b)$. Similarly,

$$\mathbf{II}_{a+8b,a} \coloneqq \bigoplus_{i=1}^{a} H \oplus \bigoplus_{j=1}^{b} (-E_8),$$

which is again even and unimodular.

Remark 11.0.6: Thus

- $\mathbf{I}_{p,q}$ is odd, unimodular, of signature (p, q).
- $\mathbf{II}_{p,q}$ is even, unimodular, of signature (p, q) only for $p \equiv q \pmod 8$.

Theorem 11.0.7 ***(Serre).***
Every unimodular lattice which is not positive or negative definite is isomorphic to either $\mathbf{I}_{p,q}$ or $\mathbf{II}_{p,q}$ with $8 \mid p - q$.

Remark 11.0.8: So there are obstructions to the existence of even unimodular lattices. Other than that, the number of (say) positive definite even unimodular lattices is

Dimension	Number of Lattices
8	1: E_8
16	2: $E_8^{\oplus 2}, D_{16}^+$
24	24: The Neimeir lattices (e.g. the Leech lattice)
32	$>8 \times 10^{16}$!!!!

Note that the signature of a definite lattice must be divisible by 8.

Remark 11.0.9: There is an isometry: $f : E_8 \to E_8$ where $f \in O(E_8)$, the linear maps preserving the intersection form (i.e. the Weyl group $W(E_8)$, given by $v \mapsto v + (v, e_i)e_i$. The Leech lattice also shows up in the sphere packing problems for dimensions $2, 4, 8, 24$. See Hale's theorem / Kepler conjecture for dimension 3! This uses an identification of L as a subset of $\mathbb{R}^n$, namely $L \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{24}$ for example, and the map $L \hookrightarrow (\mathbb{R}^{24}, \cdot)$ is an isometric embedding into $\mathbb{R}^n$ with the standard form. Connection to classification of Lie groups: root lattices.

Remark 11.0.10: If M^4 is a compact oriented 4-manifold and if the intersection form on $H^2(M; \mathbb{Z})$ is indefinite, then the only invariants we can extract from that associated lattice are

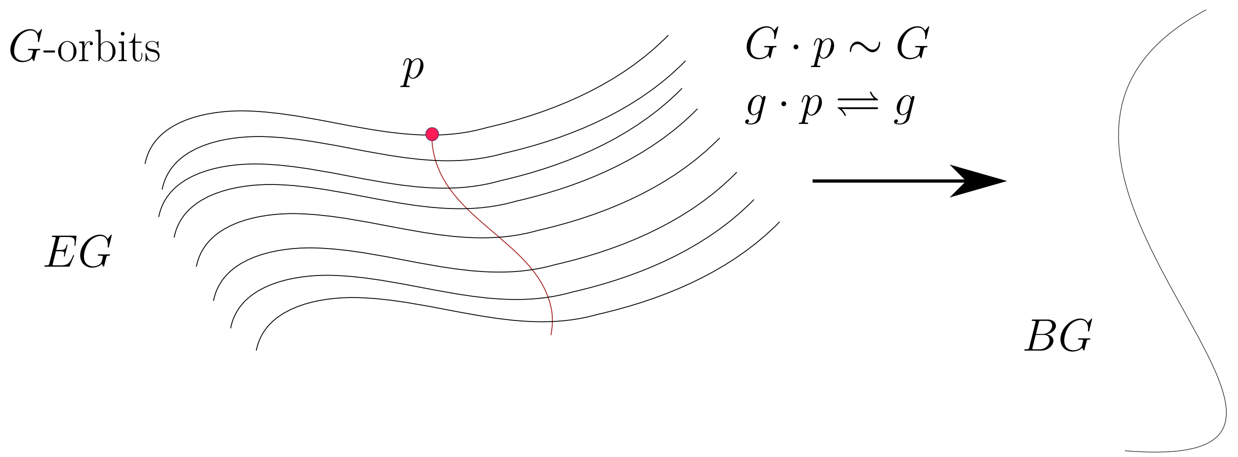
- Whether it's even or odd, and
- Its signature

If the lattice is even, then the signature satisfies $8 \mid p - q$. So Poincaré duality forces unimodularity, and then there are further number-theoretic restrictions. E.g. this prohibits $\beta_2 = 7$, since then the signature couldn't possibly be 8 if the intersection form is even.

11.1 Characteristic Classes

Definition 11.1.1 (Classifying space)
Let G be a topological group, then a **classifying space** EG is a contractible topological space admitting a free continuous G-action with a "nice" quotient.

Remark 11.1.2: Thus there is a map $EG \to BG \coloneqq EG/G$ which has the structure of a principal G-bundle.



Here we use a point p depending on U in an orbit to identify orbits $g \cdot p$ with g, and we want to take transverse slices to get local trivializations of $U \in BG$. It suffices to know where $\pi^{-1}(U) \cong U \times G$, and it suffices to consider $U \times \{e\}$. Moreover, $EG \to BG$ is a universal principal G-bundle in the sense that if $P \to X$ is a universal G-bundle, there is an $f : X \to BG$.

$$\begin{tikzcd}
P \arrow[r, dashed] \arrow[d] & EG \arrow[d] \\
X \arrow[r, "f"] & BG
\end{tikzcd}$$

Link to Diagram

Here bundles will be classified by homotopy classes of f, so

$$\left\{ \text{Principal } G\text{-bundles}_{/X} \right\} \rightleftharpoons [X, BG].$$

Warning 11.1.3
This only works for paracompact Hausdorff spaces! The line $\mathbb{R}$ with the doubled origin is a counterexample, consider complex line bundles.

Revisit this last section, had to clarify a few things for myself!

12 | Monday, February 08

Last time: BG and EG. See Milnor and Stasheff.

Example 12.0.1 *(?):* Let $G \coloneqq \mathrm{GL}_n(\mathbb{R}) = \mathbb{R}^\times$, then we can take

$$EG = \mathbb{R}^\infty \coloneqq \left\{ (a_1, a_2, \cdots) \;\middle|\; a_i \in \mathbb{R}, a_{i \gg 0} = 0, a_i \text{ not all zero} \right\}.$$

Then $\mathbb{R}^\times$ acts on EG by scaling, and we can take the quotient $\mathbb{R}^\infty \setminus \{0\} / \mathbb{R}^\times$, where $\mathbf{a} \sim \lambda \mathbf{a}$ for all $\lambda \in \mathbb{R}^\times$. This yields $\mathbb{RP}^\infty$ as the quotient. You can check that E_G is contractible: it suffices to show that $S^\infty \coloneqq \left\{ \sum |a_i| = 1 \right\}$ is contractible. This works by decreasing the last nonzero coordinate and increasing the first coordinate correspondingly. Moreover, local lifts exist, so we can identify $\mathbb{RP}^\infty \cong B\mathbb{R}^\times = BG$. Similarly $BC^\times \cong \mathbb{CP}^\infty$ with $E\mathbb{C}^\times \coloneqq \mathbb{C}^\infty \setminus \{0\}$.

Example 12.0.2 *(?):* Consider $G = \mathrm{GL}_n(\mathbb{R})$. It turns out that $BG = \mathrm{Gr}(d, \mathbb{R}^\infty)$, which is the set of linear subspaces of $\mathbb{R}^\infty$ of dimension d. This is spanned by d vectors $\{e_i\}$ in some large enough $\mathbb{R}^N \subseteq \mathbb{R}^\infty$, since we can take N to be the largest nonvanishing coordinate and include all of the vectors into $\mathbb{R}^\infty$ by setting $a_{>N} = 0$. For any $L \in \mathrm{Gr}_d(\mathbb{R}^\infty)$, since $\mathbb{R}^d$ has a standard basis, there is a natural GL_d torsor: the set of ordered bases of linear subspaces. So define

$$EG \coloneqq \left\{ \text{bases of linear subspaces } L \in \mathrm{Gr}_d(\mathbb{R}^\infty) \right\},$$

then any $A \in \mathrm{GL}_d(\mathbb{R})$ acts on EG by sending $(L, \{e_i\}) \mapsto (L, \{Le_i\})$. We can identify EG as d-tuples of linearly independent elements of $\mathbb{R}^\infty$, and there is a map

$$\begin{aligned} EG &\to BG \\ \{e_i\} &\mapsto \mathrm{span}_{\mathbb{R}} \{e_i\}. \end{aligned}$$

Thus there is a universal vector bundle over BGL_d:

$$\begin{tikzcd} \mathcal{E}_L \coloneqq L \arrow[r] & \mathcal{E} \arrow[d] \\ & BGL_d \end{tikzcd}$$

So $\mathcal{E} \subseteq BGL_d \times \mathbb{R}^\infty$, where we can define $\mathcal{E} \coloneqq \left\{ (L, p) \;\middle|\; p \in L \right\}$. In this case, $EG = \mathrm{Frame}(\mathcal{E})$ is the frame bundle of this universal bundle. The same setup applies for $G \coloneqq \mathrm{GL}_d(\mathbb{C})$, except we take $\mathrm{Gr}_d(\mathbb{C}^\infty)$.

Example 12.0.3 *(?):* Consider $G = O_d$, the set of orthogonal transformations of $\mathbb{R}^d$ with the standard bilinear form, and U_d the set of unitary such transformations. To be explicit:

$$U_d \coloneqq \left\{ A \in \mathrm{Mat}(d \times d, \mathbb{C}) \;\middle|\; \langle Av,\ Av \rangle = \langle v,\ v \rangle \right\},$$

where

$$\langle [v_1, \cdots, v_n],\ [v_1, \cdots, v_n] \rangle = \sum |v_i|^2.$$

Alternatively, $A^t A = I$ for O_d and $\overline{A^t} A = I$ for U_d. In this case, $BO_d = \mathrm{Gr}_d(\mathbb{R}^\infty)$ and $BU_d = \mathrm{Gr}_d(\mathbb{C}^\infty)$, but we'll make the fibers smaller: set the fiber over L to be

$$(EO_d)_L \coloneqq \{\text{orthogonal frames of } L\}$$

and similarly $(EU_d)_L$ the unitary frames of L. That there are related comes from the fact that GL_d retracts onto O_d using the Gram-Schmidt procedure.

Remark 12.0.4: Recall that there is a bijective correspondence

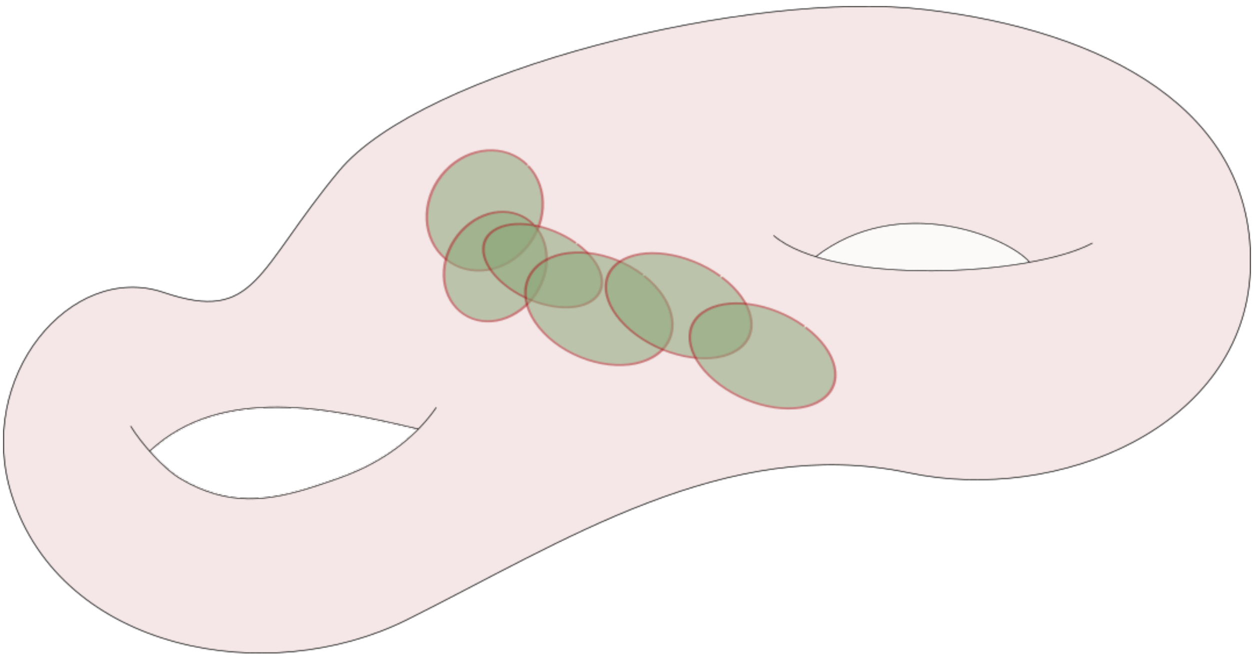
$$\left\{ \substack{\text{Principal } G\text{- bundles} \\ \text{on } X} \right\} \rightleftharpoons [X, BG]$$

and there is also a correspondence

$$\left\{ \substack{\text{Principal } \mathrm{GL}_d\text{-bundles} \\ \text{on } X} \right\} \rightleftharpoons \left\{ \substack{\text{Principal } \mathcal{O}_d\text{-bundles} \\ \text{on } X} \right\}$$

Using the associated bundle construction, on the LHS we obtain vector bundles $\mathcal{E} \to X$ of rank d, and on the RHS we have bundles with a metric. In local trivializations $U \times \mathbb{R}^d \to \mathbb{R}^d$, the metric is the standard one on $\mathbb{R}^d$. This is referred to as a **reduction of structure group**, i.e. a principal GL_d bundle admits possibly different trivializations for which the transition functions lie in the subgroup O_d.

Example 12.0.5 *(?):* Given any trivial principal G-bundle, it has a reduction of structure group to the trivial group. But the fact that the bundle is trivial may not be obvious.



Remark 12.0.6: We want to compute $H^*(BU_d;\mathbb{Z})$. Why is this important? Given any complex vector bundle $\mathcal{E} \to X$ there is an associated principal U_d bundle by choosing a metric, so we get a homotopy class $[X, BU_d]$. Given any $f \in [X, BU_d]$ and any $\alpha \in H^k(BU_d;\mathbb{Z})$, we can take the pullback $f^*\alpha \in H^k(X;\mathbb{Z})$, which are **Chern classes**.

Exercise 12.0.7 (?)
Show that $H^*(BU_d;\mathbb{Z})$ stabilizes as $d \to \infty$ to an infinitely generated polynomial ring $\mathbb{Z}[c_1, c_2, \cdots]$ with each c_i in cohomological degree $2i$, so $c_i \in H^{2i}(BU_d, \mathbb{Z})$.

Definition 12.0.8 (Chern class)
There is a map $BU_{d-1} \to BU_d$, which we can identify as

$$\begin{aligned} \mathrm{Gr}_{d-1}(C^\infty) &\to \mathrm{Gr}_d(\mathbb{C}^\infty) \\ \{v_1, \cdots, v_{d-1}\} &\mapsto \operatorname{span}\{(1,0,0,\cdots), sv_1, \cdots, sv_{d-1}\}. \end{aligned}$$

This is defined by sending a basis where $s: \mathbb{C}^\infty \to \mathbb{C}^\infty$ is the map that shifts every coordinate to the right by one.

Question: does $\mathrm{Gr}_d(\mathbb{C}^\infty)$ deformation retract onto the image of this map?

This will yield a fiber sequence

$$S^{2d-1} \to BU_{d-1} \to BU_d$$

and using connectedness of the sphere and the LES in homotopy this will identify

$$H^*(BU_d) = H^*(BU_{d-1})[c_d] \qquad \text{where } c_d \in H^{2d}(BU_d).$$

The **Chern class** of a vector bundle $\mathcal{E}$, denoted $c_k(\mathcal{E})$, will be defined as the pullback f^*c_k.

13 | Wednesday, February 10

Theorem 13.0.1 ***(Stable cohomology of BOn).***
As $n \to \infty$, we have

$$H^*(BO_n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1, w_2, \cdots] \qquad w_i \in H^i.$$

Definition 13.0.2 (Stiefel-Whitney class)
Given any principal O_n-bundle $P \to X$, there is an induced map $X \xrightarrow{f} BO_n$, so we can pull back the above generators to define the **Stiefel-Whitney classes** f^*w_i.

Remark 13.0.3: If $P \coloneqq \mathrm{OFrame}\, TX$, then f^*w_1 measures whether X has an orientation, i.e. $f^*w_1 = 0 \iff X$ can be oriented. We also have $f^*w_i(P) = w_i(\mathcal{E})$ where $P = \mathrm{OFrame}(\mathcal{E})$. In general, we'll just write w_i for Stiefel-Whitney classes and c_i for Chern classes.

Definition 13.0.4 (Pontryagin Classes)
The **Pontryagin classes** of a real vector bundle $\mathcal{E}$ are defined as

$$p_i(\mathcal{E}) = (-1)^i c_{2i}(\mathcal{E} \otimes_{\mathbb{R}} \mathbb{C}).$$

Note that the complexified bundle above is a complex vector bundle with the same transition functions as $\mathcal{E}$, but has a reduction of structure group from $\mathrm{GL}_n(\mathbb{C})$ to $\mathrm{GL}_n(\mathbb{R})$.

Observation 13.0.5
$\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$ are examples of $K(\pi, n)$ spaces, which are the unique-up-to-homotopy spaces defined by

$$\pi_k K(\pi, n) = \begin{cases} \pi & k = n \\ 0 & \text{else.} \end{cases}$$

Theorem 13.0.6 ***(Brown Representability).***

$$H^n(X; \pi) \cong [X, K(\pi, n)].$$

Example 13.0.7 ***(?):***

$$[X, \mathbb{RP}^\infty] \cong H^1(X; \mathbb{Z}/2\mathbb{Z})$$
$$[X, \mathbb{CP}^\infty] \cong H^2(X; \mathbb{Z}).$$

Proposition 13.0.8 ***(Classification of complex line bundles).***
There is a correspondence

$$\left\{ \text{Complex line bundles} \right\} \rightleftharpoons [X, \mathbb{CP}^\infty] = [X, BC^\times] \rightleftharpoons H^2(X; \mathbb{Z})$$

Importantly, note that for $X \in \mathsf{Mfd}_{\mathbb{C}}$, $H^2(X; \mathbb{Z})$ measures *smooth* complex line bundles and not holomorphic bundles.

Proof (of proposition).
We'll take an alternate direct proof. Consider the exponential exact sequence on X:

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times.$$

Note that $\underline{\mathbb{Z}}$ consists of locally constant $\mathbb{Z}$-valued functions, $\mathcal{O}$ consists of smooth functions, and $\mathcal{O}^\times$ are ???.

Can't read screenshot! :(

This yields a LES in homology:

$$\begin{tikzcd}
H^0(X;\mathbb{Z}) \arrow[r] & H^0(X;\mathcal{O}) \arrow[r] & H^0(X;\mathcal{O}^\times) \arrow[dll, hook] \\
H^1(X;\mathbb{Z}) \arrow[r] & H^1(X;\mathcal{O}) \arrow[r] & H^1(X;\mathcal{O}^\times) \arrow[dll, hook] \\
H^2(X;\mathbb{Z}) \arrow[r] & H^2(X;\mathcal{O}) \arrow[r] & H^2(X;\mathcal{O}^\times)
\end{tikzcd}$$

Link to Diagram

Since $\mathcal{O}$ admits a partition of unity, $H^{>0}(X;\mathcal{O}) = 0$ and all of the red terms vanish. For complex line bundles L, $H^1(X,\mathcal{O}^\times) \cong H^2(X;\mathbb{Z})$. Taking a local trivialization $L|_U \cong U \times \mathbb{C}$, we obtain transition functions

$$t_{UV} \in C^\infty(U \cap V, \mathrm{GL}_1(\mathbb{C}))$$

where we can identify $\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times$. We then have

$$(t_{U_{ij}}) \in \prod_{i<j} \mathcal{O}^\times(U_i \cap U_j) = C^1(X;\mathcal{O}^\times).$$

Moreover,

$$\left(t_{U_{ij}} t_{U_{ik}}^{-1} t_{U_{jk}}\right)_{i,j,k} = \partial(t_{U_{ij}})_{i,j} = 0,$$

since transitions functions satisfy the cocycle condition. So in fact $(t_{U_{ij}}) \in Z^1(X;\mathcal{O}^\times) = \ker \partial^1$, and we can take its equivalence class $[(t_{U_{ij}})] \in H^1(X;\mathcal{O}^\times) = \ker \partial^1 / \operatorname{im} \partial^0$. Changing trivializations by some $s_i \in \prod_i \mathcal{O}^\times(U_i)$ yields a composition which is a different trivialization of the same bundle:

$$\begin{tikzcd}
L|_{U_i} \arrow[r, "h_i"] \arrow[rr, bend right] & U_i \times \mathbb{C} \arrow[r, "\cdot s_i"] & U_i \times \mathbb{C}
\end{tikzcd}$$

Link to Diagram

So the $(t_{U_{ij}}$ change *exactly* by an $\partial^0(s_i)$. Thus the following map is well-defined:

$$L \mapsto [(t_{U_{ij}})] \in H^1(X;\mathcal{O}^\times).$$

There is another construction of the map

$$\begin{aligned}
\{L\} &\to H^2(X;\mathbb{Z}) \\
L &\mapsto c_1(L).
\end{aligned}$$

Take a smooth section of L and $s \in H^0(X;L)$ that intersects an $\mathcal{O}$-section of L transversely. Then

$$V(s) := \left\{ x \in X \;\middle|\; s(x) = 0 \right\}$$

is a submanifold of real codimension 2 in X, and $c_1(L) = [V(s)] \in H^2(X;\mathbb{Z})$.

■

Theorem 13.0.9 *(Splitting Principle for Complex Vector Bundles).*

1. Suppose that $\mathcal{E} = \bigoplus_{i=1}^{r} L_i$ and let $c(\mathcal{E}) \coloneqq \sum c_i(\mathcal{E}$. Then

$$c(\mathcal{E}) = \prod_{i=1}^{r} (1 + c_i(L_i))\,.$$

2. Given any vector bundle $\mathcal{E} \to X$, there exists some Y and a map $Y \to X$ such that $f^* : H^k(X;\mathbb{Z}) \hookrightarrow H^k(Y;\mathbb{Z})$ is injective and $f^*\mathcal{E} = \bigoplus_{i=1}^{r} L_i$.

Slogan 13.0.10
To verify any identities on characteristic classes, it suffices to prove them in the case where $\mathcal{E}$ splits into a direct sum of line bundles.

Example 13.0.11 *(?):*

$$c(\mathcal{E} \oplus \mathcal{F}) = c(\mathcal{E})c(\mathcal{F}).$$

To prove this, apply the splitting principle. Choose Y, Y' splitting $\mathcal{E}, \mathcal{E}'$ respectively, this produces a space Z and a map $f : Z \to X$ where both split. We can write

$$\begin{aligned} f^*\mathcal{E} &= \bigoplus L_i & \qquad c(f^*\mathcal{E}) &= \prod (1 + c_1(L_i)) \\ f^*\mathcal{F} &= \bigoplus M_j & \qquad c(f^*\mathcal{E}) &= \prod (1 + c_1(M_j))\,. \end{aligned}$$

We thus have

$$\begin{aligned} c(f^*\mathcal{E} \oplus f^*\mathcal{F}) &= \prod (1 + c_1(L_i))\,(1 + c_1(M_j)) \\ &= c(f^*\mathcal{E})c(f^*\mathcal{F}), \end{aligned}$$

and $f^*(c(\mathcal{E} \oplus \mathcal{F}) = f^*(c(\mathcal{E})c(\mathcal{F}))$. Since f^* is injective, this yields the desired identity.

Example 13.0.12 *(?):* We can compute $c(\operatorname{Sym}^2 \mathcal{E})$, and really any tensorial combination involving $\mathcal{E}$, and it will always yield some formula in the $c_i(\mathcal{E})$.

14 | Friday, February 12

Remark 14.0.1: Last time: the splitting principle. Suppose we have $\mathcal{E} = L_1 \oplus \cdots \oplus L_r$ and let $x_i \coloneqq c_i(L_i)$. Then $c_k(\mathcal{E})$ is the degree $2k$ part of $\prod_{i=1}^{r}(1 + x_i)$ where each x_i is in degree 2. This is equal to $e_k(x_1, \cdots, x_r)$ where e_k is the kth elementary symmetric polynomial.

Example 14.0.2 *(?)*: For example,

- $e_1 = x_1 + \cdots x_r$.
- $e_2 = x_1x_2 + x_1x_3 + \cdots = \sum_{i<j} x_i x_j$
- $e_3 = \sum_{i<j<k} x_i x_j x_k$, etc.

Remark 14.0.3: The theorem is that any symmetric polynomial is a polynomial in the e_i. For example, $p_2 = \sum x_i^2$ can be written as $e_1^2 - 2e_2$. Similarly, $p_3 = \sum x_i^3 = e_1^3 - 3e_1e_2 - 3e_3$ Note that the coefficients of these polynomials are important for representations of S_n, see *Schur polynomials.*

Remark 14.0.4: Due to the splitting principle, we can pretend that $x_i = c_i(L_i)$ exists even when $\mathcal{E}$ doesn't split. If $\mathcal{E} \to X$, the individual symbols x_i don't exist, but we can write '

$$x_1^3 + \cdots + x_r^3 = e_1^3 - 3e_1e_2 - 3e_3 \coloneqq c_1(\mathcal{E})^3 + 3c_1(\mathcal{E})c_2(\mathcal{E}) + \cdots,$$

which is a well-defined element of $H^6(X;\mathbb{Z})$. So this polynomial defines a characteristic class of $\mathcal{E}$, and this can be done for any symmetric polynomial. We can change basis in the space of symmetric polynomials to now define different characteristic classes.

Definition 14.0.5 (Chern Character)
The **Chern character** is defined as

$$\begin{aligned}
\operatorname{ch}(\mathcal{E}) &\coloneqq \sum_{i=1}^r e^{x_i} \in H^*(X;\mathbb{Q}) \\
&\coloneqq \sum_{i=1}^r \sum_{k=0}^\infty \frac{x_i^k}{k!} \\
&= \sum_{k=0}^\infty \frac{p_k(x_1, \cdots, x_r)}{k!} \\
&= \operatorname{rank}(\mathcal{E}) + c_1(\mathcal{E}) + \frac{c_1(\mathcal{E}) - c_2(\mathcal{E})}{2!} + \frac{c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}) - 3c_3(\mathcal{E})}{3!} + \cdots \\
&\quad \in H^0 + H^2 + H^4 + H^6 \\
&= \operatorname{ch}_0(\mathcal{E}) + \operatorname{ch}_1(\mathcal{E}) + \operatorname{ch}_2(\mathcal{E}) + \cdots, \\
&\quad \operatorname{ch}_i(\mathcal{E}) \in H^{2i}(X;\mathbb{Q}).
\end{aligned}$$

Definition 14.0.6 (Total Todd class)
The **total Todd class**

$$\operatorname{td}(\mathcal{E}) \coloneqq \prod_{i=1}^r \frac{x_i}{1 - e^{-x_i}}.$$

Note that

$$\frac{x_i}{1-e^{-x_i}} = 1 + \frac{x_i}{2} + \frac{x_i^2}{12} + \frac{x_i^4}{720} + \cdots = 1 + \frac{x_i}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1} B_i}{(2i)!} x^{2i}.$$

where L'Hopital shows that the derivative at $x_i = 0$ exists, so it's analytic at zero and the expansion makes sense, and the B_i are Bernoulli numbers.

Remark 14.0.7 ***(Very important and useful!!):*** $\mathrm{ch}(\mathcal{E} \oplus \mathcal{F}) = \mathrm{ch}(\mathcal{E}) + \mathrm{ch}(\mathcal{F})$ and $\mathrm{ch}(\mathcal{E} \otimes \mathcal{F}) = \sum_{i,j} e^{x_i + y_j} = \mathrm{ch}(\mathcal{E})\,\mathrm{ch}(\mathcal{F})$ using the fact that $c_1(L_1 \otimes L_2) = c_1(L_1)c_1(L_2)$. So ch is a "ring morphism" in the sense that it preserves multiplication $\otimes$ and addition $\oplus$, making the Chern character even better than the total Chern class.

Definition 14.0.8 (Todd Class)
Let $X \in \mathsf{Mfd}_{\mathbb{C}}$, then define the **Todd class** of X as $\mathrm{td}_{\mathbb{C}}(X) := \mathrm{td}(TX)$ where TX is viewed as a complex vector bundle. If $X \in \mathsf{Mfd}_{\mathbb{R}}$, define $\mathrm{td}_{\mathbb{R}} = \mathrm{td}(TX \otimes_{\mathbb{R}} \mathbb{C})$.

14.1 Section 5: Riemann-Roch and Generalizations

Remark 14.1.1: Let $X \in \mathsf{Top}$ and let $\mathcal{F}$ be a sheaf of vector spaces. Suppose $h^i(X; \mathcal{F}) := \dim H^i(X; \mathcal{F}) < \infty$ for all i and is equal to 0 for $i \gg 0$.

Definition 14.1.2 (Euler Characteristic of a Sheaf)
The **Euler characteristic** of $\mathcal{F}$ is defined as

$$\chi(X; \mathcal{F}) := \chi(\mathcal{F}) := \sum_{i=0}^{\infty} (-1)^i h_i(X; \mathcal{F}).$$

⚠ **Warning 14.1.3**
This is not always well-defined!

Example 14.1.4 ***(?):*** Let $X \in \mathsf{Mfd}_{\mathrm{cpt}}$ and take $\mathcal{F} := \underline{\mathbb{R}}$, we then have

$$\chi(X; \underline{\mathbb{R}}) = h^0(X; \mathbb{R}) - h^1(X; \mathbb{R}) + \cdots = b_0 - b_1 + b_2 - \cdots := \chi_{\mathsf{Top}}(X).$$

Example 14.1.5 ***(?):*** Let $X = \mathbb{C}$ and take $\mathcal{F} := \mathcal{O} := \mathcal{O}^{\mathrm{holo}}$ the sheaf of holomorphic functions. We then have $h^{>0}(X; \mathcal{O}) = 0$, but $H^0(X; \mathcal{O})$ is the space of all holomorphic functions on $\mathbb{C}$, making $\dim_{\mathbb{C}} h^0(X; \mathcal{O})$ infinite.

Example 14.1.6 ***(?):*** Take $X = \mathbb{P}^1$ with $\mathcal{O}$ as above, $h^0(\mathbb{P}^1; \mathcal{O}) = 1$ since $\mathbb{P}^1$ is compact and the maximum modulus principle applies, so the only global holomorphic functions are constant. We can write $\mathbb{P}^1 = \mathbb{C}_1 \cup \mathbb{C}_2$ as a cover and $h^i(\mathbb{C}, \mathcal{O}) = 0$, so this is an acyclic cover and we can use it to compute $h^1(\mathbb{P}^1; \mathcal{O})$ using Čech cohomology. We have

- $C^0(\mathbb{P}^1; \mathcal{O}) = \mathcal{O}(\mathbb{C}_1) \oplus \mathcal{O}(\mathbb{C}_2)$
- $C^1(\mathbb{P}^1; \mathcal{O}) = \mathcal{O}(\mathbb{C}_1 \cap \mathbb{C}_2) = \mathcal{O}(\mathbb{C}^\times)$.
- The boundary map is given by

$$\begin{aligned} \partial_0 : C^0 &\to C^1 \\ (f(z), g(z)) &\mapsto g(1/z) - f(z) \end{aligned}$$

  and there are no triple intersections.

Is every holomorphic function on $\mathbb{C}^\times$ of the form $g(1/z) - f(z)$ with f, g holomorphic on $\mathbb{C}$. The answer is yes, by Laurent expansion, and thus $h^1 = 0$. We can thus compute $\chi(\mathbb{P}^1; \mathcal{O}) = 1 - 0 = 1$.

15 | Monday, February 15

Remark 15.0.1: Last time: we saw that $\chi(\mathbb{P}^1, \mathcal{O}) = 1$, and we'd like to generalize to holomorphic line bundles on a Riemann surface. This will be the main ingredient for Riemann-Roch.

Theorem 15.0.2 ***(Euler characteristic and homological vanishing for holomorphic vector bundles).***
Let $X \in \mathsf{Mfd}_{\mathbb{C}}$ be compact and let $\mathcal{F}$ be a holomorphic vector bundle on X [a] Then χ is well-defined and

$$h^{>\dim_{\mathbb{C}} X}(X; \mathcal{F}) = 0.$$

[a] Or more generally a finitely-generated $\mathcal{O}$-module, i.e. a coherent sheaf.

Remark 15.0.3: The locally constant sheaf $\underline{\mathbb{C}}$ is not an $\mathcal{O}$-module, i.e. $\underline{\mathbb{C}}(U) \notin \mathcal{O}(\mathsf{U})\text{-}\mathsf{Mod}$. In fact, $h^{2i}(X, \underline{\mathbb{C}}) = \mathbb{C}$ for all i.

Proof (of theorem).
We'll can resolve $\mathcal{F}$ as a sheaf by first mapping to its smooth sections and continuing in the following way:

$$0 \to \mathcal{F} \to C^\infty \mathcal{F} \xrightarrow{\bar\partial} F \otimes A^{0,1} \to \cdots,$$

where $\bar\partial f = \sum_i \frac{\partial f}{\partial \bar z_i}\, d\bar z_i$. Suppose we have a holomorphic trivialization of $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$ and we have sections $(s_1, \cdots, s_r) \in C^\infty \mathcal{F}(U)$, which are smooth functions on U. In local coordinates we have

$$\bar\partial s := (\bar\partial s_1, \cdots, \bar\partial s_r),$$

but is this well-defined globally? Given a different trivialization over $V \subseteq X$, the s_i are related by transition functions, so the new sections are $t_{UV}(s_1, \cdots, s_r)$ where $t_{UV} : U \cap V \to \mathrm{GL}_r(\mathbb{C})$.

Since t_{UV} are holomorphic, we have

$$\bar{\partial}(t_{UV}(s_1, \cdots, s_r)) = t_{UV}\bar{\partial}(s_1, \cdots, s_r).$$

This makes $\bar{\partial} : C^\infty\mathcal{F} \to F \otimes A^{0,1}$ a well-defined (but not $\mathcal{O}$-linear) map. We can thus continue this resolution using the Leibniz rule:

$$0 \to \mathcal{F} \to C^\infty\mathcal{F} \xrightarrow{\bar{\partial}} F \otimes A^{0,1} \xrightarrow{\bar{\partial}} \cdots F \otimes A^{0,2} \xrightarrow{\bar{\partial}} \cdots,$$

which is an exact sequence of sheaves since $(A^{0,-}, \bar{\partial})$ is exact.

Why? Split into line bundles?

We can identify $C^\infty\mathcal{F} = \mathcal{F} \otimes A^{0,0}$, and $\mathcal{F} \otimes A^{0,q}$ is a smooth vector bundle on X. Using partitions of unity, we have that $\mathcal{F} \otimes A^{0,q}$ is acyclic, so its higher cohomology vanishes, and

$$H^i(X; \mathcal{F}) \cong \frac{\ker(\bar{\partial} : \mathcal{F} \otimes A^{0,i} \to \mathcal{F} \otimes A^{0,i+1}}{\operatorname{im}(\bar{\partial} : \mathcal{F} \otimes A^{0,i-1} \to \mathcal{F} \otimes A^{0,i}}.$$

However, we know that $A^{0,p} = 0$ for all $p > n := \dim_{\mathbb{C}} X$, since any wedge of $p > n$ forms necessarily vanishes since there are only n complex coordinates.

■

Warning 15.0.4
This only applies to holomorphic vector bundles or $\mathcal{O}$-modules!

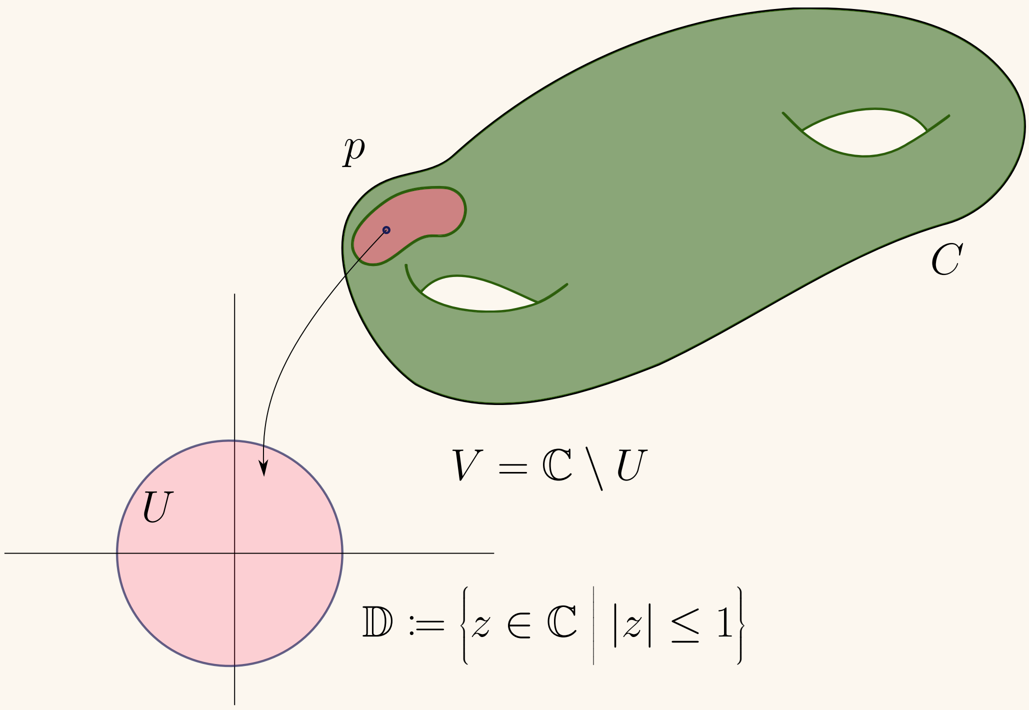
15.1 Riemann-Roch

Theorem 15.1.1 ***(Riemann-Roch).***
Let C be a compact connected Riemann surface, i.e. $C \in \mathsf{Mfd}_{\mathbb{C}}$ with $\dim_{\mathbb{C}}(C) = 1$, and let $\mathcal{L} \to C$ be a holomorphic line bundle. Then

$$\chi(C, \mathcal{L}) = \deg(L) + (1 - g) \qquad \text{where } \deg(L) := \int_C c_1(\mathcal{L})$$

and g is the genus of C.

Proof (of Riemann-Roch).
We'll introduce the notion of a "point bundle", which are particularly nice line bundles, denoted $\mathcal{O}(p)$ for $p \in \mathbb{C}$.



Taking $\mathbb{D}$ to be a disc of radius $1/2$ and V to be its complement, we have $t_{uv}(z) = z^{-1} \in \mathcal{O}^*(U \cap V)$. We can take a holomorphic section $s_p \in H^0(C, \mathcal{O}(p))$, where $s_p|_U = z$ and $s_p|_V = 1$. Then $t_{uv}(s_p|_U) = s_p|_V$ on the overlaps. We have a function which precisely vanishes to first order at p. Recall that $c_1(\mathcal{O}(p))$ is represented by $[V(s)] = [p]$, and moreover $\int_C c_1(\mathcal{O}(p)) = 1$.
We now want to generalize this to a **divisor**: a formal $\mathbb{Z}$-linear combination of points.
Example 15.1.2*(?)*: Take $p, q, r \in C$, then a divisor can be defined as something like $D := 2[p] - [q] + 3[r]$.
Define $\mathcal{O}(D) := \bigotimes_i \mathcal{O}(p_i)^{\otimes n_i}$ for any $D = \sum n_i[p_i]$. Here tensoring by negatives means taking duals, i.e. $\mathcal{O}(-[p]) := \mathcal{O}^{\otimes -1} := \mathcal{O}(p)^\vee$, the line bundle with inverted transition functions. $\mathcal{O}(D)$ has a meromorphic section given by

$$s_D := \prod s_{p_i}^{n_i} \in \mathrm{Mero}(C, \mathcal{O}(D))$$

where we take the sections coming from point bundles. We can compute

$$\int_C c_1(\mathcal{O}(D)) = \sum n_i := \deg(D).$$

.
Example 15.1.3*(?)*:

$$\deg(2[p] - [q] + 3[r]) = 4.$$

Remark 15.1.4: Assume our line bundle L is $\mathcal{O}(D)$, we'll prove Riemann-Roch in this case by induction on $\sum |n_i|$. The base case is $\mathcal{O}$, which corresponds to taking an empty divisor. Then either

- Take $D = D_0 + [p]$ with $\deg(D_0) < \sum |n_i|$ (for which we need some positive coefficient), or
- Take $D_0 = D + [p]$.

Claim: There is an exact sequence

$$\begin{gathered} 0 \to \mathcal{O}(D_0) \to \mathcal{O}(D) \to \mathbb{C}_p \to 0 \\ s \in \mathcal{O}(D_0)(U) \mapsto s \cdot s_p \in \mathcal{O}(D_0 + [p])(U), \end{gathered}$$

where the last term is the skyscraper sheaf at p.

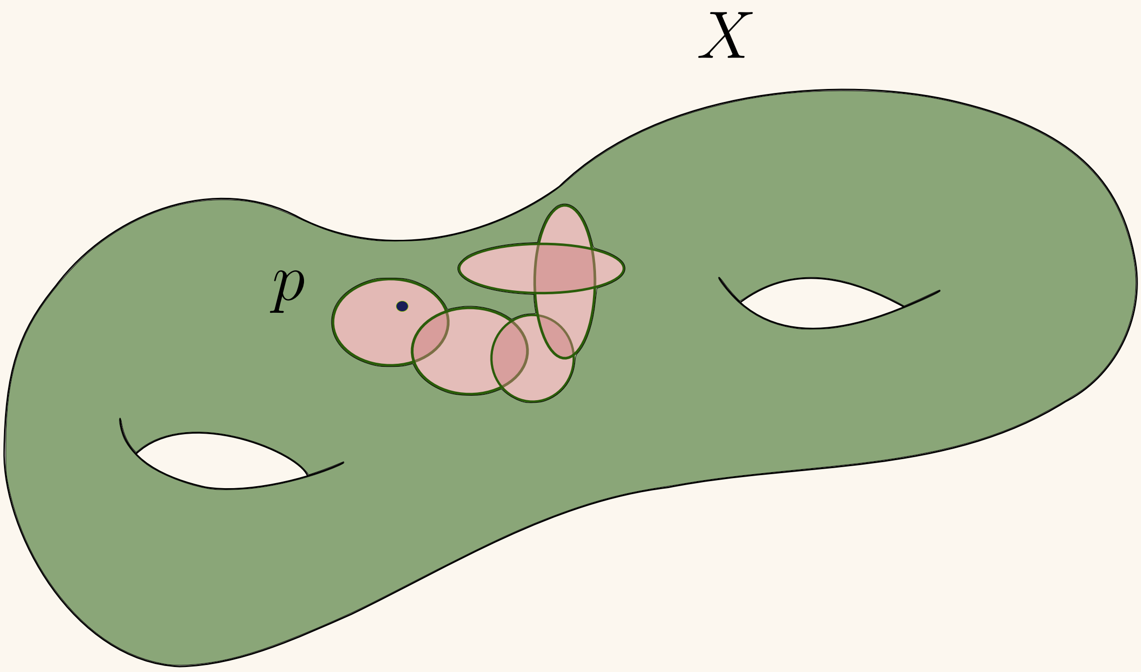
Proof (of claim).
The given map is $\mathcal{O}$-linear and injective, since $s_p \neq 0$ and $ss_p = 0$ forces $s = 0$. Recall that we looked at $\mathcal{O} \xrightarrow{\cdot z} \mathcal{O}$ on $\mathbb{C}$, and this section only vanishes at p (and to first order). The same situation is happening here. ■

Thus there is a LES

$$\begin{tikzcd}
 & & 0 \arrow[dll, out=0, in=180] \\
H^0(\mathcal{O}(D_0)) \arrow[r] & H^0(\mathcal{O}(D)) \arrow[r] & H^0(\mathcal{O}(\mathbb{C}_p)) \arrow[dll, out=0, in=180] \\
H^1(\mathcal{O}(D_0)) \arrow[r] & H^1(\mathcal{O}(D)) \arrow[r] & H^1(\mathcal{O}(\mathbb{C}_p)) = 0 \arrow[dll, out=0, in=180] \\
0 & &
\end{tikzcd}$$

Link to Diagram

We also have $h^1(\mathbb{C}_p) = 0$ by taking a sufficiently fine open cover where p is only in one open set. So just checking Čech cocycles yields $C^1_U(C, \mathbb{C}_p) := \prod_{i<j} \mathbb{C}_p(U_i \cap U_j) = 0$ since p is in no intersection.



We obtain $\chi(\mathcal{O}(D) = \chi(\mathcal{O}(D_0)) + 1$, using that it is additive in SESs

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0 \implies \qquad \chi(\mathcal{E}_2) = \chi(\mathcal{E}_\infty) + \chi(\mathcal{E}_3)$$

and thus

$$\int_C c_1(\mathcal{O}(D)) = \sum n_i = \deg(D) = \deg D_0 + 1.$$

The last step is to show that $\chi(C, \mathcal{O}) = 1 - g$, so just define g so that this is true! ■

Remark 15.1.5: Why is every $L \cong \mathcal{O}(D)$ for some D? Easy to see if L has meromorphic sections: if s is a meromorphic section of L, then the following works:

$$D = \mathrm{Div}(s) = \sum_p \mathrm{Ord}_p(s)[p].$$

Then $\mathcal{O} \cong L \otimes \mathcal{O}(-D)$ has a meromorphic section ss_{-D}, a global nonvanishing section with $\mathrm{Div}(ss_{-D}) = \emptyset$. Proving that every holomorphic line bundle has a meromorphic section is hard!

16 | Friday, February 19

16.1 Applications of Riemann-Roch

Definition 16.1.1 (Curves)
A **curve** is a compact complex manifold of complex dimension 1.

Example 16.1.2 *(?):* Let C be a curve, then Ω_C^1 is the sheaf of holomorphic 1-forms, and $\Omega_C^{\geq 1} = 0$. We also have the sheaves $A^{1,0}, A^{0,1}, A^{1,1}$, the sheaves of smooth (p,q)-forms. Here the only nonzero combinations are $(0,0), (0,1), (1,0), (1,1)$ by dimensional considerations. Let L be a holomorphic line bundle on C, then

$$\chi(C, L) = h^0(L) - h^1(L) = \deg(L) + 1 - g.$$

Remark 16.1.3: In general it can be hard to compute $h^1(L)$, since this is sheaf cohomology (sections over double overlaps, cocycle conditions, etc). On the other hand, h^0 is easy to understand, since $h^0(\Omega_C^1)$ is the dimension of the global holomorphic sections $H^0(C, L) = L(C)$. A key tool here is the following:

16.1.1 Serre Duality

Proposition 16.1.4 *(Serre Duality).*

$$H^1(C, L) \cong H^0(C, L^{-1} \otimes \Omega_C^1)^\vee,$$

noting that these are both global sections of a line bundle.

Proof (of Serre Duality).
Recall that we had a resolution of the sheaf L given by by smooth vector bundles:

$$0 \to L \hookrightarrow L \otimes A^{0,0} \xrightarrow{\bar\partial} L \otimes A^{0,1} \xrightarrow{\bar\partial} 0.$$

So we know that

$$H^1(C, L) = H^0(L \otimes A^{0,1}) / \bar\partial H^0(L \otimes A^{0,0}).$$

Choose a Hermitian metric h on L, i.e. a map $h : L \otimes \overline{L} \to \mathcal{O}$. On fibers, we have $h_p : L_p \otimes \overline{L_p} \to \mathbb{C}$. We'll also choose a metric on C, say g. Since C is a Riemann surface, we have an associated volume form ν on C (essentially the determinant), so we can define a pairing between sections of $L \otimes A^{0,0}$:

$$\langle s,\ t \rangle := \int_C h(s, \bar t)\, d\nu.$$

Note that

$$\langle s,\ s\rangle = \int_C h(s,\overline{s})\,d\nu \geq 0 \qquad\qquad \text{since } h(s,\overline{s})(p) = 0 \iff s_p = 0,$$

and moreover this integral is zero if and only if $s = 0$. So we have an inner product on $H^0(L \otimes A^{0,0})$. We can also define a pairing on sections of $L \otimes A^{0,1}$, say

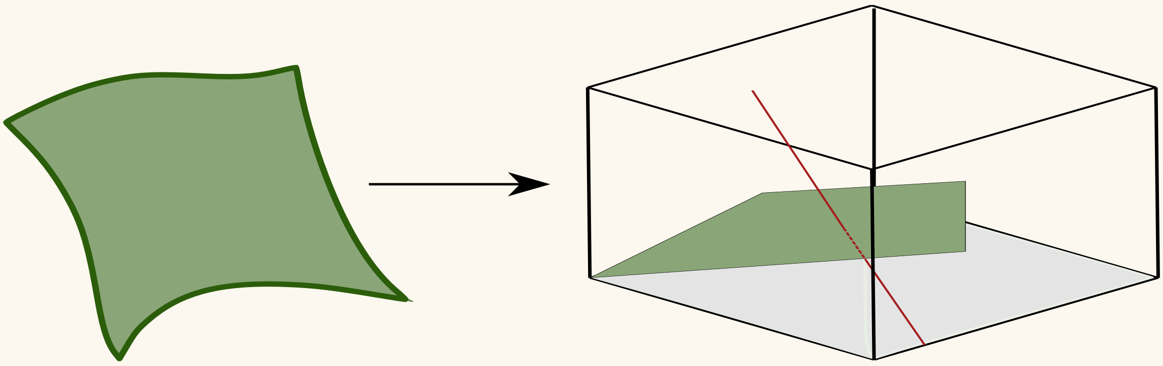
$$\langle s \otimes \alpha,\ t \otimes \beta\rangle = \int_C h(s,\overline{t})\alpha \wedge \overline{\beta}.$$

Note that h is a smooth function and $\alpha \wedge \overline{\beta}$ is a $(1,1)$-form. Moreover, this is positive and nondegenerate. We want to understand the cokernel of the linear map

$$H^0(L \otimes A^{0,0}) \xrightarrow{\overline{\partial}} H^0(L \otimes A^{0,1}).$$

To compute $\operatorname{coker}(\overline{\partial})$, we can look at the kernel of the adjoint, and it suffices to find the orthogonal complement of $\operatorname{im}(\overline{\partial})$, i.e.

$$\operatorname{coker}(\overline{\partial}) = \left\{ t \in H^0(L \otimes A^{0,1}) \;\middle|\; \left\langle \overline{\partial} s,\ t \right\rangle = 0\, \forall s \right\}.$$



So we want to understand sections $t \in H^0(L \otimes A^{0,1})$ such that

$$\int_C (\overline{\partial} s)\overline{t} = 0 \qquad\qquad \forall s \in H^0(L \otimes A^{0,0}),$$

where $\partial C = \emptyset$. We'll basically want to do integration by parts on this. Note that $h(s,t) = hst$ here where we view h as a certain section. Note that $\overline{t} \in H^0(\overline{L} \otimes A^{1,0})$, so we can replace ∂ with $d = \overline{\partial} + \partial$ and apply Stokes' theorem:

$$\begin{aligned} \int_C sd(h\overline{t}) &= 0 \qquad\qquad \forall s \in H^0(L \otimes A^{0,0}) \\ 0 &= \int_C s\overline{\partial}(h\overline{t}) \\ &= \int_C s\frac{\overline{\partial}(h\overline{t})}{d\nu}d\nu \\ &= \left\langle s,\ \overline{\frac{\overline{\partial}(h\overline{t})}{d\nu}} \right\rangle \end{aligned}$$

where $h \in C^\infty(L^{-1} \otimes \overline{L}^{-1})$ and $h\bar{t} \in C^\infty(L^{-1} \otimes A^{1,0})$. But the right-hand side is in $H^0(L \otimes A^{0,0})$ and by nondegeneracy we can conclude

$$\frac{\overline{\bar{\partial}(h\bar{t})}}{d\nu} = 0 \iff \bar{\partial}(h\bar{t}) = 0.$$

We thus have $h\bar{t} \in H^0(L^{-1} \otimes A^{1,0}$ which is a holomorphic line bundle tensored with $A^{0,0}$. Thus $\operatorname{coker}(\bar{\partial}) \cong_h H^0(L^{-1} \otimes \Omega^1)$.

■

Remark 16.1.5: We showed $\left\langle \bar{\partial}s,\ t \right\rangle = \langle s,\ Y(t) \rangle$ where Y is the adjoint given above. Then the kernel of Y wound up being where $\bar{\partial}$ vanishes, i.e. holomorphic sections of a separate bundle. Here we had

- $t \in H^0(L \otimes A^{0,1})$
- $\bar{t} \in H^0(\overline{L} \otimes A^{1,0})$
- $h \in H^0(L^{-1} \otimes \overline{L^{-1}})$

17 | Monday, February 22

Remark 17.0.1: Last time: Serre duality, and we'll review Riemann-Roch. Recall that this depended on the statement that every holomorphic line bundle $L \to C$ for C a complex curve is of the form $L = \mathcal{O}(D)$ for some divisor D. Then

$$\chi(C, L) = h^0(L) - h^1(L) = \deg L + 1 - g, \qquad \deg L = \int_C c_1(L),$$

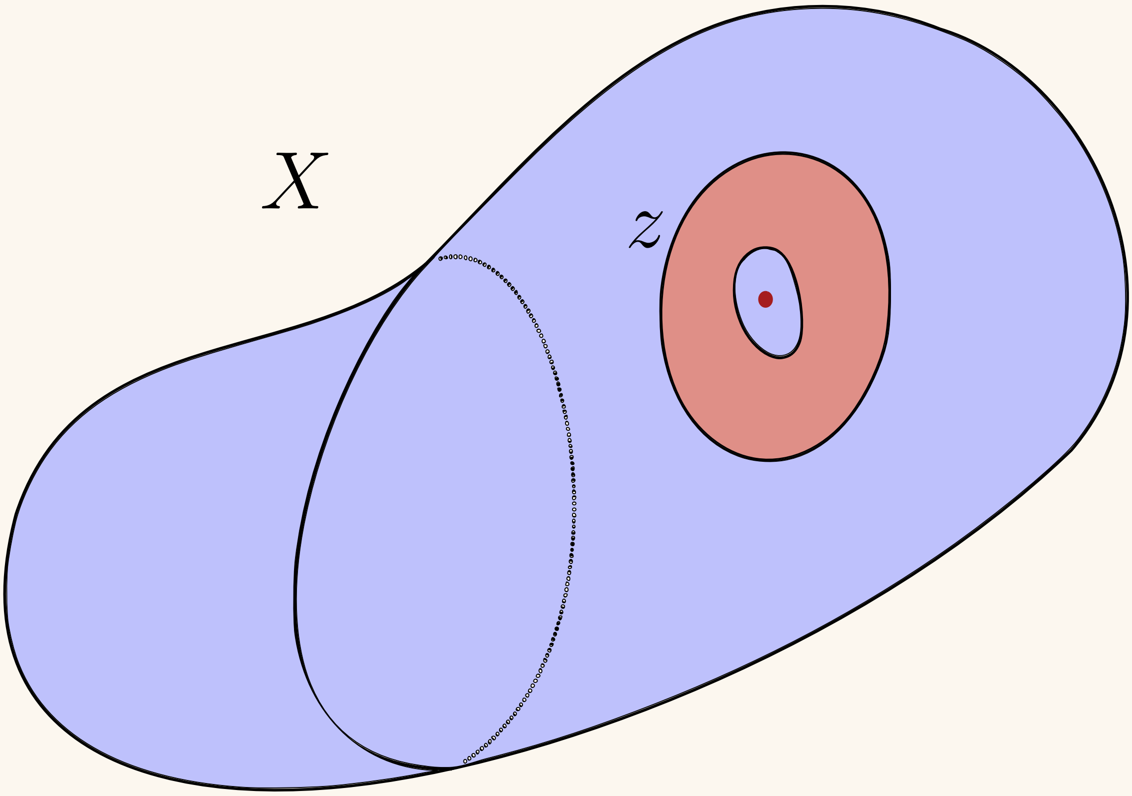
Serre duality said that the space of sections $H^1(C; L)$ is naturally isomorphic to $H^0(C, L^{-1} \otimes \Omega_C^1)^\vee$. Notation: given $X \in \mathsf{Mfd}_{\mathbb{C}}^n$ of complex, dimension n, the **canonical bundle** is written $K_X := \Omega_X^n$ and is the sheaf of holomorphic n-forms. Serre duality will generalize: if $\mathcal{E} \to X$ is a holomorphic vector bundle, then $H^i(X; \mathcal{E}) \cong H^{n-i}(X; \mathcal{E}^\vee \otimes K_X)^\vee$. Note that only H^0, H^1 are the only nontrivial degrees for a curve. For 4-manifolds, we'll have an H^2 as well.

17.1 Applications of Riemann-Roch

Proposition 17.1.1 ***(The 2-sphere has a unique complex structure).***
There is a unique complex $X \in \mathsf{Mfd}_{\mathbb{C}}$ diffeomorphic to S^2.

Proof (of proposition).
Note existence is clear, since we can take $\mathbb{CP}^1 := (\mathbb{C}^2 \setminus \{0\})/\mathbf{x} \sim \lambda\mathbf{x}$ for $\lambda \in \mathbb{C}^\times$, which is identified as the set of complex lines through 0 in $\mathbb{C}^2$. This decomposes as $\mathbb{C} \cup \mathbb{C} =$

$\{[1,*]\} \cup \{[*,1]\}$. We now want to show that any two such complex manifolds are biholomorphic. Let $X \in \mathsf{Mfd}^1_{\mathbb{C}}$ with $X \cong_{C^\infty} S^2$, and consider for $p \in X$ the point bundle $\mathcal{O}(p) \to X$. The defining property was that there exists a section $s_p \in H^0(X; \mathcal{O}(p))$ which vanishes at first order at p:



We have

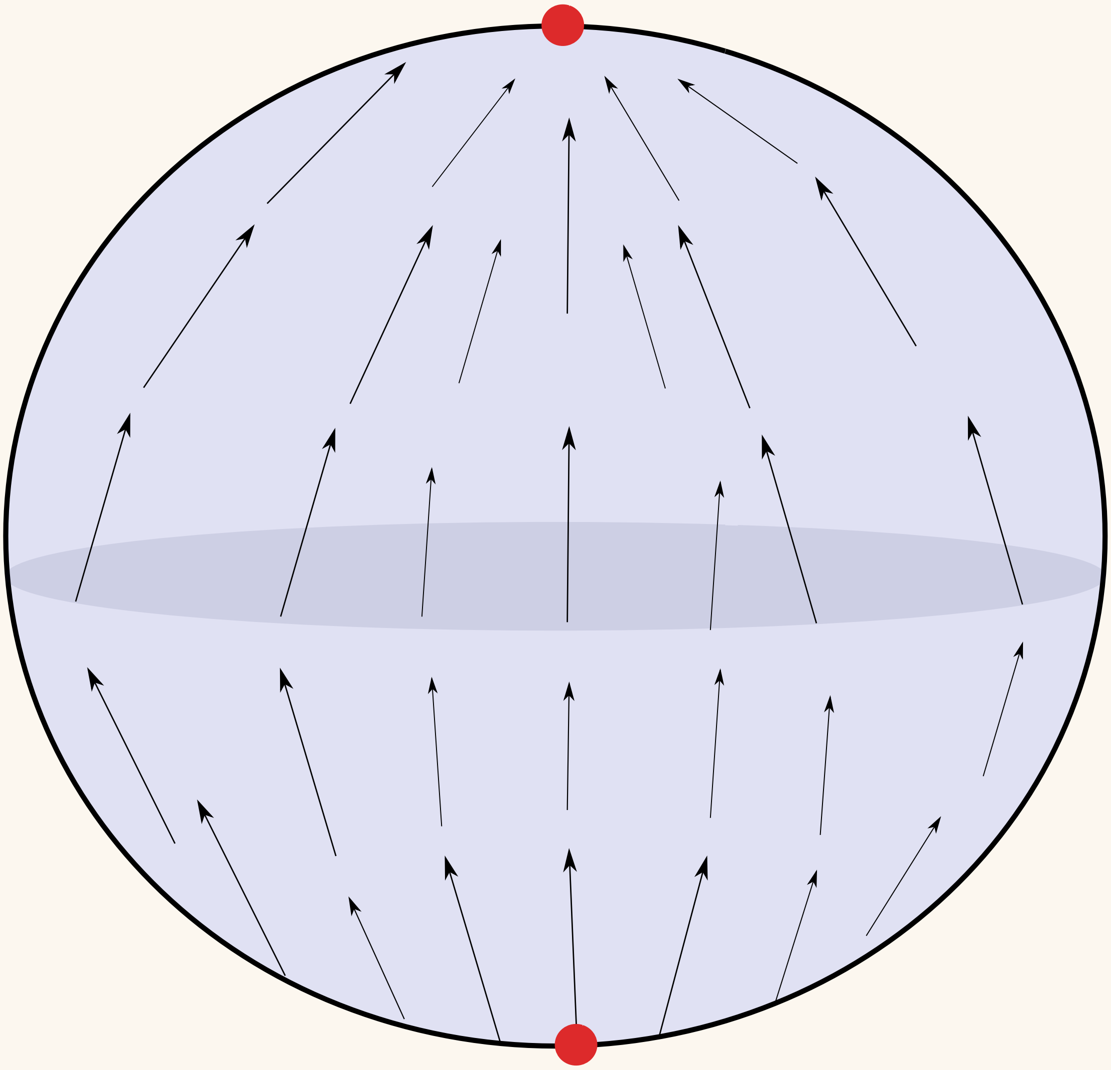
$$\chi(X; \mathcal{O}(p)) = \deg \mathcal{O}(p) + 1 - g(x) = 1 + 1 - 0 = 2.$$

Exercise *(?)*
Check that $\deg \mathcal{O}(p) = 1$.

On the other hand we have

$$\chi(X; \mathcal{O}(p)) = h^0(\mathcal{O}(p)) - h^1(\mathcal{O}(p)).$$

We have $h^1(\mathcal{O}(p)) = H60(K \otimes \mathcal{O}(-p)$, and $K_X = \Omega^1_X = T^\vee X$, so the question is: what is the degree of TX for $X \cong S^2$? We need to compute $\int_X c_1(TX)$. How many zeros does a vector field on the sphere have? You can take the gradient vector field for a height function to get 2, noting that the two zeros come in with a positive orientation



In coordinates on $\mathbb{CP}^1$, the coordinate is given by z and $z\dfrac{\partial}{\partial z} \mapsto -2\dfrac{\partial}{\partial w}$ for the coordinate $w = 1/z$. We get $\displaystyle\int_X c_1(TX) = 2$ and thus $\deg K_X = -2$ by dualizing.

Fact

$\deg K_X = 2g - 2$. Use the existence of a smooth vector field on X.

Lemma 17.1.4 ***(When h0 of a line bundle on a curve vanishes).***
If $\deg L < 0$ on C, thne $h^0(C, L) = 0$.

Proof (of lemma).
If $s \in H^0(C, L)$ is nonzero, then since s is a holomorphic section,

$$0 \leq \sum_{p \in C} \mathrm{Ord}_P(s) = \deg L.$$

■

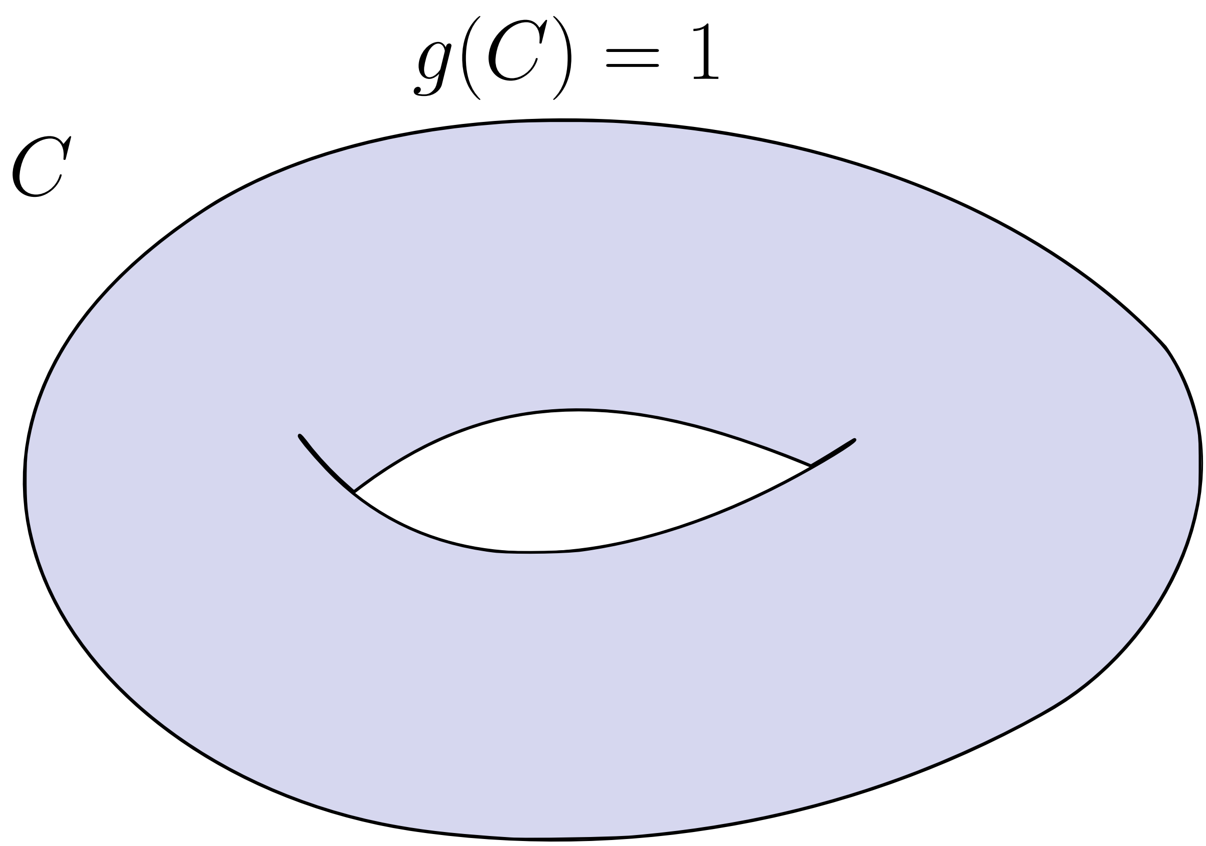
By this lemma, $h^1(\mathcal{O}(p)) = 0$. We have $H^0(X; \mathcal{O}(p)) = \mathbb{C}s_p \oplus \mathbb{C}s$ for our specific section s_p and some other section $s \neq \lambda s_p$. Note that s/s_p is a meromorphic section of $\mathcal{O}(p) \times \mathcal{O}(-p) = \mathcal{O}$, so we have a map

$$\varphi : \frac{s}{s_p} : X \to \mathbb{P}^1.$$

Note that $P \mapsto \infty \in \mathbb{P}^1$ under this φ, and it's only the ratio that is well-defined. We have $\varphi^{-1}(u) = \{s/s_p = u\} = \{s - us_p = 0\}$ which is a single point. So φ is a degree 1 map, and X is biholomorphic to $\mathbb{P}^1$ via φ.

■

Remark 17.1.5: So there is only one genus 0 Riemann surface. What about genus 1?



By Riemann-Roch we know

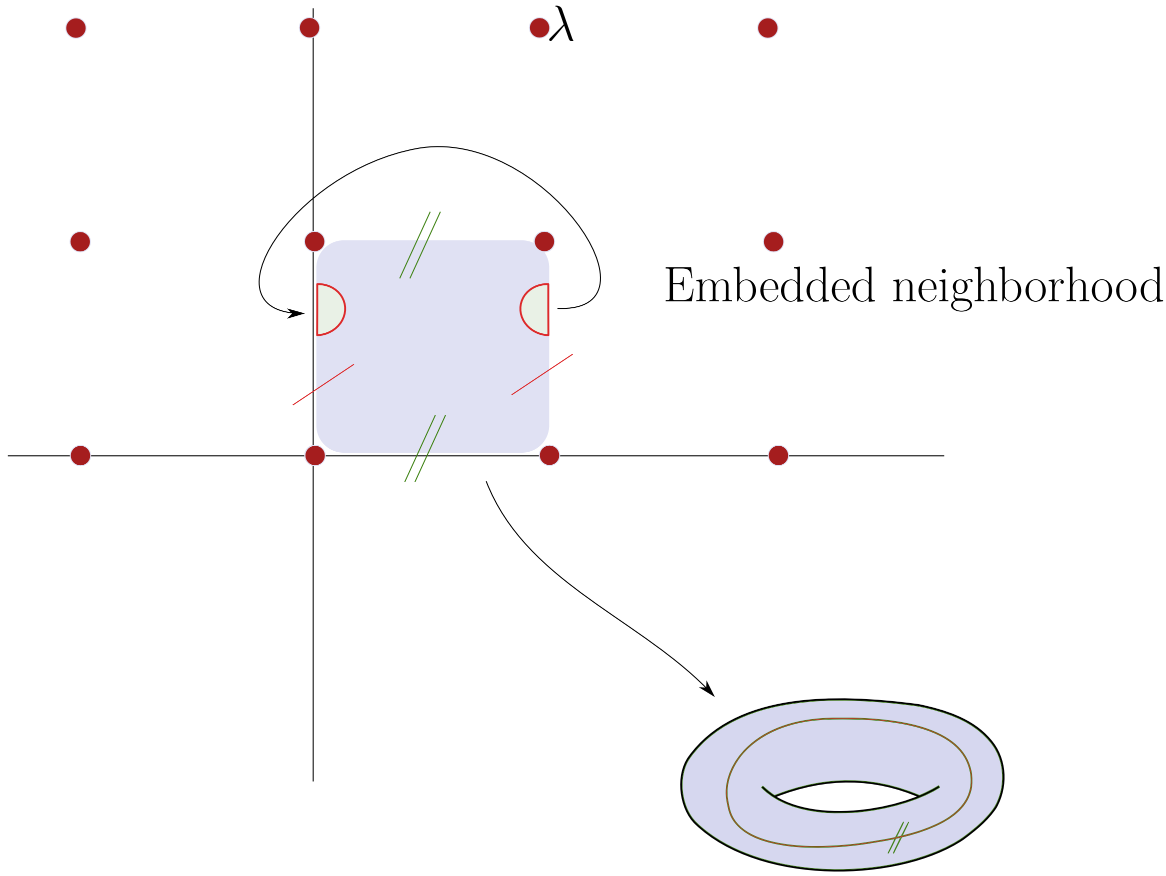
$$\chi(C;\mathcal{O}) = \deg \mathcal{O} + l - 1 = 0 = h^0(\mathcal{O}) - h^1(\mathcal{O}).$$

We know $h^0(\mathcal{O}) = 1$ by the maximum modulus principle and $h^1(C;\mathcal{O}) = 1$. By Serre duality, $h^0(C, K) = 1$, and since $\deg K = 2g - 2 = 0$. So let $s \in H^0(C, K)$ by a nonzero section, which we know exists. We then get $\mathrm{Ord}_p s = 0$ for all p, so s vanishes nowhere. But then we get an isomorphism of sheaves, since s everywhere nonvanishing implies trivial cokernel:

$$\mathcal{O} \xrightarrow{\cdot s} K.$$

So $K_C = \mathcal{O}_C$ if $g(C) = 1$, and such a Riemann surface is an **elliptic curve**.

Example 17.1.6*(?)*: Let $C := \mathbb{C}/\Lambda$ for Λ some lattice.



All transition functions are of the form $z \mapsto z + \lambda$ for some $\lambda \in \Lambda$. What is a nonvanishing section of K_C, i.e. a holomorphic one form $\omega := f(z)dz$ on $\mathbb{C}$ that descends to $\mathbb{C}/\Lambda$. We would need $f(z)dz = f(z+\lambda)d(z+\lambda)$ for all λ. Something like $f = 1$ works, so $\omega = dz$ descends. In fact, f must be constant, since $H^0(\mathbb{C}/\Lambda, \mathcal{O}) = \mathbb{C}\, dz$ by the maximum modulus principle. Now let $p, q \in C$

and apply Riemann-Roch to the line bundle $\mathcal{O}(p+q)$ yields

$$\begin{aligned}\chi(\mathcal{O}(p+q)) &= h^0(\mathcal{O}(p+q)) - h^1(\mathcal{O}(-p-q)) \\ &= h^0(\mathcal{O}(p+q)) - 0 \\ &= \deg \mathcal{O}(p+q) + 1 - 1 \\ &= 2.\end{aligned}$$

Thus there is a section $s_{p+q} \in H^0(\mathcal{O}(p+q)) \ni s$ that vanishes at $p+q$, and similarly a map

$$\frac{s}{s_{p+q}} : C \xrightarrow{\varphi} \mathbb{P}^1.$$

We can check $\varphi^{-1}(\infty) = p+q$ and $\deg \varphi = 2$. Thus genus 1 surfaces have a generically 2-to-1 map to $\mathbb{P}^1$.

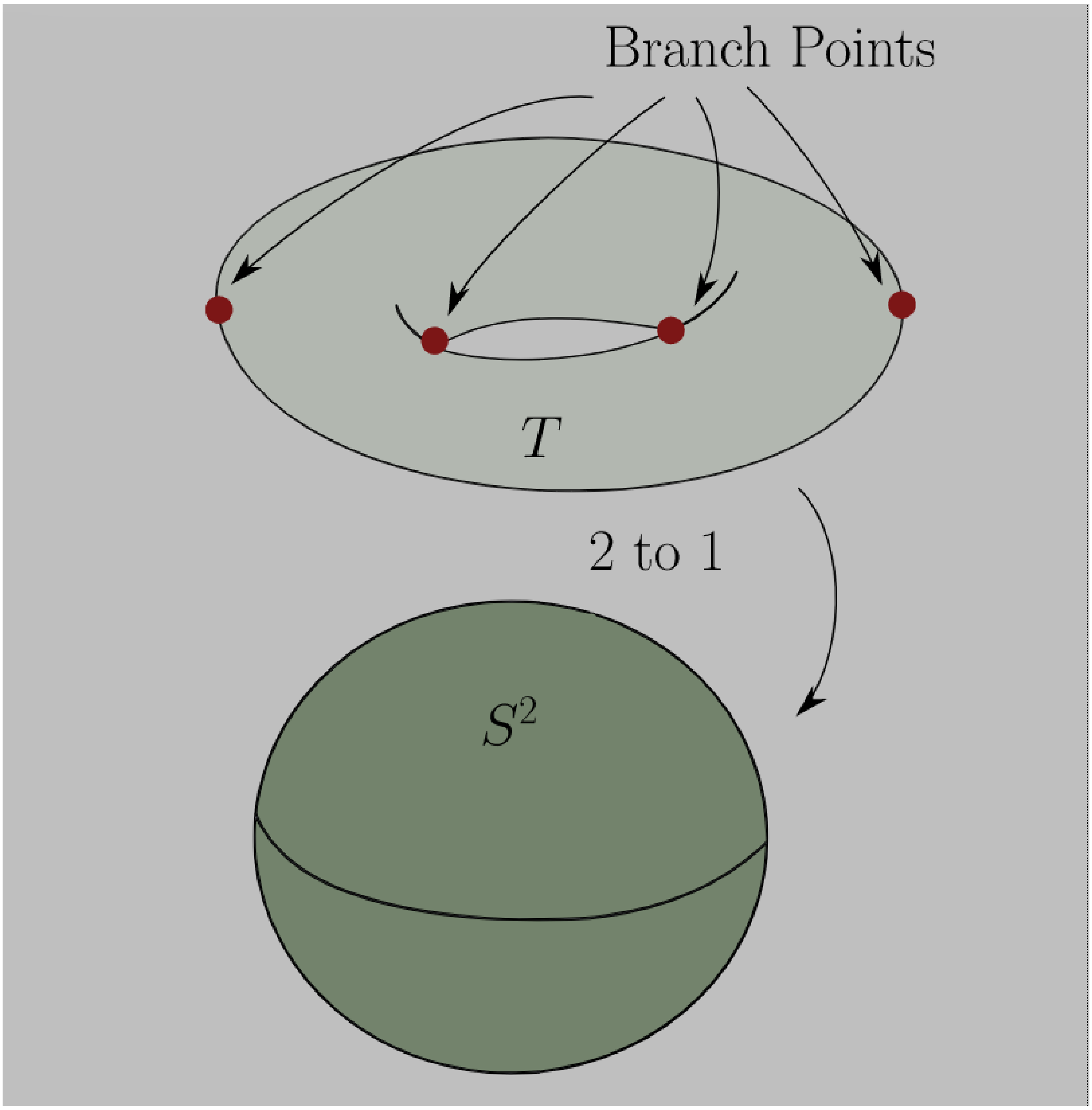


Figure 2: image_2021-02-25-20-41-53

Note that homothetic lattices define an isomorphism between the elliptic curves, and lattices mod homothety are in correspondence of elliptic curves. By acting $\mathrm{PGL}_2(C) \curvearrowright \mathbb{P}^1$ since GL_2 acts on lines since scaling an element fixes a line. This is dimension 3. So elliptic curves are also in correspondence with $\left\{ 4 \text{ points on } \mathbb{P}^1 \right\} / \mathrm{PGL}_2(\mathbb{C})$ since this is now dimension 1. Note that by applying homothety, the two basis vectors for a lattice can be rescaled so one is length 1 and the other is a complex number τ, and we can identify this space with $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$.

Exercise 17.1.7 (?)
Show that any $g(C) = 2$ curve has a degree 2 map to $\mathbb{P}^1$.

Remark 17.1.8: Similarly $g(C) = 3$ are usually a curve of degree 4 in $\mathbb{CP}^2$. Severi proof in the 50s: false! issues with building moduli space for $g \geq 23$. Need to use orbifold structure to take into account automorphisms.

18 | Wednesday, February 24

Last time:

$$\begin{aligned}\chi(C, L) &= h^0(C, L) - h^1(C, L) \\ &= h^0(C, L) - h^0(C, L^{-1} \otimes K_C) \\ &= \deg L + 1 - g,\end{aligned}$$

which is determined by purely topological information. We can generalize this to arbitrary ranks of the bundle and arbitrary dimensions of manifold:

Theorem 18.0.1 ***(Hirzebruch-Riemann-Roch (HRR) Formula).***
Let X be a compact complex manifold and let $\mathcal{E} \to X$ be a holomorphic vector bundle. Then

$$\chi(\mathcal{E}) = \int_C \mathrm{ch}(\mathcal{E})\mathrm{td}(X).$$

The constituents here:

- The **Chern character**, summed over R the *Chern roots*, which is in mixed cohomological degree.

$$\mathrm{ch}(\mathcal{E}) := \sum_{x_i \in R} e^{x_i} = \mathrm{ch}_0(\mathcal{E}) + \mathrm{ch}_1(\mathcal{E}) + \cdots + \mathrm{ch}_i(\mathcal{E}) \in H^{2i}(X; \mathbb{Q}).$$

- The **Todd class**, defined as

$$\mathrm{td}(F) := \prod_{x_i \in R} \frac{x_i}{1 - e^{-x_i}}$$

  where $\mathrm{td}(X) := \mathrm{td}(TX)$ is viewed as a complex vector bundle, which is again in mixed cohomological degree.

Remark 18.0.2: Note that integrating over cohomology classes in mixed degree is just equal to the integral over the top degree terms. Applying this to $X = C$ a curve and $\mathcal{E} := \mathcal{O}$, we obtain

$$\chi(C, \mathcal{O}) = \int_C \mathrm{ch}(\mathcal{O})\mathrm{td}(C).$$

We have

- $\mathrm{ch}(\mathcal{O}) = e^{c_1(\mathcal{O})} = e^0 = 1$
- $\mathrm{td}(C) := \mathrm{td}(TC) = c_1(TC)/(1 - e^{-c_1(TC)})$, whose Taylor coefficients are the Bernoulli numbers. We can expand $x/(1-e^{-x}) = 1 + (x/2) + (x^2/12) - x^4(720) + \cdots$, and since terms above degree 2 vanish, we have

$$\begin{aligned}
\cdots &= \int_C 1 + \left(1 + \frac{c_1(TC)}{2}\right) \\
&= \int_C \left(\frac{c_1(TC)}{2}\right) \\
&= \frac{1}{2}\chi_{\mathsf{Top}}(C) && \text{Chern-Gauss-Bonnet} \\
&= \frac{2-2g}{2} \\
&= 1 - g.
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\chi(C, L) &= \int_C \mathrm{ch}(L)\mathrm{td}(C) \\
&= \int_C (1 + c_1(L))\left(1 + \frac{c_1(L)}{2}\right) \\
&= \int_C c_1(L) + \frac{c_1(TC)}{2} \\
&= \deg L + 1 - g.
\end{aligned}$$

Remark 18.0.3: Note that this is a better definition of genus than the previous one, which was just the correction term in Riemann-Roch. Here we can define it as $g := h^1/2$.

Exercise 18.0.4 (?)
Try to state and prove a Riemann-Roch formula for vector bundles on curves.

Proposition 18.0.5 ***(Formula for Euler characteristic of a line bundle on a complex surface).***
Let S be a compact complex surface, i.e. $S \in \mathsf{Mfd}_{\mathbb{C}}^2$. An example might be $C \times D$ for C, D two complex curves, or $\mathbb{CP}^2$. Let $L \to S$ be a holomorphic vector bundle. Then

$$\chi(L) = \chi(\mathcal{O}_S) + \frac{1}{2}\left(L^2 - L \cdot K\right).$$

Note that $L^2 := \int_S c_1(L)c_1(L)$ is just shorthand for taking the intersection of L with itself. Recall that $K := \Omega_S^2$ is the space of holomorphic top forms.

Proof (?).

Let x_1, x_2 be the Chern roots of TS. By HRR, we have

$$\begin{aligned}
\chi(L) &= \int_S \operatorname{ch}(L)\operatorname{td}(S) \\
&= \int_S \left(1 + c_1(L) + \frac{c_1(L)^2}{2!}\right)\left(\frac{x_1}{1-e^{-x_1}}\frac{x_2}{1-e^{-x_2}}\right) \\
&= \int_S \left(1 + c_1(L) + \frac{c_1(L)^2}{2!}\right)\left(1 + \frac{x_1}{2} + \frac{x_1^2}{12}\right)\left(1 + \frac{x_2}{2} + \frac{x_2^2}{12}\right) \\
&= \int_S \left(1 + c_1(L) + \frac{c_1(L)^2}{2!}\right)\left(1 + \frac{x_1+x_2}{2} + \frac{x_1^2+x_2^2+3x_1x_2}{12}\right) \\
&= \int_S \left(1 + c_1(L) + \frac{c_1(L)^2}{2!}\right)\left(1 + \frac{c_1(x_1,x_2)}{2} + \frac{c_1(x_1,x_2)^2 + c_2(x_1,x_2)}{12}\right) \\
&= \int_S \left(1 + c_1(L) + \frac{c_1(L)^2}{2!}\right)\left(1 + \frac{c_1(T)}{2} + \frac{c_1(T)^2 + c_2(T)}{2}\right) \\
&= \int_S \frac{c_1(L)^2}{2} + \frac{c_1(L)c_1(T)}{2} + \frac{c_1(T)^2}{2} + \frac{c_2(T)}{12} \quad \text{Take deg 4} \\
&= \int_S \left(\frac{c_1(L)^2 + c_1(L)c_1(T)}{2}\right) + \chi(\mathcal{O}_S) \quad \text{HRR on last two terms.}
\end{aligned}$$

where we've applied HRR to $\mathcal{O}_S$. It remains to show that $c_1(T) = -c_1(K)$. We have

$$K = \Omega_S^2 = \bigwedge^2 T^\vee.$$

Note that $\bigwedge^{\text{top}} \mathcal{E} \coloneqq \det(\mathcal{E})$ for any bundle $\mathcal{E}$ since this is a 1-dimensional bundle. We have $c_1(T) = -c_1(T^\vee)$ since the Chern roots of $T^\vee$ are $-x_1, -x_2$. So it suffices to show $c_1(T^\vee) = c_1(K)$, but there is a general result that $c_1(\mathcal{E}) = c_1(\det \mathcal{E})$. This uses the splitting principle $\mathcal{E} = \bigoplus_{i=1}^r L_i$ with $x_i = c_1(L_i)$. We have $c_1(\mathcal{E}) = \sum x_i$ and $\det \mathcal{E} = \bigotimes_{i=1}^r L_i$, so $\sum x_i = c_1(L_1 \otimes \cdots \otimes L_r)$.

■

Remark 18.0.6: We want to use the following formula:

$$\chi(S, L) = \chi(\mathcal{O}_S) = \frac{1}{2}(L^2 - L \cdot K).$$

This requires knowing $\chi(\mathcal{O}_S)$. Applying HRR yields

$$\begin{aligned}
\chi(\mathcal{O}_S) &= \int_S \frac{c_1(T)^2 + c_2(T)}{12} \\
&= \int_S \frac{(-c_1(K))^2 + c_2(T)}{12} \\
&= \frac{K^2 + \int_S c_2(T)}{12},
\end{aligned}$$

so we just need to understand $\int_S c_2(T)$. But for $n = \operatorname{rank} \mathcal{E}$, $c_n(\mathcal{E})$ (the top Chern class) is the fundamental class of a zero locus of a section of $\mathcal{E}$. Note that $S \in \mathsf{Mfd}_{\mathbb{R}}^4$ is oriented, so $\int_S c_2(T)$ is the signed number of zeros of a smooth vector field.

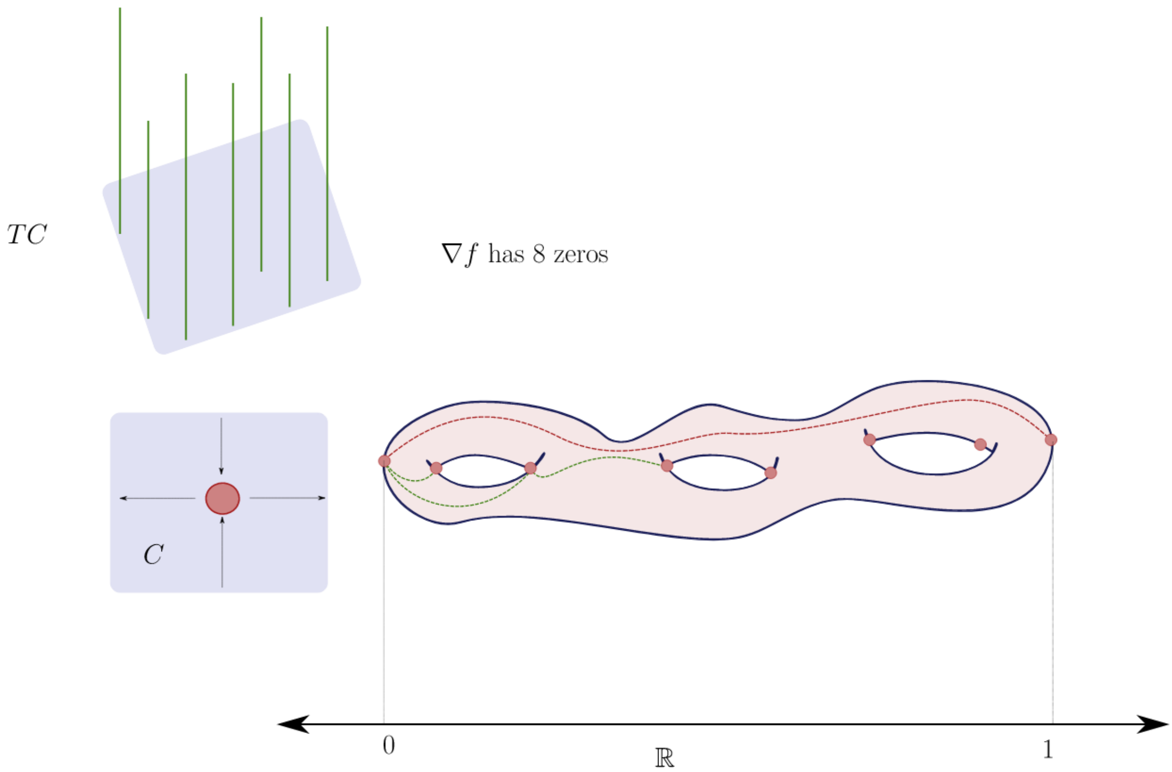


Figure 3: image_2021-02-25-20-42-49

Check.

Looking at the tangent bundle of the surface, the local sign of an intersection will be the number of incoming directions $\pmod 2$, i.e. the index of the critical point. Then the signed number of zeros here yields $1 - 6 + 1 = -4 = \chi_{\mathsf{Top}}(C)$. More generally, we have

$$\chi_{\mathsf{Top}}(M^n) = \int_C c_n(TM),$$

the **Chern-Gauss-Bonnet** formula. We can thus write

$$\chi(\mathcal{O}_S) = \frac{K^2 + \chi_{\mathsf{Top}}(S)}{12}.$$

19 | Friday, February 26

Remark 19.0.1: Last time: Riemann-Roch for surfaces, today we'll discuss some examples. Recall that if $S \in \mathsf{Mfd}_{\mathbb{C}}^2$ is closed and compact (noting that $S \in \mathsf{Mfd}_{\mathbb{R}}^4$) and $L \to S$ is a holomorphic line bundle then

$$\chi(S, L) = \chi(\mathcal{O}_S) + \frac{1}{2}(L^2 - L \cdot K)$$

where $K = c_1(K_S)$ for $K_S := \Omega_S^2$ the canonical bundle and $L = c_1(L)$. We also saw

$$\chi(\mathcal{O}_S) = \frac{1}{12}(K^2 + \chi_{\mathsf{Top}}(S)),$$

where χ_{Top} is the Euler characteristic and is given by

$$\chi_{\mathsf{Top}}(S) = 2h^0(S; \mathbb{C}) - 2h^1(S, \mathbb{C}) + h^2(S; \mathbb{C}).$$

Example 19.0.2*(?)*: Let $S = \mathbb{CP}^2$, which can be given in local coordinates by

$$\left\{ [x_0 : x_1 : x_2] \;\middle|\; (x_0, x_1, x_2) \in \mathbb{C}^3 \setminus \{0\} \right\}$$

where we only take equivalence classes of ratios $[x, y, z] = [\lambda x, \lambda y, \lambda z]$ for any $\lambda \in \mathbb{C}^\times$. This decomposes as

$$\mathbb{CP}^2 \cup \mathbb{C} \cup \{\text{pt}\} = \{[1 : x_1 : x_2]\} \cup \{[0 : x_1 : x_2]\} \cup \{[0 : 0 : 1]\},$$

i.e. we take $x_0 \neq 0$, then $x_0 = 0, x_1 \neq 0$, then $x_0 = x_1 = 0$. Note that

$$h^i(\mathbb{CP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & 0 \leq i \leq 2n \text{ even} \\ 0 & \text{else.} \end{cases}$$

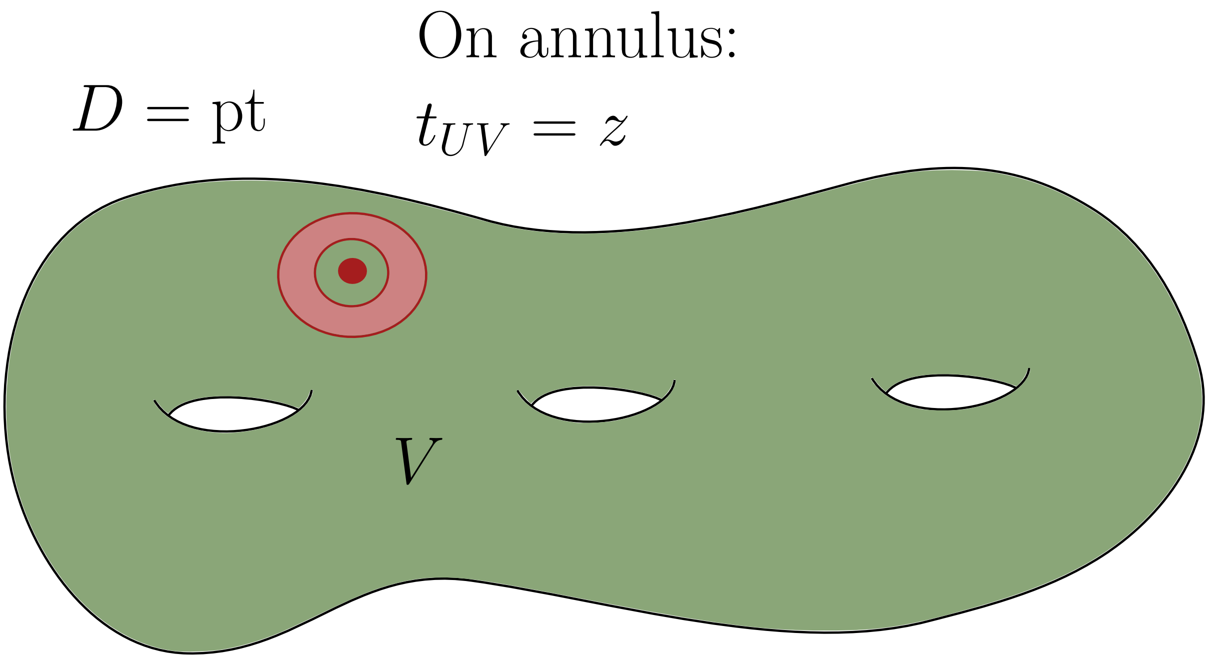
We can use this to conclude that $\chi_{\mathsf{Top}}(\mathbb{CP}^n) = n + 1$ and $\chi_{\mathsf{Top}}(\mathbb{CP}^2) = 3$. Over $\mathbb{CP}^n$ we have a **tautological line bundle** $\mathcal{O}(-1)$ given by sending each point to the corresponding line in $\mathbb{C}^{n+1}$, i.e. $\mathcal{O}(-1) \to \mathbb{CP}^n$ given by

$$\lambda(x_0, \cdots, x_n) \mapsto [x_0 : \cdots : x_n].$$

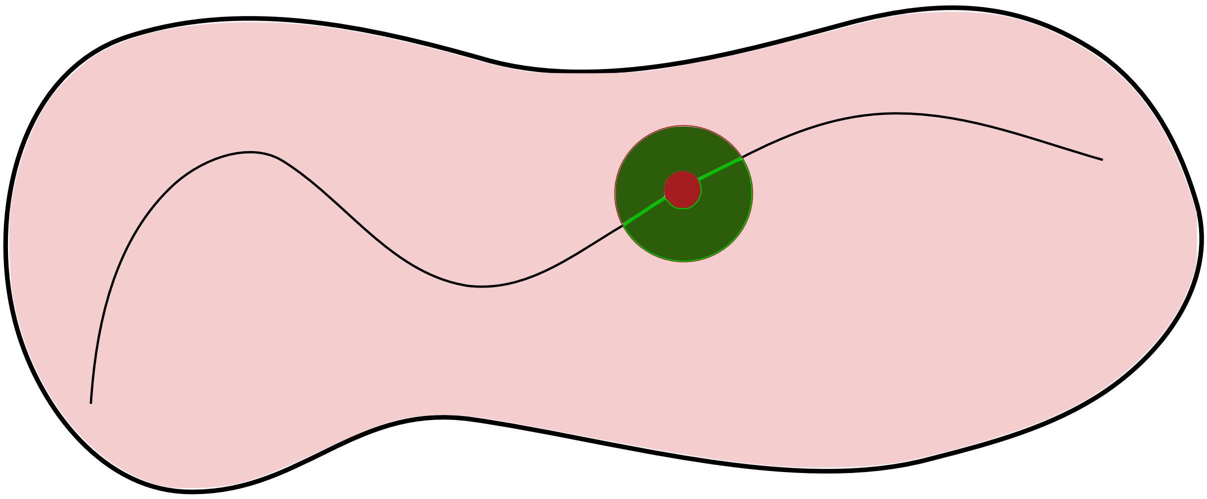
Note that the total space is $\underset{0}{\mathrm{Bl}}(\mathbb{C}^{n+1})$ is the **blowup** at zero, which separates the tangents at 0.

Remark 19.0.3: Let X be an algebraic variety, i.e. spaces cut out by polynomial equations, for example $\{xy = 0\} \subseteq \mathbb{C}^2$ which has a singularity at the origin. A **divisor** is a $\mathbb{Z}$-linear combination of subvarieties of codimension 1. Note that for a curve X, this recovers the definition involving points. For D a divisor on X, we associated a bundle $\mathcal{O}_X(D)$ which had a meromorphic section with a zero/pole locus whose divisor was precisely D.

Recall the construction: we chose a point, then a trivializing neighborhood where the transition functions where V.



For a higher dimensional algebraic variety or complex manifold, for D a complex submanifold, pick a chart around a point that the nearby portion of D to a coordinate axis in $\mathbb{C}^n$, which e.g. can be given by $\{z_1 = 0\}$.



As before there's a distinguished section $s_D \in H^0(X; \mathcal{O}_X(D))$ vanishing along D. Note that a line bundle is a free rank 1 $\mathcal{O}$-module, and analogously here the functions vanishing along D are $\mathcal{O}$-modules generated by (here) z_1.

Definition 19.0.4 (Hyperplane)

A **hyperplane** in $\mathbb{CP}^n$ is any set of the form

$$H = \left\{ [x_0 : \cdots : x_1] \;\middle|\; \sum a_i x_i = 0 \right\} \cong \mathbb{CP}^{n-1}.$$

Example 19.0.5 *(?)***:** Take $\mathbb{CP}^{n-1} \subseteq \mathbb{CP}^n$, e.g. $\{x_0 = 0\}$. This is an example of a **divisor** on $\mathbb{CP}^n$, i.e. a complex codimension 1 “submanifold”. We can take the line bundle constructed above to get $\mathcal{O}_{\mathbb{CP}^n}(\mathbb{CP}^{n-1})$ which vanishes along $\mathbb{CP}^{n-1}$. More generally, for any hyperplane H we can take $\mathcal{O}_{\mathbb{CP}^n}(H)$, and these are all isomorphic, so we'll denote them all by $\mathcal{O}_{\mathbb{CP}^n}(1)$. The implicit claim is that is the inverse line bundle of the tautological bundle, so $\mathcal{O}(1) \otimes \mathcal{O}(-1)$ is the trivial bundle since the transition functions are given by reciprocals and multiplying them yields 1. We can classify complex line bundles on $\mathbb{CP}^n$ using the SES

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to 1.$$

We know that $H^1(X; \mathcal{O}^\times)$ were precisely holomorphic line bundles, since they were functions agreeing on double overlaps with a cocycle condition. We have a LES coming from sheaf cohomology:

$$\begin{tikzcd}
 & & \cdots \arrow[dll, rounded corners] \\
H^1(X;\mathcal{O}) \arrow[r] & H^1(X;\mathcal{O}) \arrow[r] & H^1(X;\mathcal{O}^\times) \arrow[dll, "c_1", rounded corners] \\
H^2(X;\mathcal{O}) \arrow[r] & \cdots &
\end{tikzcd}$$

Link to Diagram

Applying this to $X := \mathbb{CP}^n$, we have $H^1(\mathcal{O}) = H^2(\mathcal{O}) = 0$. This can be computed directly using that $\mathbb{CP}^n = \cup_{n \geq 1} \mathbb{C}^n$ by taking charts $x_i \neq 0$, and this yields an acyclic cover. Thus c_1 is an isomorphism above, and $\operatorname{Pic}(\mathbb{CP}^n) \cong \mathbb{Z}$, where Pic denotes isomorphism classes of line bundles. We can identify $\operatorname{Pic}(\mathbb{CP}^n) = \left\{ \mathcal{O}_{\mathbb{CP}^n}(k) \;\middle|\; k \in \mathbb{Z} \right\}$.

20 | Monday, March 01

Remark 20.0.1: Last time: we defined $\operatorname{Pic}(\mathbb{CP}^n)$ as the set of line bundles on $\mathbb{CP}^n$.

Definition 20.0.2 (Picard Group of a Manifold)
Given any $X \in \mathsf{Mfd}_{\mathbb{C}}$, define $\operatorname{Pic}(X)$ as the set of isomorphism classes of holomorphic line bundles on X. This is an abelian group given by $L \otimes L'$ and inversion $L \to L^{-1}$.

Remark 20.0.3: We saw that $\operatorname{Pic}(X) \cong H^1(X; \mathcal{O}^\times)$ as groups, noting that H^1 has a natural group structure here. We defined a **tautological bundle** on $\mathbb{CP}^n$ and saw it was isomorphic to $\mathcal{O}(-1)$, and moreover $\mathcal{O}(H) \cong \mathcal{O}(1)$ for H a hyperplane. The fiber was given by

$$\begin{aligned} \text{Taut} &\to \mathbb{CP}^n \\ \left\{\lambda(x_0, \cdots, x_n) \;\middle|\; \lambda \in \mathbb{C}\right\} &\mapsto [x_0 : \cdots : x_n], \end{aligned}$$

i.e. the entire line corresponding to the given projective point. We also have $\mathcal{O}(H)(U)$ is the sect of rational homogeneous functions φ on U of degree 1 such that $\operatorname{Div}\varphi + H \geq 0$ where $H \coloneqq \{x_0 = 0\}$. We want φ/x_0 to be a well-defined function, so φ should scale like x_0 in the sense that

$$\varphi(\lambda x_0, \cdots, \lambda x_n) = \lambda \varphi(x_0, \cdots, x_n).$$

Note that there is a natural map

$$\text{Taut} \otimes \mathcal{O}(H) \to \mathcal{O},$$

given by taking the line over a point and evaluating the homogeneous function on that line. Thus Taut is the inverse of $\mathcal{O}(H)$.

Remark 20.0.4: We want to understand what Noether's formula says for $\mathbb{CP}^2$, which requires understanding the canonical bundle $K_{\mathbb{CP}^n}$. We'll do this by writing down a meromorphic section ω (since it's a meromorphic volume form) which will yield $K_{\mathbb{CP}^n} = \mathcal{O}(\operatorname{Div}\omega)$. So take

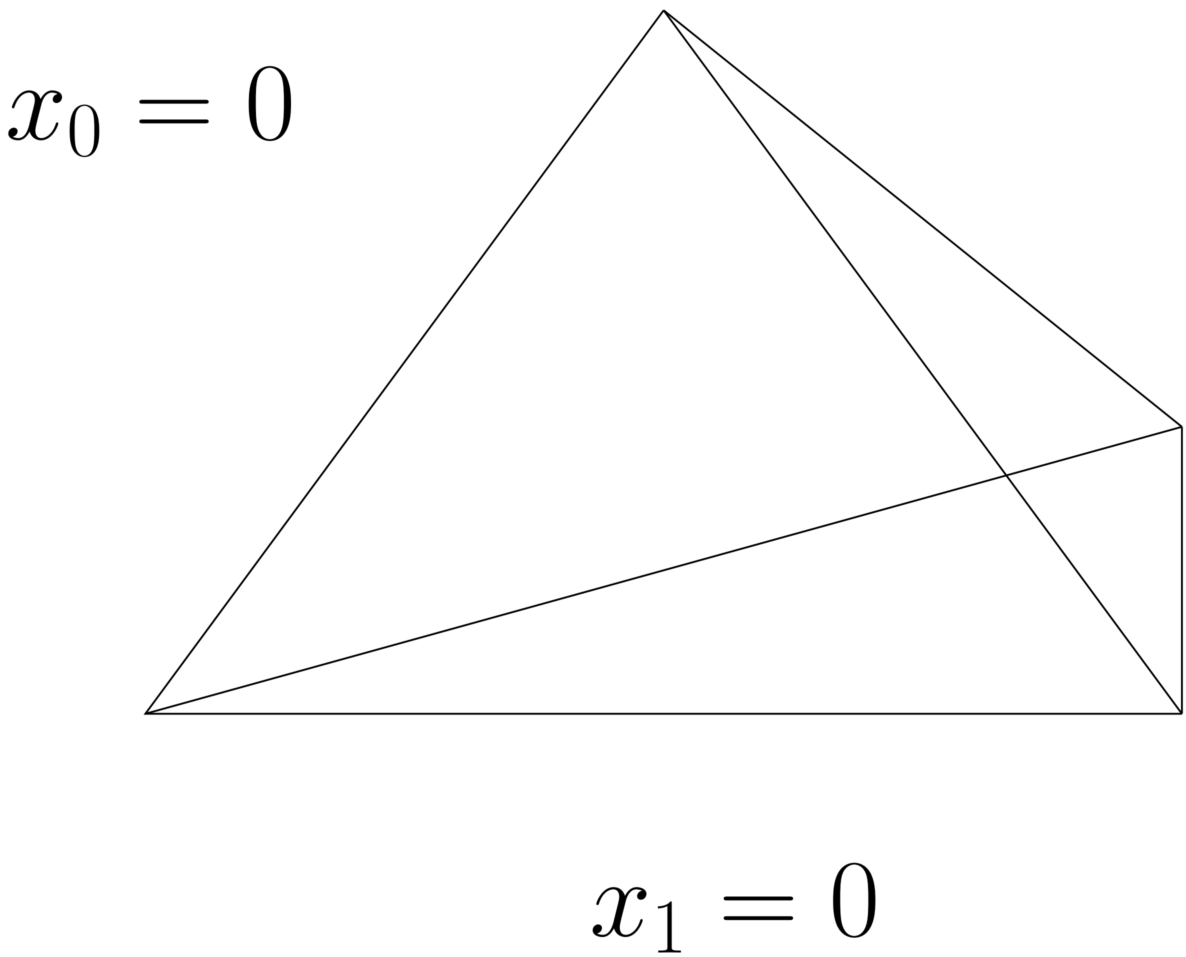
$$\omega \coloneqq x_1^{-1} dx_1 \wedge \cdots \wedge x_n^{-1} dx_n,$$

noting that we leave out the first coordinate x_0 and divide by coordinates to make this scale-invariant. Here we work in a $\mathbb{C}^n$ chart of points of the form $[1 : x_1 : \cdots : x_n]$. Where does ω have poles? Along $x_i = 0$ for any $1 \leq i \leq n$, and similarly in any other coordinate chart. We also have a 1st order pole along $x_0 = 0$. We then get

$$K_{\mathbb{CP}^n} = \mathcal{O}(\operatorname{Div}\omega) = \mathcal{O}(-H_0 - H_1 - \cdots - H_n) = \mathcal{O}(-n-1),$$

where $H_i = \{x_i = 0\}$.

Note that $\mathbb{CP}^n$ is like a simplex:



Applying this to $\mathbb{CP}^2$, we obtain

$$K_{\mathbb{CP}^2} = \mathcal{O}(-3).$$

What is the intersection form? We know $H^2(\mathbb{CP}^2;\mathbb{Z}) \cong \mathbb{Z}$ and the intersection form is unimodular. So write $\mathbb{Z} \coloneqq \mathbb{Z}\alpha$ for α some generator. Then $\alpha \cdot \alpha = \pm 1$ since $\det G = \pm 1$ for the Gram matrix for this to be unimodular. Note that $(-\alpha) \cdot (-\alpha) = \pm 1$ with the same sign.

Claim: $\mathcal{O}(1) = \mathcal{O}(H)$ generates $\operatorname{Pic}(\mathbb{CP}^2) = H^2(\mathbb{CP}^2;\mathbb{Z})$.

This is because $c_1\mathcal{O}(H) \cdot c_1\mathcal{O}(H) = H \cdot H = \{x_0 = 0\} \pitchfork \{x_1 = 0\} = \{[0:0:1]\}$ here we note that the two hyperplanes can be oriented transversely and intersected. This is an oriented intersection.

Recall Noether's formula, which was HRR applied to $\mathcal{O}$ and the Chern-Gauss-Bonet theorem:

$$\begin{aligned}\chi(\mathcal{O}) &= \frac{1}{12}(K^2 + \chi_{\mathsf{Top}}) \\ &= h^0(\mathcal{O}) - h^1(\mathcal{O}) + h^2(\mathcal{O}) \\ &= 1 - 1 + 1 \\ &= 1.\end{aligned}$$

The right-hand side can be written as

$$\frac{1}{12}((-3H)\cdot(-3H)+3) = \frac{1}{12}(9+3) = 1.$$

Proposition 20.0.5 ***(The 4-sphere has no complex structure).***
S^4 has no complex structure.

Proof (?).
We know that $\chi_{\mathsf{Top}}(S^4) = 2$. If S^4 had a complex structure, then $c_1(K_{S^4}) \in H^2(S^4;\mathbb{Z}) = 0$. Thus would make $K_{S^4}^2 = 0$, and so

$$\chi(\mathcal{O}_{S^4}) = \frac{1}{12}(0+2) = \frac{1}{6} \notin \mathbb{Z},$$

which is a contradiction. ↯

■

Example 20.0.6 ***(?):*** Consider $\overline{\mathbb{CP}}^2$, a 4-manifold diffeomorphic to $\mathbb{CP}^2$ with the opposite orientation. What is the intersection form? Taking $H\cdot H = -1$ since the orientations aren't compatible, and more generally the Gram matrix is negated when the orientation is reversed.

Proposition 20.0.7 ***(Barred projective 2-space is not orientably diffeomorphic to a complex surface).***
$\overline{\mathbb{CP}}^2$ is not diffeomorphic to a complex surface by an orientation-preserving diffeomorphism (or any homeomorphism).

Proof (?).
We have $\chi_{\mathsf{Top}} = 3$, and $K_{\overline{\mathbb{CP}}^2} = -c_1(T\overline{\mathbb{CP}}^2) = \pm 3H$. Then

$$\chi(\mathcal{O}) = \frac{1}{12}\left(K_{\overline{\mathbb{CP}}^2}^2 + \chi_{\mathsf{Top}}\right) = \frac{1}{12}(-9+3) \notin \mathbb{Z}.$$

■

Remark 20.0.8: Consider $\mathcal{O}_{\mathbb{CP}^n}(d)$, what are its global sections $H^0(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(d))$. Locally we have $\mathcal{O}_{\mathbb{CP}^n}(d)(U)$ given by holomorphic functions in $(x_0,\cdots,x_n) \in \pi^{-1}(U)$ where $\pi : \mathbb{C}^{n+1} \to \mathbb{CP}^n$ and the functions satisfy $f(\lambda\mathbf{x}) = \lambda^d f(\mathbf{x})$. The global sections will be the homogeneous degree d polynomials in the coordinates of $\mathbf{x}$.

Remark 20.0.9: Why does a holomorphic function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ such that $f(\lambda \mathbf{x}) = \lambda^d f(\mathbf{x})$ necessarily a polynomial? Use the result that any such function with at most polynomial growth is itself a polynomial. If $f|_{S^{2d+1}}$ is bounded by C, we have $\|f\|_{L^2} \leq C|x|^{2d}$. Since $(\partial_{x_1} \cdots \partial_{x_k})^d f$ is globally bounded $k \geq 2d$, applying Liouville's theorem makes it constant, and so a finite number of derivatives kill f and this forces it to be polynomial.

Remark 20.0.10: So how many homogeneous degree d functions are there? Here $h^0(\mathbb{CP}^n, \mathcal{O}(d)) =$ will be the number of linearly independent degree d polynomials in the variables $x_0, \cdots, x_n$, which is $\left(\!\binom{n+1}{d}\!\right) = \binom{n+d}{n}$, using the fact that monomials span this space.

Exercise 20.0.11 (?)
Using that $h^0(\mathbb{CP}^2; \mathcal{O}(k)) = h^2(\mathbb{CP}^2; \mathcal{O}(-3-k))$ by Serre duality and Riemann-Roch, compute $h^i(\mathbb{CP}^2; \mathcal{O}(k))$ for all i, k.

Fact 20.0.12
$h^i(\mathbb{CP}^n; \mathcal{O}(k)) = 0$ unless $i = 0, n$.

21 | Wednesday, March 03

Find first 5m.

Remark 21.0.1: When we considered $\overline{\mathbb{CP}}^2$, we implicitly assumed $T\overline{\mathbb{CP}}^2$ was a complex rank 2 vector bundle with some purported complex structure.

Claim:

$$c_1(T\overline{\mathbb{CP}}^2) = \pm 3H,$$

although it's not clear that $c_1(K) \in H^2(\overline{\mathbb{CP}}^2; \mathbb{Z}) \cong (\mathbb{Z}, [-1])$.

Remark 21.0.2: We had $\chi(\mathcal{O}) = \frac{1}{12}\left(K^2 + \chi_{\mathsf{Top}}\right) = \frac{1}{12}(3 - n^2)$, and since $3 - n^2 \in 12\mathbb{Z}$, we have $n^2 \in 3 + 12\mathbb{Z} \subset 3 + 4\mathbb{Z}$ and this forces $n^2 \equiv 3 \pmod 4$.

Definition 21.0.3 (Differential Complex)
Let

$$0 \to \mathcal{E}^0 \xrightarrow{d_0} \mathcal{E}^1 \xrightarrow{d_1} \cdots \to \mathcal{E}^n \to 0$$

be a complex (so $d^2 = 0$) of smooth vector bundles on a smooth manifold X im $\mathsf{Mfd}_{\mathbb{R}}^{C^\infty}$. Suppose that the d_i are **differential operators**, i.e. in local trivializing charts over U we have

$$\mathcal{E}^i \cong \mathcal{O}^{\oplus r_i} \mathcal{O}^{\oplus r_{i+1}} \cong \mathcal{E}^{i+1}$$

where in every matrix coordinate, d_i is of the form $\sum_{|I|<N} g_I \partial_I$ where $\partial_I := \partial_{i_1} \cdots \partial_{i_N}$ is a partial derived and the g_I are smooth functions.

Example 21.0.4 *(?)*: For $X \in \mathsf{Mfd}_{\mathbb{R}}^{C^\infty}$, we can take

$$0 \to \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots .$$

In local coordinates,

- Ω^1 is spanned over $\mathcal{O}$ by $dx_1, \cdots, dx_n$ where $n = \dim_{\mathbb{R}}(X)$
- Ω^2 is spanned over $\mathcal{O}$ by $dx_i \wedge dx_j$ for $1 \leq i, j \leq n$.

Then the component of d sending $dx_i \to dx_i \wedge dx_j$ is of the form

$$f dx_i \mapsto -\frac{\partial f}{\partial x_j} dx_i \wedge dx_j.$$

Example 21.0.5 *(?)*: For $X \in \mathsf{Mfd}_{\mathbb{C}}$ and $\mathcal{E} \to X$ a holomorphic vector bundle, take

$$\mathcal{E} \otimes A^{0,0} \xrightarrow{\bar\partial} \mathcal{E} \otimes A^{0,1} \xrightarrow{\bar\partial} \mathcal{E} \otimes A^{0,2} \to \cdots .$$

This is because for s_i local holomorphic sections and ω a smooth form we have

$$\bar\partial \left((s_1, \cdots, s_r) \otimes \omega \right) = (s_1, \cdots, s_r) \otimes \bar\partial \omega.$$

Definition 21.0.6 (Order of an operator)
The maximal N that appears in $\sum_{|I| \leq N} g_I \partial_I$ is the **order**.

Definition 21.0.7 (Symbol Complex)
The **symbol complex** is a sequence of vector bundles on $T^\vee X$. Noting that we have $\pi : T^\vee X \to X$, and using pullbacks we can obtain bundles over the cotangent bundle:

$$0 \to \pi^* \mathcal{E}_0 \xrightarrow{\sigma(d_0)} \pi^* \mathcal{E}_1 \xrightarrow{\sigma(d_1)} \cdots \to \pi^* \mathcal{E}_n \to 0.$$

The **symbol** of the differential operator d_i is $\sigma(d_i)$. It is defined by replacing ∂_i in $\sum_{|I|=N} g_I \partial_I$ with y_i where

$$y_i : T^\vee U \to \mathbb{R}$$

is the coordinate function on the second factor of $T^\vee U = U \times \mathbb{R}^n$ associated to the local coordinate i. Using that $TU = (T^\vee)^\vee U$, we can view ∂_i as functions on the cotangent bundle, $\sigma(d_i)$ is given in local trivializations by multiplication by a smooth function $\sum_{|I|=N} g_I y^I$.

Example 21.0.8*(?)*: Consider $\mathcal{O} \xrightarrow{d} \Omega^1$. In local coordinates, this is given by $d = (\partial_1, \cdots, \partial_n)$, i.e. coordinate-wise differentiation, since we can write a local trivialization $\Omega^1 = \mathcal{O}dz_1 \oplus \cdots \oplus \mathcal{O}dz_n$. Then the symbol of d is given by

$$\begin{aligned}\sigma(d) : \pi^*\mathcal{O} &\to \pi^*\Omega^1 \\ 1 &\mapsto (y_1, \cdots, y_n),\end{aligned}$$

thought of as vector bundles over $T^\vee X$, and this is projection onto to cotangent factor. Locally, the image of 1 is given by $y_1 dx_1 + \cdots y_n dx_n$, which is a point in ${T_p}^\vee X$ for all $(p, \alpha) \in T^\vee X$ which is an assignment to every point $(p, \alpha) \in {T_p}^\vee X$ a point in $(\pi^*\Omega^1)_{p,\alpha} \cong {T_p}^\vee X$. There is a tautological section $(p, \alpha) \to \alpha \in {T_p}^\vee X \in (\pi^*\Omega^1)_{p,\alpha}$, or really $(p, \alpha) \mapsto ((p, \alpha), \alpha)$.

Remark 21.0.9: See similarly to the canonical symplectic structure of the cotangent bundle.

Remark 21.0.10: More generally, for $d : \Omega^p \to \Omega^{p+1}$, $\sigma(d)$ acts on the frame $dx_{i_1} \wedge \cdots dx_{i_p}$ in the following way:

$$\sigma(d)(dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = \sum_y y_y dx_j \wedge dx_{i_1} \wedge \cdots dx_{i_p}$$

where

$$d : f dx_{i_1} \wedge \cdots \wedge dx_{i_p} \mapsto \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge \left(dx_{i_1} \wedge \cdots \wedge dx_{i_p}\right).$$

The symbol complex is

$$\pi^*\mathcal{O} \xrightarrow{\sigma(d)} \pi^*\Omega^1 \xrightarrow{\sigma(d)} \pi^*\Omega^2 \to \cdots \to \pi^*\Omega^n \to 0$$

for n the dimension. In this case, $\sigma(d)$ has the same formula everywhere, since it's C^∞-linear:

$$\sigma(d) = \sum_j y_j dx_j \wedge (\cdots).$$

> **Definition 21.0.11** (Elliptic Complex)
> A differential complex $(\mathcal{E}_*, d)$ is **elliptic** if the symbol complex $(\pi^*\mathcal{E}_*, \sigma(d))$ is an exact sequence of sheaves (importantly) on $T^\vee X \setminus \{s_z\}$ for s_z the zero section.

Claim: (Ω_*, d) is elliptic. To check exactness of a sequence of vector bundles, it suffices to check exactness on every fiber. Fix $(p, \alpha) \in T^\vee X \setminus \{s_z\}$, then

$$0 \to \mathbb{C} \xrightarrow{\wedge\alpha} T_p^\vee X \xrightarrow{\wedge\alpha} \bigwedge^2 {T_p}^\vee X \xrightarrow{\wedge\alpha} \bigwedge^3 {T_p}^\vee X \to \cdots.$$

Moreover, if $\alpha \wedge \beta = 0$ implies that $\beta = \alpha \wedge \gamma$ for some γ, which implies that this sequence is exact.

22 | Friday, March 05

Remark 22.0.1: Recall that we set up a differential complex, whose objects were vector bundles and differentials were differential operators (i.e. linear combinations of partial derivatives) in local trivializations. We pulled back to tangent bundles (?) and defined the *symbol* of an operator, and saw that when taking the symbol complex of the deRham complex. the sequence of maps was given by wedging against a tautological one-form. This was an *elliptic complex* because the maps became wedging with a covector.

Example 22.0.2 ***(of an elliptic complex):*** Let $X \in \mathsf{Mfd}_{\mathbb{C}}$ and $\mathcal{E} \to X \in \mathsf{Bun}_{\mathrm{GL}_r\mathbb{C}}$ be holomorphic. There is a resolution

$$0 \to \mathcal{E} \xrightarrow{i} \mathcal{E} \otimes A^{0,0} \xrightarrow{\bar\partial} \mathcal{E} \otimes A^{0,1} \xrightarrow{\bar\partial} \cdots .$$

What is the symbol complex? Consider the projection $\pi : T^\vee X \to X$, and use pullbacks to get a sequence

$$0 \to \pi^*\mathcal{E} \otimes A^{0,0} \xrightarrow{\sigma(\bar\partial)} \pi^*\mathcal{E} \otimes A^{0,1} \xrightarrow{\sigma(\bar\partial)} \cdots .$$

Here the symbol $\sigma(\bar\partial)$ replace $\dfrac{\partial}{\partial t\bar z_i}$ with the corresponding function on $T^\vee X$, say $\bar y_i$. Then $\sigma(\bar\partial) = \sum_i \bar y_i \, d\bar z_i \wedge (-) = \bar\alpha \wedge (-)$. As before, at a point (p, α) where $\alpha \neq 0$ in $T^\vee X$, we get

$$0 \to \mathcal{E}_p \xrightarrow{\bar\alpha\wedge(-)} \mathcal{E}_p \otimes \bigwedge_p^{0,1} X \xrightarrow{\bar\alpha\wedge(-)} \mathcal{E}_p \otimes \bigwedge^{0,2} X \to \cdots ,$$

which is an exact sequence of vector spaces. So $(\mathcal{E} \otimes A^{0,p}, \bar\partial)$ is an elliptic complex.

Slogan 22.0.3
The symbol being exact is approximately the top-order part being nowhere-vanishing.

Remark 22.0.4: The next theorem computes the cohomology of an elliptic complex using Chern and Todd classes.

Theorem 22.0.5 ***(Atiyah-Singer Index Theorem).***
If $(\mathcal{E}_*, d)$ is an elliptic complex of smooth vector bundles on a compact oriented $X \in \mathsf{Mfd}_{\mathbb{R}}^n$, then

$$\chi(\mathcal{E}_*, d) = \sum (-1)^i \dim \left(\frac{\ker d^i}{\operatorname{im} d^{i-1}} \right) = (-1)^{\binom{\dim(X)}{2}} \int_X \frac{\mathrm{ch}}{\mathrm{eul}} (\mathcal{E}_*) \mathrm{td}(TX \otimes_{\mathbb{R}} \mathbb{C}).$$

Remark 22.0.6: Here we define $\mathrm{ch}(\mathcal{E}_*) := \sum_i (-1)^i \mathrm{ch}(\mathcal{E}^i)$. What does it mean to divide by the Euler class? Let $\{x_i, -x_i\}$ be the Chern roots of the complexified tangent bundle $TX \otimes \mathbb{C}$, then

$\mathrm{eul}(X) := \prod x_i$ is the product where we pick one of each of the Chern roots from each of the pairs. The preferred sign to choose is the one for which $\int_X \prod x_i = \chi_{\mathsf{Top}}(X)$. Dividing just means to take the Chern character, then if it's divisible by $\prod x_i$, we do so. We have

$$\mathrm{td}(TX \otimes \mathbb{C}) = \prod_i \left(\frac{x_i}{1-e^{-x_i}}\right)\left(\frac{-x_i}{1-e^{-x_i}}\right).$$

Thus

$$\frac{\mathrm{td}(TX \otimes \mathbb{C})}{\mathrm{eul}(X)} = \prod_i \frac{1}{x_i}\left(\frac{x_i}{1-e^{-x_i}}\right)\left(\frac{-x_i}{1-e^{-x_i}}\right),$$

but note that this doesn't necessarily make sense. However, all all computations we'll see, there will be enough cancellation to make this well-defined.

Exercise 22.0.7 (Chern character of the de Rham complex)
$\mathrm{ch}(\Omega_* X \otimes \mathbb{C}) = \prod_i (1-e^{x_i})(1-e^{-x_i})$ for $X \in \mathsf{Mfd}_{\mathbb{R}}^{2n}$ even dimensional.

Example 22.0.8*(?)*: Supposing $X \in \mathsf{Mfd}_{\mathbb{R}}^2$ is a genus g surface, we have

$$\mathcal{O} \to \Omega^1 \otimes \mathbb{C} \to \Omega^2 \otimes \mathbb{C},$$

and $\mathrm{ch}(\Omega_*) = \mathrm{ch}(\mathcal{O}) - \mathrm{ch}(\Omega^1 \otimes \mathbb{C}) + \mathrm{ch}(\Omega^2 \otimes \mathbb{C})$. The Chern roots of $TX \otimes \mathbb{C}$ are $\{x_i, -x_i\}$, which come in pairs. So

$$\mathrm{ch}(\Omega_*) = 1 - e^{x_i} - e^{x_i} + e^{-x_i + x_i} = (1-e^{-x_i})(1-e^{x_i}).$$

From the theorem, we're supposed to have

$$\begin{aligned}
\chi(\Omega_*, d) &= (-1)^{\frac{n(n-1)}{2}} \int_X \frac{\prod_i (1-e^{-x_i})(1-e^{x_i})}{\prod_{i=1}^n x_i} \prod_i \left(\frac{x_i}{1-e^{-x_i}}\right)\left(\frac{-x_i}{1-e^{-x_i}}\right) \\
&= (-1)^{\frac{n(n-1)}{2}} \int_X \prod_{i=1}^n (-x_i) \\
&= \int_X \prod_i x_i \\
&= \chi_{\mathsf{Top}}(X) && \text{C-G-B.}
\end{aligned}$$

Letting $d = \dim X = 2n$, we have

$$(-1)^n (-1)^{\frac{d(d-1)}{2}} = (-1)^n (-1)^{n(2n-1)} = (-1)^2 n = 1.$$

Example 22.0.9*(?)*: We can prove HRR using this theorem: we have

$$\chi(X, \mathcal{E}) = \chi(\mathcal{E} \otimes A^{0,-}, \bar{\partial}) \overset{\text{ASIT}}{=} \int_X \frac{\mathrm{ch}(\mathcal{E} \otimes A^{0,-})}{\mathrm{eul}(X)} \mathrm{td}(TX \otimes_R \mathbb{C}).$$

We have $\mathrm{ch}(\mathcal{E} \otimes A^{0,-}) = \mathrm{ch}(\mathcal{E})\,\mathrm{ch}(A^{0,-})$ where $\mathrm{ch}(A^{0,1}) = \sum_I (-1)^i \mathrm{ch}(\bigwedge^i A^{0,1})$. The Chern roots of

- TX are $\{x_i\}$
- $A^{1,0} = T^\vee X$ are $\{-x_i\}$
- $A^{0,1}$ are $\{-x_i\}$

So we obtain

$$\begin{aligned}
\chi(\mathcal{E}) &= (-1)^n \int_X \frac{\prod(1-e^{x_i})}{\prod x_i} \prod_i \left(\frac{x_i}{1-e^{-x_i}}\right)\left(\frac{-x_i}{1-e^{-x_i}}\right) \\
&= \int_X \mathrm{ch}(\mathcal{E}) \prod_i \frac{x_i}{1-e^{-x_i}} \\
&= \int_X \mathrm{ch}(\mathcal{E}) \mathrm{td}(TX),
\end{aligned}$$

which is HRR.

23 | Monday, March 08

Remark 23.0.1: Recall that given a differential complex $(\mathcal{E}_*, d)$ we had a symbol complex $(\pi^* \mathcal{E}_*, \sigma(d))$ where $\pi : T^\vee X \to X$ and

$$\sigma\left(\sum_{|I| \leq N} f_I \partial_I\right) \coloneqq \sum_{|I|=N} f_I y^I,$$

where we take the top-order differentials, $\dfrac{\partial}{\partial x_j} \mapsto y_j$ and

$$\begin{aligned}
T^\vee X &\to \mathbb{R} \\
\alpha &\mapsto \alpha\left(\frac{\partial}{\partial x_j}\right).
\end{aligned}$$

We say that $(\mathcal{E}_*, d)$ is **elliptic** if the symbol complex is exact on $T^\vee X \setminus \{0\}$ where we delete the zero section. The Atiyah-Singer index theorem stated

$$\chi(\mathcal{E}_*, d) = \int_X \frac{\mathrm{ch}(\mathcal{E}_*)}{\mathrm{eul}(X)} \mathrm{td}(TX \otimes_{\mathbb{R}} \mathbb{C}).$$

What's the connection to elliptic operators? Given a 2-term complex

$$0 \to \mathcal{E}^0 \xrightarrow{D} \mathcal{E}^1 \to 0,$$

then D is an **elliptic operator** if this is an elliptic complex. This means the symbol complex is an isomorphism, i.e.

$$0 \to \pi^* \mathcal{E}^0 \xrightarrow{\sigma(D)} \pi^* \mathcal{E}^1 \to 0$$

where $\sigma(D)$ is an isomorphism away from the zero section.

Remark 23.0.2: Every elliptic complex can be converted into a 2-term complex using a hermitian metric. Given

$$\mathcal{E}^0 \xrightarrow{d^0} \mathcal{E}^1 \xrightarrow{d^1} \mathcal{E}^2 \to \cdots ,$$

we map this to

$$0 \to \mathcal{E}^{\text{even}} := \bigoplus_{i \text{ even}} \mathcal{E}^i \underset{D^{\text{odd}}}{\overset{D^{\text{even}}}{\rightleftharpoons}} \mathcal{E}^{\text{odd}} := \bigoplus_{i \text{ odd}} \to 0$$

where

$$D := ((d^{2i-1})^\dagger, d^{2i}) : \mathcal{E}^{2i} \to \mathcal{E}^{2i-1} \oplus \mathcal{E}^{2i+2}$$

and $(d^{2i-1})^\dagger$ is defined by the following property: for $\alpha \in \mathcal{E}^{2i-1}$ and $\beta \in \mathcal{E}^{2i}(X)$,

$$\left\langle d^{2i-1}\alpha,\ \beta \right\rangle_h = \left\langle \alpha,\ ((d^{2i-1})^\dagger \beta \right\rangle_h.$$

Here this pairing depends on a hermitian metric h, which is a hermitian form on each fiber:

$$h_i : \mathcal{E}^i \otimes \overline{\mathcal{E}^i} \to \mathbb{C}.$$

Using this, we can fix a volume form dV on X and define

$$\langle u,\ v \rangle_h := \int_X h_i(u, \overline{v})\, dV \qquad\qquad u, v \in \mathcal{E}^i(X).$$

This yields the desired two-term complex, and $(\mathcal{E}_*, d)$ is elliptic if and only if $D^e \circ D^o : \mathcal{E}^o \circlearrowleft$ and $D^o \circ D^e : \mathcal{E}^e \circlearrowleft$ are elliptic operators.

Example 23.0.3 ***(?):*** Taking the de Rham complex

$$0 \to \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to \cdots ,$$

one can define

$$\Omega^{\text{even}} \underset{d+d^\dagger}{\overset{d+d^\dagger}{\rightleftharpoons}} \Omega^{\text{odd}}.$$

Then using adjoint properties, we have

$$\left\langle \alpha,\ d^\dagger d^\dagger \beta \right\rangle = \left\langle d\alpha,\ d^\dagger \beta \right\rangle = \left\langle d^2\alpha,\ \beta \right\rangle = 0,$$

using that $d^2 = 0$, and since this is true for all α, β we have $(d^\dagger)^2 \beta = 0$ for all β. Noting that $dd^\dagger + d^\dagger d : \Omega^i(X) \circlearrowleft$, and this operator is **the Laplacian**. Moreover $\ker(dd^\dagger + d^\dagger d)$ is the space of **harmonic i-forms**.

Remark 23.0.4: Note that this space of harmonic forms depended on the Hermitian metrics on $\mathcal{E}^i$ and the volume form dV. In the case $\mathcal{E}^i := \Omega^i$, there is a natural metric determined by any Riemannian metric on X. Recall that this is given by a metric

$$g : TX \otimes TX \to \mathbb{R}.$$

This determines an isomorphism

$$\begin{aligned} T_pX &\xrightarrow{\sim} {T_p}^\vee X \\ v &\mapsto g(v, -), \end{aligned}$$

which we can invert to get a metric on the cotangent bundle $T^\vee X$. This induces a metric on i-forms using the identification $\Omega^i := \bigwedge^i T^\vee X$ and induces a volume form

$$dV := \sqrt{\det g} : \bigwedge^{\text{top}} TX \to \mathbb{R}.$$

In this case, $dd^\dagger + d^\dagger d$ on $\Omega^i(X)$ is called the **metric Laplacian**.

Remark 23.0.5: Let (X, g) be a Riemannian manifold. We thus have a symmetric bilinear form on $\Omega^p(X)$ given by pairing sections:

$$\langle \alpha,\ \beta \rangle := \int_X g(\alpha, \beta).$$

Note that we have orthonormal frames on $\Omega^p(X)$ of the form $e_{i_1} \wedge \cdots \wedge e_{i_p}$ where the $\{e_i\}$ are orthonormal frames on $T^\vee X$.

Definition 23.0.6 (Hodge Star Operator)
Let $n := \dim(X)$. The **Hodge star** operator is a map

$$\star : \Omega^p \to \Omega^{n-p}.$$

defined by the property

$$\alpha \wedge \star\beta = g(\alpha, \beta) dV.$$

Concretely, we have

$$\begin{aligned} \star\left(\sum f_I dx_{i_1} \wedge \cdots \wedge dx_{i_p}\right) &= \star\left(\sum f_I e_{i_1} \wedge \cdots \wedge e_{i_p}\right) \\ &= (-1)^\ell \sum_{j_k \in \{1, \cdots, n\} \setminus I} f_I e_{j_1} \wedge \cdots \wedge e_{j_{n-p}} \end{aligned}$$

for some sign ℓ.

Example 23.0.7*(?)*: Let $X := \mathbb{R}^4$ and g the standard metric, i.e. $d = dx_1^2 + \cdots + dx_4^2$. Take an orthonormal basis of $T^\vee \mathbb{R}^4$, say $\{e_1, e_2, e_3, e_4\}$ where $e_i := dx_i$. Then the induced volume form is $dV := e_1 \wedge e_2 \wedge e_3 \wedge e_4$. We can then compute $\star(e_1 \wedge e_2)$ which is defined by the property

$$\alpha \wedge \star(e_1 \wedge e_2) = g(\alpha, e_1 \wedge e_2) dV.$$

On the right-hand side, $g(\alpha, e_1 \wedge e_2) = c_{12}(\alpha) e_1 \wedge e_2 \wedge e_3 \wedge e_4$ where c_{12} is the coefficient of $e_1 \wedge e_2$. To extract that coefficient, we can take $\alpha(e_3 \wedge e_4$, writing $\alpha = \sum c_{ij} e_i \wedge e_j$. Similarly, $\star)e_1 \wedge e_3) = -e_2 \wedge e_4$. This follows from writing

$$\alpha \wedge \star(e_1 \wedge e_3) = c_{13}(\alpha) e_1 \wedge e_2 \wedge e_3 \wedge e_4 = (-1) c_{13}(\alpha) e_1 \wedge e_3 \wedge e_2 \wedge e_4.$$

From this, $\star : \Omega^p \to \Omega^{n-p}$ is defined fiber-wise as

$$\langle \alpha,\ \beta \rangle = \int_X \alpha \wedge \star\beta.$$

Exercise 23.0.8 (?)
Show that $\star^2 = (-1)^{p(n-p)}$.

Proposition 23.0.9 ***(Formula for the adjoint of the Hodge star).***
Let $d^\dagger := (-1)^{n(p-1)+1} \star d\star$. Then

$$\langle \alpha,\ d\beta \rangle = \left\langle d^\dagger\alpha,\ \beta \right\rangle \qquad\qquad \alpha \in \Omega^p(X), \beta \in \Omega^{p-1}(X).$$

Proof (?).
A slick application of Stokes' theorem! Using that $\star$ is an isometry, we have

$$\begin{aligned}
\langle \alpha,\ d\beta \rangle &= \int_X \alpha \wedge \star d\beta \\
&= \int_X \star\alpha \wedge d\beta(-1)^{p(n-p)} && \text{applying } \star \text{ to both} \\
&= -\int_X d(\star\alpha) \wedge \beta(-1)^{p(n-p)} && \text{Stokes/IBP} \\
&= (-1)^{p(n-p)+1} \int_X \star d \star \alpha \wedge \star\beta && \text{isometry} \\
&= (-1)^{p(n-p)+1} \langle \star d \star \alpha,\ \beta \rangle,
\end{aligned}$$

which shows that the term in the left-hand side of the inner product above is the adjoint of $d^\dagger$. ∎

24 | Wednesday, March 10

⚠ **Warning 24.0.1**
Missing some stuff from the first few minutes here!

Remark 24.0.2: Can we always get a Hermitian metric? Let $X \in \mathsf{Mfd}_{C^\infty(\mathbb{R})}$ and $\mathcal{E} \to X \in \mathsf{Bun}_{\mathrm{GL}_r\mathbb{C}}$ a smooth complex vector bundle. Then any section $h \in \mathcal{E}^\vee \otimes \overline{\mathcal{E}}^\vee(X)$, we have

$$\begin{aligned}
h : \mathcal{E} \otimes \overline{\mathcal{E}} &\to \mathcal{O} \\
&h(e \otimes f).
\end{aligned}$$

for $e, f \in \mathcal{E}_p$ is a Hermitian form for all p. In local trivializations, $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}$, and one can take the standard Hermitian form here. Then for $(f_1, \cdots, f_r) \in \mathcal{O}^{\oplus r}(U)$, we have $\sum f_i \overline{f}_i \in \mathcal{O}(U)$. This can be extended to all of X using a partition of unity subordinate to the coordinate charts.

The thing to check here is that on $\mathbb{C}^r$, for any collection $h_1, \cdots, h_n$, any positive linear combination $\sum a_i h_i$ is again a Hermitian metric for any $a_i \in \mathbb{R}^+$. One can regard these as skew-symmetric matrices, which are closed under addition, and the positive-definite property ensures it's still a metric since $h(v,v) = \sum a_i h_i(v,v) > 0$ for $v \neq 0$.

Remark 24.0.3: Recall that we start with a Riemannian manifold (X, g) where $g : TX^{\otimes 2} \to \mathcal{O}$ is a metric on the tangent bundle. Locally choose $f_1, \cdots, f_n$ an orthogonal frame of TX, then setting $e_i := f_i^\vee$ yields an orthogonal frame of $T^\vee X$ and thus an orthogonal frame $e_{i_1} \wedge \cdots e_{i_p}$ of $\bigwedge^p T^\vee X := \Omega^p X$. So we get a metric on the smooth p-forms $\Omega^p X$. We defined the Hodge star operator

$$\begin{aligned}\star : \Omega^p &\to \Omega^{n-p} \\ e_{i_1} \wedge \cdots e_{i_p} &\mapsto \pm e_{j_1} \wedge \cdots \wedge e_{j_{n-p}}.\end{aligned}$$

where $\{i_1, \cdots, i_p, j_1, \cdots, j_{n-p}\} = \{e_1, \cdots, e_n\}$. We saw that

$$\begin{aligned}e_{i_1} \wedge \cdots \wedge e_{i_p} \star (e_1 \wedge \cdots e_{i_p}) &= e_1 \wedge \cdots \wedge e_n \\ \star \left(\sum_{|I|=p} f_I e_I \right) &= \sum_{|I|=p} e_{I^c} (-1)^{\operatorname{sign}(I)}.\end{aligned}$$

Moreover,

$$\langle \alpha,\ \beta \rangle = \int_X g(\alpha, \beta) dV = \int_X \alpha \wedge (\star \beta),$$

and we showed that

$$\langle \alpha,\ d\beta \rangle = \pm \left\langle d^\dagger \alpha,\ \beta \right\rangle \qquad d^\dagger := \star d \star, \beta \in \Omega^{p-1}(X), \alpha \in \Omega^p(X),$$

yielding an adjoint operator

$$d^\dagger : \Omega^p(X) \to \Omega^{p-1}(X).$$

Definition 24.0.4 (Laplacian)
The **Laplacian** is the differential operator

$$\Delta := dd^\dagger + d^\dagger d : \Omega^p(X) \to \Omega^p(X).$$

Definition 24.0.5 (Harmonic Forms)
A p-form ω is **harmonic** if and only if $\Delta \omega = 0$. We define $\mathcal{H}^p(X)$ as the space of harmonic p-forms.

Remark 24.0.6: This operator is $\mathbb{R}$-linear, so $\mathcal{H}^p(X) \in \mathsf{Vect}_{\mathbb{R}}$. Note that this whole construction can be made to work over $\mathbb{C}$ by adding conjugates in appropriate places.

Proposition 24.0.7 ***(Characterization of when a smooth p-form is harmonic).***
A smooth p-form ω is harmonic if and only if $d\omega = d^\dagger\omega = 0$.

Proof (?).
$\Longleftarrow$: This direct is easy, since $\Delta\omega := (dd^\dagger + d^\dagger d)\omega = d(0) + d^\dagger 0 = 0$.
$\Longrightarrow$: A nice trick! Using the adjunction $d, d^\dagger$ we have

$$\begin{aligned}\langle\Delta\omega,\ \omega\rangle &= \left\langle dd^\dagger\omega,\ \omega\right\rangle + \left\langle d^\dagger\omega,\ \omega\right\rangle \\ &= \left\langle d^\dagger\omega,\ d^\dagger\omega\right\rangle + \langle d\omega,\ d\omega\rangle.\end{aligned}$$

We now use that since g is positive definite, it is a non-negative smooth function, and

$$\langle\alpha,\ \alpha\rangle := \int_X g(\alpha,\alpha)\,dV \geq 0 \text{ with equality } \iff \alpha \equiv 0 \text{ on } X.$$

So we can conclude that $d^\dagger\omega = d\omega = 0$.

■

⚠ **Warning 24.0.8**
Note that we've used that the inner product is symmetric over $\mathbb{R}$. Over $\mathbb{C}$, there are bars introduced from conjugation when swapping the variables.

Proposition 24.0.9 ***(Orthogonal decomposition of p-forms).***
The following three subspaces of $\Omega^p(X)$ are mutually orthogonal:

$$d\Omega^{p-1}(X), \mathcal{H}^p(X), d^\dagger\Omega^{p+1}(X).$$

Proof (?).
We can write

$$\left\langle d\alpha,\ d^\dagger\right\rangle = \left\langle d^2\alpha,\ \beta\right\rangle = \langle 0,\ \beta\rangle,$$

showing that the 1st and 3rd spaces are orthogonal. If $\alpha \in \mathcal{H}^p(X)$ then by the above proposition, $d\alpha = d^\dagger\alpha = 0$, and so

$$\begin{aligned}\langle\alpha,\ d\beta\rangle &= \left\langle d^\dagger\alpha,\ \beta\right\rangle = 0 \\ \left\langle\alpha,\ d^\dagger\beta\right\rangle &= \langle d\alpha,\ \beta\rangle = 0.\end{aligned}$$

Thus the 2nd space is orthogonal to the 1st and 3rd.

■

Observation 24.0.10
Suppose something false (⚠): that $\Omega^p(X)$ is a *complete* vector space with respect to the inner product. Remember that it is **not**! But if it were, there would be a decomposition

$$\Omega^p(X) = d\Omega^{p-1}(X) \oplus \mathcal{H}^p(X) \oplus d^\dagger\Omega^{p+1}(X).$$

Let $\alpha \in \left(d\Omega^{p-1}(X) \oplus d^\dagger \Omega^{p+1}(X) \right)^\perp$ where we take the orthogonal complement with respect to the inner product. Then

$$\begin{aligned} \langle \alpha,\ d\beta \rangle &= 0 \forall \beta \\ \left\langle \alpha,\ d^\dagger \gamma \right\rangle &= 0 \forall \gamma \end{aligned}$$

$$\begin{aligned} \implies \left\langle d^\dagger \alpha,\ \beta \right\rangle &= 0 \forall \beta \\ \implies d^\dagger \alpha &\equiv 0 && \text{setting} \beta := d^\dagger \alpha. \end{aligned}$$

Similarly, $d\alpha = 0$ and so $\alpha \in \mathcal{H}^p(X)$.

The conclusion (which is true *without* the false assumption) is that

$$\left(d\Omega^{p-1}(X) \oplus d^\dagger \Omega^{p+1}(X) \right)^\perp = \mathcal{H}^p.$$

However, this doesn't yield the full direct sum decomposition: if $W \subseteq V$, then it's not necessarily true that $V \cong W \oplus W^\perp$, which only holds if

- V is complete,
- W is closed.

Fact 24.0.11
For smooth p-forms, this decomposition **does** hold despite the false assumption:

$$\Omega^p(X) = d\Omega^{p-1}(X) \oplus \mathcal{H}^p(X) \oplus d^\dagger \Omega^{p+1}(X).$$

Corollary 24.0.12 ***(p-forms have harmonic representatives).***
Thus $\mathcal{H}^p(X)$ represents $H^p(X; \mathbb{R})$.

Remark 24.0.13: We have

$$\begin{aligned} H^p(X; \mathbb{R}) &= \frac{\ker d}{\operatorname{im} d} \\ &= \frac{d\Omega^{p-1}(X) \oplus \mathcal{H}^p(X)}{d\Omega^{p-1}(X)} \\ &= \mathcal{H}^p(X). \end{aligned}$$

Note that there is a map

$$\mathcal{H}^p(X) \to H^p(X; \mathbb{R})$$

since $\alpha \in \mathcal{H}^p(X)$ satisfies $d\alpha = 0$ in addition to $d^\dagger \alpha = 0$.

Remark 24.0.14: Note that one can complete these spaces using Sobolev spaces, but there are issues. Take S^1, then

$$L_2(S^1) := \left\{ \sum a_n e^{2\pi i n z} \;\middle|\; \sum |a|_i < \infty \right\},$$

but for $f \in L_2(S^1)$ we have $df = \sum 2\pi i n a_n e^{2\pi i n z}$ which may not converge.

25 | Review (Monday, March 15)

Remark 25.0.1: Recall that a *sheaf of rings* $\mathcal{F}$ on $X \in \mathsf{Top}$ is an assignment of a ring $\mathcal{F}(U)$ to each open set $U \subseteq X$ and restriction maps $\mathcal{F}(U) \xrightarrow{\rho_{UV}} \mathcal{F}(V)$ for $V \subseteq U$ that is a presheaf, so

1. This diagram commutes:

$$\begin{tikzcd}
U \arrow[r, "\rho_{UV}"] \arrow[rr, bend right, "\rho_{UW}"'] & V \arrow[r, "\rho_{VW}"] & W
\end{tikzcd}$$

Link to Diagram

2. $\varphi_{UU} = \mathbb{1}_{\mathcal{F}(U)}$ and $\mathcal{F}(\emptyset) = 0$.

That additionally satisfies unique gluing on double overlaps.

Example 25.0.2*(?)*: Any reasonable class of functions whose behavior is only locally restricted. Examples are being smooth or continuous, but e.g. being constant is a global condition. Other examples include $X \in \mathsf{Mfd}^n(C^\infty(-, \mathbb{R}))$, denoting $\mathcal{O}$ the sheaf of smooth functions. This also carries a sheaf of *abelian groups* Ω^p. In the special case where U is a coordinate chart, we have functions $\varphi_U : U \to \mathbb{R}^n$. Writing $S := \varphi_U(U)$, we can define

$$\Omega^p(U) \cong \Omega^p(S) := \left\{ \sum f_I(\mathbf{x}) dx_I \;\middle|\; f_I \in C^\infty(\mathbb{R}^n, \mathbb{R}) \right\}.$$

:::{.remark} More generally, for an arbitrary open U, cover it by coordinate charts $\{U_i\} \rightrightarrows U$. Then we want $\omega_i \in \Omega^p(U_i)$ which are compatible on double overlaps, so such a collection defines a section $\left\{ \omega_i \;\middle|\; i \in I \right\} \in \Gamma(\Omega^p(U))$. The compatibility is given by taking coordinate charts $\varphi_i : U_i \to \mathbb{R}^n$ with $\omega_i \in \Omega^p(U_i)$, we consider

$$t_{ij} : \varphi_i \circ \varphi_2^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j),$$

and we require that the pullback satisfies $t_{ij}^*(\omega_1) = \omega_2$ This pullback can be thought of as a coordinate change for the forms. Writing x_I as coordinates on U_i and y_J on U_j, we can write

$$\begin{aligned} x_1 &= h_1(y_J) \\ x_2 &= h_2(y_J) \\ &\vdots \\ x_n &= h_n(y_J) \end{aligned}$$

which expresses t_{ij} in coordinates. This allows us to give meaning to the formal symbols dx_I:

$$\begin{aligned} dx_1 &:= \sum_{i=1}^n \frac{\partial h_1}{\partial y_i} dy_i \\ dx_2 &:= \sum_{i=1}^n \frac{\partial h_2}{\partial y_i} dy_i \\ &\vdots \\ dx_k &:= \sum_{i=1}^n \frac{\partial h_k}{\partial y_i} dy_i \\ &, \end{aligned}$$

and under these substitutions in the original expression we obtain

$$\omega_1 = \sum_{|I|=p} f_I(\mathbf{x}) dx_I \mapsto \omega_2.$$

Remark 25.0.3: For $X \in \mathsf{Mfd}(\mathrm{Hol}(-, \mathbb{C}))$ such that $\varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \to \varphi_V(U \cap V)$ is holomorphic, so $\bar{\partial} z_i = 0$. Then $\Omega^p(U) = \left\{ \sum_{|I|=p} f_I(\mathbf{z}) dz_I \right\}$, and the *key difference* is that the f_I be holomorphic. This matters since POUs exist in the smooth setting but not the complex setting. Note that $\mathcal{O}, \Omega^p$ denote smooth/holomorphic functions and smooth/holomorphic p-forms in the smooth/complex settings. So we need a new notation for *smooth holomorphic* p-forms in the complex setting. We defined $A^{p,0}$ to be the smooth p-forms, and $A^{p,q}$ the smooth (p, q)-forms. In local coordinates, these look like

$$A^{p,q}(U) = \left\{ \sum_{|I|=p, |J|=q} f_{I,J}(\mathbf{z}) dz_I \wedge d\bar{z}_J \right\}.$$

Example 25.0.4*(?)*:

- $\Re(z)\, dz \in A^{1,0}(\mathbb{C})$ is a smooth $(1, 0)$-form.
- $z\, dw - w\, dz \in \Omega^1(\mathbb{C}^2)$ is a holomorphic 1-form.
- On $\mathbb{C}^3$, $z_1 dz_2 \wedge d\bar{z}_3 - \Re(z_3) dz_1 d\bar{z}_1 \in A^{1,1}(\mathbb{C}^3)$.

Remark 25.0.5: Why are these $A^{p,q}$ useful? They give a resolution of Ω^p on a complex manifold. There are maps of sheaves

$$0 \to \Omega^p \xrightarrow{i} A^{p,0},$$

where being a map of sheaves means there are maps $\Omega^p(U) \to A^{p,0}(U)$ for all opens U which are compatible with restriction:

$$\begin{tikzcd}
\Omega^p(U) \arrow[r, "i_U"] \arrow[d, dash, "\rho^*_{UV}"'] & A^{p,0}(U) \arrow[d, dash, "\rho^*_{UV}"] \\
\Omega^p(V) \arrow[r, "i_V"] & A^{p,0}(U)
\end{tikzcd}$$

Link to Diagram

It's clear that this works for i, since any holomorphic function simply *is* smooth. We could continue this resolution:

$$0 \to \Omega^p \xrightarrow{i} A^{p,0} \xrightarrow{\bar{\partial}} A^{p,1}$$

where

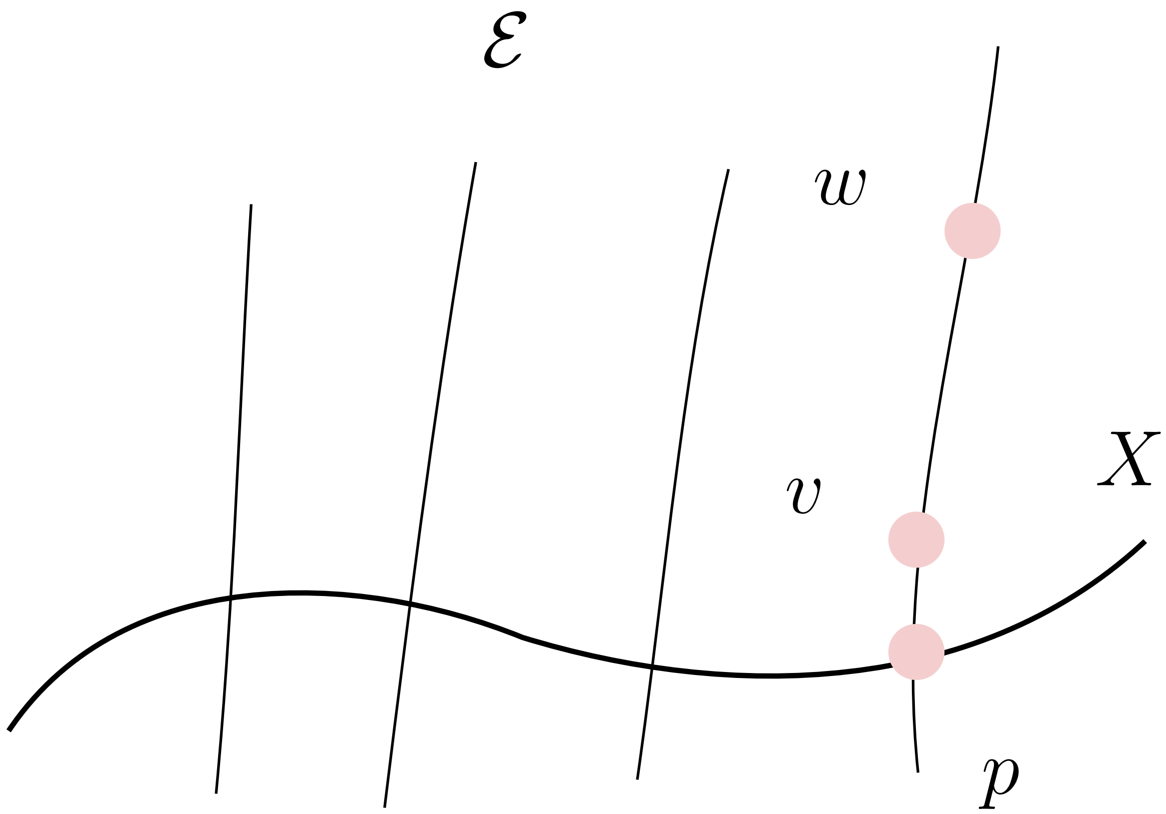
$$\bar{\partial}\left(\sum_{I,J} f_{I,J} dz_I \wedge d\bar{z}_J\right) := \sum_{I,J,K} \frac{\partial f_{I,J}}{\partial z_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J.$$

We then defined Dolbeaut cohomology, $H^q(X, \Omega^p) = \ker \bar{\partial}_{p,q} / \operatorname{im} \bar{\partial}_{p,q-1}$.

26 | Wednesday, March 17

26.1 Inverting Bundles

Remark 26.1.1: Continuing review: let $\mathcal{E} \to X \in \mathsf{Bun}(\mathbb{R}^n)$. A **metric** on $\mathcal{E}$ is a smoothly varying positive definite inner product on the fibers.



Fix this diagram! Need to remember what it was demonstrating.

For $v, w \in \mathcal{E}_p$, we want a pairing $g_p(v, w) : \mathcal{E}_p^{\otimes 2} \to \mathbb{R}$. To think about this globally, this should be a map

$$g : \mathcal{E}^{\otimes 2} \to \mathcal{O}.$$

where $g_p : \mathcal{E}_p^{\otimes 2} \to \mathbb{R}$. Note that this map is $\mathcal{O}$-linear, which follows from the fact that it's $\mathbb{R}$-linear on each fiber, or equivalently it is a map of vector bundles. We should also have that $g(s \otimes s) \in \mathcal{O}(X)$ is a smooth function, and we require $g(s \otimes s) \geq 0$. We also require $g(s \otimes s)(p) = 0 \iff s_0 = 0$ and $g(s \otimes t) = g(t \otimes s)$. This implies that $g \in (\mathcal{E}^{\otimes 2})^{\vee} \otimes \mathcal{O} = (\mathcal{E}^{\vee})^{\otimes 2}(X)$. The symmetric condition means that $g \in \operatorname{Sym}^2 \mathcal{E}^{\vee}(X)$.

Remark 26.1.2: For Hermitian forms, we take

$$h : (\mathbb{C}^n)^{\otimes 2} \to \mathbb{C}$$

where h is conjugate linear, so $h(cv, c'w) = \bar{c}c' h(v, w)$. Note that we can write $h(v, w) = \overline{v}^t H w$ where H is Hermitian, so $\overline{H}^t = H$. This implies that $h(v, v) \in \mathbb{R}^{\geq 0}$ and $h(v, v) = 0 \iff v = 0$ with $h(v, w) = \overline{h(v, w)}$ The great thing about metrics: we can identify zero sections by self-pairing, multiplying by a volume form, and integrating. For $\mathcal{E} \to X \in \mathsf{Bun}(\mathbb{C})$, there is another bundle $\overline{\mathcal{E}} \to X \in \mathsf{Bun}(\mathbb{C})$. Supposing that $\mathcal{E}|_U \xrightarrow{\varphi_U} \mathcal{O}_U^{\oplus n}$ in a local trivialization, conjugating all of the transition functions gives the transition functions $\overline{\mathcal{E}}\big|_U \xrightarrow{\text{conj}\circ\varphi_U} \mathcal{O}_U^{\oplus n}$. This yields a map

$$h : \overline{\mathcal{E}} \otimes_{\mathbb{C}} \mathcal{E} \to \mathcal{O} \in (\overline{\mathcal{E}} \otimes \mathcal{E})^\vee.$$

In local trivializations we have $\mathcal{E}|_U = \mathcal{O}_U^{\oplus n} = \mathbb{C}^n \times U$, and h is described by $h_U \in (\overline{\mathcal{O}}^{\oplus n} \otimes \mathcal{O}^{\oplus n})(U)$.

Remark 26.1.3: When $\operatorname{rank}\mathcal{E} = 1$ we abuse notation! For $h \in (\overline{\mathcal{E}}^\vee \otimes \mathcal{E}^\vee)(X)$, this is locally a 1×1 Hermitian matrix, thus of the form $[a]$ for $a \in \mathbb{R}^{\geq 0}$. So we write

$$h(s,t) = hs\bar{t} \coloneqq h \otimes s \otimes \bar{t} \in (\overline{\mathcal{E}}^\vee \otimes \mathcal{E}^\vee) \otimes \mathcal{E} \otimes \overline{\mathcal{E}} = \mathcal{O}$$

if $\mathcal{E}$ is a line bundle. Why is $V \otimes V^\vee = \mathcal{O}$ in this case? There is a pairing $v \otimes \lambda \mapsto \lambda(v)$, or more generally a trace pairing.

26.2 Serre Duality Revisited

Remark 26.2.1: Let X be a Riemann surface, so $X \in \mathsf{Mfd}^1(\mathbb{C})$. Let $L \to X \in \mathsf{Bun}^1(\mathrm{Hol})$, then we have a resolution

$$0 \to L \hookrightarrow L \otimes A^{0,0} \xrightarrow{\bar\partial} L \otimes A^{0,1} \to 0,$$

where the first map is inclusion of smooth holomorphic sections into smooth sections. What is this cut out by? We had $s \mapsto \bar\partial s$ and thus $f \mapsto \dfrac{\partial f}{\partial \bar z}\, d\bar z$. Note that $H_1(L) = \operatorname{coker}\bar\partial$.

Remark 26.2.2: Serre duality said that

$$h^1(L) = \dim H^1(L) = h^0(L^\vee \otimes K) \qquad\qquad K = \Omega^1,$$

where Ω^1 is the sheaf of holomorphic 1-forms. Choose a metric to identify $H^1(L)$ and $H^0(L^\vee \otimes K)$. Choose a hermitian metric on L and take $s,t \in H^0(L\otimes A^{0,0}) = C^\infty(L;\mathbb{C})$, then we get $h(s,t) \in C^\infty(X;\mathbb{C})$ a smooth complex function. We abuse notation by writing this as $h(s,t) = hs\bar{t}$, viewing $h \in C^\infty(L^\vee \otimes \overline{L}^\vee)$ locally. Note that we can't integrate a function on a manifold without a form, so choosing a volume for dV we can define a pairing on sections

$$\langle s,\ t\rangle \coloneqq \int_X hs\bar{t}dV.$$

Now for two sections $\alpha, \beta \in H^0(L \otimes A^{0,1})$ we can write

$$\int_X h\alpha\overline{\beta} = \int_X \omega,$$

where ω is a smooth $(1,1)$-form since $h \in \overline{L}^\vee \otimes L^\vee$, $\alpha \in L \otimes A^{0,1}$, and $\overline{\beta} \in \overline{L} \otimes A^{1,0}$. We now have metric on both the source and target spaces here:

$$H^0(L \otimes A^{0,0}) \xrightarrow{\bar\partial} H^0(L \otimes A^{0,1}),$$

where on the left-hand side we take $(s,t) \mapsto \displaystyle\int_X hs\bar{t}dV$ and on the right-hand side we have $(\alpha,\beta) \mapsto \displaystyle\int_X h\alpha\overline{\beta}$.

Remark 26.2.3: Given a map of metric vector spaces $V \xrightarrow{\varphi} W$, the *adjoint* $\varphi^\dagger$ satisfies

$$\langle \varphi(v),\ w\rangle = \left\langle v,\ \varphi^\dagger(w)\right\rangle.$$

and $\operatorname{coker}(\varphi) = \ker(\varphi^\dagger)$. So $H^1(L) = \operatorname{coker}\bar\partial = \ker\bar\partial^\dagger$, and after integrating by parts we have

$$\begin{aligned}
\left\langle \alpha,\ \bar\partial s\right\rangle &:= \int_X \alpha\overline{\bar\partial s} h \\
&= \int_X \alpha\partial(\bar s)h \\
&= -\int_X \bar s\partial(\alpha h) && \text{IBP} \\
&= -\int_X \bar s\frac{\partial(\alpha h)}{dV}dV \\
&= \left\langle -\frac{\partial(\alpha h)}{dV},\ s\right\rangle.
\end{aligned}$$

So we could define

$$\bar\partial^\dagger\alpha = \frac{\overline{-\bar\partial(\bar\alpha h)}}{dV}.$$

Note that $\alpha \mapsto \bar\alpha h$, so $\alpha \in \ker\bar\partial^\dagger \iff \bar\alpha h \in \ker\bar\partial$. Then $\ker(\bar\partial^\dagger) = H^0(L^\vee \otimes K)$.

27 | Friday, March 19

Remark 27.0.1: Recall Serre duality: let $C \in \mathsf{Mfd}_{\mathbb{C}}(\text{compact}, \text{oriented})$ and $L \to C \in \mathsf{Bun}(\text{Hol})$. Then

$$h^1(L) = h^0(L^\vee \otimes K_C).$$

We also have Riemann-Roch, a very important tool:

$$h^0(L) - h^1(L) = \deg L + 1 - g(C),$$

where $\deg L = \int_C c_1(L)$, which is also equal to $\deg[\{s = 0\}] = \deg(\operatorname{Div} s)$. Note that c_1 is the most important Chern class to know, thanks to the splitting principle. How was it defined? There are several definitions:

1. L defines an element of

$$\begin{aligned}
H^1(C, \mathcal{O}^\times) &= \left\{t_{UV} : U \cap V \to \mathbb{C}^\times \;\middle|\; t_{UV}t_{UW}^{-1}t_{VW} = 1\right\} / \partial\left\{h_u : U \to \mathbb{C}^\times\right\} \\
&= \ker\partial^1 / \operatorname{im}\partial^0
\end{aligned}$$

in Čech cohomology. By definition $\partial\left\{h_U \;\middle|\; U \in \mathcal{U}\right\} = \left\{h_u h_v^{-1} \;\middle|\; U, V \in \mathcal{U}\right\}$, where $\partial^2 = 1$ since

$$(h_U h_V)^{-1}\left(h_U h_W^{-1}\right)^{-1}(h_V h_W^{-1}) = 1 \qquad \text{on } U \cap V \cap W.$$

By assigning L to its transition functions, we get a map $L \to H^1$. We have the exponential exact sequence:

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to 1,$$

which induces a map

$$\begin{aligned} H^1(C, \mathcal{O}^\times) &\to H^2(C, \mathbb{Z}) \\ L &\mapsto c_1(L). \end{aligned}$$

2. L defines an element $\operatorname{Fr} L \in \operatorname{Bun}^{\text{prin}}(\mathbb{C}^\times)$ (which only works for line bundles), which is defined by $\operatorname{Fr} L = L \setminus s_0$ where s_0 is the zero section of L. By topology, we get a classifying map

$$C \xrightarrow{\varphi_L} B\mathbb{C}^\times = \mathbb{CP}^\infty = (\mathbb{C}^\infty \setminus \{0\})/\mathbb{C}^\times.$$

There is a universal $c_1 \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$, so we take the pullback to define $c_1(L) := \varphi_L^*(c_1)$. We can use that there is a cell decomposition $\mathbb{CP}^\infty = \mathbb{C}^0 \cup \mathbb{C}^1 \cup \mathbb{C}^2 \cup \cdots$, and so there is a unique generator in its H^2.

3. Consider a smooth section $s \in C^\infty(L)$, then we can define $c_1(L) := [\{s = 0\}]$ by taking the fundamental class, assuming that s is transverse to the zero section s_z of L. Here we view the zero set as an oriented submanifold. See picture: in this case $[\{s = 0\}] = [p] - [q] + [r]$.

Add picture.

Remark 27.0.2: Applying Serre duality to the left-hand side in Riemann-Roch yields the dimension of the space of holomorphic sections of some *other* bundle, $L^\vee \otimes K$.

Example 27.0.3 ***(The structure sheaf):*** Applying Riemann-Roch to $L := \mathcal{O}$, we get

$$\chi(\mathcal{O}) = h^0(\mathcal{O}) - h^1(\mathcal{O}) = 0 + 1 - g,$$

which is equal to $h^0(\mathcal{O}) - h^0(K)$. But the only holomorphic functions on $\mathbb{C}$ are constant, so $h^0(\mathcal{O}) = 1$. In particular, $h^0(K) = g$, so any Riemann surface of genus g has a g-dimensional space of holomorphic 1-forms.

Example 27.0.4 ***(The Canonical Bundle):*** Applying Riemann-Roch to $L := K$, we get

$$\chi(K) = h^0(K) - h^0(K^\vee \otimes K) = \deg(K) + 1 - g.$$

Since $K^\vee \otimes K = \mathcal{O}$, we obtain $g - 1 = \deg(K) + 1 - g$, so $\deg(K) = 2g - 2$.

We also proved this using that K was the dual of holomorphic vector fields, i.e. $\int_C c_1(K) = -\int_C c_1(T)$, which by Gauss-Bonnet equals $-\chi_{\mathsf{Top}}(C) = -(2 - 2g) = 2g - 2$.

Example 27.0.5 ***(Genus 2 Riemann Surfaces):*** Taking C of genus 2, we have $h^0(K_C) = g = 2$, so $\deg K_C = 2(2) - 2 = 2$. Thus there exist linearly independent sections $s, t \in H^0(K_C)$, i.e. two linearly independent holomorphic 1-forms. We can take the ratio s/t, which defines a map

$$\frac{s}{t} : C \to \mathbb{P}^1.$$

Locally we have $s = f(z)\, dz$ for z a local holomorphic coordinate on C and $f \in \mathrm{Hol}(C, \mathbb{C})$, and similarly $t = g(z)\, dz$. So $s/t = f(z)/g(z)$ is meromorphic in this chart. Choosing a new coordinate chart w, this yields a transition function $z(w)$ – not of L, but from the atlas on C. We can write $s = f(z(w))\, d(z(w)) = f(z(w))z'(w)\, dw$ by the chain rule. Thus

$$\frac{s}{t}(z) = \frac{f(z(w))z'(w)\, dw}{g(z(w))z'(w)\, dw} = \frac{s}{t}(w).$$

So although s/t was only defined in a coordinate chart, it winds up being independent of coordinates. This works in general for any holomorphic line bundle: for $s, t \in H^0(L)$, there is a map $\dfrac{s}{t} : C \to \mathbb{P}^1$ since writing $s_V = \varphi_{UV} s_U, t_V = \varphi_{UV} t_U$ where φ_{UV} is the transition function for L.

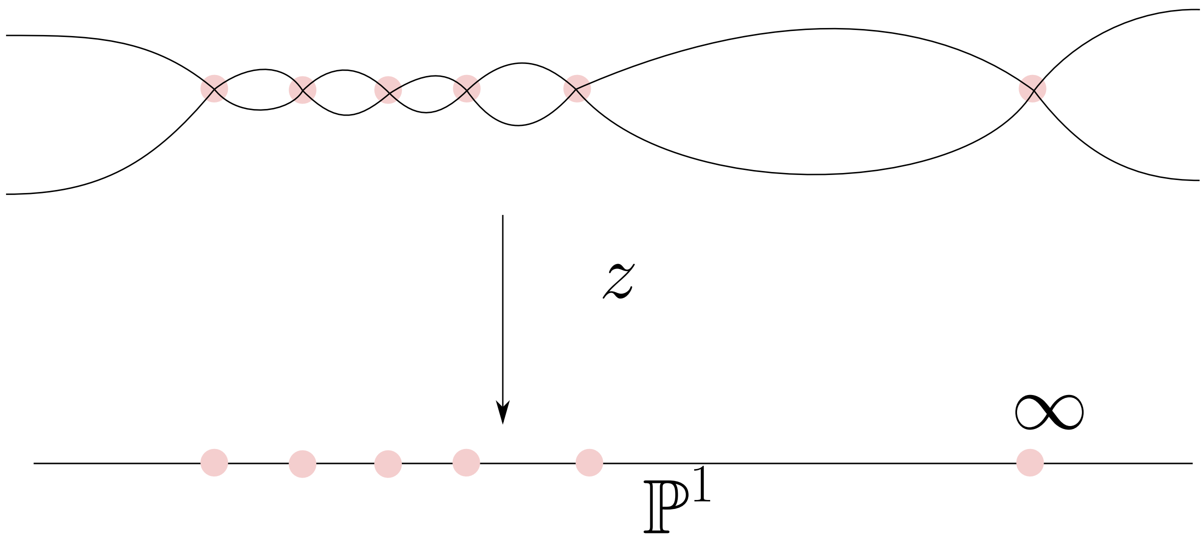
Fact 27.0.6
Important fact: we can take these ratios to get maps to $\mathbb{P}^1$.

Slogan 27.0.7
The canonical bundle is the line bundle whose transition functions are the Jacobians of the change of variables for the atlas.

Question 27.0.8
What is the degree of this map generically? I.e. given $[x_0 : x_1] \in \mathbb{P}^1$ fixed, what is the size of the inverse image $\left(\dfrac{s}{t}\right)^{-1}([x_0 : x_1])$?

Answer 27.0.9
Writing $s/t = x_1/x_0$, we have $x_0 s - x_1 t = 0$. This is in $H^0(K_C)$, and we computed $\deg K_C = 2$, meaning there are two zeros of this function. Thus is $g(C) = 2$, there is a generically 2-to-1 map $C \to \mathbb{P}^1$, a degree 2 meromorphic function. Note that this section could have a double zero.

Example 27.0.10 ***(?):*** Consider the curve $y^2 = (z-1)(z-2)\cdots(z-5)$, where we think of $z, y \in \mathbb{P}^1$. This has roots $z = 1, \cdots, 5$, and is equal to ∞ if $z = \infty$. These are the only points of $\mathbb{P}^1$ with just one square root, all other points have two square roots.


z
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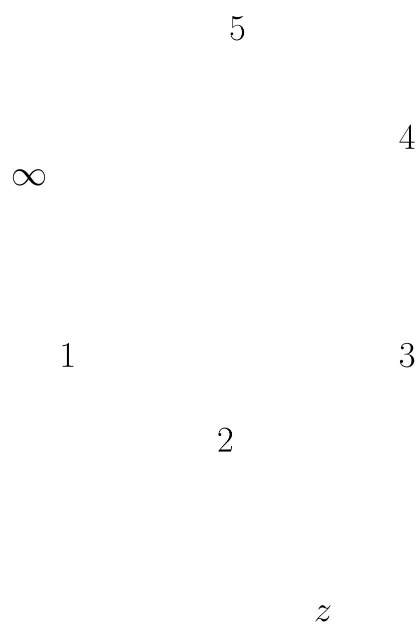
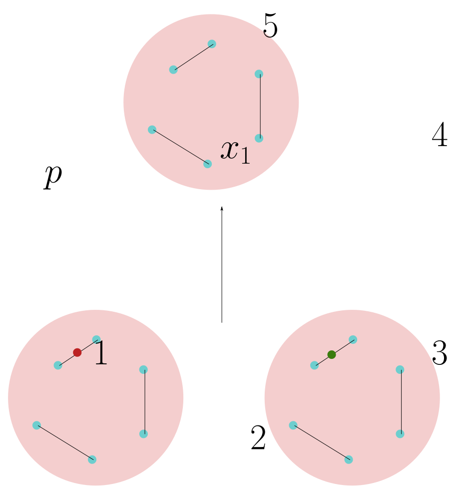
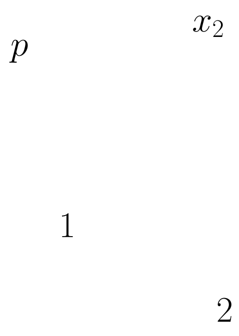
$\mathbb{P}^1$







28 | Monday, March 22

Remark 28.0.1: Last time: we reviewed Riemann-Roch, Serre duality, sheaves of p-forms. Recall a theorem from a few weeks ago:

Theorem 28.0.2 ***(The Hodge Theorem).***
If (X, g) is a compact oriented Riemannian manifold, then there is a decomposition of the smooth p-forms on X:

$$\Omega^p(X) = d\Omega^{p-1}(X) \oplus \mathcal{H}^p(X) + d^\dagger \Omega^{p+1}(X).$$

Remark 28.0.3: Note that $\mathcal{H}$ was the space of harmonic p-forms, and $d^\dagger ::= (-1)^? \star d\star$ where

$$\begin{aligned} \star : \Omega^p(X) &\to \Omega^{n-p}(X) \\ e_{i_1} \wedge \cdots \wedge e_{i_p} &\mapsto \pm e_{j_1} \wedge \cdots e_{j_{n-p}} \end{aligned}$$

where $\{e_i\}$ is an orthonormal basis of basis of $T^\vee X$. Note that this formula is replacing the e_i that do appear with the e_i that don't appear, up to a sign. The harmonic forms were defined as $\mathcal{H}^p(X) = \ker(dd^\dagger + d^\dagger d) = \ker(d) \cap \ker(d^\dagger)$. We proved that assuming this decomposition, there is an isomorphism

$$\mathcal{H}^p(X) \cong H^p_{\mathrm{dR}}(X; \mathbb{R}).$$

Example 28.0.4 ***(The circle S^1):*** There's a standard flat metric g_{std} on S^1 where $g_{\mathrm{std}} = dx^2$ with x the coordinate on $\mathbb{R}$ which is the universal cover of S^1. We can write

$$\Omega^1(S^1) = \left\{ f(x)\, dx \;\middle|\; f \in C^\infty(S^1, \mathbb{R}) \right\},$$

since every 1-form ω looks like this. Then $d\omega = 0$ since this is a 2-form on S^1. On the other hand, what is $d^\dagger$? We know that $\star\omega$ is a 0-form, so a function. The volume form is given by $\sqrt{\det g_{\mathrm{std}}} = \sqrt{[\,dx^2]}$, and you can wedge $1 \wedge dx = dx$, so $\star\omega = f(x)$. Then $d \star \omega = f'(x)\, dx$ and $d^\dagger x\omega = f'(x)$. If this is zero, $f'(x) = 0$ and f is a constant function. So in this metric, $\mathcal{H}^1(S^1) = \mathbb{R}\langle dx \rangle \cong H^1(S^1; \mathbb{R})$.

Remark 28.0.5 ***(Important):*** The harmonic forms $\mathcal{H}^p(X)$ depend on the metric g, despite mapping isomorphically to de Rham cohomology.

Remark 28.0.6: This was just in the case of a real smooth Riemannian manifold. What extra structure to we have for $X \in \mathsf{Mfd}(\mathrm{Hol}(-, \mathbb{C}))$?

Definition 28.0.7 (Kähler Forms (Important!))
Let $X \in \mathsf{Mfd}(\mathrm{Hol}(-, \mathbb{C}))$ be a complex manifold. A **Kähler form** $\omega \in \Omega^2(X_\mathbb{R})$ is a closed real (possibly needed: J-invariant) 2-form on the underlying real manifold of X for which $\omega(v, Jw) := g(v, w)$ is a metric on $TX_\mathbb{R}$ where J is an almost complex structure. The associated **hermitian metric** is $h := g + i\omega$, which defines a hermitian form on $TX \in \mathsf{Vect}_\mathbb{C}$.

Example 28.0.8 ***(?):*** Take $X := \mathbb{C}^n$ and $J(v) := i \cdot v$. Note that $X_\mathbb{R} = \mathbb{R}^{2n}$, so write its coordinates as x_k, y_k for $k = 1, \cdots, n$ where $z_k = x_k + iy_k$ are the complex coordinates. Consider $g = g_{\mathrm{std}}$ on $\mathbb{R}^{2n}$ – does this come from a closed 2-form $g_{\mathrm{std}} = \sum (\,dx_k)^2 + (dy_k)^2$? Using $\omega(v, Jw) = g(v, w)$, we have $\omega(v, J^2 w) = g(v, Jw)$. The left-hand side is equal to $-\omega(v, w)$ and the right-hand side is

$\omega(v, w) = -g(v, Jw)$. What 2-form does this give? We have

$$
\begin{aligned}
\omega\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell}\right) &= -g\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_\ell}\right) = 0 \\
\omega\left(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial x_\ell}\right) &= -g\left(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_\ell}\right) = 0 \\
\omega\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_\ell}\right) &= -g\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_\ell}\right) = 0 && \forall k \neq \ell \\
\omega\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_k}\right) &= -g\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_k}\right) \\
&= (-1)^2 g\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_k}\right) \\
&= 1 \\
\omega\left(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial x_k}\right) &= -1.
\end{aligned}
$$

So we can write this in block form using blocks

$$
M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \omega = \begin{bmatrix} M & & \\ & M & \\ & & M \end{bmatrix},
$$

which is a closed ($d\omega = 0$) antisymmetric 2-form, i.e. a symplectic form, and

$$
\omega_{\text{std}} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n,
$$

Remark 28.0.9: So the Kähler geometry is determined by the data $(\mathbb{C}^n, g_{\text{std}}, J, \omega_{\text{std}})$, i.e. a metric, an almost complex structure, and a symplectic form. Note that the relation $\omega(x, y) = g(x, Jy)$ can be used to determine the 3rd piece of data from any 2. This is the fiberwise/local model, i.e. every tangent space at a point looks like this.

⚠ **Warning 28.0.10**
But note that a form being closed is not a tensorial property! So this local data (looking at a single fiber) is not quite enough to determine the global geometry.

Remark 28.0.11: Given g and J, ω is automatically a 2-form. That it's antisymmetric follows from

$$
\begin{aligned}
-\omega(w, v) &= -g(w, Jv) \\
&= -g(Jv, w) \\
&= -g(J^2 v, Jw) \\
&= g(v, Jw) \\
&= \omega(v, w).
\end{aligned}
$$

Conversely, we can always define $g(v, w) := -\omega(v, Jw)$, but a priori this may not be a metric. This will be symmetric, but potentially not positive-definite.

Definition 28.0.12 (ω-tame almost complex structures)
An almost complex structure J is **ω-tame** if $g(v, w) = -\omega(v, Jw)$ is positive definite.

Remark 28.0.13: Next time: we'll see that if X is Kähler, then

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X),$$

so this is compatible with the Hodge decomposition. This is what people usually call the Hodge decomposition theorem, and gives some invariants of complex manifolds. By a miracle, this decomposition only depends on g and the complex structure.

Remark 28.0.14: Note that there is a notion of *hyperkähler* manifolds, which have 3 complex structures I, J, K such that $I^2 = J^2 = K^2 = IJK = -\mathbb{1}$, yielding 3 "parallel" 2-forms $\omega_I, \omega_J, \omega_K$ such that the covariant derivative vanishes, i.e. $\nabla_g \{\omega_I, \omega_J, \omega_K\} = 0$. With respect to the complex structure I, $\omega_J + \omega_K$ is a holomorphic 2-form. There is a sphere's worth of almost complex structures, and there is an action $\mathrm{SO}(4, b_2 - 4) \curvearrowright H^*(X)$. There's no known example where the hyperkähler metric has been explicitly written down.

29 | Wednesday, March 24

Remark 29.0.1: Last time: we defined a **Kähler manifold**: $X \in \mathsf{Mfd}(\mathbb{C})_{\text{compact}}$ and $\omega \in \Omega^2(X_\mathbb{R})$ a closed real 2-form such that $g(x, y) := \omega(x, Jy)$ is a metric. By the Hodge theorem, we have a space $\mathcal{H}^k(X)$ of harmonic k-forms for (X, g) which represents $H^k_{\mathrm{dR}}(X; \mathbb{R})$. We can consider the $\mathbb{C}$-valued harmonic forms $\mathcal{H}^k_\mathbb{C} := \mathcal{H}^k(X) \otimes_\mathbb{R} \mathbb{C}$, which represents $H^k_{\mathrm{dR}}(X; \mathbb{C})$

Question 29.0.2
How does this interact with the decomposition of the smooth k-forms

$$\Omega^k(X_\mathbb{R}) \otimes_\mathbb{R} \mathbb{C} = \bigoplus_{p+q=k}^{K} A^{p,q}(X),$$

where $\mathcal{H}^k_\mathbb{C}(X)$ is contained in this. Note that this is a small finite dimensional space in an infinite dimensional space! The following miracle occurs:

Theorem 29.0.3 ***(Kähler manifolds admit a Hodge decomposition?).***
If $X \in \mathsf{Mfd}(\text{Kähler})$,

$$\mathcal{H}^k_\mathbb{C} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X),$$

where

$$\mathcal{H}^{p,q}(X) := \left(\mathcal{H}^K(X) \otimes_\mathbb{R} \mathbb{C}\right) \cap A^{p,q}(X) \subseteq \Omega^k(X_\mathbb{R}).$$

Example 29.0.4 *(?)*: Let $X = \mathbb{C}/\Lambda$ be an elliptic curve where Λ is a lattice. The standard metric $dx^2 + dy^2$ on $\mathbb{C}$ descends to a metric on X since translation is an isometry on the metric space $(\mathbb{C}, dx^2 + dy^2)$. Let $z = x + iy$ be a complex coordinate on $\mathbb{C}$ so $dz = dx + idy$ and $d\bar{z} = dx - idy$, then $dx^2 + dy^2 = dzd\bar{z} \in \mathrm{Sym}^2(\mathsf{T}\mathbb{C})$. The symplectic form is given by

$$\omega(v, w) = \pm g(v, Jw) = i\, dz\, d\bar{z}(v, w)$$

since J is given by i on $\mathbb{C}$. Then $\omega(v, w) = i\, dz(v)\, d\bar{z}(w)$, i.e. $\omega = i\, dz \wedge d\bar{z}$. So

$$\bar{\omega} = \bar{i}\, d\bar{z} \wedge dz = -i\, d\bar{z} \wedge dz = i\, dz\, d\bar{z} = \omega,$$

and this determines the Kähler geometry on X. What are the harmonic 1-forms on X, $\mathcal{H}^1(X) \otimes_{\mathbb{R}} \mathbb{C}$? Note that $\omega = dV$ is the volume form. The smooth 1-forms are given by

$$\Omega^1(X_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C} = A^{1,0}(X) \oplus A^{0,1}(X) = \{f(z, \bar{z})\, dz\} \oplus \{g(z, \bar{z})\, d\bar{z}\},$$

where f, g are smooth and Λ-periodic on $\mathbb{C}$ to make them well-defined. We can find the Hodge star:

$$\begin{aligned} \star : ? &\to ? \\ dz &\mapsto i\, d\bar{z} \\ d\bar{z} &\mapsto -i\, dz. \end{aligned}$$

Writing $\alpha := f(z, \bar{z})\, dz + g(z, \bar{z})\, d\bar{z}$, this is harmonic if $d\alpha = 0$ and $\star d \star \alpha = 0$. The first implies $\partial_{\bar{z}} f - \partial_z g = 0$. What does the second imply? We can compute

$$\begin{aligned} \star\alpha &= if(z, \bar{z})\, d\bar{z} - ig(z, \bar{z})\, dz \\ \implies \partial_z f + \partial_{\bar{z}} g &= 0, \end{aligned}$$

and so $\partial_{\bar{z}} f = \partial_z g$ and $\partial_{\bar{z}}^2 f = \partial_{\bar{z}} \partial_z g = -\partial_z^2 f$, so

$$\begin{aligned} \left(\partial_{\bar{z}}^2 + \partial_z^2\right) f &= 0 \\ \left(\partial_{\bar{z}}^2 + \partial_z^2\right) g &= 0. \end{aligned}$$

Note that this recovers the usual notion of harmonic functions on $\mathbb{C}$, i.e. being in the kernel of the Laplacian. The only biperiodic functions that satisfy these equations are constants, since there is a maximum modulus principle for harmonic functions. Thus

$$\mathcal{H}^1(X) \otimes_{\mathbb{R}} \mathbb{C} = \{c_1\, dz + c_2\, d\bar{z}\} = \mathbb{C}\, dz \oplus \mathbb{C}\, d\bar{z} = H^{1,0}(X) \oplus H^{0,1}(X).$$

Remark 29.0.5: There is a generalization to higher genus curves. Recall the following theorem:

> **Theorem 29.0.6** *(Uniformization)*.
> Let $C \in \mathsf{Mfd}^1(\mathbb{C})_{\text{compact}}$ of genus $g \geq 2$. Then the universal cover admits a biholomorphism
>
> $$\tilde{C} \cong \mathbb{H} := \left\{ z \in \mathbb{C} \;\middle|\; \Im(z) > 0 \right\}.$$

Remark 29.0.7: This essentially follows from the Riemann mapping principle.

Corollary 29.0.8 ***(Every curve of genus g>1 is the plane mod a subgroup of biholomorphisms).***
Any curve C of genus $g \geq 2$ is of the form $C = \mathbb{H}/\Gamma$ where $\Gamma \leq \mathrm{BiHol}(\mathbb{H})$ is a subgroup that acts freely. By covering space theory, $\Gamma = \pi_1(C)$, and it's known that $\mathrm{BiHol}(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbb{R})$ by the map

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z \mapsto \frac{az+b}{cz+d}.$$

Proposition 29.0.9 ***(The upper half-plane admits a PSL-invariant hyperbolic metric).***
The upper half plane $\mathbb{H}$ admits a **hyperbolic metric** which is invariant under $\mathrm{PSL}_2(\mathbb{R})$ given by

$$g_{\text{hyp}} = \frac{dx^2 + dy^2}{y^2} = \frac{dz\, d\bar{z}}{\Im(z)^2}.$$

Proof (?).
This follows from a computation:

$$\begin{aligned}
d\left(\frac{az+b}{cz+d}\right) &= \frac{a\,dz}{cz+d} - \frac{c(az+b)\,dz}{(dz+d)^2} \\
&= \frac{a(cz+d) - c(az+b)\,dz}{(cz+d)^2} \\
&= \frac{(ad-bc)\,dz}{(cz+d)^2} \\
&= \frac{dz}{(cz+d)^2} \\
&= \frac{d\left(\frac{az+b}{cz+d}\right) d\left(\overline{\frac{az+b}{cz+d}}\right)}{\Im\left(\frac{az+b}{cz+d}\right)^2} \\
&= \frac{dz\,d\bar{z}}{(cz+d)^2(c\bar{z}+d)^2\Im\left(\frac{az+b}{cz+d}\right)} \\
&= \frac{dz\,d\bar{z}}{\Im(z)^2}.
\end{aligned}$$

■

Remark 29.0.10: It's miraculous! The biholomorphisms of $\mathbb{H}$ preserve a metric. So C has a canonical metric, g_{hyp}, which descends along the quotient map $\mathbb{H} \to \mathbb{H}/\Gamma \cong \mathbb{C}$.

Question 29.0.11
What are the harmonic 1-forms on (C, g_{hyp})?

Remark 29.0.12: By lifting we can write

$$\Omega^1(C_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C} = A^{1,0}(C) \oplus A^{0,1}(C) = \left\{ f(z, \bar{z})\, dz + g(z, \bar{z})\, d\bar{z} \;\middle|\; z \in \mathbb{H},\ f, g \in C^\infty(\mathbb{C}, \mathbb{R}) \right\}$$

But dz is *not* invariant under the map $z \mapsto \dfrac{az+b}{cz+d}$, since $dz \mapsto \dfrac{dz}{(cz+d)^2}$. In order to descend $f(z)$ to C, we need

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 f(z) \qquad\qquad \text{for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$$

This says that f is a **modular form of weight 2**.

Exercise 29.0.13 (?)
Check that this implies that f must be holomorphic and g must be antiholomorphic.

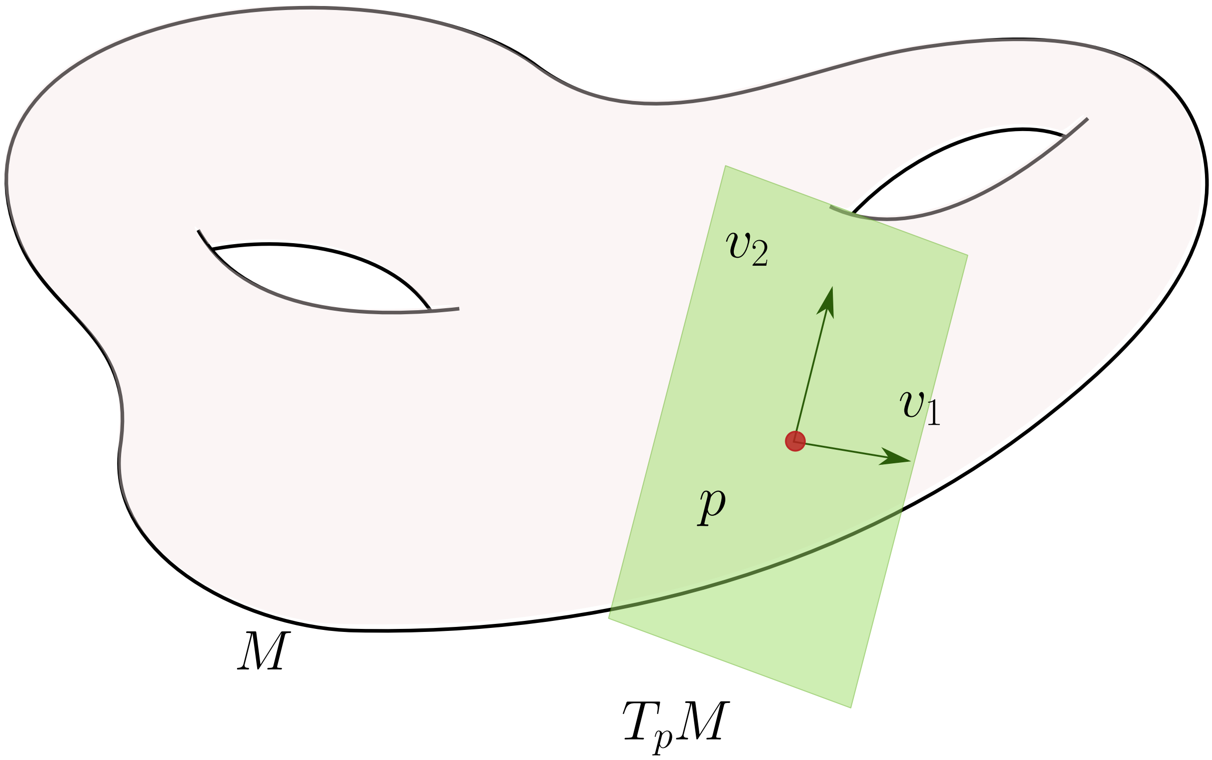
Fact 29.0.14
There is a decomposition

$$\mathcal{H}^1(C_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{H}^{1,0}(C) \oplus \mathcal{H}^{0,1}(C),$$

and the first space will be the space of holomorphic 1-forms $H^0(K_C)$, and the second term will be $\overline{H^0(K_C)}$. This shows the power of the Hodge decomposition theorem!

30 | Friday, March 26th

Remark 30.0.1: Recall the Hodge decomposition theorem. Let $(M, g) \in \mathsf{Mfd}^n_{\mathbb{R}}(\mathsf{Riem}, \text{compact})$, then choosing an orthonormal basis $\{v_j\}$ for T_pM yields a corresponding orthonormal basis in ${T_p}^\vee M := \underset{\mathbb{R}}{\text{Hom}}(T_pM, \mathbb{R})$ given by taking $\left\{ e_i \;\middle|\; e_i(v_j) = \delta_{ij} \right\}$.



There is a map

$$\begin{aligned}\star : \bigwedge^{k} T_p{}^{\vee} M &\to \bigwedge^{n-k} T_p{}^{\vee} M \\ \bigwedge_{j=1}^{k} e_{i_j} &\mapsto \pm \bigwedge_{\ell=1}^{n-k} e_{j_\ell}\end{aligned}$$

where the e_j are defined such that $\bigwedge_{j=1}^{k} e_{i_j} \wedge \bigwedge_{\ell=1}^{n-k} e_{j_\ell} := dV$, where dV is the volume form on M at p.

Thus we have a map

$$\begin{aligned}\star : \Omega^k &\to \Omega^{n-k} \\ 1 &\mapsto dV.\end{aligned}$$

We defined $d^\dagger := \star d \star$, and said a form ω was *harmonic* iff $\Delta\omega = 0$, where $\Delta := dd^\dagger + d^\dagger d$. The space of such forms was denoted $\mathcal{H}^k(M) \subseteq \Omega^k(M)$.

Theorem 30.0.2 ***(Hodge Theorem).***

$$\mathcal{H}^k(M) \cong H^k_{\mathrm{dR}}(M; \mathbb{R}).$$

Question 30.0.3
What kinds of extra structure can we put on a complex manifold?

Definition 30.0.4 (Kähler Form)
A **Kähler form** is a closed 2-form $\omega \in \Omega^2_{\mathbb{R}}$ such that the following equation defines a metric on T_pM:

$$g(u, v) := \omega(u, iv).$$

I.e., this is a closed symplectic form that defines a metric.

Example 30.0.5 *(?)*: Consider $M = \mathbb{C}^n$ with holomorphic coordinates $z_1, z_2, \cdots, z_n$, where $z_j := x_j + iy_j$. Then take

$$\omega := \sum_{j=1}^{n} dx_j \wedge dy_j.$$

Note that multiplication by i induces a map

$$\begin{aligned} \cdot i : T_p\mathbb{C}^n &\circlearrowleft \\ \frac{\partial}{\partial x_j} &\mapsto \frac{\partial}{\partial y_j} \\ \frac{\partial}{\partial y_j} &\mapsto -\frac{\partial}{\partial x_j} \end{aligned}.$$

Moreover, $\omega(u, iv)$ recovers the standard metric on $\mathbb{C}^n$ given by

$$g_{\text{std}} = \sum (\,dx_j)^2 + (\,dy_j)^2 \in \operatorname{Sym}^2 T^\vee \mathbb{C}^n,$$

which is incidentally positive-definite, where $(\,dx)^2(u, v) := (\frac{\partial}{\partial x_j})u \cdot *(\frac{\partial}{\partial y_j})v$. Is this closed? We need to check to see if $d\omega = 0$, but this is true: applying d to all of the coefficients yields the constant 1.

Remark 30.0.6: So for $M \in \mathsf{Mfd}(\mathbb{C})$ a complex manifold, we have a decomposition

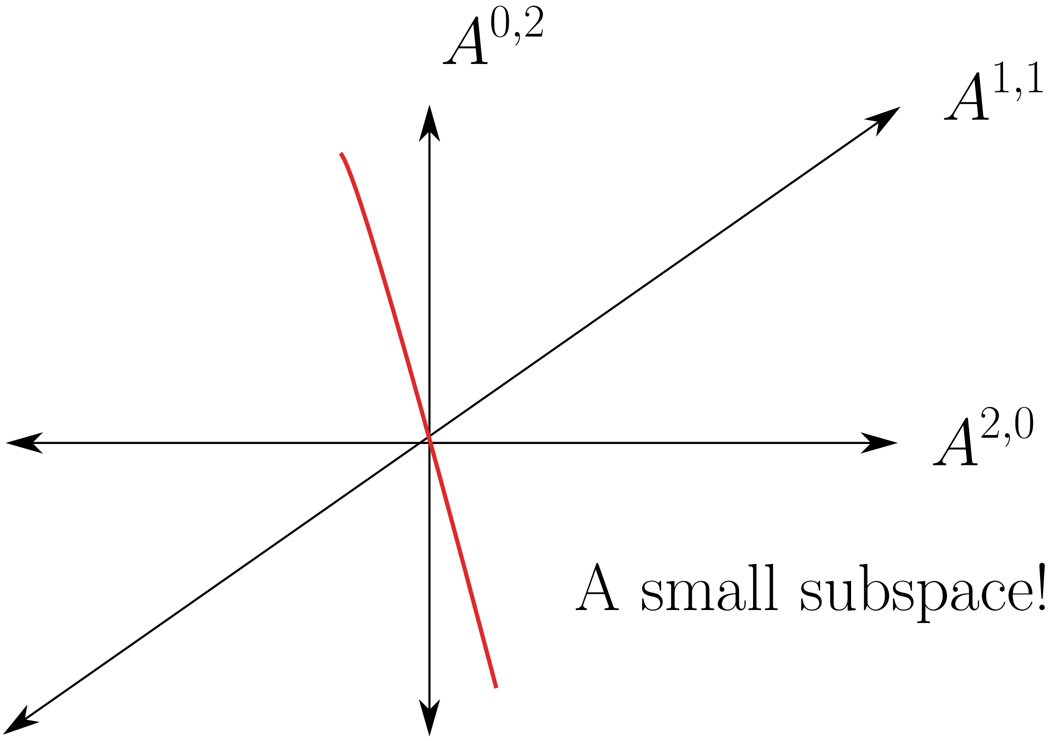
$$\Omega^k(M) = \bigoplus_{p+q=k} A^{p,q}(M)$$

$$A^{p,q} := \left\{ \sum_{\substack{|I|=p \\ |J|=q}} (\,dz_{i_1} \wedge \cdots dz_{i_p}) \wedge (\,dz_{j_1} \wedge \cdots dz_{j_q}) \right\}.$$

For M a Kähler manifold, we have

$$\mathcal{H}^k(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M)$$

$$\mathcal{H}^{p,q}(M) = \mathcal{H}^k(M) \cap A^{p,q}(M).$$



Remark 30.0.7: Why is this true? We have a map

$$d : A^{p,q}(M) \to A^{p+1,q}(M) \oplus A^{p,q+1}(M),$$

where for example if $f(z) := z\bar{z} \in A^{0,0}(\mathbb{C})$, we have $df = \bar{z}\, dz + z\, d\bar{z}$ where the first is a $(1,0)$ form and the latter is a $(0,1)$ form. Write $d = \partial + \bar{\partial}$ where $\partial := \sum dz_j$ and $\bar{\partial} = \sum d\bar{z}_j$, as well as

$$d^\dagger : A^{p,q}(M) \to A^{p-1,q}(M) \oplus A^{p,q-1}(M).$$

Now $\star$ of a (p,q) form is an $(n-p, n-q)$ form, and so

$$\star(dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_q) := \star(dz_I \wedge d\bar{z}_J) = \pm\, dz_{I^c} \wedge d\bar{z}_{J^c},$$

and we have $d^\dagger = \partial^\dagger + \bar{\partial}^\dagger$. We can thus move around the bigraded group in several ways:

$$\begin{tikzcd}
A^{2,0} \arrow[r, dashed, "\bar{\partial}"] \arrow[d, bend right, "\partial^\dagger"'] & A^{2,1} \arrow[r, dashed, "\bar{\partial}"] & A^{2,2} \\
A^{1,0} \arrow[u, dashed, "\partial"] \arrow[r, dashed, "\bar{\partial}"] \arrow[d, bend right, "\partial^\dagger"'] & A^{1,1} \arrow[u, dashed, "\partial"] \arrow[r, dashed, "\bar{\partial}"] & A^{1,2} \arrow[u, dashed, "\partial"] \\
A^{0,0} \arrow[u, dashed, "\partial"] \arrow[r, dashed, "\bar{\partial}"] & A^{0,1} \arrow[u, dashed, "\partial"] \arrow[r, dashed, "\bar{\partial}"] \arrow[l, bend left, "\bar{\partial}^\dagger"] & A^{0,2} \arrow[u, dashed, "\partial"] \arrow[l, bend left, "\bar{\partial}^\dagger"]
\end{tikzcd}$$

Link to Diagram

Theorem 30.0.8 ***(Kähler Identities).***
Let

$$\begin{aligned}\Delta_{\bar\partial} &:= \bar\partial \bar\partial^\dagger + \bar\partial^\dagger \bar\partial \\ \Delta_{\partial} &:= \partial \partial^\dagger + \partial^\dagger \partial \\ \Delta_d &:= dd^\dagger + d^\dagger d.\end{aligned}$$

Then

$$\frac{1}{2}\Delta_d = \Delta_{\bar\partial} = \Delta_\partial.$$

Remark 30.0.9: See Griffiths-Harris for details. Note that this is a local statement, i.e. it can be checked in coordinate charts.

Remark 30.0.10: The upshot:

$$\mathcal{H}^k(M) = \ker \Delta_d = \ker \Delta_{\bar\partial},$$

and moreover

$$\Delta_{\bar\partial} : A^{p,q}(M) \circlearrowleft$$

which implies that on $\Omega^k(M)$,

$$\ker \Delta_{\bar\partial} \circ \bigoplus_{p+q=k} \ker \left(A^{p,q}(M) \xrightarrow{\Delta_{\bar\partial}} A^{p,q}(M) \right) = \bigoplus_{p+q=k} \ker \Delta_d,$$

which yields the Hodge decomposition theorem

$$\mathcal{H}^k(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M).$$

Remark 30.0.11: This is a strong restriction on what manifolds can admit a Kähler structure. Moreover, since Δ_d is a real operator, we obtain $\overline{\mathcal{H}^{p,q}(M)} \cong \mathcal{H}^{p,q}(M)$.

Remark 30.0.12: Some consequences:

For M a Kähler manifold, the odd Betti numbers $\beta_{2i+1}(M) := \dim H^{2i+1}_{\mathrm{dR}}(M; \mathbb{C})$ are even. This is because

$$\bigoplus_{p+q=k} \mathcal{H}^{p,q} \cong \mathcal{H}^{2i+1}(M) \cong H^{2i+1}_{\mathrm{dR}}(M).$$

If we define $h^{p,q}(M) := \dim_{\mathbb{C}} \mathcal{H}^{p,q}(M)$, we clearly have

$$\beta_{2i+1} = \sum_{p+q=2i+1} h^{p,q}(M).$$

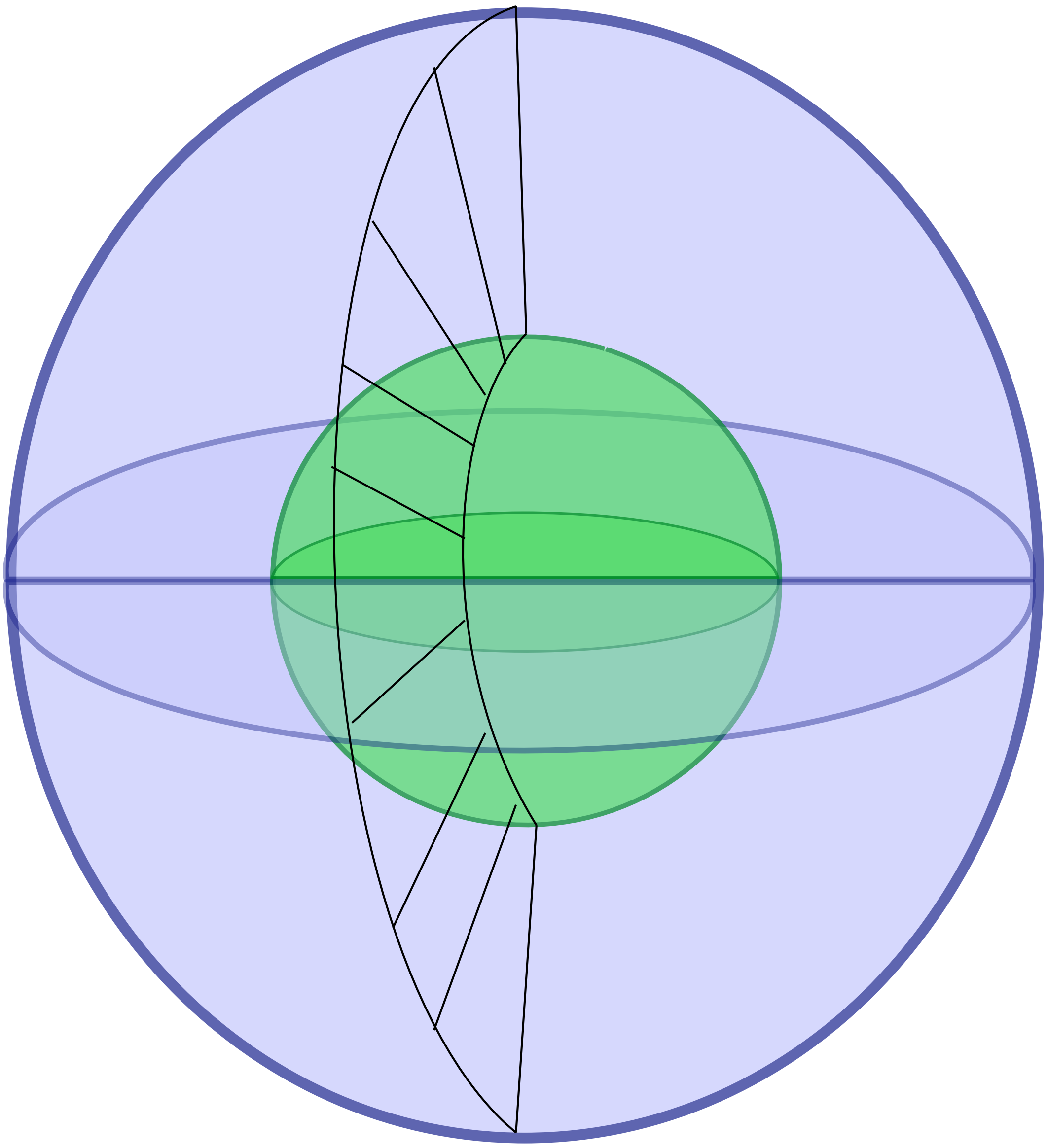
Now using that $\overline{\mathcal{H}} \cong \mathcal{H}$, we can rewrite this as

$$\begin{aligned}\beta_{2i+1} &= \sum_{p+q=2i+1} h^{p,q}(M) \\ &= 2 \sum_{\substack{p+q=2i+1 \\ p<q}} h^{p,q}(M).\end{aligned}$$

Remark 30.0.13: Is this just some fact about arbitrary complex manifolds, with no extra structure? The answer is no, and the counterexample is the *Hopf surface*

$$X := \left(\mathbb{C}^2 \setminus \{\mathbf{0}\}\right) / (x, y) \sim (2x, 2y),$$

which we can roughly identify as $\mathbb{R}^4$ "modulo doubling". We can take a fundamental domain $1 \leq |r| \leq 3$, this yields an annulus-like sphere with the inner shell glued to the outer:



This is homeomorphic to $S^1 \times S^3$, but $\beta_1(M) = 1$, so this won't yield a Kähler structure.

31 | Monday, March 29

Remark 31.0.1: Last time: the Hodge decomposition theorem. Let $(X, g) \in \mathsf{Mfd}_{\mathbb{C}}^{\text{compact}}(\text{Kähler})$, then the space of harmonic k-forms $\mathcal{H}^k(X) \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as $\bigoplus_{p+q=k} \mathcal{H}^{p,q}(X)$. There is also a symmetry $\overline{\mathcal{H}^{p,q}(X)} = \mathcal{H}^{q,p}(X)$. We have an isomorphism to the de Rham cohomology $\mathcal{H}^k(X) \otimes_{\mathbb{R}} \mathbb{C} \cong H^k_{\text{dR}}(X; \mathbb{C})$. We know the constituent pieces as well, as well as several relationships:

$$\begin{aligned}\mathcal{H}^{p,q}(X) &= \ker(\Delta_d : A^{p,q}(X) \circlearrowleft) \\ \Delta_{\bar\partial} &= \bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial \\ \Delta_d &= 2\Delta_{\bar\partial}.\end{aligned}$$

There was a proposition that $\ker(\Delta_d) = \ker(d) \cap \ker(d^\dagger)$, and the same proposition holds for $\Delta_{\bar\partial}$. In this case we have $\ker(\Delta_{\bar\partial}) = \ker(\bar\partial) \cap \ker(\bar\partial^\dagger)$ on $A^{p,q}(X)$, and this is isomorphic to $\ker(\bar\partial)/\operatorname{im}(\bar\partial)$. Recall that we resolved the sheaf Ω^p of holomorphic p-forms by taking the Dolbeault resolution

$$0 \to \Omega^p \to A^{p,0} \xrightarrow{\bar\partial} A^{p,1} \xrightarrow{\bar\partial} A^{p,2} \to \cdots.$$

Thus we can identify $\ker(\bar\partial)/\operatorname{im}(\bar\partial) \cong \mathcal{H}(X; \Omega^p)$ as sheaf cohomology. We defined $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X)$.

> **Corollary 31.0.2** ***(Homology is independent of the choice of Kähler form).***
> $h^{p,q}(X)$ is independent of the Kähler form, noting that the isomorphism to sheaf cohomology doesn't involve taking adjoints, and $\dim_{\mathbb{C}} \mathcal{H}^q(X; \Omega^p)$ doesn't depend on the complex structure.

Remark 31.0.3: A priori, one could vary the Kähler form and have some $h^{p,q}$ jump or drop dimension. It also turns out that varying the complex structure will also not change these dimensions.

Remark 31.0.4: Whenever the Hodge-de Rham spectral sequence degenerates, one generally gets $\sum_{p+q} h^{p,q} = h^k$. Note that there is a resolution:

$$0 \to \underline{\mathbb{C}} \to \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots,$$

which is not acyclic and thus has homology. In general, the spectral sequence is

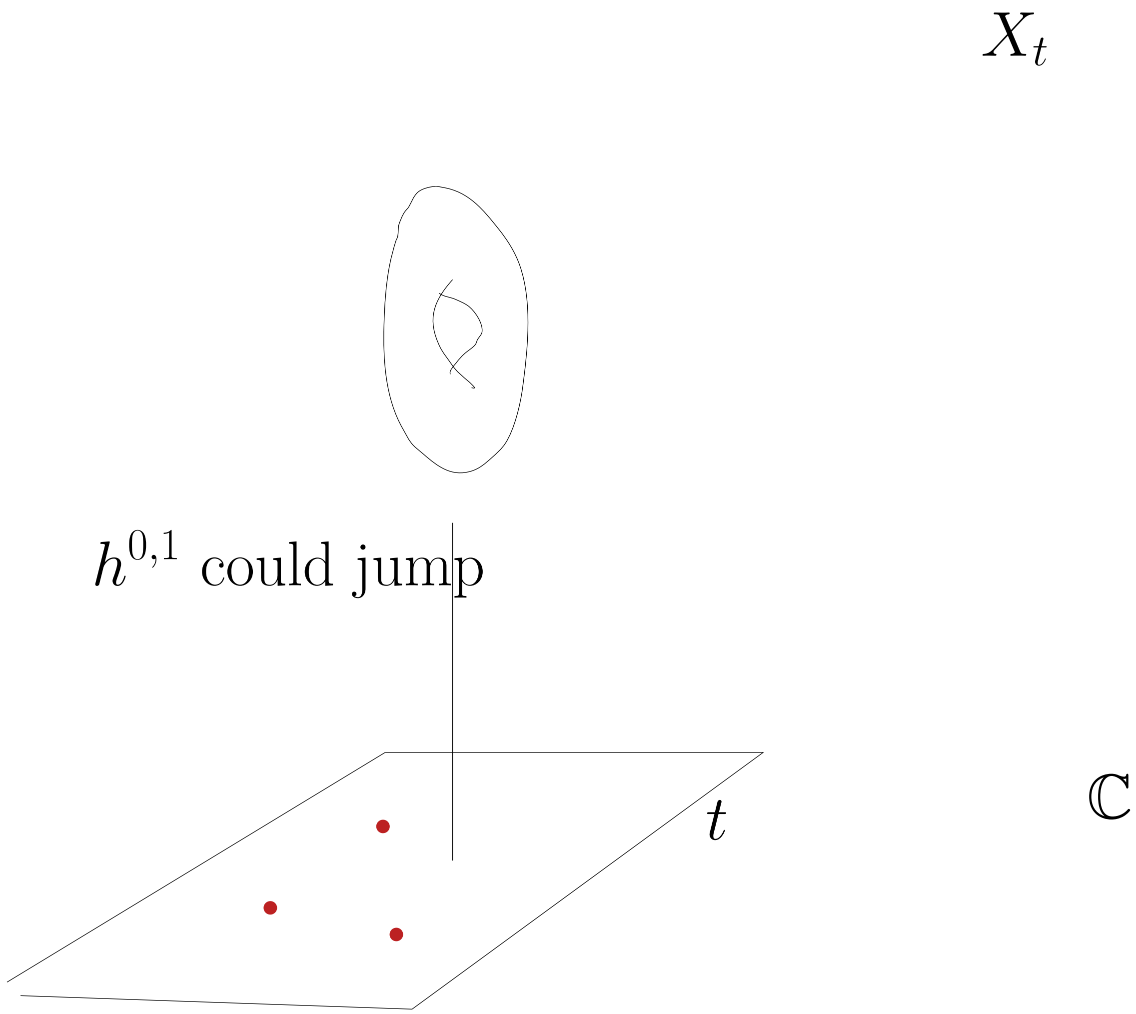
$$E^1_{p,q} = \mathcal{H}^q(X; \Omega^p) \Rightarrow \mathcal{H}^{p+q}(X; \underline{\mathbb{C}}).$$

Fact 31.0.5
A fact about the cohomology of vector bundles: given a family of Kähler manifolds X_t, one can consider $H^q(X_t; \mathcal{E}_t$ where $\mathcal{E}_t$ is a family of holomorphic vector bundles. This can only jump upward in dimension, i.e. $\dim_{\mathbb{C}} H^q(X_t; \mathcal{E}_t)$ is **lower semicontinuous**.

Example 31.0.6 ***(?):*** Consider

$$X_t := \left\{x^3 + y^3 + z^3 + txyz = 0\right\} \subseteq \mathbb{CP}^2,$$

where t varies in $\mathbb{C}$. These all admit a line bundle $\mathcal{L}_t := \mathcal{O}(1)|_{X_t}$, the anti-tautological line bundle on $\mathbb{P}^2$.



The real points of this vanishing locus form an elliptic curve, and each X_t is a Riemann surface of genus 1. Note that $h^{0,1}$ can jump on closed sets, but H^1 is constant since Riemann-Roch involves genus and degree. What is $\deg \mathcal{O}(1)|_{X_t}$? Take a section $s \in H^0(\mathbb{P}^2; \mathcal{O}(1))$ which vanishes on a line in $\mathbb{P}^2$. How many points lie in a line intersected with X_t? Looking at fundamental classes, we have $[X_t] = 3\ell$, and by Bezout $3\ell \cdot \ell = 3$.

The point is that $H^q(X_t; \Omega^p)$ can only possibly increase at special values of t. Assuming the X_t are all diffeomorphic, then $h^k(X_t)$ is constant and $h^{p,q}(X_t)$ can't jump. So the $h^{p,q}$ are invariants of families.

Definition 31.0.7 (Hodge Diamond)
The **Hodge Diamond** of $X \in \mathsf{Mfd}(\text{Kähler})$ (which won't depend on the choice of Kähler form) is given by

$$
\begin{tikzcd}
 & & & h^{n,n} & & & \\
 & & h^{n-1,n} \arrow[dddd, dotted, leftrightarrow, "z\mapsto\bar{z}"'] \arrow[ddddrr, dotted, leftrightarrow, "\star"] & & h^{n,n-1} \arrow[dddd, dotted, leftrightarrow, "z\mapsto\bar{z}"] \arrow[ddddll, dotted, leftrightarrow, "\star"'] & & \\
 & \ddots & & & & \ddots & \\
 \ddots & & & \vdots & & & \ddots \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{tikzcd}
$$

Link to Diagram

Note that there are symmetries, e.g. $\star$ takes $h^{1,0} = h^{n-1,n}$ and $\overline{h^{p,q}} = h^{q,p}$.

Proposition 31.0.8 ***(CYs have extra Hodge diamond symmetry).***
If X is **Calabi-Yau**, so $K_X = \mathcal{O}_X$ (i.e the canonical bundle is trivial), then the Hodge diamond has an orientation preserving $(\mathbb{Z}/2)^2$ symmetry, i.e. there is a rotation by $\pi/2$.

Note: this isn't extra symmetry! Just a proof of the symmetry in this case.

Proof (?).
Let Ω^k_X be the sheaf of holomorphic k-forms, then there is a map

$$
\begin{aligned}
\Omega^k_X \otimes \Omega^{n-k}_X &\to \Omega^n_X \coloneqq K_X \\
\alpha \otimes \beta &\mapsto \alpha \wedge \beta.
\end{aligned}
$$

Fiberwise, this is a perfect pairing. If one takes $\alpha \coloneqq e_{i_1} \wedge \cdots e_{i_k} \in \bigwedge^k T_x{}^\vee X$, there is a unique basis wedge $\beta \coloneqq e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$ then $\alpha \wedge \beta$ is a basis wedge $e_1 \wedge \cdots \wedge e_n$. So $\Omega^k_X \cong (\Omega^{n-k}_X)^\vee$

if X is Calabi-Yau. By Serre duality,

$$\mathcal{H}^p(X;\Omega_X^q)^\vee \cong \mathcal{H}^{n-p}(X;(\Omega_X^q)^\vee \otimes K_X).$$

■

Example 31.0.9*(?)*: In dimension 3, take

$$X := \left\{ x_0^5 + \cdots + x_4^5 = 0 \right\} \subseteq \mathbb{P}^4 \in \mathsf{Mfd}^3(\mathbb{C}).$$

See Hodge diamond.

Remark 31.0.10: Note that $K3$s are special CYs. An example is $\mathbb{C}^2/\Lambda$ for Λ a rank 4 lattice. This is diffeomorphic to $(S^1)^4$, for example $E \times E$.

32 | Wednesday, March 31

32.1 Polyvector Fields

Remark 32.1.1: We have a perfect pairing

$$\Omega^k \otimes \Omega^{n-k} \to K,$$

and thus $\Omega^{n-k} \cong K \otimes (\Omega^k)^\vee$. So we have

$$H^p(\Omega^k)^\vee \cong H^{n-p}((\Omega^k)^\vee \otimes K) = H^{n-p}(\Omega^{n-k}),$$

and thus $h^{p,k} = h^{n-p,n-k}$, which recovers what we knew about $\star : \mathcal{H}^{p,q} \to \mathcal{H}^{n-p,n-q}$.

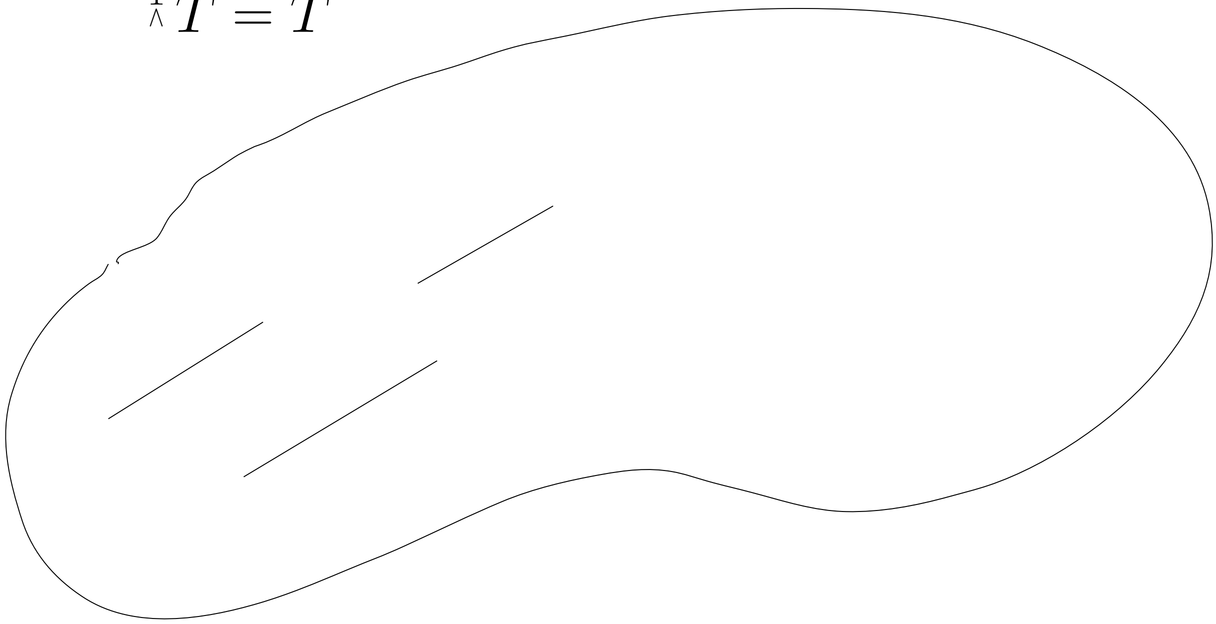
So we don't get anything new from the Serre duality argument.

What is special when $X \in \mathrm{CY}$ is that

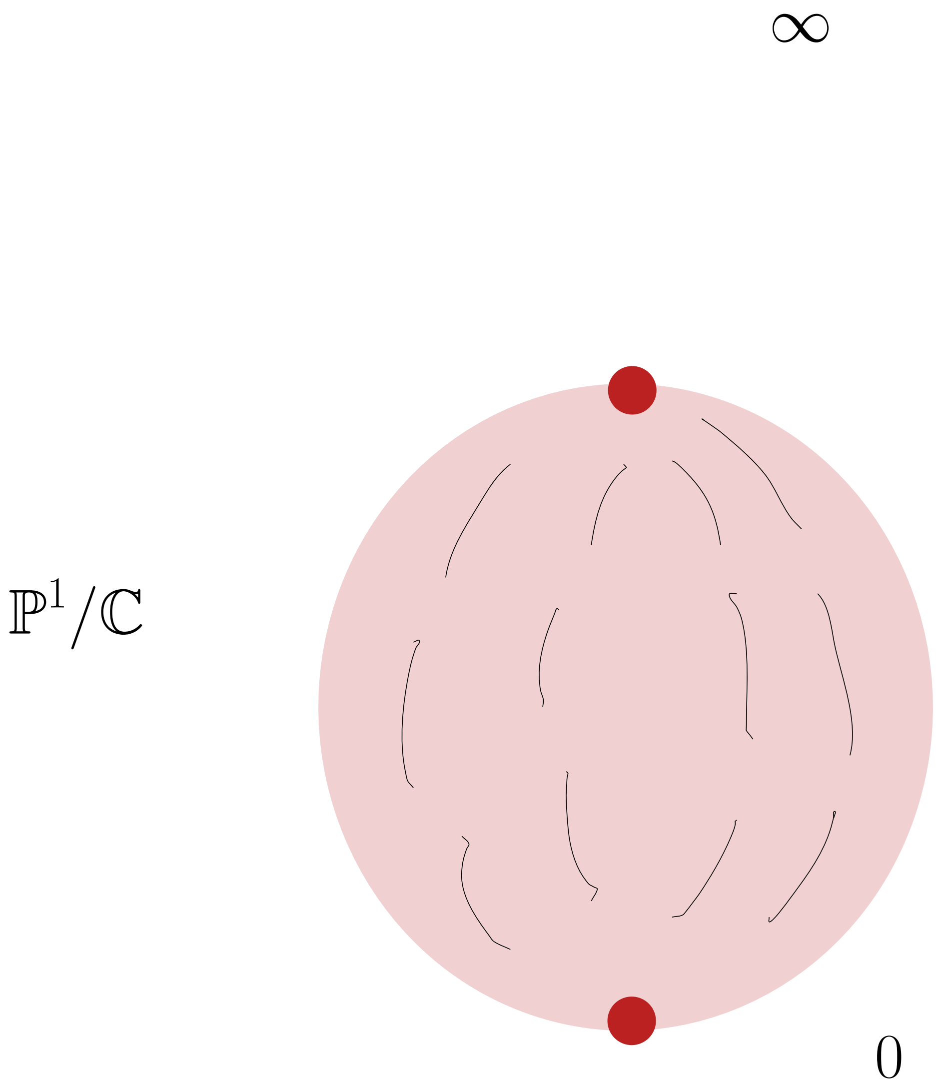
$$\Omega^{n-k} \cong (\Omega^k)^\vee = \bigwedge^k TX$$

for TX the tangent bundle. Note that taking the cotangent bundle gives forms, and instead this gives a bundle of *polyvector fields*. For $k = 1$, we get a holomorphic vector field, which one might think of as an infinitesimal biholomorphism.

$$\overset{1}{\wedge} T = T$$



Example 32.1.2 ***(?):*** $\mathbb{P}^1$ has a holomorphic vector field in coordinate charts $\mathbb{C} \cong \left\{ [z:1] \in \mathbb{P}^1 \right\}$ which we'll write as $z \dfrac{\partial}{\partial z}$. The coordinate chart is $\mathbb{P}^1 \setminus \infty$, so we obtain



Does this vector field V extend over ∞? The local coordinate at ∞ is $w = 1/z$, so $z = 1/w$ and we can compute

$$\frac{1}{w}\frac{\partial}{\partial\frac{1}{w}} = \frac{1}{w}\frac{\partial}{\frac{-1}{w^2}\partial w} = -w\frac{\partial}{\partial w}.$$

We have $\mathrm{Ord}_0 V = 1$ and $\mathrm{Ord}_\infty V = 1$, and so $\deg T\mathbb{P}^1 = 2$.

Example 32.1.3*(?):* For $\bigwedge^2 T$, the local sections are of the form $\sum f_I \frac{\partial}{\partial x_I} \wedge \frac{\partial}{\partial x_J}$ instead of e.g. $\frac{d}{dx_I}$. This yields a **Poisson structure** $H^0(X, \bigwedge^2 T)$, which is a generalization of symplectic structure, which would be a section $\omega \in H^0(X, \bigwedge^2 T^\vee)$ which is nondegenerate. This would yield an isomorphism $\omega : T \xrightarrow{\sim} T^\vee$ which is alternating, in which case $\omega^{-1} : T^\vee \xrightarrow{\sim} T$ which is also alternating, so $\omega^{-1} \in H^0(X, \bigwedge^2 T)$. However the Poisson structure need not be nondegenerate.

Remark 32.1.4: Polyvector fields show up in Hochschild homology!

32.2 Algebraic Surfaces

Definition 32.2.1 (Algebraic Surface)
An **algebraic surface** is a compact complex 2-fold (so of complex dimension and real dimension 4, admitting local charts to $\mathbb{C}^2$) which admits a holomorphic embedding into $\mathbb{CP}^N$ for some N.

Remark 32.2.2: This implies that S is a **projective variety** cut out by homogeneous polynomials in $N+1$ variables in $\mathbb{CP}^N$.

Example 32.2.3*(?):* A non-example would be $\mathbb{C}^2 \setminus \{(0,0)\} / (x,y) \sim (2x, 2y)$, The *Hopf surface*. This is a complex manifold of complex dimension 2. It is compact, but has no projective embedding!

Example 32.2.4*(?):* Another non-example is $\mathbb{C}^2 \setminus \{0\} / (x,y) \sim (2x, 2e^{i\theta} y)$, a *twisted Hopf surface*. This admits no nontrivial holomorphic line bundles.

Remark 32.2.5: What makes having a projective embedding special? If $S \hookrightarrow \mathbb{CP}^N$, it admits a line bundle: $\mathcal{O}_S(1) := \mathcal{O}_{\mathbb{CP}^N}(1)\big|_S$.

Proposition 32.2.6*(Existence of the Fubini-Study form/metric).*
$\mathbb{CP}^N$ is a Kähler manifold, and admits a distinguished 2-form $\omega := \omega_{\mathrm{FS}}$ the **Fubini-Study form** which induces the Fubini-Study metric g_{FS}.

Remark 32.2.7: This can be written down as $\dfrac{i}{2}\partial\bar\partial \log(\sum_{i=1}^N z_i \bar z_i)$, which is well-defined since scaling comes out as a constant. Being closed follows from $\partial\bar\partial = d\bar\partial$ since $\bar\partial^2 = 0$, which implies $d(\partial\bar\partial \cdots) = d^2\bar\partial(\cdots) = 0$. This defines a metric: this follows from checking in local coordinate charts, say $z_0 = 1$, and checking that $g(x,y) := \omega(x, Jy)$ yields a metric. This involves taking a fussy derivative!

Remark 32.2.8: Thus given $S \xhookrightarrow{\varphi} \mathbb{CP}^N$, we can restrict or take the pullback of ω_{FS} to S. Then $\omega := \varphi^* \omega_{\mathrm{FS}}$ is still Kähler:

1. ω is closed: this is true for any smooth map at the level of smooth manifolds because of the chain rule.

2. ω defines a metric: this is true because S is a complex submanifold. Suppose $v, w \in T_p S$, and we want to check if $g(v,w) := \omega(v, Jw)$. This equals $\omega_{\mathrm{FS}}(v, JW)$, viewing $T_p S \subseteq T_p \mathbb{CP}^N$, so this is equal to $g_{\mathrm{FS}}(v,w)$.

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Remark 32.2.9: Note that a submanifold of a *symplectic* manifold is not necessarily a symplectic submanifold, since there are Lagrangian submanifolds for which the symplectic form restricts to 0 and isn't nondegenerate. However, Kähler forms do restrict.

Remark 32.2.10: So we get a Hodge diamond:

$$
\begin{array}{ccccc}
 & & h^{2,2} & & \\
 & h^{2,1} & & h^{1,2} & \\
h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & h^{1,0} & & h^{0,1} & \\
 & & h^{0,0} & &
\end{array}
$$

Link to Diagram

Here $h^{2,0} = h^0(\Omega^2) = h^0(K) = g$ is called the *genus* in analogy with curves. Similarly, $h^{1,0} = h^0(\Omega^1)$ is the space of holomorphic 1-forms, sometimes referred to as the *irregularity*. There is some symmetry:

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & q & & q & \\
g & & h^{0,0} & & g \\
 & q & & q & \\
 & & 1 & &
\end{array}
$$

Link to Diagram

Exercise 32.2.11 (?)
Solve for $h^{1,1}$ in terms of q and g.

33 | Friday, April 02

33.1 When Line Bundles are $\mathcal{O}$ of a Divisor

Remark 33.1.1: Last time: if we have such a Hodge diamond, can we solve for $h^{1,1}$?

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & q & & q & \\
p & & h^{1,1} & & p \\
 & q & & q & \\
 & & 1 & &
\end{array}
$$

Link to Diagram

Recall Noether's formula

$$
\begin{aligned}
\chi(S, \mathcal{O}_S) &= \int \mathrm{ch}(\mathcal{O}_S)\mathrm{td}(S) \\
&= \int_S \frac{x_1}{1-e^{-x_1}} \frac{x_2}{1-e^{-x_2}} \\
&= \frac{K^2 + \chi_{\mathsf{Top}}(S)}{12},
\end{aligned}
$$

where $c_1(TS) = -K$ and χ_{Top} is due to the Chern-Gauss-Bonet formula. We have

$$
\chi(\mathcal{O}_S) = h^0(\mathcal{O}_S) - h_1(\mathcal{O}_S) + h^2(\mathcal{O}_S) = 1 - q + p.
$$

On the other hand,

$$
\chi_{\mathsf{Top}}(S) = 1 - 2q + (2p + h^{1,1}) - 4q = 1 - 4q + 2^p + h^{1,1},
$$

so

$$
12(1 - q + p) = K^2 + 2 - 4g + 2p + h^{1,1} \implies h^{1,1} = 110 - 8q + 10p - K^2.
$$

Remark 33.1.2: Recall the extraordinarily important exact sequence

$$0 \to \mathcal{O}(-p) \to \mathcal{O} \to \mathcal{O}_p \to 0,$$

where the right-hand side is the sheaf of holomorphic functions vanishing at p and this is an inclusion into the sheaf of holomorphic functions, and the right-hand term is the skyscraper sheaf. There is a similar exact sequence for an embedded curve $C \hookrightarrow S$ in a surface:

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0,$$

where the left term is the sheaf of holomorphic functions vanishing on C. Note that this has no global sections! Any function vanishing along a compact subset (?) are constant (?). Locally on an open set U, one can write $C \cap U = V(f_u)$, since algebraically this ring is locally a PID. So this is a line bundle, where we can map into the trivial bundle by $\varphi \mapsto \varphi/f_u$. Thus

$$\mathcal{O}_S(U)/\mathcal{O}_S(-C)(U) \cong \mathcal{O}_C(C \cap U).$$

We then get surjectivity since every holomorphic function on C extends to a holomorphic function on S.

Now letting $\mathcal{E} \in \mathsf{Vect}(\mathrm{Hol})$, we can tensor this exact sequence to get

$$0 \to \mathcal{E}(-C) \to \mathcal{E} \to \mathcal{E}|_C \to 0,$$

which is also exact since locally we have the splitting principle.

Proposition 33.1.3 ***(Every line bundle over a smooth projective complex manifold is O of a divisor).***
Let X be a smooth projective [a] complex manifold. Then every line bundle over X is of the form $L = \mathcal{O}_X(D)$ for some divisor $D = \sum n_i D_i \in \mathbb{Z}[\mathrm{SubMfds}(\mathrm{codim}_1)]$.

[a] So X admits an embedding into some $\mathbb{CP}^N$.

33.2 Proof

Proof (?).
Let H be a **hyperplane section**, i.e. an intersection of X with a generic hyperplane in $\mathbb{CP}^N$.

Lemma 33.2.1 ***(Serre Vanishing Theorem).***
For any vector bundle $\mathcal{E}$ and all $i > 0$, for $k \gg 0$ we have

$$h^i(X, \mathcal{E} \otimes \mathcal{O}(kH)) = 0.$$

Remark 33.2.2: We'll not prove this! It requires some heavy analysis and the Kähler identities, see Huybrechts complex geometry Prop 5.27.

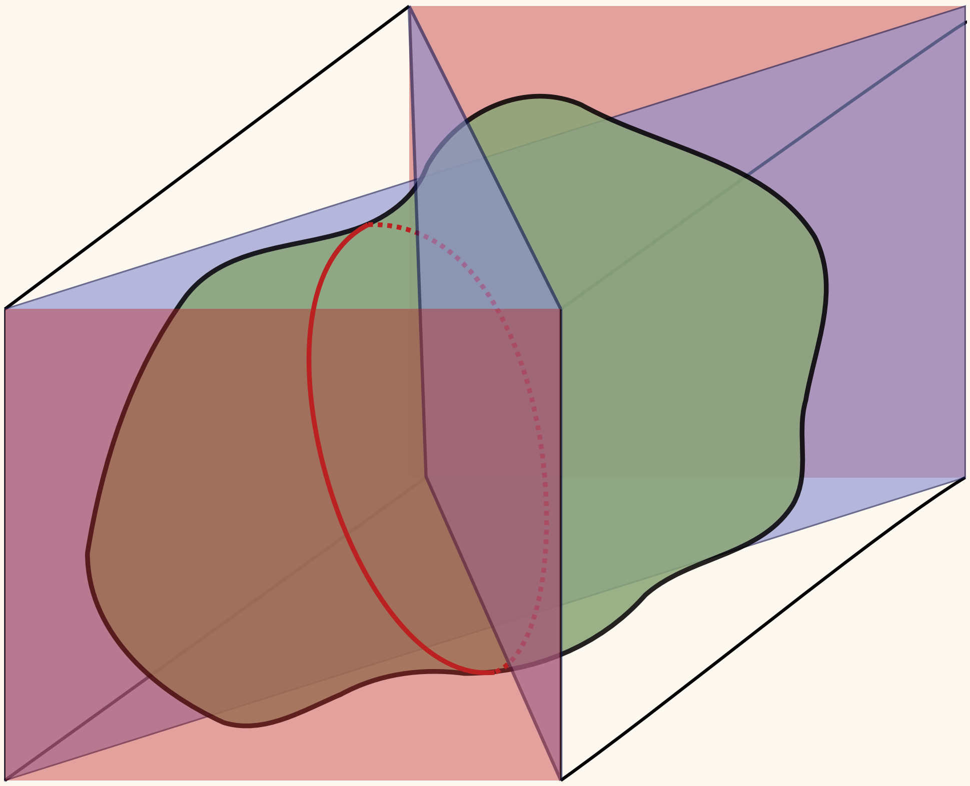
We can write

$$\begin{aligned}\chi(L\otimes\mathcal{O}(kH)) &= \int_X \mathrm{ch}(L\otimes\mathcal{O}(kH))\mathrm{td}(X)\\ &= \int_X \mathrm{ch}(L)\,\mathrm{ch}(H)^k\mathrm{td}(X)\\ &= \int_X \left(1+c_1(L)+\frac{c_1(L)^2}{2}+\cdots\right)\cdot\left(1+kh+\frac{(kh)^2}{2}+\cdots+\frac{(kh)^{\dim X}}{(\dim X)!}\right)\cdot(1+\mathrm{td}_1(X)+\mathrm{td}_2(\end{aligned}$$

where h is the restriction of the generator of $H^2(\mathbb{CP}^N;\mathbb{Z})$ to X. Note that for k large, the dominating term grows like $(kh)^{\dim X}$, so asymptotically we have

$$\cdots\sim\int_X \frac{k^{\dim X}h^{\dim X}}{(\dim X)!}.$$

What is this $\dim(X)$-fold intersection?



We can slice X by multiple hyperplanes, each homologically perturbed, and so $\int_X h^{\dim X}$ is the number of points where $\dim X$ generic hyperplanes intersect X, which is called the **degree** $\deg X$. This roughly follows from $\int_X \omega_{\mathrm{FS}}^{\dim X}>0$. Alternatively, suppose $X\cap H=\emptyset$, then

$X \hookrightarrow H^c = \mathbb{A}^N$. Then each holomorphic coordinate restricts to a constant on X by the maximal principle.
Back to what we were proving: we have

$$\chi(L \otimes \mathcal{O}(kH)) \sim ck^{\dim X},$$

for c some constant. By Serre Vanishing, $h^i(L \otimes \mathcal{O}(kH)) = 0$ for $k \gg 0$, and so we obtain

$$h^0(L \otimes \mathcal{O}(kH)) \sim ck^{\dim X} \implies \exists k \text{ s.t. } h^0(L \otimes \mathcal{O}(kH)) > 0.$$

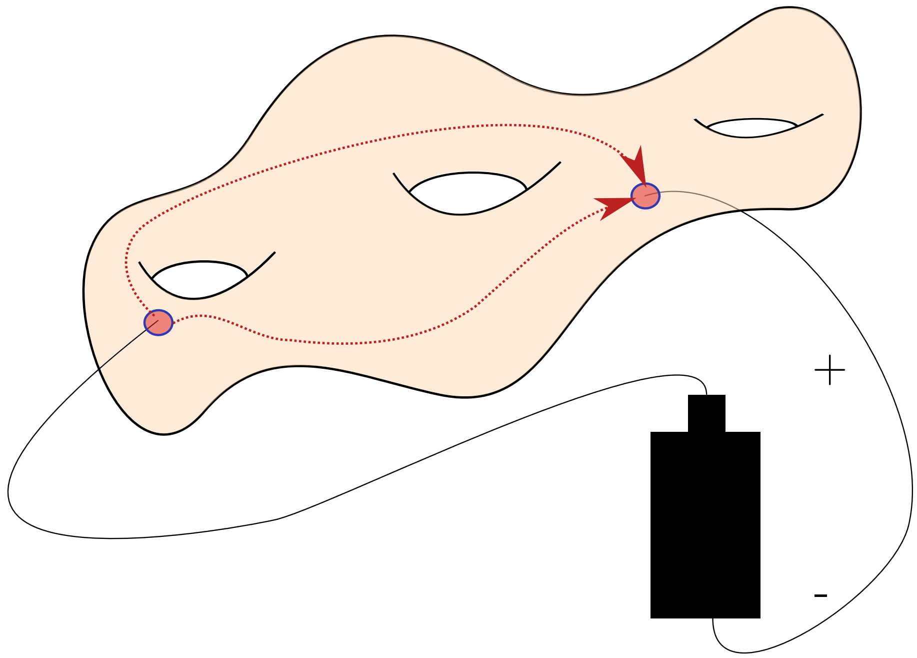
We conclude that there is some nonzero section $s \in \mathcal{H}^0(X; L \otimes \mathcal{O}(kH))$ for which $\mathcal{O}(\operatorname{Div} s) \cong L \otimes \mathcal{O}(kH)$. Thus $L \cong \mathcal{O}(\operatorname{Div} s - kH)$, where $\operatorname{Div} s - kH$ is some divisor.

■

Remark 33.2.3: With some more work, one can show $L \cong \mathcal{O}(C - D)$ for C, D *smooth* divisors.

33.3 Aside

Remark 33.3.1: Felix Klein has a "proof" of the existence of a meromorphic function on a Riemann surface. The argument roughly goes as follows: take your Riemann surface and make it out of metal. Attach it to a battery:



This induces an electric potential function $V : C \to \mathbb{R}$, where V is the real part of the meromorphic function. Here V is a harmonic function away from p and q.

34 | Monday, April 05

Remark 34.0.1: Last time: line bundles are of the form $\mathcal{O}(D)$ for D a divisor, and the extremely important SES

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0.$$

We now want to discuss an alternative characterization of the intersection form on an algebraic surface. The next result comes from Beauville's "Complex Algebraic Surfaces":

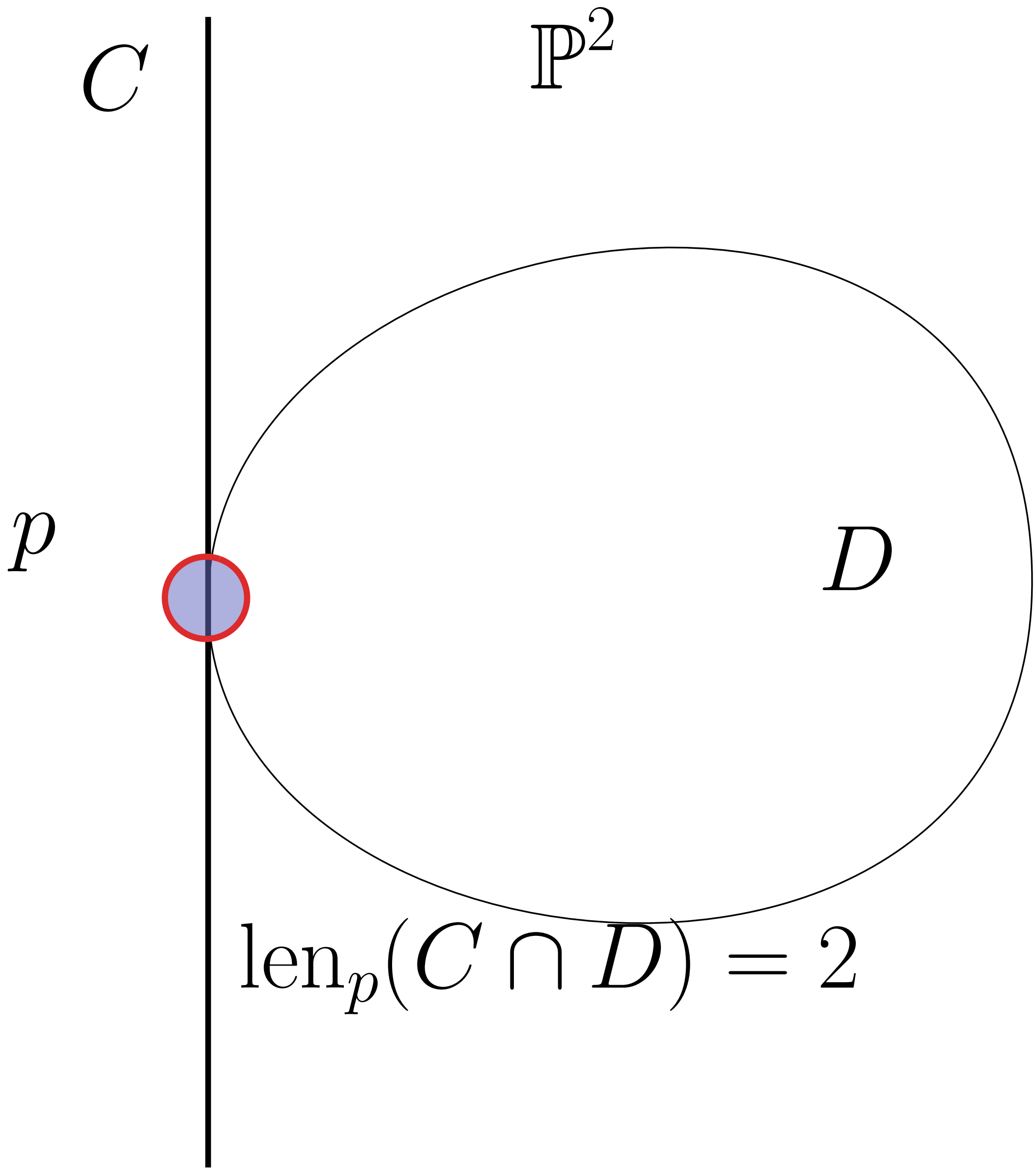
Proposition 34.0.2 ***(Formula for computing intersection numbers between complex curves).***
Let $S \in \mathsf{Mfd}^2(\mathbb{C})^{\text{compact}}$, then the intersection number between complex curves C, D can be

computed in the following ways:

$$C \cdot D = \deg \mathcal{O}_S(C)\,|_D = \sum_{p \in C \cap D} \operatorname*{len}_p(C \cap D),$$

where we'll define $\operatorname*{len}_p$ soon.

Remark 34.0.3: This will count intersection points after a small perturbation. Note that not every two curves will intersect transversely: consider $\mathbb{P}_2$ with a line C and a tangent conic D:



Proof (?).
We have the first equality because

$$C \cdot D = \int_S [C] \frown [D] = \int_C i^*[D],$$

where $i : C \hookrightarrow S$ is the inclusion. This equality holds because if $\alpha \in \Omega^2$ is a 2-form,

$$\int_S [C] \cdot \alpha = \int \alpha|_C.$$

Using the pullback commutes with taking Chern classes, we can write the

$$\int_C i^*[D] = \int_C i^*(c_1(\mathcal{O}(D))) = \int_C c_1(i^*\mathcal{O}(D)) = \int_C \mathcal{O}(D)|_C = \deg \mathcal{O}(D)|_C.$$

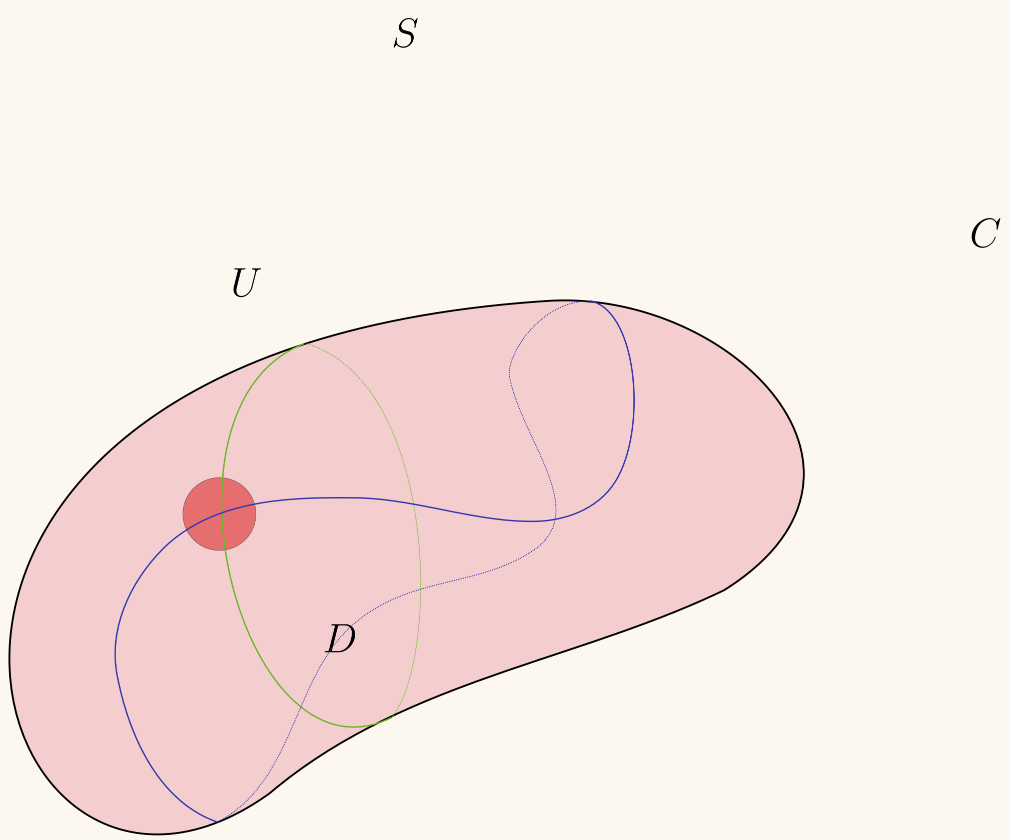
Note that this formula was symmetric, so we could have done this the other way to obtain $\deg \mathcal{O}_S(C)|_D = \deg \mathcal{O}_S(D)|_C$.
For the second equality, consider the following 4-term exact sequence:

$$0 \to \mathcal{O}_S(-C-D) \xrightarrow[p_1]{[s_D, s_C]} \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-D) \xrightarrow[p_2]{[s_D, -s_C]^t} \mathcal{O}_S \xrightarrow[p_3]{} \mathcal{O}_{C\cap D} \to 0.$$

For the first map, we have

$$\{\text{Functions vanishing on } C+D\} \hookrightarrow \{\text{Functions vanishing on } C\} \oplus \{\text{Functions vanishing on } D\}.$$

Locally we can write $C = V(f)$ and $D = V(g)$ for some holomorphic functions $f, g \in \mathrm{Hol}(U, \mathbb{C})$.
We have the following picture:



We have $s_C \in H^0(S; \mathcal{O}_S(C))$ and $s_D \in H^0(S; \mathcal{O}_S(D))$ as global sections where $V(s_c) = C, V(s_D) = D$. In a local trivialization, we can assume $s_C|_U = f$ and $s_D|_U = g$. So the first map is (s_D, s_C). The next map is $[s_C, -s_D]^t$ as a column vector, i.e. given a section we map it in the following way:

$$(\varphi_1, \varphi_2) \in H^0(U, \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-D)) \mapsto \varphi_1 \cdot s_D - \varphi_2 \cdot s_C.$$

Why is this exact? Considering the composition, we have

$$\varphi \xrightarrow{p_1} (\varphi s_D, \varphi s_C) \xrightarrow{p_2} (\varphi s_D) s_C - (\varphi s_C) s_D = 0.$$

So we get $\operatorname{im} p_1 \subseteq \ker p_2$. Why do we have the reverse containment for exactness? Looking locally, given a pair $\varphi_1, \varphi_2 \in \operatorname{Hol}(U; \mathbb{C})$ such that $\varphi_1 \varphi - \varphi_2 g = 0$ and locally $(\varphi_1, \varphi_2) \in \ker p_2$, we want to show that $\varphi_1 = g\varphi, \varphi_2 = f\varphi$ for some $f, g \in \operatorname{Hol}(U; \mathbb{C})$. Equivalently, we want to show that

$$\varphi_1 f = \varphi_2 g \implies g \mid \varphi_1.$$

If this is true, then we can set $\varphi := \dfrac{\varphi_1}{g}$, since this would yield $g\varphi = \varphi_1$ and $f\varphi = \dfrac{f\varphi_1}{g} = \varphi_2$. Note that we can divide here because the ring $\operatorname{Hol}(U; \mathbb{C})$ is a domain (i.e. it has no zero divisors) on small sets.

Question

Is $\operatorname{Hol}(U, \mathbb{C})$ a PID in general?

Answer

No! Take $U \subseteq \mathbb{C}^2$ a ball around $z = 0$, then $\langle x, y \rangle$ is not principal.

However, this will form a UFD, which is weaker but still enough here. This is not obvious, but can be proved using the Weierstrass preparation theorem. This should be believable since R a UFD implies $R[x]$ is a UFD, and $\mathbb{C}[x, y] \subsetneq \operatorname{Hol}(U; \mathbb{C}) \subsetneq \mathbb{C}[[x, y]]$, and the latter is a UFD. So we do get exactness at this position.

For exactness at the next position $\mathcal{O}_S(-C) \oplus \mathcal{O}_S(-D) \to \mathcal{O}_S$, locally we have $(\varphi_1, \varphi_2) \mapsto \varphi_1 f - \varphi_2 g$ where $V(f) = C \cap U$ and $V(g) = D \cap U$. We can write $\varphi_1 f - \varphi_2 g = \langle f, g \rangle$ locally, so the cokernel sheaf of p_2 is given by

$$\operatorname{coker} p_2(U) := \frac{\mathcal{O}_S(U)}{\operatorname{im} p_2} = \frac{\mathcal{O}_S(U)}{\langle f, g \rangle}.$$

By definition, this is equal to $\mathcal{O}_{V(f,g)} = \mathcal{O}_{C \cap D}$, and if $C \cap D \cap U = \emptyset$ then $\mathcal{O}_{C \cap D}(U) = 0$. So let $p \in \mathcal{O}_{C \cap D}$ and let $U_p \ni p$ which contains no other points $q \in C \cap D$, since the set of intersection points is isolated (and thus finite). Note that compactness here prevents accumulation of intersection points. In this case, $\mathcal{O}_{C \cap D}(U_p)$ will be a finite-dimensional vector space $\mathbb{C}^d$, and we'll define $\operatorname{len}_p(C \cap D) := d$.

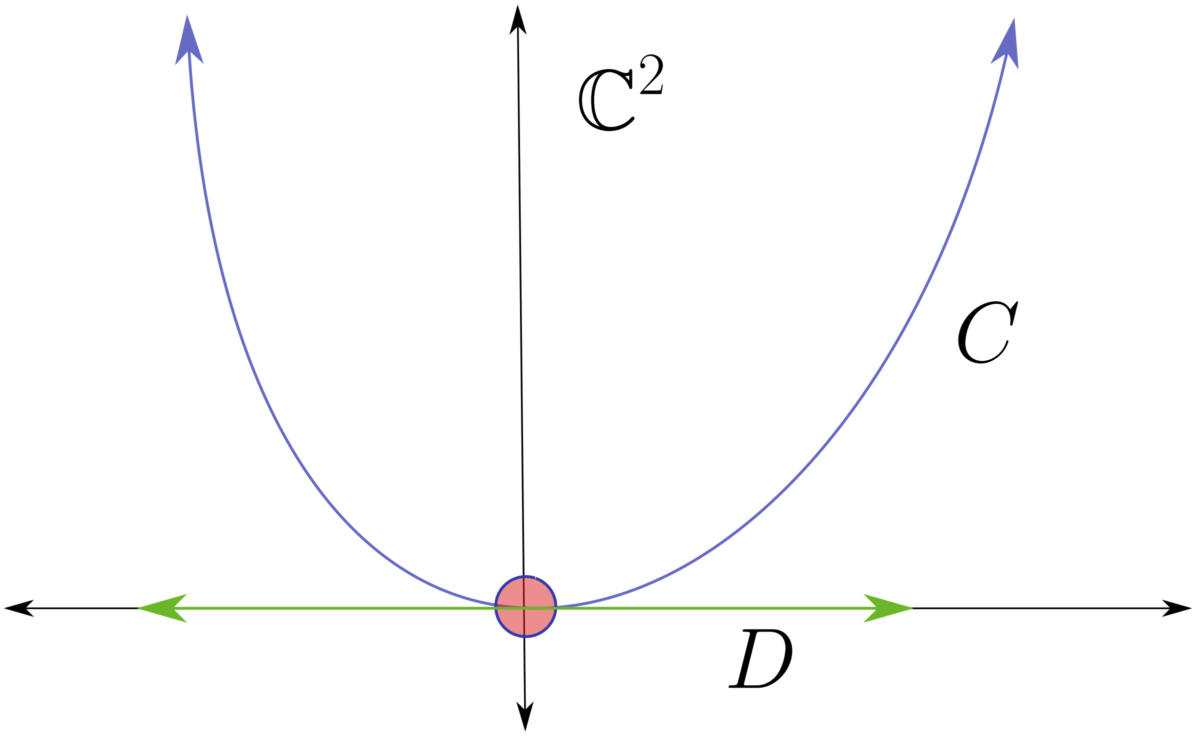
$\blacksquare$

Example 34.0.6 ***(?):*** Let $U = \mathbb{C}^2$ and take $f = y$ so $C := V(f)$ is the x-axis, and set $g = y - x^2$

so $D \coloneqq V(g)$ is a parabola. We're then considering

$$\frac{\mathrm{Hol}(\mathbb{C}^2)}{y\,\mathrm{Hol}(\mathbb{C}^2) + (y - x^2)\,\mathrm{Hol}(\mathbb{C}^2)} = \frac{\mathrm{Hol}(\mathbb{C}^2)}{\langle y, x^2 \rangle}.$$

Elements in the ideal can be expanded as power series of the form $a_{0,1}y + a_{2,0}x^2 + a_{1,1}xy + a_{2,2}y^2$, where there is no $a_{1,0} \sim x^1 y_0$ coefficient, nor any $a_{0,0} \sim x^0 y^0$ coefficient. So this quotient is isomorphic to $\mathbb{C}1 \oplus \mathbb{C}x$, which is 2-dimensional, so $\underset{(0,0)}{\mathrm{len}}\, V(y) \cap V(x) = 2$. Geometrically we have the following, where this is picking up the multiplicity 2 intersection:



Remark 34.0.7: What's the payoff of this algebraic work? We can compute the Euler characteristic as

$$\chi(\mathcal{O}_{C \cap D}) = h^0(\mathcal{O}_{C \cap D}) = \sum_{p \in C \cap D} \underset{p}{\mathrm{len}}(C \cap D).$$

But by additivity of χ over exact sequences, we also have

$$\begin{aligned}
\chi(\mathcal{O}_{C \cap D}) &= \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) - \chi(\mathcal{O}_S(-D)) + \chi(\mathcal{O}_S(-C - D)) \\
&\overset{HRR}{=} \int_S (\mathrm{ch}(\mathcal{O}_S) - \mathrm{ch}(\mathcal{O}_S(-C)) - \mathrm{ch}(\mathcal{O}_S(-D)) + \mathrm{ch}(\mathcal{O}_S(-C - D)))\,\mathrm{td}(S) \\
&= c_1(\mathcal{O}_S(-C)) \cdot c_1(\mathcal{O}_S(-D)) \\
&= (-[C]) \cdot (-[D]) \\
&= C \cdot D.
\end{aligned}$$

Remark 34.0.8: Next time: adjunction formula that allows computing genus for surfaces.

35 | Wednesday, April 07

Remark 35.0.1: Last time: let $C, D \subset S$ be distinct curves, then the intersection number is given by

$$C \cdot D = \deg \mathcal{O}_S(C)|_D = \sum_{p \in C \cap D} \operatorname{len}_p(C \cap D)$$

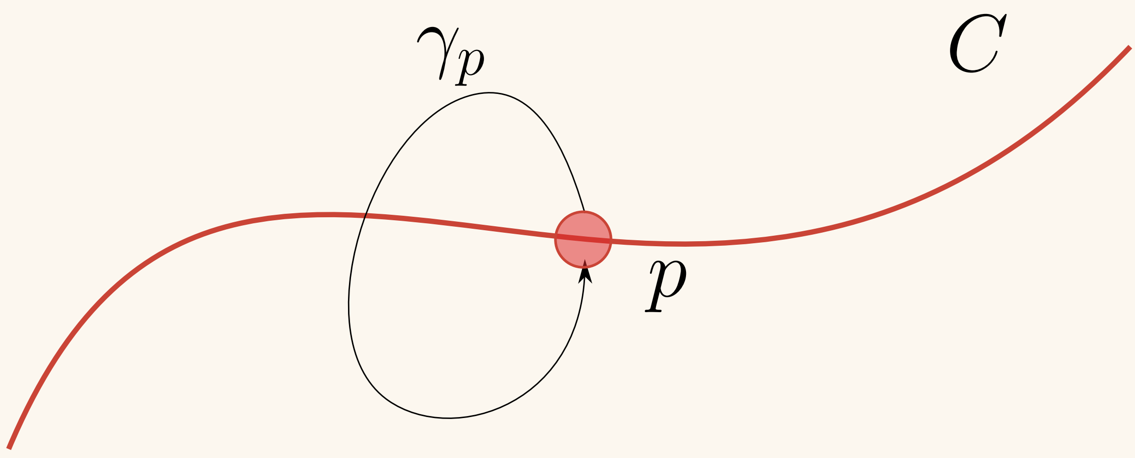
where $\operatorname{len}_p(C \cap D) := \dim_{\mathbb{C}} \mathcal{O}(U) / \langle f, g \rangle$ where $V(f) = C \cap U$ and $V(g) = D \cap U$ with $f, g \in \mathcal{O}(U) = \operatorname{Hol}(U)$. Here we're also assuming that $C \cap D \cap U = \{p\}$.

35.1 Adjunction Formula

Remark 35.1.1: We'll now discuss a way to compute the genus of a curve in a surface.

Proposition 35.1.2 ***(Adjunction Formula).***
Let $C \subset S$ be a smooth curve, then $K_C = (K_S \otimes \mathcal{O}_S(C))|_C$, which is restriction of a line bundle. Note that $K_C = \Omega_C^1$ is the sheaf of holomorphic 1-forms, but $K_S = \Omega_S^2$ since we take the sheaf of top forms.

Proof (?).
Let $s \in \Omega_S^2 \otimes \mathcal{O}(C)(U)$ be a section, then s_C is a section of $\mathcal{O}_C$ vanishing along c and have s/s_C a meromorphic section of $\Omega_S^2(U)$. Here dividing by s_C is like tensoring with $\mathcal{O}(-C)$. This can have poles along $\{s_C = 0\} = C$ up to first order.
There is a residue map: let $p \in C$ be a point and $\gamma_p(r)$ be an oriented loop in $S \setminus C$ around $p \in C$ of radius r (a meridian):



We can assemble a 1-form from the following contour integral:

$$\operatorname*{Res}_C \frac{s}{s_C} := \lim_{r \to 0} \frac{1}{2\pi i} \oint_{\gamma_p(r)} \frac{s}{s_C} \in \Omega^1(U \cap C).$$

Locally $C = V(x)$ in a coordinate chart of $\mathbb{C}^2$ where $s_C = x$, so this is roughly of the form $\oint_{|x|=r} \frac{f(x,y)}{x}\, dx \wedge dy$, which is a one form in the variable y. Note that if f were analytic, writing $f = a_{0,0} + a_{0,1}y + a_{0,2}y^2 + \cdots + a_{1,0}x + \cdots$, we would have

$$\operatorname*{Res}_C \frac{s}{s_C} = (a_{0,0} + a_{0,1}y + a_{0,2}y^2 + \cdots)\, dy = f(0,y)\, dy \text{locally},$$

which picks out all components not involving x. This defines an $\mathcal{O}$-linear map

$$\begin{aligned} \Omega^2_S \otimes \mathcal{O}_C &\to \Omega^1_C \\ s &\mapsto \operatorname*{Res}_C \frac{s}{s_C}, \end{aligned}$$

since it doesn't involve any derivatives of f. Note that this only depends on the restriction of s to C. What is the kernel of $\operatorname*{Res}_C$? We claim it is $\Omega 2_S$, which follows from the fact that the contour integral of any holomorphic form ω will integrate to zero. We thus get a SES of sheaves

$$0 \to \Omega^2_S \xrightarrow{\cdot s_C} \Omega^2_S \otimes \mathcal{O}(C) \to \Omega^1(C) \to 0.$$

where we send holomorphic forms to meromorphic forms with at most order 1 poles along C to holomorphic 1-forms on C. The residue map is surjective since we can take

$$\operatorname*{Res}_{x=0} \frac{g(y)}{x}\, dx \wedge dy = g(y)\, dy,$$

so locally an arbitrary 1-form is a residue of some 2-form with simple poles along C. We have a SES

$$0 \to \mathcal{O}(-C) \xrightarrow{\cdot s_C} \mathcal{O} \to \mathcal{O}_C \to 0,$$

and tensoring with the line bundle $\Omega^2 \otimes \mathcal{O}(C)$ we obtain

$$0 \to \Omega^2_S \to \Omega^2_S \otimes \mathcal{O}(C) \to \Omega^2_S \otimes \mathcal{O}(C)\,|_C \to 0.$$

Since cokernels are unique, we have $\Omega^1_C \cong \Omega^2_S \otimes \mathcal{O}(C)\,|_C$, which yields the adjunction formula. ■

Corollary 35.1.3 ***(The Genus Formula).***
We have

$$\deg \Omega^2_S \otimes \mathcal{O}(C)\,|_C = \deg \Omega^1_C = 2g - 2$$

where $g = g(C)$ is the genus of C. On the other hand, the left-hand side is equal to

$$(K_S + C) \cdot C = 2g(C) - 2.$$

Example 35.1.4 ***(?):*** We showed $K_{\mathbb{P}^n} = \mathcal{O}(-n-1)$ where $\mathcal{O}(-1)$ was the tautological line bundle over $\mathbb{P}^n$. So for example $K_{\mathbb{P}^2} = \mathcal{O}(-3) = -3L$ where $L \in H^2(\mathbb{P}^2, \mathbb{Z})$ is a hyperplane (here a line) in $\mathbb{P}^2$.

Corollary 35.1.5 ***(Formula for genus of a curve in terms of degree).***
Let f be a degree d homogeneous polynomial in x, y, z, then $V(f) \subseteq \mathbb{P}^2 = \{[x : y : z]\}$. If $C := V(f)$ is a smooth complex curve, then applying the genus formula yields

$$2g(C) - 2 = (-3L + dL) \cdot dL.$$

Using that $L^2 = 1$, this equals $d(d-3)$ and thus

$$g(C) = \frac{d^2 - 3d + 2}{2} = \binom{d-1}{2}.$$

Example 35.1.6 ***(?):*** If $d = 3$ and say $f(x, y, z) = x^3 + y^3 + z^3$, then $V(f) \subseteq \mathbb{P}^2$ has genus $\binom{3-1}{2} = 1$. So this is diffeomorphic to a torus.

Example 35.1.7 ***(?):*** If $d = 2$ then $g(C) = 0$, so conics in $\mathbb{P}^2$ have genus zero, and we proved that every genus zero curve is isomorphic to $\mathbb{P}^1$. So conics in $\mathbb{P}^2$ are isomorphic to $\mathbb{P}^1$ (as are lines of course!).

Example 35.1.8 ***(?):*** If $d = 4$ then $g(C) = 3$

Theorem 35.1.9 ***(Harnack Curve Theorem).***
Noting that $\mathbb{RP}^2 \subset \mathbb{CP}^2 = \mathbb{P}^2$, the number n_C of connected components of a curve $C \cap \mathbb{RP}^2$ satisfies

$$n_C \leq 1 + g(C).$$

Remark 35.1.10: See the Trott curve:

$$144(x^4 + y^4) - 225(x^2 + y^2) + 350x^2y^2 + 81 = 0,$$

whose plot looks like the following:

```
f(x,y) = 12^2*(x^4 + y^4) - 15^2*(x^2 + y^2) + 350*x^2*y^2 + 81
implicit_plot(f, (x,-1,1), (y,-1,1))
```

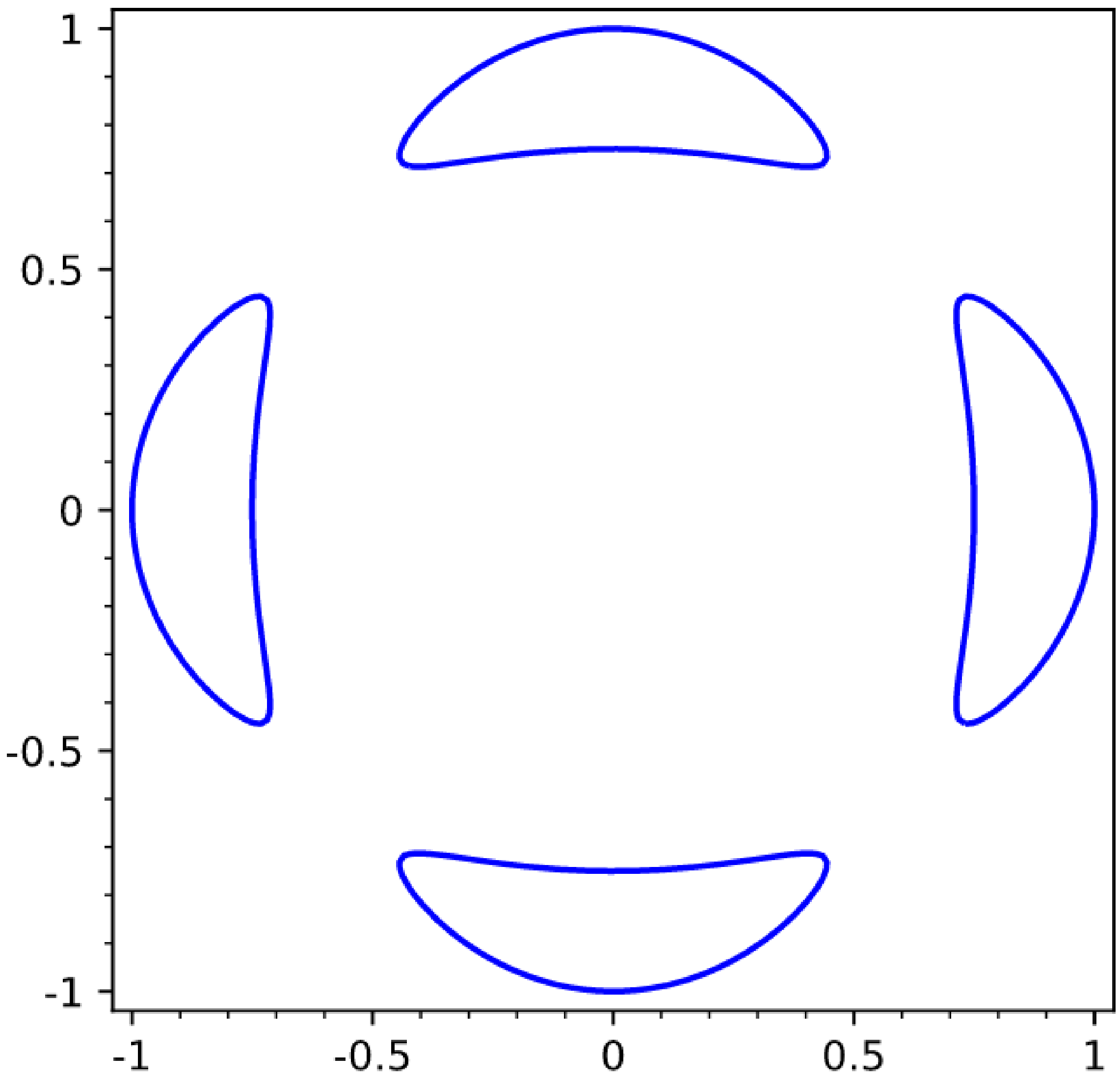
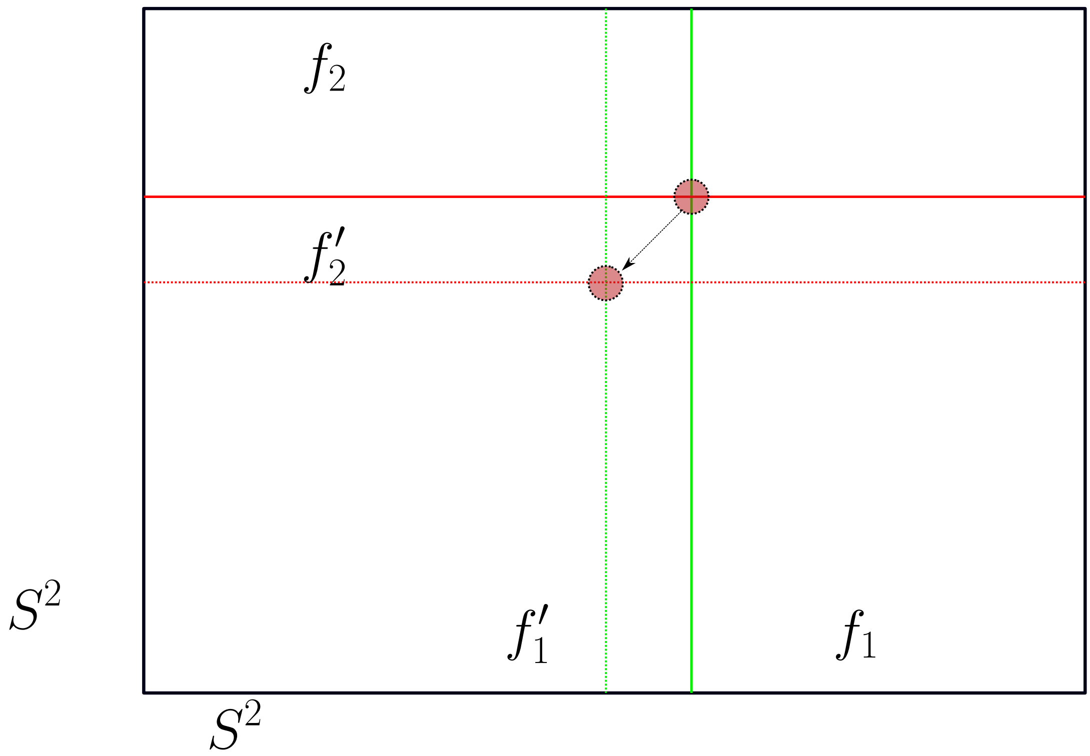


Figure 4: image_2021-04-09-16-40-49

Example 35.1.11 ***(?):*** Consider $S := \mathbb{P}^1 \times \mathbb{P}^1$, which is homeomorphic to $S^2 \times S^2$. The homology is given by $\mathbb{Z}$ in degrees 0 and 4, $\mathbb{Z}^{\oplus 2}$ in degree 3, and 0 elsewhere. What is the intersection form on $\mathbb{Z}^{\oplus 2} = H^2(\mathbb{P}^1 \times \mathbb{P}^1; \mathbb{Z})$? The two generators are $f_1 = [S^2 \times \mathrm{pt}], f_2 = [\mathrm{pt} \times S^2]$. We can compute

- $f_1 \cdot f_1 = 0$
- $f_1 \cdot f_2 = 1$
- $f_2 \cdot f_2 = 0$

This is because we can perturb these to be transverse:



Since $f_2 \cap f_2' = \emptyset$, we have $f_2 \cdot f_2' = f_2 \cdot f_2 = 0$, and similarly with 1. So the Gram matrix is

$$G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

So setting $C = \mathbb{P}^1 \times \mathbb{P}^1 = V(f_{2,3})$, a function of bidegree $(2,3)$, writing the coordinates as $[x : y], [z : w]$, we can write this as $x^2 z^3 + y^2 z^2 w + xyw^3 = 0$. We get

$$2g(C) - 2 = (K_{\mathbb{P}^1 \times \mathbb{P}^1} + 2f_1 + 3f_2) \cdot (2f_1 + 3f_2) = f_2 \cdot (2f_1 + 3f_2) = 2,$$

since $K_{\mathbb{P}^1 \circ} = -2f_1 - 2f_2$ and so $g(C) = 2$.

36 | Friday, April 09

Remark 36.0.1: Recall the adjunction formula: for $D \subset X \in \mathsf{Mfd}_{\mathbb{C}}$ a codimension 1 complex submanifold, we have

$$K_D = (K_x + \mathcal{O}_x(0))\,|_D.$$

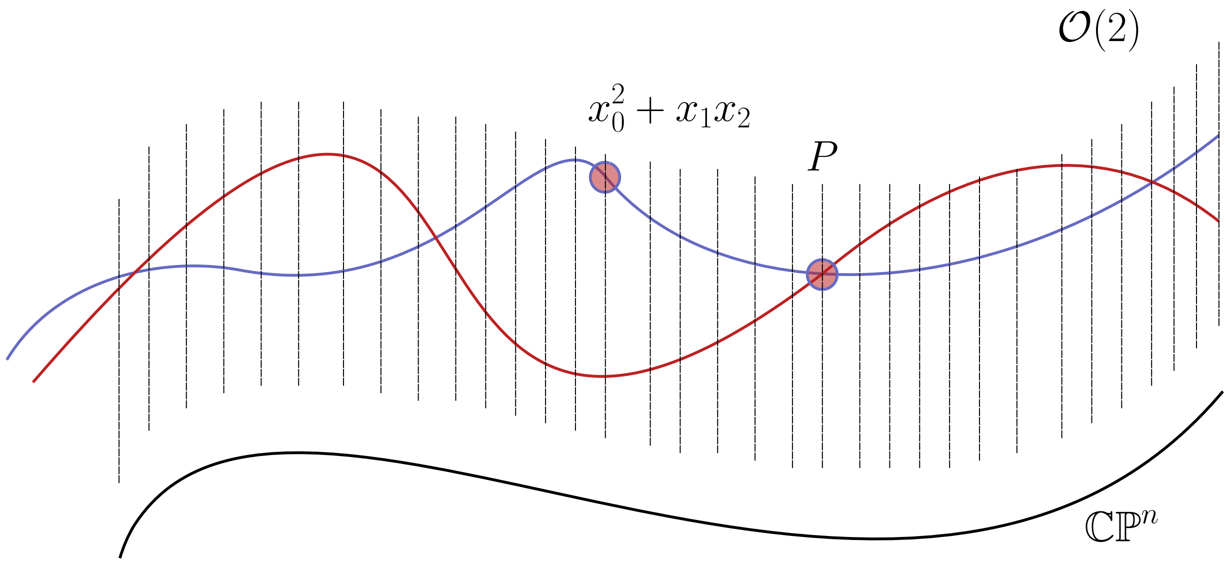
We'll apply this to curves C in a surface S. Recall the genus formula, which was given by $2g(C) - 2 = (C + K_S) \cdot C$. For example, a degree 4 equation in $\mathbb{P}^2$ carves out a genus $g(C) = 3$ complex curve.

Remark 36.0.2: Recall that line bundles on $\mathbb{CP}^n$ were in bijection with $\mathbb{Z}$, where send d to a bundle $\mathcal{O}(d) := \mathcal{O}_{\mathbb{CP}^N}(d)$. We produced the tautological line bundle $\mathcal{O}(-1)$ whose fiber over $\mathbf{x} \subseteq \mathbb{CP}^n$ is the line in $\mathbb{C}^n$ spanned by its coordinates. We have $\mathcal{O}(-1)^\vee := \mathcal{O}(1)$, and $\mathcal{O}(n) := \mathcal{O}(1)^{\otimes n}$. Alternatively, it was characterized in terms of homogeneous functions, where the fiber $\mathcal{O}(n)_{\mathbf{x}}$ are the linear functions L on lines $\{\lambda \mathbf{x}\} \to \mathbb{C}$ such that $L(\lambda p) = \lambda^n L(p)$. Noting that these are linear functions, such L form a 1-dimensional $\mathbb{C}$-vector space.

Example 36.0.3 ***(K3 Surfaces):*** The classic example is $x_0 \in \mathcal{O}(1)_{\mathbf{x}}$ since $x_0(\lambda p) = \lambda x_0(p)$. Similarly, $x_0^2 + x_1 x_2 \in \mathcal{O}(2)_{\mathbf{x}}$ since

$$x_0^2 + x_1 x_2(\lambda p) = \lambda^2(x_0^2 + x_1 x_2(p)).$$

Remark 36.0.4: Note that the global sections were given by $\Gamma^0(\mathbb{P}^n, \mathcal{O}(d)) = H^0(\mathbb{P}^n; \mathcal{O}(d))$ was the span of degree d monomials in $x_0, \cdots, x_n$. For example $x_0^2 + x_1 x_2$ is a well-defined element of $\mathcal{O}(2)_p$ which varies holomorphically with p, yielding a section:



Example 36.0.5 ***(?):*** For a K3 surface, consider $S = \left\{ \sum_{i=0}^4 x_i^4 = 0 \right\} \subset \mathbb{CP}^3$. By the adjunction formula,

$$K_S = (K_{\mathbb{CP}^3} \otimes \mathcal{O}_{\mathbb{CP}^3}(S))|_S.$$

Note that if $s \in H^0(\mathcal{L})$, we can recover $\mathcal{O}(\operatorname{Div} S) = \mathcal{L}$. Moreover, $K_{\mathbb{CP}^3} = \mathcal{O}(-4)$ and $\mathcal{O}_{\mathbb{CP}^3}(S) = \mathcal{O}(4)$ since we can view the formula as a function on the tautological line, which yields a section. So we get $K_S = \mathcal{O}(-4) \otimes \mathcal{O}(4) = \mathcal{O}(0) = \mathcal{O}$, i.e. these yield actual functions on $\mathbb{CP}^n$ since they're products of functions that scale by λ^{-4} and functions that scale by λ^4. We're using the fact that $\mathcal{O}_{\mathbf{p}=[x_0:\cdots:x_n]}$ are functions L such that $L(\lambda p) = \lambda^0 L(p) = L(p)$, which yields a well-defined function on $\mathbb{CP}^n$. So quartics in $\mathbb{P}^3$ have trivial canonical bundle, i.e. $K_S = \mathcal{O}_S$ for $S = V(x_0^4 + x_1^4 + x_2^4 + x_3^4)$.

Remark 36.0.6: We know that $H^0(S, K_S)$ are the globally holomorphic 2-forms on S, and here this is isomorphic to $H^0(S, \mathcal{O}_S) = \mathbb{C}\Omega_S$ for some single holomorphic 2-form. Moreover $\operatorname{Div}(\Omega_S) = 0$ since $\mathcal{O}(\operatorname{Div}(\Omega_S)) = K_S = \mathcal{O}_S$. So these are the analogs of elliptic curves in dimension 2, since for example $E := \mathbb{C}/\Lambda$ has a nonvanishing section $dz \in H^0(E, K_E)$, and we can write $E = V(f)$ for f a cubic in $\mathbb{P}^3$, and we computed the genus of cubics. Moreover, every genus 1 curve is $\mathbb{C}$ mod a lattice.

Remark 36.0.7: Recall an exercise from the notes: computing the Hodge diamond of a genus 5 curve. We'll compute the diamond for a K3 surface:

$$\begin{array}{ccccc}
 & & h^{2,2} & & \\
 & h^{3,1} & & h^{1,3} & \\
h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & h^{1,0} & & h^{0,1} & \\
 & & h^{0,0} & &
\end{array}$$

Link to Diagram

We know $h^{2,0} = H^0(S, \Omega_S^2)$, which yields 1s:

$$\begin{array}{ccccc}
 & & 1 & & \\
 & h^{3,1} & & h^{1,3} & \\
1 & & h^{1,1} & & 1 \\
 & h^{1,0} & & h^{0,1} & \\
 & & 1 & &
\end{array}$$

Link to Diagram

We'll use the following theorem:

Theorem 36.0.8 ***(Lefschetz Hyperplane Theorem).***
Let $X \subset \mathbb{P}^n$ with $\dim X > 3$. Then $\pi_1(X) \cong \pi_1(X \cap H)$ for H a hypersurface intersection X at a smooth codimension 1 complex manifold.

Remark 36.0.9: Applying this to $X = \mathbb{P}^3$, we have $V(x_0^4 + \cdots + x_3^4) = S$, we have $\pi_1(\mathbb{P}^3) = \pi_1(S)$. We can write $\mathbb{P}^3 = \mathbb{C} \cup \mathbb{C}^2 \cup \mathbb{C}^4$, which is a cell decomposition with cells only in degrees 0,2,4, and so in fact $\pi_1(\mathbb{P}^n) = 0$.

Corollary 36.0.10 ***(h1 of K3 surfaces).***
K3 surfaces are simply connected, and $h^1(S; \mathbb{C}) = 0$.

Note that anything embedded in projective space as a complex submanifold is Kähler by restricting the Fubini-Study form. Using simple connectedness and Serre duality, we have

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & h^{1,1} & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Link to Diagram

We know $\chi(\mathcal{O}_S) = (1/12)(K^2 + \chi_{\mathsf{Top}})$, and since $K_S = \mathcal{O}_S$ is trivial, we have $c_1(\mathcal{O}_S) = 0$. Noting that $h^{p,q} = H^(\Omega^p)$, so we can sum the lower-right part of the diamond to get $\chi(\mathcal{O}_S) = 1 - 0 + 1 = 2$, since we take $p = 0$ to get $\Omega^p = \mathcal{O}$. Computing χ_{Top}, we get $h_{1,1} = 20$.

37 | Monday, April 12

Remark 37.0.1: Last time: the Lefschetz hyperplane theorem. Intersecting a projective variety of dimension $d \geq 3$ with a hypersurface S, the map $\pi_1(\mathbb{P}^3) \to \pi_1(S)$ is an isomorphism. We saw that K3 surfaces were thus simply connected, and $h^1(S; \mathbb{C}) = 0$, so we could compute the Hodge diamond.

Example 37.0.2 ***(?):*** What is the Hodge diamond for a cubic surface $S \subseteq \mathbb{P}^3$, such as $\sum x_i^3 = 0$? We first need to compute the canonical bundle K, for which we have a useful tool: the adjunction formula. This say $K_S = (K_{\mathbb{P}^3} \otimes \mathbb{P}_{\mathbb{P}^3}(S))\,|_S = (\mathcal{O}(-4) \otimes \mathcal{O}(3))\,|_S = \mathcal{O}(-1)|_S$.

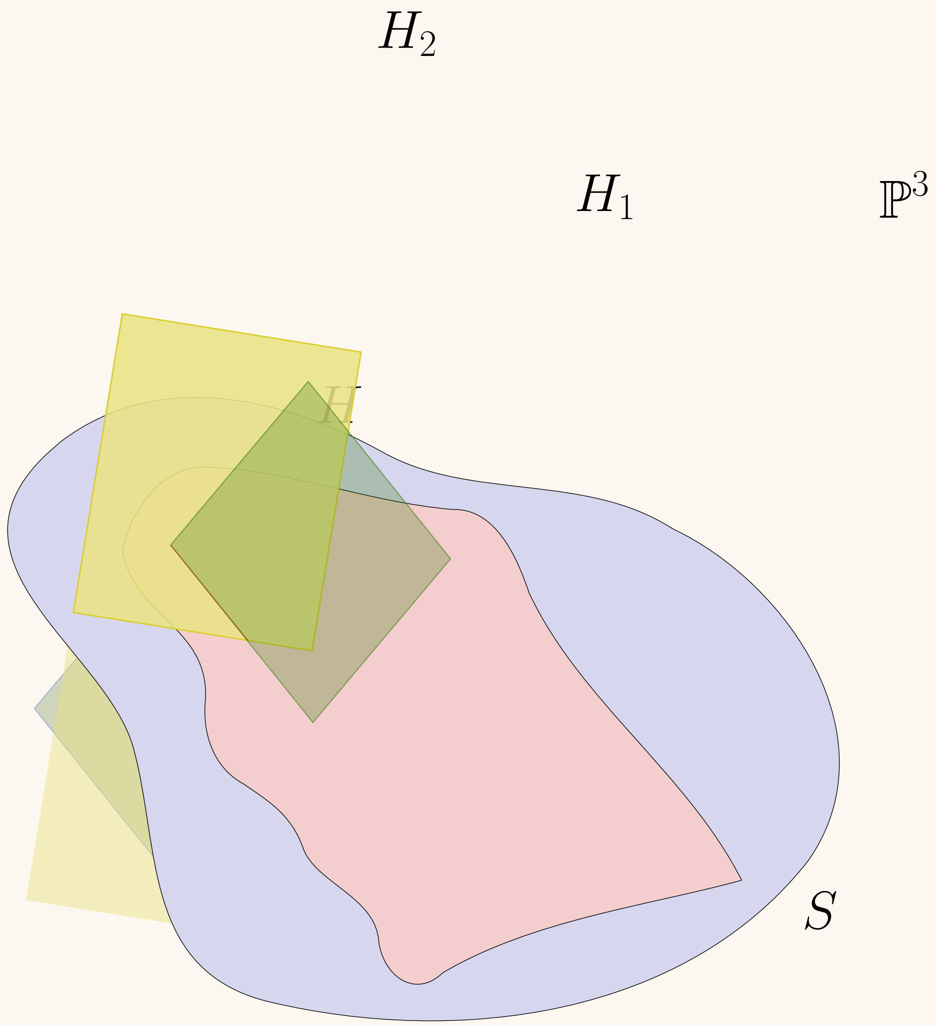
Proposition 37.0.3 ***(If a holomorphic line bundle has a section, its inverse doesn't).***
Let $\mathcal{L} \to X$ be a holomorphic line bundle. If $h^0(\mathcal{L}^{-1}) > 0$, then either $\mathcal{L} = \mathcal{O}$ or $h^0(\mathcal{L}) = 0$.

Slogan 37.0.4
If a line bundle has a section, its inverse does not.

Proof (?).
Suppose that both $\mathcal{L}, \mathcal{L}^{-1}$ have a section, so $h^0(\mathcal{L}), h^0(\mathcal{L}) > 0$. Let s, t be sections of each, then $st \in H^0(\mathcal{L} \otimes \mathcal{L}^{-1}) = H^0(\mathcal{O}) = \mathbb{C}$. So taking zero loci yields $\operatorname{Div}(s) + \operatorname{Div}(t) = 0$ Writing these as $\operatorname{Div}(s) \coloneqq \sum n_D D, \operatorname{Div}(t) \coloneqq \sum n_C C$, we have $n_D, n_C \geq 0$, which implies that $\operatorname{Div}(s) = \operatorname{Div}(t) = 0$. So s, t are nowhere vanishing, making $\mathcal{O} \xrightarrow{\cdot s} \mathcal{L}$ is an isomorphism. ■

Corollary 37.0.5 ***(H0 of cubic surfaces).***
For S a cubic surface, $H^0(S; K_S) = 0$.

Proof (?).
This follows because $K_S = \mathcal{O}_S(-1)$, so $K_S^{-1} = \mathcal{O}_S(1)$ which has a nontrivial section: namely $\mathcal{O}_{\mathbb{CP}^1}(1)$ which has sections vanishing along hyperplanes.



Letting H be a hyperplane containing S, there exists an $f \in H^0(\mathbb{P}^3; \mathcal{O}_{\mathbb{CP}^3}(1))$. Since $\mathrm{Div}(f) = H$, the restriction $f|_S$ is a section of $\mathcal{O}_S(-1) = K_S^{-1}$ which is not identically zero and vanishes along $H \cap S$.

■

We now know $h^0(S; K_S) = 0$, and this equals $h^0(S, \Omega^2) = h^{2,0}(S)$, so we have the following Hodge diamond:

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
0 & & h^{1,1} & & 0 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

Link to Diagram

We have $h^{0,1} + h^{1,0} = h^1 = 0$ since S is simply connected. We can now apply Noether's formula as before: $\chi(\mathcal{O}_S) = \frac{1}{12}(K_S^2 + \chi_{\mathsf{Top}}(S))$. We have $K_S = \mathcal{O}_S(-1)$, so $K_S^2 = c_1(\mathcal{O}(-1))^2$, and $\chi(\mathcal{O}_S) = 1 - 0 + 1 = 1$. We now want to compute $\int_S (-c_1(\mathcal{O}_S(1)))^2$. We know $c_1(\mathcal{L}) = [\operatorname{Div} s]$ where $s \in H^0(\mathcal{L})$ is a section of a line bundle. This equals $[H \cap S]$. On the other hand, $\int_S c_1 (\mathcal{O}_S(1))^2$ is the self-intersection number of $H \cap S$.

Take $H_1 \coloneqq \{x_0 = 0\}$ and $H_2 \coloneqq \{x_1 = 0\}$. Points in this intersection are of the form $[0 : 0 : 1 : \zeta_6^a]$ where $a = 1, 3, 5$ since this is in the triple intersection $H_1 \cap H_2 \cap S$. So there are exactly 3 points here, and in fact $\deg S = 3$. This is the same as integrating $\int_{\mathbb{P}^3} c_1(S) c_1(\mathcal{O}(1)) c_1(\mathcal{O}(2))$, which contains 3 elements in H^2 and lands in H^6, so this yields a number.

We thus have $K_S = \mathcal{O}_S(-1) \coloneqq \mathcal{O}_{\mathbb{CP}^3}(-1)|_S$. Thus $\chi_{\mathsf{Top}}(S) = 9$ and $h^{1,1} = 7$.

Example 37.0.6 ***(Hypersurfaces):*** Note that a degree 5 surface (a quintic) such as $x_0^5 + x_3^5 = 0$ would be harder, since $h^{2,0} \neq 0$. We would get $K_S = \mathcal{O}(-4) \otimes \mathcal{O}(5)|_S = \mathcal{O}_S(1)$, and there are nontrivial sections so $h^0(K_S) = \operatorname{span} x_0, x_1, x_2, x_3$. This follows because there is a map given by restriction which turns out to be an isomorphism

$$
\begin{gathered}
0 \to H^0(\mathbb{P}^3; \mathcal{O}(1)) \xrightarrow{\operatorname{res}_S} H^0(S; \mathcal{O}(1)) \to 0 \\
f \mapsto f|_S.
\end{gathered}
$$

Injectivity isn't difficult, surjectivity is harder. We have a SES

$$
0 \to \mathcal{O}_{\mathbb{CP}^3}(-S) \to \mathcal{O}_{\mathbb{CP}^3} \to \mathcal{O}_S \to 0.
$$

Tensor all of these with $\mathcal{O}(1)$ to obtain

$$0 \to \mathcal{O}_{\mathbb{CP}}^3(-4) \to \mathcal{O}_{\mathbb{CP}^3}(1) \to \mathcal{O}_S(1) \to 0.$$

Taking the associated LES yields

$$\begin{tikzcd}
{} \arrow[r] & H^1(\mathcal{O}_{\mathbb{CP}^3}(-4)) =_? 0 & & \\
& H^0(\mathcal{O}_{\mathbb{CP}^3}(-4)) = 0 \arrow[r] & H^0(\mathcal{O}_{\mathbb{CP}^3}(-1)) = \mathbb{C}\langle x_0, x_1, x_2, x_3 \rangle \arrow[r] & {} \\
& 0 \arrow[u] & &
\end{tikzcd}$$

Link to Diagram

This gives us a way to relate things back to the cohomology of $\mathbb{CP}^3$. Showing that the indicated term is zero involves computing Čech cohomology.

It turns out that $h^0(K_S) = 4$ here, and it turns out that the Hodge diamond is the following:

$$\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
4 & & h^{1,1} = 45 & & 4 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}$$

Link to Diagram

Here $K_S^2 = c_1(\mathcal{O}_S(1))^2 = 5$ and $\chi_{\mathsf{Top}} = 55$.

Example 37.0.7 ***(Products):*** Consider now a product of curves $C \times D$ of genera g, h respectively. Computing the Hodge diamond is easy here due to the Kunneth formula:

$$H^k(S; \mathbb{C}) = \bigoplus_{i+j=k} H^i(C; \mathbb{C}) \otimes H^j(D; \mathbb{C}).$$

What is the actual map? Take cohomology classes $[\alpha], [\beta]$, closed i and j forms respectively. The surface has two maps:

$$\begin{tikzcd}
S \arrow[r, "\pi_C"] \arrow[d, "\pi_D"'] & C \\
D &
\end{tikzcd}$$

Here we send $[\alpha] \otimes [\beta] \mapsto [\pi_C^* \alpha \wedge \pi_D^* \beta]$ where we take pullbacks. Note that π_D, π_C are holomorphic maps, and pullbacks of (p, q) forms are still (p, q) forms. Thus the Kunneth formula gives a decomposition

$$H^{p,q}(S; \mathbb{C}) = \sum_{\substack{i_1 + j_1 = p \\ i_2 + j_2 = q}} H^{i_1, j_1}(C) \oplus H^{i_2, j_2}(D).$$

So we can "tensor" the Hodge diamonds:

$$\begin{array}{ccc}
 & 1 & \\
g & & g \\
 & 1 &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & 1 & \\
h & & h \\
 & 1 &
\end{array}$$

$$\begin{array}{ccccc}
 & & 1 & & \\
 & g+h & & g+h & \\
gh & & 2+2gh & & gh \\
 & g+h & & g+h & \\
 & & 1 & &
\end{array}$$

Link to Diagram

Remark 37.0.8: Check out complete intersections.

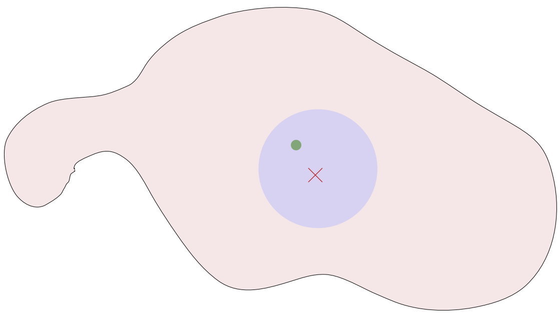
38 Blowups and Blowdowns (Wednesday, April 14)

Definition 38.0.1 (Blowup)
Let $S \in \mathsf{Mfd}_{\mathbb{C}}^2$ be a complex surface and $p \in S$ a point, and let (x, y) be local holomorphic coordinates on a neighborhood of U containing p. Without loss of generality, $p = (0,0)$ in these coordinates. Set $U^* := U \setminus \{p\}$, and consider the holomorphic map

$$\begin{aligned} \varphi : U^* &\to U \times \mathbb{CP}^2 \\ (x, y) &\mapsto ((x, y), [x : y]). \end{aligned}$$

We'll define the **blowup at** p to be $\underset{p}{\mathrm{Bl}}(U)\mathrm{cl}(\varphi(U^*))$ to be the closure of the image of U^*.

Observation 38.0.2
There is a map $\underset{p}{\mathrm{Bl}}(U) \to U$ given by projection onto the first coordinate which is the identity on U^*.



Here q maps to the pair (q, s) where s is the slope of a line through q, and this will be continuous.

? Missed part

We claim that $\pi_U^{-1}(0,0) \subset \underset{p}{\mathrm{Bl}}(U) = \{p\} \times \mathbb{CP}^1$, and for a fixed $9x_0, y_0) \in U^*$, considering $\varphi(x_0 t, y_0 t)$

as $t \to 0$, we can write

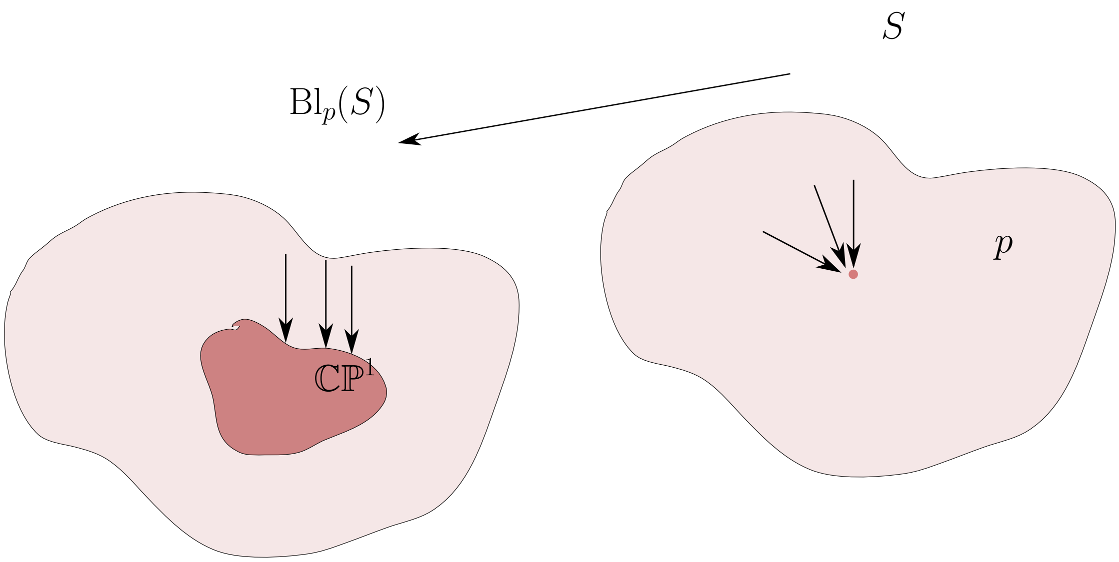
$$
\begin{aligned}
&((x_0 t, y_0 t), [x_0 : y_0]) \in U \times \mathbb{CP}^1 \\
&\overset{t \to 0}{(} \;(0,0)[x_0 : y_0]) \subset \mathrm{cl}(\varphi(U^*)).
\end{aligned}
$$

So approaching $(0,0)$ along any slope s just yields the point $(0,s)$ in the blowup.

Remark 38.0.3: We can thus write

$$\underset{p}{\mathrm{Bl}}\, S S \setminus \{p\} \coprod_{U^*} \underset{p}{\mathrm{Bl}}\, U.$$

Writing $\pi : \underset{p}{\mathrm{Bl}}\, S \to S$, we have $\pi^{-1}(p) \cong \mathbb{CP}^1$ and $\pi^{-1}(q)$ is a point for all $q \neq p$. Then all limits approaching p in S turn into distinct limit points in $\underset{p}{\mathrm{Bl}}(S)$

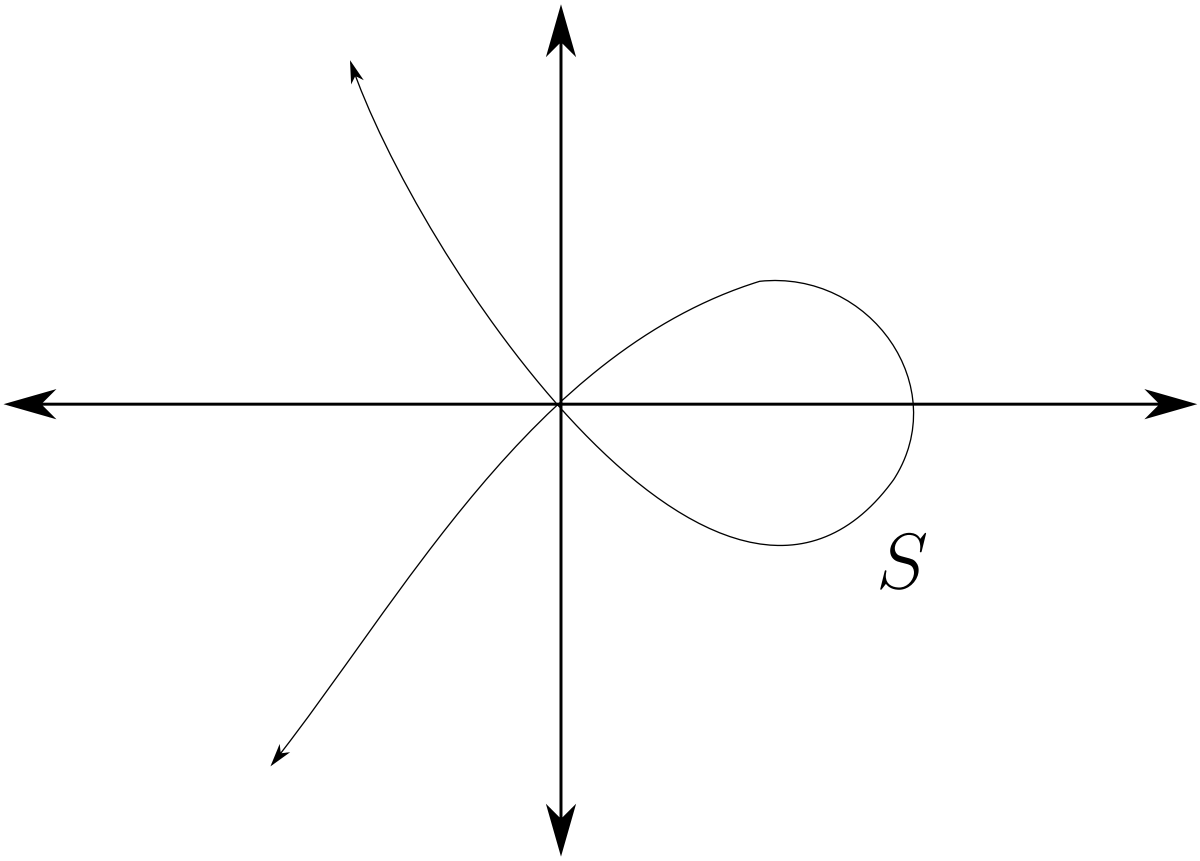


Slogan 38.0.4
The blowup separates all tangent directions at p.

Example 38.0.5 ***(?):*** Consider

$$\left\{ y^2 = x^3 - x^2 \right\} \subseteq \mathbb{C}^2.$$

This yields a nodal curve with a double-point:

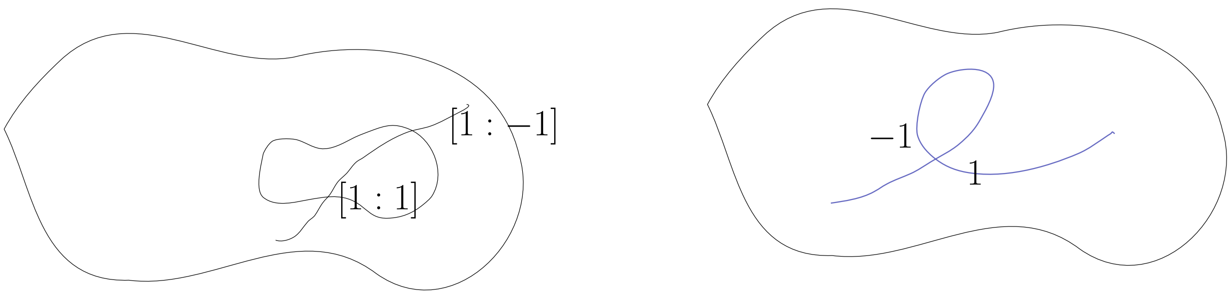


Here we'll consider $\operatorname*{Bl}_{(0,0)} \mathbb{C}^2$.

Definition 38.0.6 ***(Strict Transform)***
Letting $C \subset S$ be a curve, define the **strict transform**

$$\widehat{C} := \operatorname{cl}(\pi^{-1}(C \setminus \{p\})).$$

Note that approaching by different sequences yields different limiting slopes



The curve in the blowup is called the **exceptional divisor**.

Example 38.0.7*(?):* Consider all lines in $\mathbb{CP}^2$ through $[0:0:1]$, which we can model in the following way:

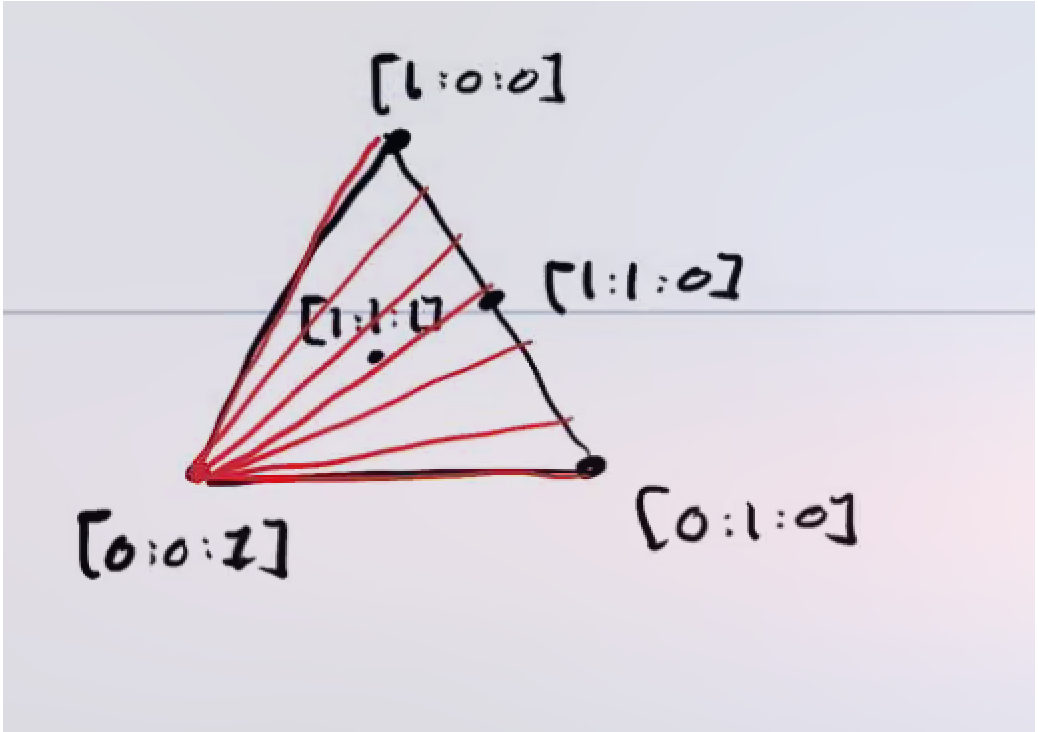


Figure 5: image_2021-04-14-14-18-15

These are in bijection with $\mathbb{CP}^1$ since there is always a unique line through $[0:0:1]$ and $[s:t:0]$, where the latter is a copy of $\mathbb{CP}^1$ as s,t are allowed to vary. So consider $\underset{p}{\mathrm{Bl}}\, \mathbb{CP}^2$ for $p = [0:0:1]$, and consider the strict transforms of the lines L to obtain $\widehat{L} \subset \underset{p}{\mathrm{Bl}}\, \mathbb{CP}^2$. Any two are disjoint since they pass through different slopes of the exceptional divisor. Thus the red lines in the blowup go through distinct slopes, yielding a fibration of $\mathbb{CP}^1$s:

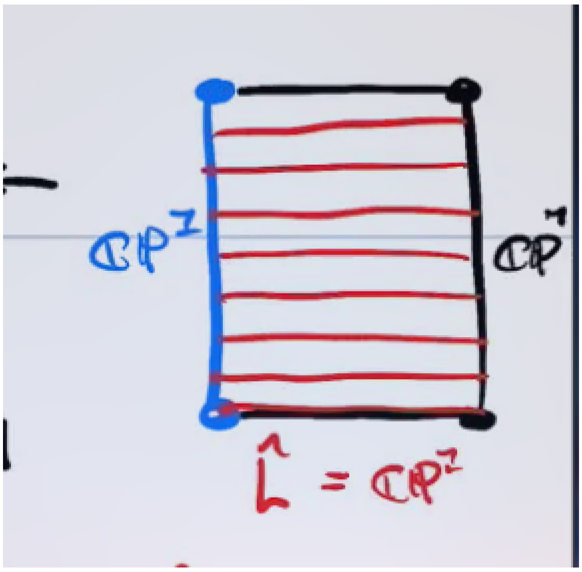
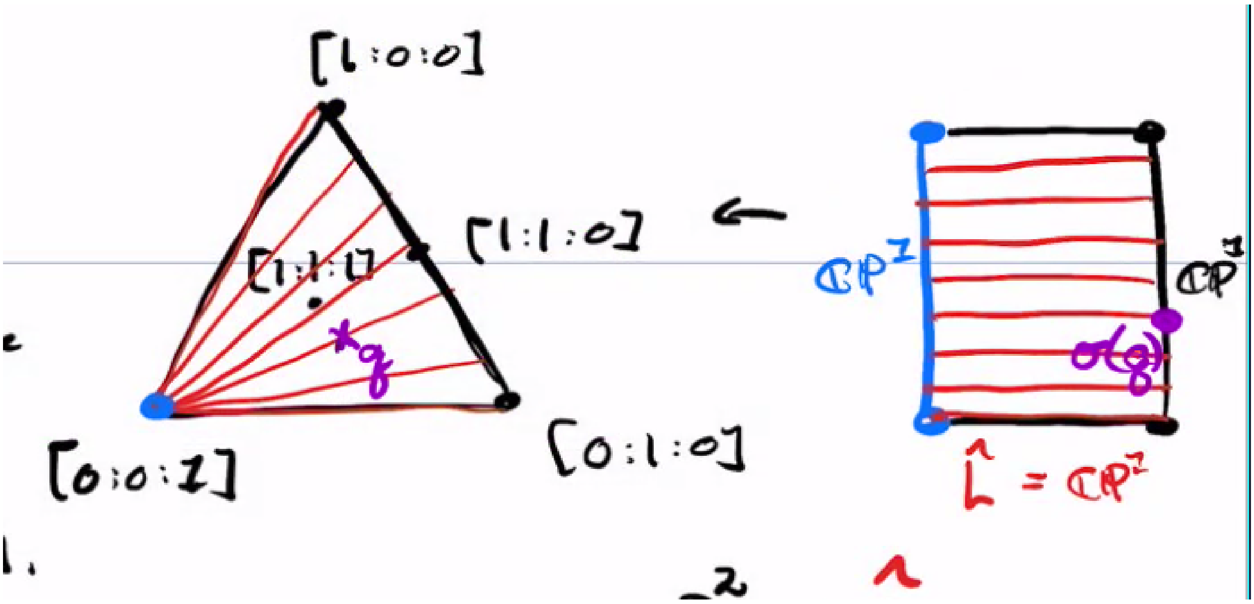


Figure 6: image_2021-04-14-14-24-31

So consider the map

$$\begin{aligned} \sigma : \operatorname*{Bl}_p \mathbb{CP}^2 &\to \mathbb{CP}^2 \\ p \in \widehat{L} &\mapsto [0 : s : t]. \end{aligned}$$

which projects points to the boundary copy of $\mathbb{CP}^1$:



We can't necessarily project from the blue point itself, but if we add in the data of a tangent vector at that point, the map becomes well-defined. Thus the blowup makes projecting from a point in $\mathbb{CP}^2$ to a line in $\mathbb{CP}^2$ a well-defined map on $\operatorname{Bl}\mathbb{CP}^2$.

Remark 38.0.8: This is referred to as $\mathbb{F}_1$, the **first Hirzebruch surface**.

Proposition 38.0.9 ***(Blowup for smooth manifolds is connect-sum with CP2).***
For $S \in \mathsf{Mfd}_{\mathbb{R}}(C^\infty)$ a smooth manifold, we can identify

$$\operatorname*{Bl}_p S = S \# \overline{\mathbb{CP}^2}.$$

Proof (?).
It suffices to work in coordinate charts and prove this for $p = 0$.

Claim:

$$\operatorname*{Bl}_0 \mathbb{C}^2 = \operatorname{Tot}(\mathcal{O}_{\mathbb{CP}^1}(-1)).$$

Recall that this was the tautological line bundle that whose fibers at a point $p \in \mathbb{CP}^1$ was the line in $\mathbb{C}^2$ spanned by p. We can write this as $\left\{ [x:y] \;\middle|\; (x,y) \in L_{[x:y]} \right\}$:

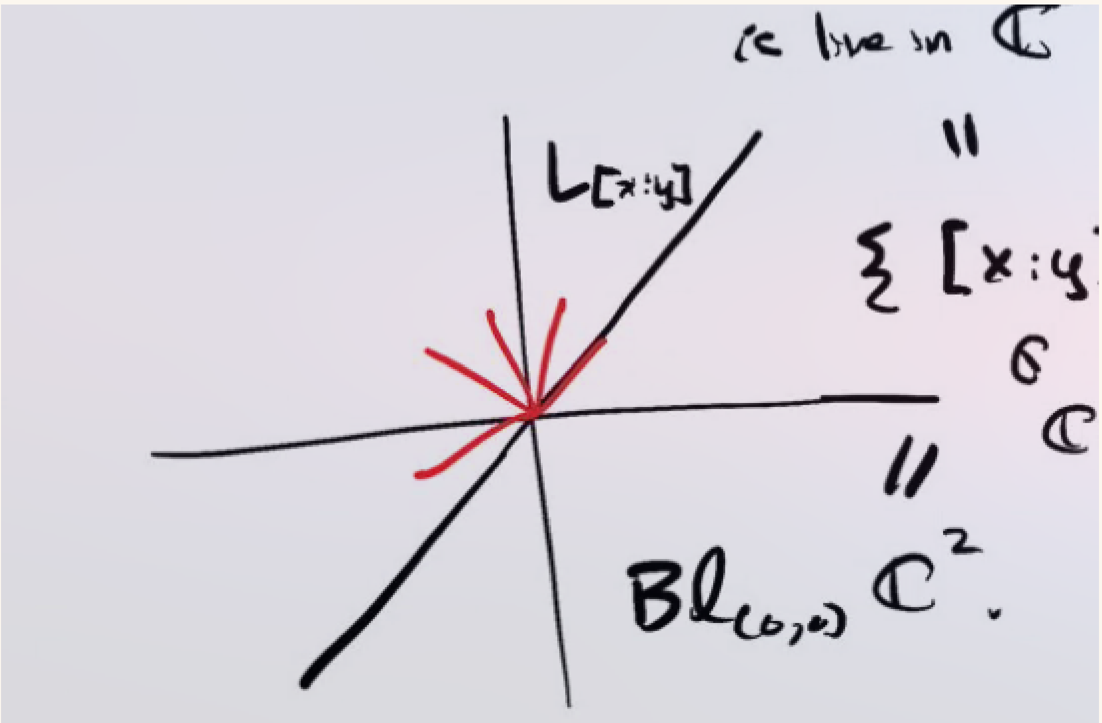


Figure 7: image_2021-04-14-14-32-58

We have $\mathcal{O}(-1) \xrightarrow{\sim} \overline{\mathcal{O}(1)}$, where this map is a diffeomorphism that can be constructed using a Hermitian metric. However we can identify $\mathcal{O}(1)$ with the set of lines in $\mathbb{CP}^2$ through $[0:0:1]$, leaving out the point $[0:0:1]$ itself. This follows by checking that there exists a section that vanishes at only one point. In fact $\operatorname{Tot}\mathcal{O}(1)$ is diffeomorphic to the complement of a ball in $\mathbb{CP}^2$, which ends up precisely being taking a connect-sum. So we obtain $\operatorname*{Bl}_0 \mathbb{C}^2 \cong \mathbb{C}^2 \# \overline{\mathbb{CP}^2}$. ■

Proof (Alternative).
Cut out a ball $B^4 \subseteq \mathbb{C}^2$, so $\partial B^4 = S^3 = \left\{ |x|^2 + |y|^2 = \varepsilon \right\}$. Then $\operatorname*{Bl}_0 \mathbb{C}^2$ is the result of collapsing S^3 along an S^1-foliation $(e^{i\theta}x, e^{i\theta}y)$. This has an S^2 quotient, yielding the Hopf fibration

$$S^1 \hookrightarrow S^3 \to S^2.$$

■

Exercise 38.0.10 (?)
Show that the blowup over $\mathbb{R}$ is gluing in a mobius strip.

See the Tate curve!

39 | Friday, April 16

Remark 39.0.1: Last time: we defined the blowup $\operatorname*{Bl}_0 \mathbb{C}^2$ as the closure of

$$\operatorname*{Bl}_0 \mathbb{C}^2 \coloneqq \operatorname{cl} \left\{ (x,y), [x:y] \;\middle|\; (x,y) \neq 0 \right\} \subseteq \mathbb{C}^2 \times \mathbb{CP}^2.$$

This had the effect of adding in all limits of slopes as points approach $(0,0) \in \mathbb{C}^2$. We defined this using local holomorphic coordinate charts to $\mathbb{C}^2$. Why is this a complex manifold? We can cover it with charts: given a point (x, μ) where $\mu = \frac{y}{x} \in \mathbb{P}^1$ is a slope, we can form a first chart by sending

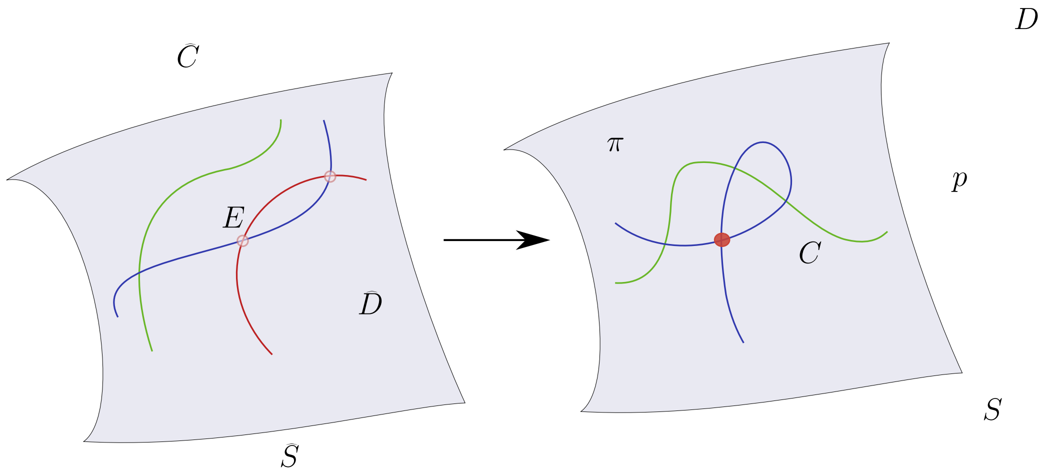
$$(x, \mu) \mapsto \{ (x, x\mu), [1 : \mu] \}.$$

This yields the first chart, as long as the slope is not infinite, so this applies to all finite slopes. The second chart will work for all nonzero slopes, where we take

$$(v, y) \in \mathbb{C}^2 \mapsto \{ (yv, y), [v : 1] \}.$$

Note that restricting to $(x,y) = (0,0)$, these give the standard $\mathbb{C}$-charts on $\mathbb{CP}^2$. How do these two charts glue? When $\mu, \nu \neq 0$, we have well-defined transition functions $\mu = \nu^{-1}$ and $x = y\nu$.

Remark 39.0.2: Recall that for a complex curve $C \in \mathsf{Mfd}^2_{\mathbb{C}}$, we have the blowup morphism $\pi : \operatorname*{Bl}_p S \to S$ and we defined the **strict transform** $\widehat{C} \coloneqq \operatorname{cl} \pi^{-1}(C \setminus \{\mathrm{pt}\})$.



Here $E = \mathbb{CP}^1$ is the exceptional curve of the blowup, and intersects the curve twice. This has the effect of changing D into an embedded curve.

Note that here $\pi^* D = \widehat{D} + 2E$, where we'll define this next.

Definition 39.0.3 (Pullback of a Curve)
The **pullback** of C, denoted $\pi^* C$, is constructed by writing $C = V(f)$ locally. We then set $\pi^* C \coloneqq V(\pi^* f)$.

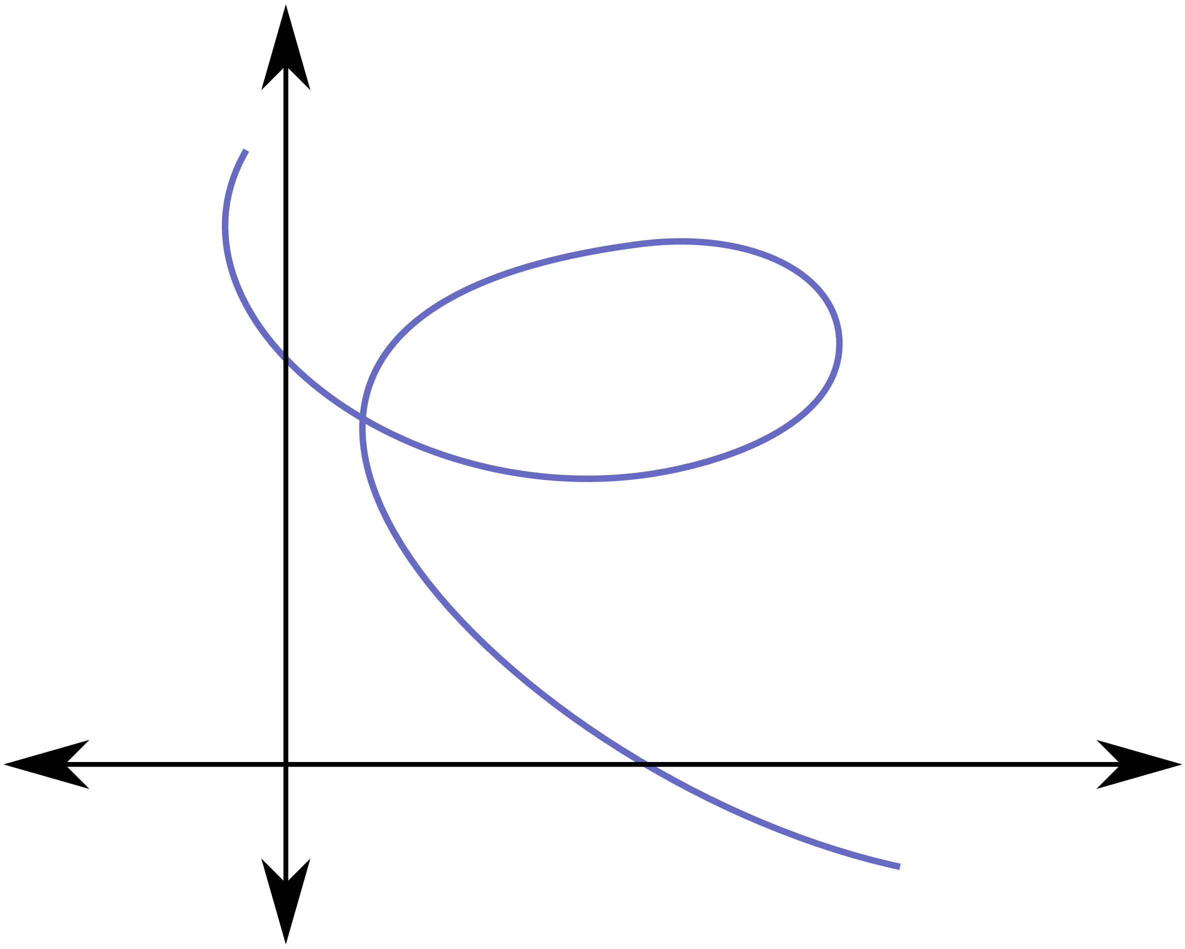
Example 39.0.4 ***(?):*** Take $C \coloneqq \{y = x\} \subset \mathbb{C}^2$ and consider $\underset{0}{\mathrm{Bl}}\,\mathbb{C}^2$. Then

$$\widehat{C} \coloneqq \mathrm{cl}\left\{((x,x),[x:x]) \;\middle|\; x \neq 0\right\} = \mathrm{cl}\left\{((x,x),[1:1]) \;\middle|\; x \neq 0\right\} \subset \underset{0}{\mathrm{Bl}}\,\mathbb{C}^2.$$

By projecting onto the first component, $\pi : \widehat{C} \xrightarrow{\sim} C$ is an isomorphism. We can compute the pullback: we first have $\pi^* C = \pi^* V(y-x) = V(\pi^*(y-x))$, so consider $\pi^*(y-x)$ in the coordinate chart (x, μ). In this chart, $y = x\mu$, and so $\pi^*(y-x) = x\mu - x = x(\mu - 1)$, and so

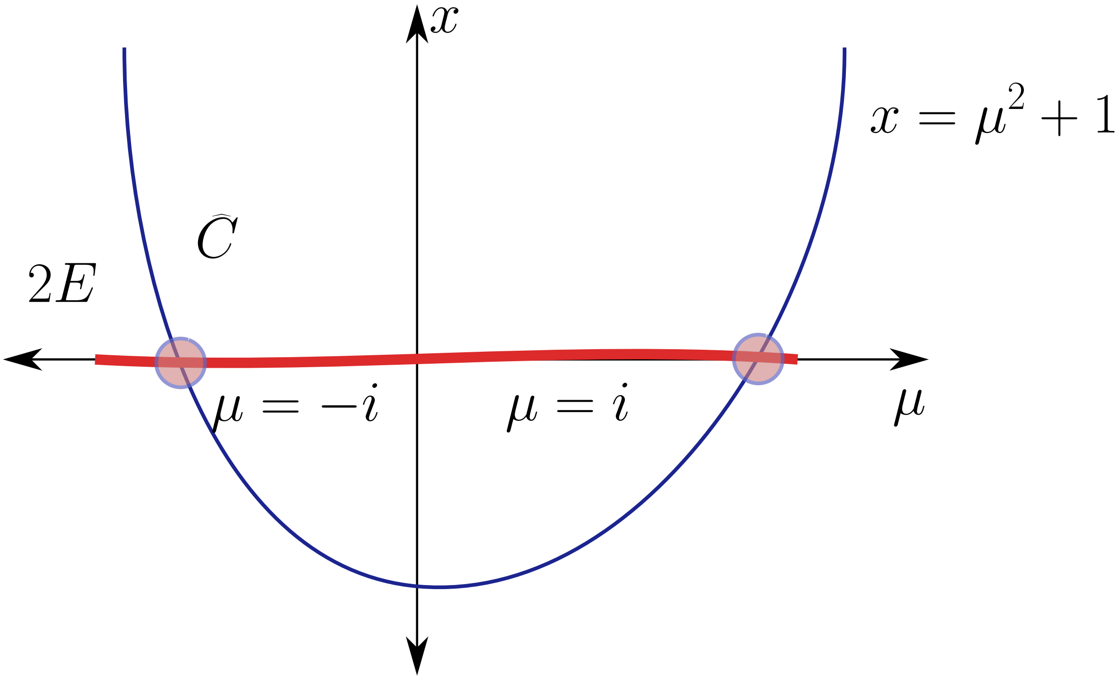
$$V(\pi^*(y-x)) = V(x) + V(\mu - 1) \implies \pi^* C = E + \widehat{C} \text{ as a divisor.}$$

Example 39.0.5 ***(A nodal curve):*** Take the nodal curve $C = \left\{y^2 - x^3 + x^2\right\}$:



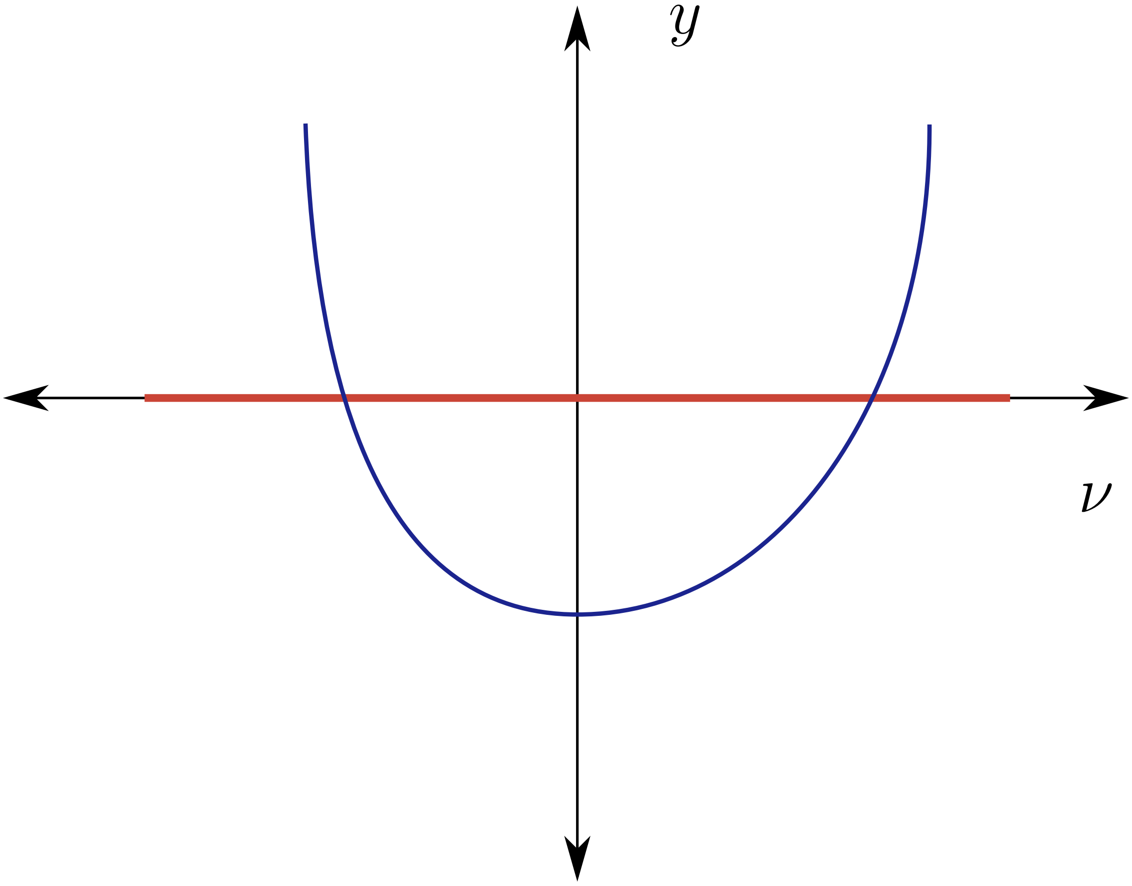
The pullback is then given by

$$\begin{aligned}\pi^* C &= V(\pi^*(y^2 - x^3 + x^2)) \\ &= V(\mu^2 x^2 - x^3 + x^2) \\ &= V(x^2) + V(\mu^2 - x + 1) \\ &= 2V(x) + V(\mu^2 - x + 1).\end{aligned}$$

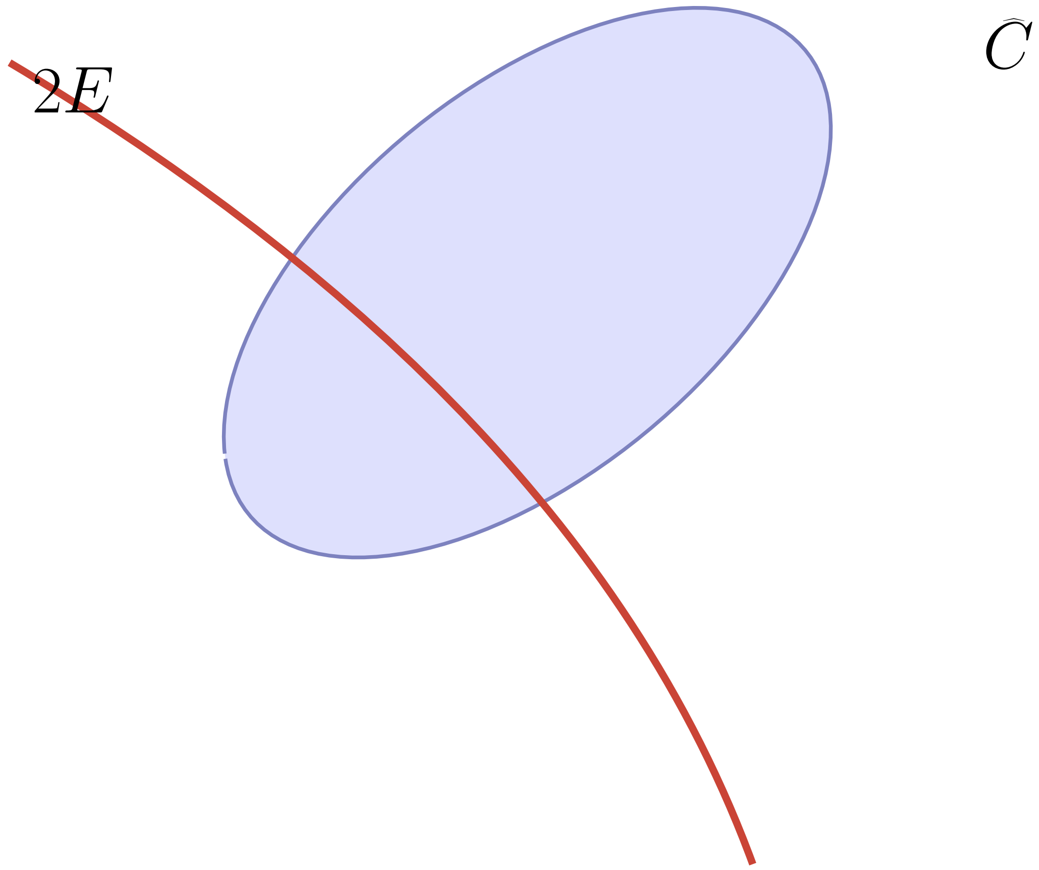


In the second coordinate chart, we have

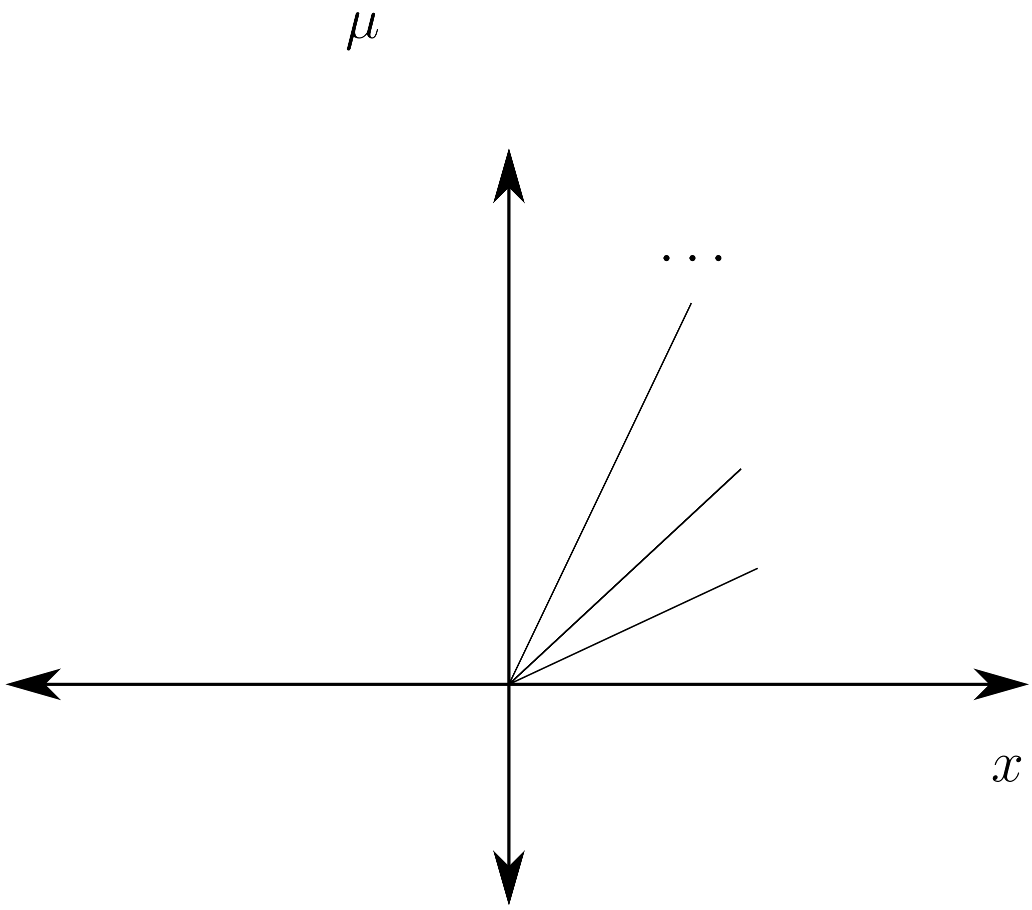
$$\pi^* C = V(y^2 - y^4\nu^3 + y^2\nu^2) = 2V(y) + V(1 - y\nu^3 + \nu^2.$$



Gluing along $\mu, \nu \neq 0$ we get the following picture for π^*C:



Writing $C = \{x = 0\}$, note that $\widehat{C}$ doesn't intersect the first coordinate chart. In the μ, x coordinate chart, for example, we can't get an infinite slop:



39.1 Change in Canonical Bundle Formula

Question 39.1.1
Given $\Omega_S^2 = K_S \to S$ the canonical line bundle, can we relate $K_{\mathrm{Bl}_p S}$ to K_S?

Proposition 39.1.2 ***(Canonical of a blowup).***

$$K_{\mathrm{Bl}_p S} = \pi^* K_S \otimes \mathcal{O}_S(E).$$

Proof (?).
We'll abbreviate $\widehat{S} := \underset{p}{\mathrm{Bl}}(S)$. Let ω be a local section of K_S near p, and in coordinate charts (x, y), write $\omega = dx \wedge dy$. In the first coordinate chart on the blowup, we can write

$$\pi^* \omega = dx \wedge d(x\mu) = dx \wedge (\mu dx + x d\mu) = x\, dx \wedge d\mu.$$

Note that $V(x) = E$, and that pulling back the canonical bundle yields something vanishing to order 1 (?). So $\pi^* K_S$ is isomorphic to the subsheaf of $K_{\widehat{S}}$ whose sections vanish along E, which is isomorphic to $K_{\widehat{S}} \otimes \mathcal{O}(-E)$, since the latter are the functions which vanish along E. Tensoring both sides with $\mathcal{O}(E)$ yields

$$K_{\widehat{S}} = \pi^* K_S \otimes \mathcal{O}_{\widehat{S}}(E)$$

as a line bundle, or in divisor notation $K_{\widehat{S}} = \pi^* K_S + E$ where we take the divisor representing the line bundle instead.

■

Remark 39.1.3: Using $\pi : \widehat{S} \to S$, we get pullback maps

$$\pi^* : H^2(S;\mathbb{Z}) \to H^2(\widehat{S};\mathbb{Z})$$
$$\pi^* : \mathrm{Div}(S) \to \mathrm{Div}(\widehat{S}).$$

These are compatible in the sense that

$$[\pi^* C] = \pi^*[C].$$

. This can be seen by expressing $\mathcal{O}_S(C) \cong \mathcal{O}_S(A - b)$ for A, B hyperplane section. We can assume A, B avoid p in their projective embeddings, making $[C] = [A] - [B]$ since $c_1(\mathcal{O}_S(c)) = [C]$ is the fundamental class of C. So it suffices to prove the formula for curves *not* passing through p, but this is obvious! It follows from the fact that $\pi : \widehat{S} \setminus E \xrightarrow{\sim} S \setminus \{p\}$ is an isomorphism.

Remark 39.1.4: In fact,

$$H^2(\widehat{S};\mathbb{Z}) \cong \pi^* H^2(S, \mathbb{Z}) \oplus \mathbb{Z}[E].$$

, which follows from Mayer-Vietoris. So this adds one to the rank.

40 | Monday, April 19

Remark 40.0.1: Recall that we have the following:

$$H^2(\widehat{S};\mathbb{Z}) = \pi^* H^2(S;\mathbb{Z}) \oplus \mathbb{Z}[E]$$

where E is the exceptional curve, which follows from Mayer-Vietoris. We can write $\widehat{S} = S \# \overline{\mathbb{CP}^2}$, and by excision $H^2(S \setminus \mathbb{B}^4) = H^2(S)$. So we get a LES

$$\begin{tikzcd}
H^3(S, S \setminus B) & & \\
H^2(S \setminus \mathbb{B}^4) \arrow[r, "\sim"] & H^2(S) \arrow[r] & H^2(S, S \setminus B) = 0 \arrow[ull] \\
& H^1(S) \arrow[r] & H^1(S, S \setminus B) = 0 \arrow[ull]
\end{tikzcd}$$

Link to Diagram

We have $H^i(S, S \setminus \mathbb{B}^4) = H^i(T, T \setminus \mathbb{B}^4) = H^i(\mathbb{B}^4, \partial)$, and by Poincaré-Lefschetz duality, this is isomorphic to $H_{4-i}(\mathbb{B}^4)$. This is equal to 0 if $i \neq 0$ or 4. Writing $\widehat{S} = (S \setminus \mathbb{B}^4) \coprod_{S^3} (\overline{\mathbb{CP}^2} \setminus \mathbb{B}^4)$ and applying Mayer-Vietoris yields

$$\begin{tikzcd}
H^2(\widehat{S}) \arrow[r, "\sim"] & H^2(S \setminus \mathbb{B}^4) \oplus H^2(\overline{\mathbb{CP}^2} \setminus \mathbb{B}^4) \arrow[r] & H^2(S^3) = 0 \\
& \cdots \arrow[r] & H^1(S^3) = 0 \arrow[ull]
\end{tikzcd}$$

Link to Diagram

Combining this with the isomorphisms from earlier, we can write the direct sum as $H^2(S) \oplus H^2(\overline{\mathbb{CP}^2})$ where the latter is equal to $\mathbb{Z}\ell = [E]$ for ℓ a line class.

Question 40.0.2
What is the intersection form on $H^2(\widehat{S}; \mathbb{Z})$?

Remark 40.0.3: Using the proposition, along with the fact that

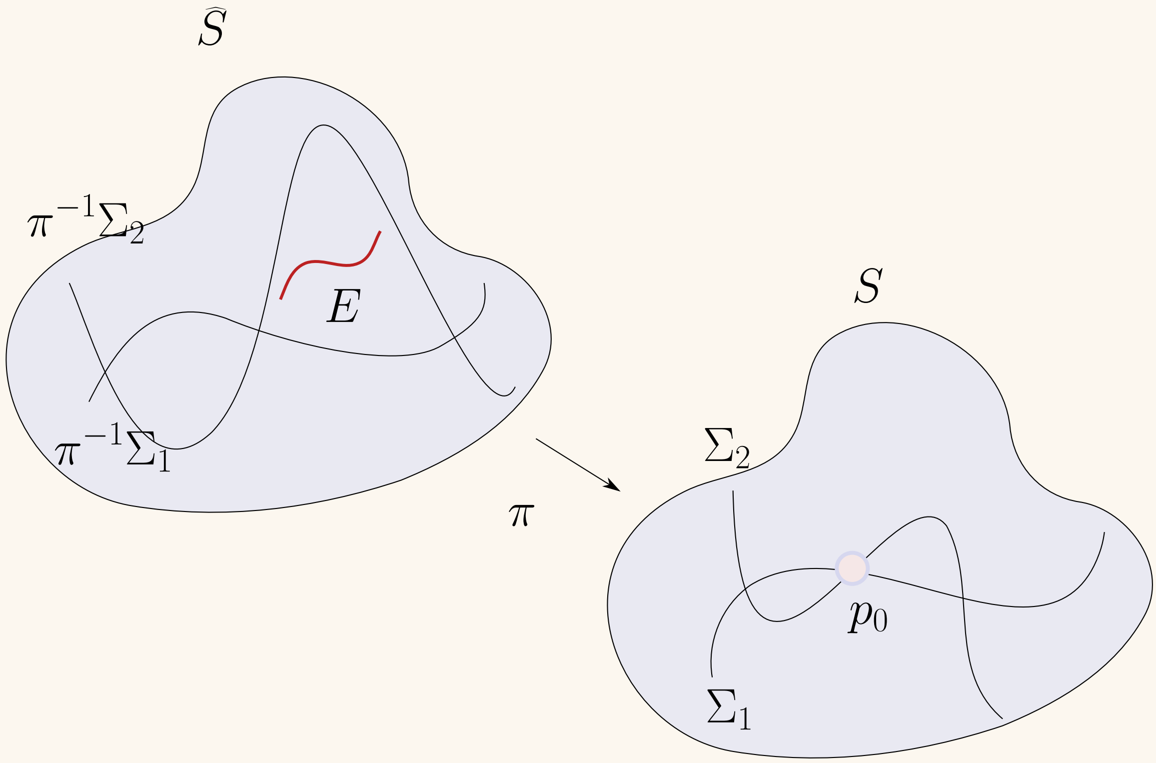
1. its an orthogonal decomposition,
2. π^* is an isometry, and
3. $[E]^2 = -1$,

we know that the Gram matrix for $H^2(\widehat{S})$ is the same as that for $H^1(S) \oplus [-1]$, i.e. it is of the form

$$\begin{bmatrix} A & 0 \\ 0 & -1 \end{bmatrix}.$$

Proof (of 2).
Consider $[\Sigma_1], [\Sigma_2] \in H^2(S;\mathbb{Z})$ where the Σ_i are real surfaces, and suppose $\Sigma_1 \pitchfork \Sigma_2$ and $p \notin \Sigma_1, \Sigma_2$. We then have

$$[\pi^{-1}(\Sigma_i)] = \pi^*[\Sigma_i].$$



The intersection number is preserved because π is generically injective. ■

Proof (of 1).
It also follows that if $p \notin \Sigma$, $\pi^*[\Sigma] = [\pi^{-1}\Sigma]$ where the latter is disjoint from E. So $\pi^*[\Sigma] \cdot E = 0$. ■

Proof (of 3).
Since $[E] \sim [\text{line}] \in \overline{\mathbb{CP}^2} \setminus \mathbb{B}^4$, and $E^2 = [E] \cdot [E] = -1$ since the orientations disagree in $\overline{\mathbb{CP}^2}$. ■

Proposition 40.0.4 ***(Computing the pullback of a curve).***
Let $C \subset S$ be a curve on a surface and suppose C is locally cut out by

$$f(x,y) = a_{m,0}x^m + a_{n-1,1}x^{m-1}y + \cdots + a_{0,m}y^m + O(x^{m+1}, y^{m+1}),$$

near $p \in S$, so the lowest order terms in the Taylor expansion are degree m. Then

$$\pi^* C = \widehat{C} + mE.$$

Proof (?).
On the blowup, take local coordinates (x, μ) where $y = x\mu$ and write

$$\begin{aligned}V(\pi^* f) &= V(x^m \left(a_{m,0} + a_{m-1,1}\mu + \cdots + a_{0,m}\mu^m + O(x^{m+1}, \mu^{m+1})\right)) \\ &= mV(x) + V(a_{m,0} + \cdots) \\ &= E + \widehat{C}.\end{aligned}$$

■

Example 40.0.5 ***(?):*** Take

$$C = \left\{y^2 = x^3 - x^2\right\} \subseteq \mathbb{C}^2,$$

where $\underset{0}{\mathrm{Bl}}\,\mathbb{C}^2 \to C$. Then $\pi^* C = \widehat{C} + 2E$, so

$$C = V(x^2 + y^2 + O(\deg(3)).$$

.

Corollary 40.0.6 ***(Computing the square of the strict transform).***
$\widehat{C}^2 = C^2 - m^2$.

Proof (?).
Write $\pi^* C = \widehat{C} + mE$, then $\widehat{C} = \pi^* C - mE$ implies that $\widehat{C}^2 = (\pi^* C - mE)^2$. This equals

$$\begin{aligned}(\pi^* C)^2 - 2m\pi^* C \cdot E + m^2 E^2 &= C^2 - 0 - m^2 \\ &= C^2 - m^2,\end{aligned}$$

where we've used (2), (1), and (3) respectively to identity these terms.

■

Example 40.0.7 ***(?):*** Let

$$C := \left\{zy^2 = x^3 - x^2 z\right\} \subset \mathbb{CP}^2,$$

then $C^2 = (3\ell)^2 = 9$. The multiplicity of C at the point $[0 : 0 : 1]$ is 2. Taking the coordinate chart $\{z = 1\} \cong \mathbb{C}^2$, we recover the curve $y^2 = x^3 - x^2$ which has multiplicity 2 at $(0, 0)$. We can conclude $\widehat{C} = \underset{[0:0:1]}{\mathrm{Bl}}\,\mathbb{CP}^2$ has self-intersection number $\widehat{C}^2 = 9 - 2^2 = 5$.

Theorem 40.0.8 ***(Castelnuovo Contractibility Criterion).***
Let S be a complex surface and let $E \subset S$ be a holomorphically embedded $\mathbb{CP}^2$ such that $E^2 = -1$ Then there exists a smooth surface $\overline{S}$ and $p \in \overline{S}$ such that $S = \underset{p}{\mathrm{Bl}}\,\overline{S}$ with E as the exceptional curve.

Definition 40.0.9 (Blowdown)
This $\overline{S}$ is called the **blowdown** of S along E.

Remark 40.0.10: Note that this is the exact situation when we blow things up. This is a converse: if we have something that looks like a blowup, we can find something that blows up to it.

Exercise 40.0.11 (?)
Show that the category $\mathsf{Mfd}_{\mathbb{C}}$ is not closed under blowdowns, i.e. there is no blowdown of a holomorphically embedded $\mathbb{CP}^1$, say E, with $E^2 = 1$.

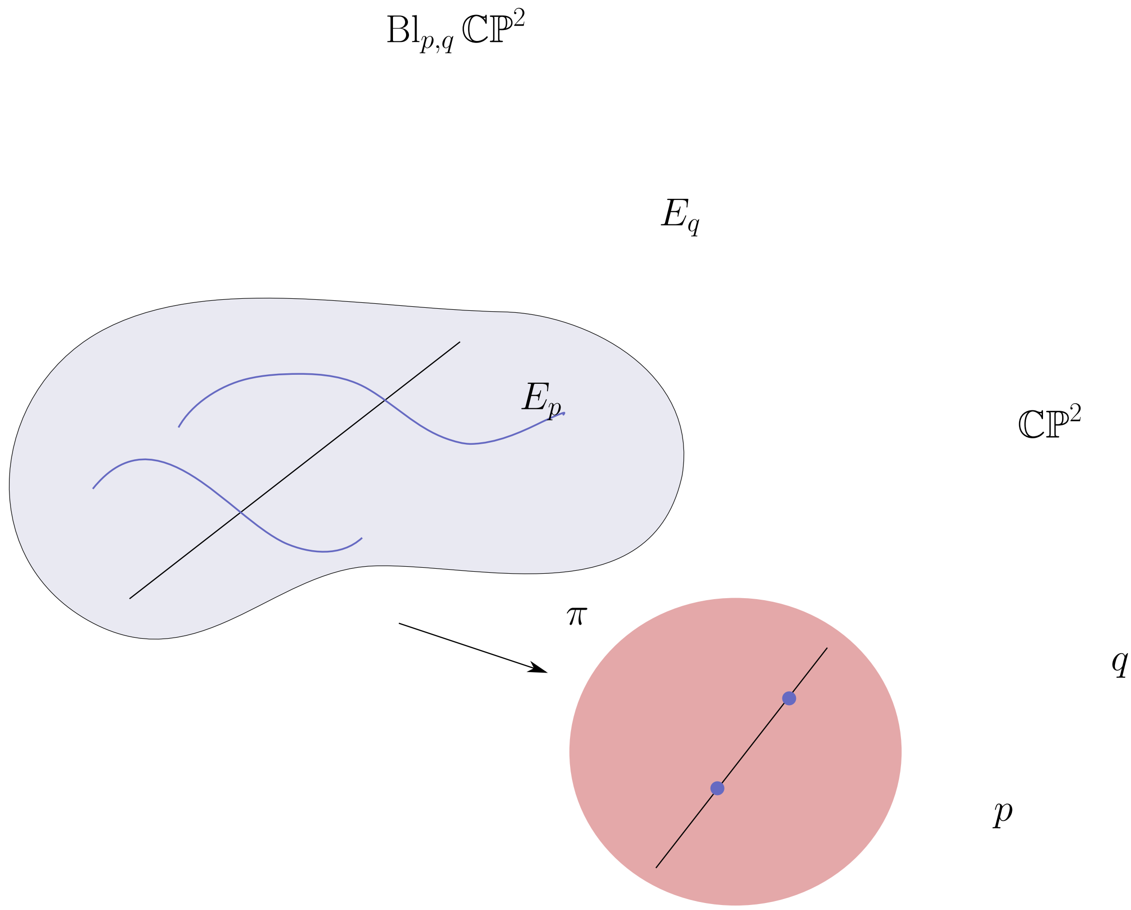
Hint: think about $\mathbb{CP}^2$.

Remark 40.0.12: This is interesting because there does exist a blowdown in the smooth category $\mathsf{Mfd}(C^\infty(\mathbb{R}))$. This is because $S \to S\#\overline{\mathbb{CP}^2}$ and $S \to S\#\mathbb{CP}^2$ are indistinguishable here. One can just reverse orientations.

Example 40.0.13*(?):* A complex surface with a holomorphically embedded $\mathbb{CP}^1$ of self intersection -1. Let $p, q \in \mathbb{CP}^2$ be distinct points, and let $\operatorname*{Bl}_{p,q} \mathbb{CP}^2 := \operatorname*{Bl}_p \operatorname*{Bl}_q \mathbb{CP}^2$. Note that these two operations commute since these are distinct points and blowing up is a purely local operation. Let $\ell \subset \mathbb{CP}^2$ be the unique line through p and q. Viewing p, q as lines in $\mathbb{C}^3$, they span a unique plane, which is a line in projective space, so this makes sense and we can write $\ell \approx \operatorname{span}\{p, q\}$. Since ℓ is defined by a linear equation in local coordinates near p, q, we have $\operatorname{mult}_p \ell = \operatorname{mult}_q \ell = 1$. We hve

$$\begin{aligned} \widehat{\ell} &= \pi^* \ell - E_p - E_q \\ \widehat{\ell}^2 &= \ell^2 - 1^2 - 1^2 = 1 - 1 - 1 = -1. \end{aligned}$$

Under $\pi : \operatorname*{Bl}_{p,q} \mathbb{CP}^2 \to \mathbb{CP}^2$, we have $\widehat{\ell} \xrightarrow{\sim} \ell$.



Here since all of the lower order terms have degree 1, there is a well-defined tangent line. Since $\ell \cong \mathbb{CP}^2$, we have $\widehat{\ell} \cong \mathbb{CP}^2$. Letting σ be the blowdown of $\widehat{\ell}$, we have

$$\begin{tikzcd}
 & \underset{p,q}{\mathrm{Bl}}\,\mathbb{CP}^2 \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & \\
\mathbb{CP}^1 \times \mathbb{CP}^2 & & \mathbb{CP}^2
\end{tikzcd}$$

Link to Diagram

Remark 40.0.14: There's a way to do this with Kirby Calculus.

41 | Wednesday, April 21

Remark 41.0.1: Why can't one blow down a curve $E \cong \mathbb{CP}^1$ with $E^2 = 1$ in a complex surface? Disproof: consider $S := \mathbb{CP}^2$ and E a line, where $E^2 = 1$. If there were a blowdown in the complex analytic category

$$\begin{aligned} S &\to \overline{S} \\ E &\mapsto \text{pt}. \end{aligned}$$

But $\overline{S} \cong_{\mathsf{Top}} S^4$, since $S^4 \# \mathbb{CP}^2 \cong \mathbb{CP}^2$, and this would yield a complex structure on S^4 – a contradiction. This also follows because $\overline{S} \in \mathbb{Z}\,\mathrm{HS}^4$, and Noether's formula implies that every $\mathbb{Z}\,\mathrm{HS}^4$ has no complex structure.

Remark 41.0.2: Recall that we were considering the following:

$$\begin{tikzcd}
 & \underset{p,q}{\mathrm{Bl}}\,\mathbb{CP}^2 \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & \\
\mathbb{CP}^1 \times \mathbb{CP}^2 & & \mathbb{CP}^2
\end{tikzcd}$$

Link to Diagram

Let $\bar{\ell} \subset \underset{p,q}{\mathrm{Bl}}(\mathbb{CP}^2)$ the strict transform of a line through p, q with $\widehat{\ell}^2 = -1$. Goal: we want to construct the map σ sending $\widehat{\ell}$ to a single point. Let $r \in \underset{p,q}{\mathrm{Bl}}\,\mathbb{CP}^2$, then there are three possibilities:

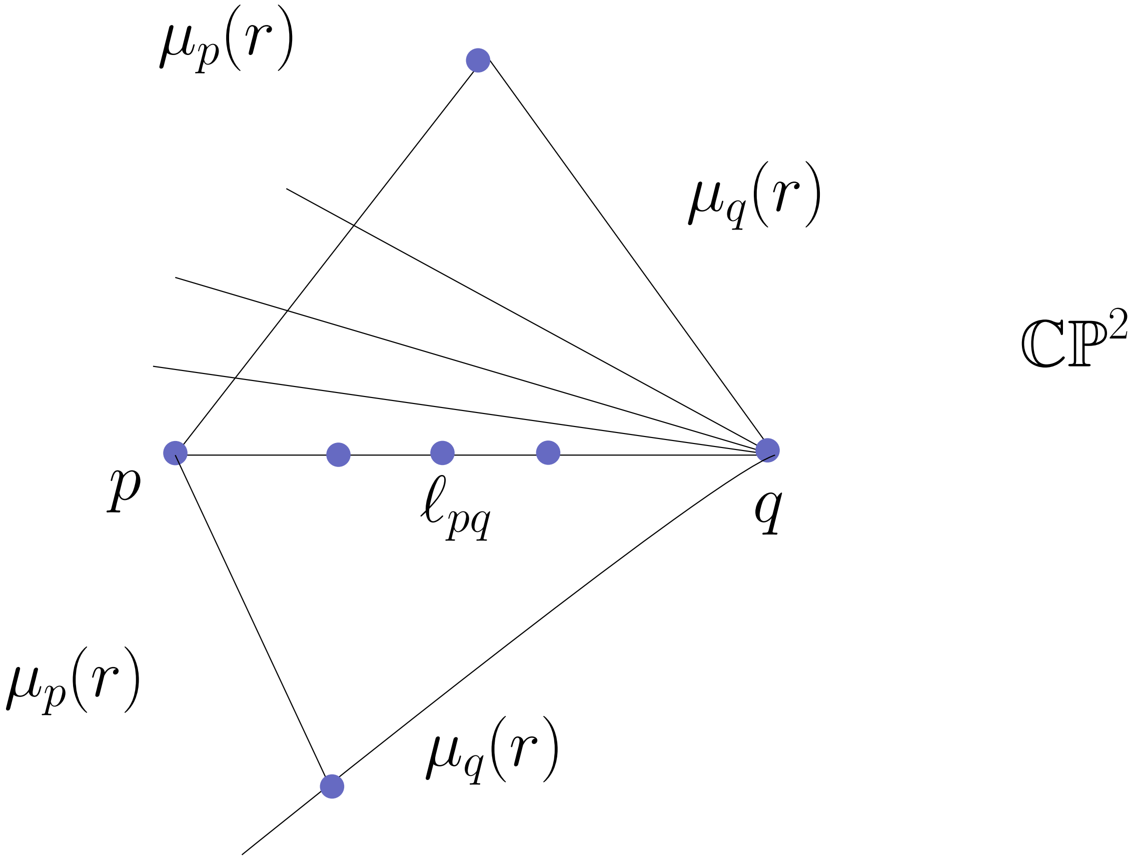
1. $r \in \mathbb{CP}^2 \setminus \{p, q\}$
2. $r \in E_p$
3. $r \in E_q$

If a point $r \neq p, q$, we can take lines $\ell_{pr}.\ell_{qr}$. We can take slopes of these lines to get points in $\mathbb{CP}^1$, and in fact it's the exceptional divisor (since these are sets of slopes through a point).

So we can map

$$r \mapsto \begin{cases} (\mathrm{slope}_p \ell_{pr}, \mathrm{slope}_q \ell_{qr}) \in \mathbb{CP}^2 \times \mathbb{CP}^2 & \text{Case 1} \\ (r, \mathrm{slope}_q \ell_{qp}) & \text{Case 2} \\ (\mathrm{slope}_p \ell_{pq}, r) & \text{Case 3.} \end{cases}$$

This is clearly continuous, is this injective? The outputs will be the same for any point on the line between p and q:



So this realizes the blowdown map, since $\Phi\widehat{\ell_{pq}}) = \text{pt}$ and restricting it to the complement of the line is injective.

41.1 Spin and Spinc Groups

Remark 41.1.1: Goal: show that $3[\ell]$ can't be realized by a sphere, we'll need Rohklin's theorem for this. Let $(V, \langle -,\ -\rangle)$ be an inner product space, and assume the inner product is positive-definite. Recall that the tensor algebra is defined as $T(V) := \bigoplus_{n\geq 0} V^{\otimes n}$.

Definition 41.1.2 (Clifford Algebra)
Define the **Clifford Algebra** of V as

$$\mathrm{Cl}(V) := T(V) / \left\langle v \otimes v + \|v\|^2 1 \right\rangle.$$

Example 41.1.3 ***(The reals):*** Take $\mathbb{R}$ with the standard inner product, so $\langle x,\ y\rangle := xy$. Then $T(\mathbb{R}) = \bigoplus_{n\geq 0} \mathbb{R}$. Letting $\{e\}$ be a basis of $\mathbb{R}$, we have $T(\mathbb{R}) = \mathbb{R} \oplus \mathbb{R}e \oplus \mathbb{R}(e^2) \oplus \cdots \cong \mathbb{R}[x]$ by sending

$e^n \mapsto x^n$. Since $\|e\| = 1$, and we mod out by $e^2 + \|e\|^2 1$ where $e^2 = -1$ and thus

$$\mathrm{Cl}(\mathbb{R}, \langle -,\ - \rangle_{\mathrm{std}}) \cong \mathbb{R}[x] / \left\langle x^2 = -1 \right\rangle \cong \mathbb{C}.$$

The denominator is referred to as the **Clifford relation**.

Example 41.1.4 ***(More reals):*** Take $\mathbb{R}^2$ with the standard inner product and an orthonormal basis $\{e_1, e_2\}$. Then

$$T(\mathbb{R}) = \mathbb{R} \oplus \mathbb{R}\langle e_1, e_2 \rangle \oplus \mathbb{R}\left\langle e_1^2, e_1 e_2, e_2 e_1, e_2^2 \right\rangle \oplus \cdots .$$

Note that there are 2^k terms in the kth graded piece. It suffices to mod out only by the relations on the orthonormal basis. This is of the form $(v+w)^2 = -\|v+w\|^2 = -\|v\|^2 - 2\langle v,\ w \rangle - \|w\|^2$. On the other hand, this equals $v^2 + vw + wv + w^2$. So we obtain

$$vw + wv = 2\langle v,\ w \rangle,$$

and setting $v = w$ and dividing by 2 yields the original Clifford relation.

For $\mathbb{R}^2$, we can explicitly check

1. $e_1^2 = -1$,
2. $e_2^2 = -1$,
3. $e_1 e_2 + e_2 e_1 = -2 e_1 e_2 = 0$,
4. $e_1 e_2 = -e_2 e_1$.

Here (1), (2), and (4) generate all of the relations, so

$$\mathrm{Cl}(\mathbb{R}^2) - \mathbb{R}\langle e_1, e_2 \rangle / \left\langle e_1^2 = -1, e_2^2 = -1, e_1 e_2 = -e_2 e_1 \right\rangle \cong HH.$$

We can form this map by

$$\begin{aligned} 1 &\mapsto 1 \\ e_1 &\mapsto i \\ e_2 &\mapsto j \\ e_1 e_2 &\mapsto k, \end{aligned}$$

and then checking that the appropriate relations hold. These hold since $i^2 = j^2 = -1$ and $ij = -ji = k$. These suffice, but you can check the rest: for example, does $jk = i$ hold? We can write this as

$$e_2(e_1 e_2) = -e_2(e_2 e_1) = -e_2^2 e_1 = -(-1)e_1 = e_1.$$

Exercise 41.1.5 (?)
Check that $\dim_{\mathbb{R}} \mathrm{Cl}(V) = 2^{\dim V} < \infty$.

42 | Friday, April 23

Remark 42.0.1: Given $(V, \cdot)$ an inner product space, we defined

$$\mathrm{Cl}(V) := \frac{\bigoplus_{n\geq 0} V^{\otimes n}}{\langle v\otimes w + w \otimes v = 2v\cdot w\rangle}.$$

Example 42.0.2*(?)*: We saw that

$$\begin{aligned}\mathrm{Cl}(\mathbb{R}, \cdot) &\cong \mathbb{R}[e]/e^2 = -1 \cong \mathbb{C} \\ \mathrm{Cl}(\mathbb{R}^2, \cdot) &= \mathbb{R}\left\langle e_1, e_2\right\rangle / \left\langle e_1^2 = e_2^2 = -1, e_1e_2 = -e_2e_1 - \right\rangle \cong \mathbb{H}\end{aligned}$$

where $e_1 \mapsto i, e_2 \mapsto j, e_3 = e_1e_2 \mapsto k$. Can we describe $\mathrm{Cl}(\mathbb{R}^n, \cdot)$ in general? Choose an orthonormal basis $\{e_i\}$, then

$$\mathrm{Cl}(\mathbb{R}^n, \cdot) = \frac{\mathbb{R}\left\langle e_1, \cdots, e_n\right\rangle}{\left\langle e_i^2 = -1, e_ie_j = -e_je_i \,\middle|\, i\neq j\right\rangle}.$$

We saw that replacing 2 with ϵ in the defining relation recovers $\bigwedge^* V$.

Definition 42.0.3 (Degree Filtration)
Define the **degree filtration** on $\mathrm{Cl}(V, \cdot)$ as the filtration induced by the degree filtration on $T(V) := \bigoplus_{n\geq 0} V^{\otimes n}$.

Example 42.0.4*(?)*: Consider $\mathrm{Cl}(\mathbb{R}^2, \cdot)$. Then

- Degree 0: $\mathbb{R}$.
- Degree 1: $\mathbb{R}\oplus \mathbb{R}e_1 \oplus \mathbb{R}e_2$
- Degree 2: $\mathbb{R}\oplus \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_1e_2$

Definition 42.0.5 (Grading and Filtration)
Recall that there's a distinction between gradings and filtration:

- Gradings: $R^iR^j \subset R^{i+j}$ and $R = \bigoplus_i R^i$.
- Filtrations: $F^1 \subset F^2 \subset \cdots$ with $F^iF^j \subseteq F^{i+j}$

An algebra equipped with a grading is a **graded algebra**, and similarly an algebra equipped with a filtration is a **filtered algebra**.

Remark 42.0.6: Note that

- $k[x_1, \cdots, x_n]$ is graded (by monomials of uniform degree) and filtered (by polynomials of a bounded degree)
- $T(V)$ is graded and filtered, since multiplying a pure p tensor with a pure q tensor yields a pure $p+q$ tensor
- $\mathrm{Cl}(V)$ is a quotient of $T(V)$, but one can't simply define $\mathrm{Cl}(V,\cdot)^i = \operatorname{im} T(V)^i$ since the relations have mixed degree: for example $e_1^2 = -1$ So $\mathrm{Cl}(V)$ isn't graded, but is still filtered: take the filtration F on $T(V)$ defined by $F^i := \bigoplus_{j \leq i} V^{\otimes j}$ and descend it through the quotient map. The relations can only decrease degree, so this is well defined.

Definition 42.0.7 (Filtration on the Clifford Algebra)
Define a filtration F^- on $\mathrm{Cl}(V)$ by the following:

$$F^i \mathrm{Cl}(V) := \operatorname{span}\left\{e_{j_1}, e_{j_2}, \cdots, e_{j_i}\right\}.$$

Definition 42.0.8 (The associated graded)
The **associated graded** ring $\mathsf{gr}_{F^-} R$ is the graded ring defined by

$$(\mathsf{gr}_{F^-})^i := F^i R / F^{i-1} R.$$

This induces a decomposition

$$\mathsf{gr}_{F^-} \cong \bigoplus_{i \geq 0} F^i R / F^{i-1} R = \bigoplus_{i \geq 0} (\mathsf{gr}_{F^-})^i,$$

which has a multiplicative structure

$$F^i / F_{i-1} \cdot F^j / F_{j-1} \to F^{i+j} / F^{i+j-1}.$$

Remark 42.0.9: Note that if $R \in \mathsf{gr}\,\mathsf{Ring}$, then $\mathsf{gr}\,(R) = R$, so taking the associated graded recovers the ring itself. What's happening: taking the smallest homogeneous ideal.

Fact 42.0.10
If one has relations of mixed degree, the associated graded also has the top degree part of each relation.

Remark 42.0.11: In our case, the Clifford relation relates degree k pieces to degree $k-2$ pieces, so we obtain

$$\mathsf{gr}_{F^-} \mathrm{Cl}(V) \cong T(V) / \langle v \otimes w + w \otimes v = 0 \rangle := \bigwedge^* V.$$

There is an isomorphism of k-vector spaces

$$\begin{aligned} \mathrm{Cl}(V) &\xrightarrow{\sim} \mathsf{gr}\,\mathrm{Cl}(V) \\ x \in F^i &\mapsto \bar{x} \in F^i / F^{i-1}. \end{aligned}$$

This is because $F^0 \subseteq \cdots \subseteq \cdots$ with $\cup_i F^i = \mathrm{Cl}(V)$. We can conclude $\dim_{\mathbb{R}} \mathrm{Cl}(V) = \dim_{\mathbb{R}} \bigwedge^* V =$

$2^{\dim_k V}$ and use this to construct a basis for $\mathrm{Cl}(V)$. The relevant map is

$$e_{j1}, e_{j2}, \cdots, e_{ji} \mapsto e_{j_1} \wedge \cdots \wedge e_{j_i}.$$

Corollary 42.0.12 ***(of the fact).***
The following set forms an $\mathbb{R}$-basis for $\mathrm{Cl}(\mathbb{R}^n, \cdot)$:

$$\left\{ e_{j1}, e_{j2}, \cdots, e_{ji} \;\middle|\; j_1 < j_2 < \cdots < j_i,\ i \leq n \right\}.$$

Example 42.0.13 ***(?):*** Consider

$$\mathrm{Cl}(\mathbb{R}^3, \cdot) \cong \mathrm{span}_{\mathbb{R}} \left\{ 1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_1e_2e_3 \right\}.$$

Then

$$\begin{aligned} e_1e_2 \cdot e_1e_3 &= -e_1e_1e_3e_3 & \qquad e_2e_1 &= -e_1e_2 \\ &= e_2e_3 & e_1^2 &= -1. \end{aligned}$$

Exercise 42.0.14 (?)
Show that $\mathrm{Cl}(\mathbb{R}^3) \cong \mathbb{H} \oplus \mathbb{H}$.

Definition 42.0.15 (Even and odd parts of the Clifford algebra)
$\mathrm{Cl}(V)$ has a $\mathbb{Z}/2$ ("super") grading, so

$$\mathrm{Cl}(V) \circ \mathrm{Cl}_0(V) \oplus \mathrm{Cl}_1(V) \qquad\qquad \mathrm{Cl}_i(V) \cdot \mathrm{Cl}_j(V) \subset \mathrm{Cl}_{i+j \pmod 2}(V).$$

The **even** subalgebra is given by

$$\mathrm{Cl}_0(V) = \mathrm{span}_k \left\{ e_{i1}, e_{i2}, \cdots, e_{i2k} \;\middle|\; 2k \leq n \right\},$$

where we take an even number of basis elements, which makes sense because the Clifford relation $vw + 2v = -2v \cdot w$ preserves degree mod 2. This is still an algebra. The **odd** sub-vector space (not an algebra) is given by

$$\mathrm{Cl}_1(V) = \mathrm{span}_k \left\{ e_{i1}, e_{i2}, \cdots, e_{i2k+1} \;\middle|\; 2k+1 \leq n \right\}.$$

Example 42.0.16 ***(?):***

$$\mathrm{Cl}(\mathbb{R}^3) = \mathrm{span}_{\mathbb{R}} \left\{ 1, e_1e_2, e_1e_3, e_2e_3 \right\},$$

and we saw $e_1e_2 = e_1e_3 = e_2e_3$. This product has degree 4, and when we applied the relation $e_1^2 = 1$ we dropped the degree by 2. For the odd part, $e_3 \in Cl_1(\mathbb{R}^3)$ and $e_1e_2 \in \mathrm{Cl}_0(\mathbb{R}^3)$, and we have

$$e_3 \cdot (e_1e_2) = -e_1e_3e_2 = e_1e_2e_3 \in \mathrm{Cl}_1(\mathbb{R}^3).$$

Proposition 42.0.17 *(Decomposing the Clifford algebra of V).*

$$\mathrm{Cl}(V) \cong \mathrm{Cl}_0(V \oplus \mathbb{R}).$$

Proof (?).
Let $e \in \mathbb{R}$ be a unit vector. Given $x \in \mathrm{Cl}(V)$, decompose $x = x_0 + x_1 \in \mathrm{Cl}_0(V) \oplus \mathrm{Cl}_1(V)$. Define an isomorphism

$$\begin{aligned} \varphi : \mathrm{Cl}(V) &\to \mathrm{Cl}_0(V \oplus \mathbb{R}) \\ x &\mapsto x_0 + x_1 e, \end{aligned}$$

which is well-defined since x_0 was odd degree, and both x_1, e were odd degree and thus $x_1 e$ is even. One checks that this preserves multiplication:

$$x \cdot y = (x_0 + x_1) \cdot (y_0 + y_1) = (x_0 y_0 + x_1 y_1) + (x_0 y_1 + x_1 y_0) \in \mathrm{Cl}_0(V) \oplus \mathrm{Cl}_1(V),$$

and so

$$\begin{aligned} \varphi(x) \cdot \varphi(y) &= (x_0 + x_1 e)(y_0 + y_1 e) \\ &= x_0 y_0 + x_0 y_1 e + x_1 e y_0 + x_1 e y_1 e_1. \end{aligned}$$

The question is if this equals

$$\varphi(xy) \coloneqq (x_0 y_0 + x_1 y_1) + (x_0 y_1 + x_1 y_0) e.$$

But for example, $x_1 e y_0 = (-1)^{|y_0|} x_1 y_0 e$, and y_0 is even. Similarly, $x_1 e y_1 e = -x_1 y_1 e^2 = x_1 y_1$. ■

43 | Wednesday, April 28

Remark 43.0.1: Last time: we defined $\mathrm{Pin}(n) \subseteq \mathrm{Cl}(\mathbb{R}^n)$ which was generated by $S^1(\mathbb{R}^n)$. These were units because $v^2 = -\|v\|^2 = -1$, so $v^{-1} = -v$, and formed a group contained in $\mathrm{Cl}(\mathbb{R}^n)^\times$. There is a decomposition $\mathrm{Cl}(V) = \mathrm{Cl}_0(V) \oplus \mathrm{Cl}_1(V)$ with a $\mathbb{Z}/2$-grading, and we defined

$$\mathrm{Spin}(V) \coloneqq \mathrm{Pin}(V) \cap \mathrm{Cl}_0(V) = \left\langle vw \;\middle|\; v, w \in S^1(\mathbb{R}^n) \right\rangle.$$

There is a map

$$\begin{aligned} \mathrm{Pin}(n) &\twoheadrightarrow O(n) \\ v &\mapsto (u \mapsto vuv^{-1}) = -R_{v^\perp}, \end{aligned}$$

which preserves $V^{\otimes 1} \subset \mathrm{Cl}(V)$, and was reflection about the hyperplane $v^\perp$. There is also a SES

$$0 \to \mathbb{Z}/2 \to \mathrm{Spin}(n) \xrightarrow{\pi} \mathrm{SO}(n) \to 0,$$

where we used the fact that $\ker \pi \subset Z\mathrm{Cl}(\mathbb{R}^n)$. It turns out that $\mathrm{Spin}(n) = \overline{\mathrm{SO}(n)}$, using that $\pi_1(\mathrm{SO}(n), \mathrm{pt}) = \mathbb{Z}/2$ and checking that $\pm 1 \in \mathrm{Spin}(n)$, yielding a nontrivial kernel.

Remark 43.0.2: This is local, at a single vector space, so we'll now try to globalise this to the tangent space of a manifold.

Definition 43.0.3 (Clifford Bundle)
Let (V, g) be an oriented smooth Riemannian manifold where g is a metric on TX. Define the **Clifford bundle** of X by

$$\mathrm{Cl}(X) := \mathrm{Cl}(T^\vee X, g^\vee),$$

where we've used the dual metric $g^\vee$ on the cotangent bundle.

Remark 43.0.4: We showed that $\mathsf{gr}\,\mathrm{Cl}(\mathbb{R}^n) = \bigwedge \mathbb{R}^n$, and so there is a bundle isomorphism

$$\mathrm{Cl}(X) \xrightarrow{\sim} \bigwedge\nolimits^* T^\vee X,$$

but the ring structure is different. On the right, we have a way of multiplying sections, namely $\omega_1 \wedge \omega_2$, but on the left we have the Clifford multiplication $\alpha_1 \cdot \alpha_2$. Note that $\omega^{\wedge 2} = 0$, but $\alpha^{\cdot 2} \in \mathbb{R}$ is some scalar. We define $\omega \cdot \omega = g^*(\omega, \omega)$, so we use the metric fiberwise to define a Clifford multiplication.

Definition 43.0.5 (The principal oriented frame bundle)
Given an oriented bundle with a metric, there is a principal $\mathrm{SO}(n)$ bundle $P := \mathrm{OFrame}$, the space of orthogonal oriented frames.

Remark 43.0.6: This is principal since any two elements are related by a unique element of $\mathrm{SO}(n)$. Recall that we had an *associated bundle* construction, so taking the standard representation $\rho : \mathrm{SO}(n) \to (\mathbb{R}^n, g)$ where elements act by their transformations (?), there is an oriented bundle $P \underset{\rho}{\times} \mathbb{R}^n$. If the bundle is TX with a metric g, this yields a distinguished $\mathrm{SO}(n)$ bundle $P \to X$.

Definition 43.0.7 (Spin Structures)
A **spin structure** is a lift $\tilde{P}$ of P to a principal $\mathrm{Spin}(n)$ bundle.

Proposition 43.0.8 ***(Spin iff nontrivial w_2).***
X admits a spin structure iff the second Stiefel–Whitney class $w_2(X) = 0$ in $H^2(X; \mathbb{Z}/2)$. If $w_2(X) = 0$, then the spin structures are torsors over $H^1(X; \mathbb{Z}/2)$.

Remark 43.0.9: Recall that a G-torsor is a set with a free transitive G-action. For example, the fibers of a principal bundle are torsors. Given any two torsors, we can compare them using elements of G, but there is no distinguished element. For example, $\mathbb{A}_n$ is a torsor over the vector space k^n.

Proof (?).

Consider transitions for $P \to X$:

$$t_{ij} : U_i \cap U_j \to \mathrm{SO}(n)$$

,

where $t_{ij} = t_{ji}^{-1}$ and the cocycle condition $t_{ij}t_{jk}t_{ki} = 1$ is satisfied. We want a lift:

$$\begin{tikzcd}
 & \mathbb{Z}/2 \in Z\mathrm{Spin}(n) \arrow[d, hook] \\
U_i \cap U_j \arrow[r, "\tilde{t}_{ij}"] \arrow[d, "\mathbb{1}"] & \mathrm{Spin}(n) \arrow[d, "2:1"] \\
U_i \cap U_j \arrow[r, "t_{ij}"] & \mathrm{SO}(n)
\end{tikzcd}$$

Link to Diagram

We can always lift to *some* $\tilde{t}_{ij}$ using the path-lifting property of covers if $U_i \cap U_j$ is contractible, using that $\mathbb{Z}/2$ is discrete. We can arrange $\tilde{t}_{ij} = \tilde{t}_{ji}^{-1}$ since $U_i \cap U_j = U_j \cap U_i$, but we may not have the cocycle condition on the lift. We have $t_{ij}t_{jk}t_{ki} = 1$, so

$$\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} \in \ker(\mathrm{Spin}(n) \to \mathrm{SO}(n)) = \{\pm 1\},$$

using that everything in sight needs to be a group morphism. So define

$$\tilde{t}_{ijk} := (\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki})_{i,j,k} \in \check{C}^2_{\mathcal{U}}(X, \underline{\mathbb{Z}/2}).$$

The claim is that $\partial^2(\tilde{t}_{ijk}) = 0$, but it turns out that regardless of choice of lift we obtain

$$\partial^2(\tilde{t}_{ijk}) = \tilde{t}_{ijk}\tilde{t}_{ikl}^{-1}\tilde{t}_{ijl}\tilde{t}_{ijk}^{-1} = 0 \implies [\tilde{t}_{ijk}] \in \check{H}^2(X, \underline{\mathbb{Z}/2}).$$

Is this class well-defined? Consider replacing $\tilde{t}_{ij}$ with $-\tilde{t}_{ij}$. In general, we have

$$i, j \in \{a, b, c\} \implies \tilde{t}_{abc} \mapsto -\tilde{t}_{abc},$$

and so this is a Čech coboundary in $\partial^1(1, \cdots, 1, -1, 1, \cdots, 1)$ where the -1 occurs in the t_{ij} coordinate. Thus $\tilde{t}_{ijk}$ is well-defined moduli $\partial^1 C^1_{\mathcal{U}}(X, \underline{\mathbb{Z}/2})$.

Note that $w_2(X)$ was produced from the pair (X, g), but the space of metrics is connected and thus $w_2(X)$ depends only on X. Suppose $w_2(X) = 0$, then $[\tilde{t}_{ijk}] = 0$ which implies that there is some (s_{ij}) with $\partial^1(s_{ij}) = (\tilde{t}_{ijk})$. So replace each $\tilde{t}_{ij}$ with $\tilde{\tilde{t}}_{ij} := s_{ij}\tilde{t}_{ij}$ is a new lift which satisfies the cocycle condition. Thus they define the transition functions of a principal $\mathrm{Spin}(n)$ bundle lifting $P \to X$.

To see the claim about torsors, given any $\ell_{ij} \in \ker \partial^1$, note that any $\tilde{\tilde{t}}_{ij}\ell_{ij}$ also satisfies the cocycle condition. There is a map

$$\begin{aligned} \{\text{Spin structures}\} &\leftarrow \ker \partial^1 \\ \tilde{\tilde{t}}_{ij}\ell_{ij} &\hookleftarrow \ell_{ij}, \end{aligned}$$

which is a torsor because we needed to start with a given lift $\tilde{\tilde{t}}_{ij}$. Then $\tilde{P}_1 \cong \tilde{P}_2$ iff there exists an $(m_i) \in \check{C}^0_{\mathcal{U}}(X, \underline{\mathbb{Z}/2})$ such that $(\ell_{ij})_1 = (\ell_{ij})_2 + \partial^0(m_i)$, which are different trivializations of the same bundle.

■

Remark 43.0.10: This is a nice example to get a hang of the use and importance of Čech cohomology. We then use the isomorphism $\check{H} \to H_{\mathrm{Sing}}$.

Theorem 43.0.11 ***(Existence of spin representation of Clifford algebras in even dimension).***
Assume $n := \dim V$ is even, then $\mathrm{Cl}(V)$ has a unique nontrivial irreducible finite dimensional complex representation S of dimension $\dim S = 2^{n/2}$, the **spin representation**.

Remark 43.0.12: It turns out that $\mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{End}(S)$. The left-hand side contains $\mathrm{Spin}(n)$, so given $\rho : \mathrm{Cl}(V) \to \mathrm{End}(S)$ a representation (i.e. a ring homomorphism) in matrices, we can restrict ρ to $\mathrm{Spin}(n)$ to get $\rho|_{\mathrm{Spin}(n)} : \mathrm{Spin}(n) \to \mathrm{GL}(S)$. Next time: spin representations. Spinor bundle will be sections of associated bundle of the Clifford bundle.

44 | Friday, April 30

Remark 44.0.1: Last time: we defined

$$\begin{aligned}
\mathrm{Cl}(V, \cdot) &:= \bigoplus_n V^{\otimes n} / \left\langle v \otimes v = -\|v\|^2 1 \right\rangle \\
\mathrm{Pin}(V) &:= \left\langle v \,\middle|\, \|v\| = 1 \right\rangle \subseteq \mathrm{Cl}(V).
\end{aligned}$$

There is a $\mathbb{Z}/2$ grading $\mathrm{Cl}(V) = \mathrm{Cl}_0(V) \oplus \mathrm{Cl}_1(V)$ where $\mathrm{Cl}_0(V)$ is the image of even tensors and $\mathrm{Cl}_1(V)$ is the image of odd tensors. We also had

$$\mathrm{Spin}(V) := \mathrm{Pin}(V) \cap \mathrm{Cl}_0(V) = \left\langle v \cdot w \,\middle|\, v, w \in V,\ \|v\| = \|w\| = 1 \right\rangle.$$

There was a map

$$\begin{aligned}
\mathrm{Pin}(V) &\to O(V) \\
v &\mapsto -R_v,
\end{aligned}$$

where R_v was reflection about $v^\perp$, where we identified this as an action on $V^{\otimes 1} \subset \mathrm{Cl}(V)$ where $u \to vuv^{-1}$. For any Riemannian manifold (X, g), we could define the Clifford bundle $\mathrm{Cl}(X) = \mathrm{Cl}(T^\vee X, g^\vee)$ to globalise this from vector spaces to bundles with metrics. We defined a spin structure on X as any lift of the principal $\mathrm{SO}(n)$ bundle over $(T^\vee X, g)$ (namely $\mathrm{Frame}(X)$) to a $\mathrm{Spin}(n)$ bundle.

⚠ **Warning 44.0.2**
Each fiber is a metric space, so what happens if you just try to define

$$Y \coloneqq \coprod_{x \in X} \left\langle v \,\middle|\, \|v\|^2 = 1,\, v \in {T_x}^\vee X \right\rangle \text{ ?}$$

This seems to be isomorphic to a spin structure, but we do not have a distinguished action of any *fixed* group $\mathrm{Spin}(n)$. We would have to choose isomorphisms to the standard spin group at each fiber, but the isomorphisms are not unique – there is ambiguity up to the entire spin group. So this does not define a spin structure.

Remark 44.0.3: We showed that there exists a spin structure iff some cohomology class $w_2(K) \in H^2(X; \mathbb{Z}/2)$ vanishes.

Theorem 44.0.4 ***(Classification of complex representations of Clifford algebras).*** If $\dim_k V$ is even, there is a unique finite-dimensional complex irreducible $\mathrm{Cl}(V)$ representation of dimension $2^{n/2}$. If $\dim_k V$ is odd, there are two complex conjugate representations of dimension $2^{\lfloor n/2 \rfloor}$.

Example 44.0.5 ***(?):*** Consider $\mathrm{Cl}(\mathbb{R}^2) \cong \mathbb{H}$. There is an irreducible complex representation of dimension 2:

$$\begin{aligned}
1 &\mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
i &\mapsto \sigma_1 \coloneqq \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \\
j &\mapsto \sigma_2 \coloneqq \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \\
k &\mapsto \sigma_3 \coloneqq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\end{aligned}$$

Definition 44.0.6 (Pauli matrices)
The σ_i defined above are referred to as the **Pauli matrices**.

Example 44.0.7 ***(?):*** Consider $\mathrm{Cl}(\mathbb{R}^4)$. By the theorem, there is a unique complex representation of $2^{4/2} = 2^2 = 4$, although the 4 here matching the dimension of $\mathbb{R}^4$ is coincidental. We'd like to find an isomorphism

$$\mathrm{Cl}(\mathbb{R}^4) \xrightarrow{\sim} \mathrm{End}((\mathbb{C}^2)^{\otimes 2}) \cong \mathrm{End}(\mathbb{C}^4) = \mathrm{Mat}(4 \times 4; \mathbb{C}).$$

Note that $\mathrm{Cl}(\mathbb{R}^4) \xrightarrow{\sim} \mathrm{End}((\mathbb{C}^2)^{\otimes 3})$, which is why the dimensions multiply. We can write

$$\mathrm{Cl}(\mathbb{R}^4) = \frac{\mathbb{R}\left\langle e_1, e_2, e_3, e_4 \right\rangle}{e_i e_j + e_j e_i = 2\delta_{ij}}.$$

So define a map

$$\begin{aligned}
e_1 &\mapsto \gamma_1 \coloneqq 1 \otimes \sigma_1 \\
e_2 &\mapsto \gamma_2 \coloneqq 1 \otimes \sigma_2 \\
e_3 &\mapsto \gamma_3 \coloneqq \sigma_1 \otimes i\sigma_3 \\
e_4 &\mapsto \gamma_4 \coloneqq \sigma_2 \otimes i\sigma_3.
\end{aligned}$$

Definition 44.0.8 *(Dirac matrices)*
The matrices appearing above are called the **Dirac matrices**.

Exercise 44.0.9 *(?)*
Determine a similar map for $\mathrm{Cl}(\mathbb{R}^6)$ continuing this pattern.

We can check that this is a representation. Note that we can tensor matrices in a simple way:

$$
\begin{array}{ccc}
 & e_1 & e_2 \\
e_1 & a & b \\
e_2 & c & d
\end{array}
\qquad
\begin{array}{ccc}
 & f_1 & f_2 \\
f_1 & e & f \\
f_2 & g & h
\end{array}
$$

Link to Diagram

Checking $e_2 \cdot e_2 = -1$, we have

$$
\begin{aligned}
(1 \otimes \sigma_2) \cdot (1 \otimes \sigma_2) &= ? \\
1_2 \otimes \sigma_2^2 &= -I_2 \oplus I_2 \\
\gamma_2 \gamma_3 &= -\gamma_3 \gamma_2 .
\end{aligned}
$$

Todo: messed up!

One can similarly check

$$(1 \otimes \sigma_2) \cdot (\sigma_1 \otimes i\sigma_3) = -(\sigma_1 \otimes i\sigma_2)(1 \otimes \sigma_2).$$

Remark 44.0.10: We thus have $\mathrm{Cl}(\mathbb{R}^4) \curvearrowright \mathbb{C}^4$ by sending $e_i \mapsto \delta_i$, the Dirac matrices. Using that $\mathrm{Pin}(4) \cap \mathrm{Cl}(\mathbb{R}^4) = \mathrm{Spin}(4) \subseteq \mathrm{Cl}(\mathbb{R}^4)$, we can a spin representation, but this may no longer be irreducible. In fact, as a $\mathrm{Spin}(4)$ representation this splits into two irreducible representations. We know that $\mathrm{Spin}(4) \subseteq \mathrm{Cl}_0(\mathbb{R}^4) = \mathrm{Cl}(\mathbb{R}^3)$ which has two complex conjugate irreducible representations. The key is to define an element $\omega_{\mathbb{C}} \in \mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$ with $\omega_{\mathbb{C}}^2 = 1$, which yields a decomposition of $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ as the ± 1 eigenspaces of the action. Here $\omega_C := -e_1 e_2 e_3 e_4 \mapsto \gamma_5$. One can define

$$\gamma_5 := \mathrm{im}(\omega_{\mathbb{C}}) = -\gamma_1\gamma_2\gamma_3\gamma_4 = -\sigma_3 \otimes \sigma_3$$

and one obtains the matrix

$$-\sigma_3 \otimes \sigma_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

One can check that γ_5 anticommutes with the δ_i for $1 \leq i \leq 4$, and thus commutes with $\mathrm{Cl}_0(\mathbb{R}^4)$. We can write $\mathbb{S}^+$, the positive 1 eigenspace of γ_5, as $\mathbb{C}(s_1 - s_4) \oplus \mathbb{C}(s_1 + s_2)$. So we have $\mathrm{Spin}(4) = \mathrm{Cl}(\mathbb{R}^4) \curvearrowright \mathbb{C}^2 \oplus \mathbb{C}^2 = \mathbb{S}$, which splits into γ_5-eigenspaces $\mathbb{S}^+ \oplus \mathbb{S}^-$, the **positive and negative spinors**. This means that γ_5 commutes with the image of $\mathrm{Spin}(4) \hookrightarrow \mathrm{GL}(\mathbb{C}^2 \oplus \mathbb{C}^2)$.

Fact 44.0.11
If the action commutes with everything in the representation, the representation splits. (??? missed)

Remark 44.0.12: Let $g \in \mathrm{Spin}(4)$, and $v^+ \in \mathbb{S}^+ \subseteq \mathbb{S}$. Question: is it true that $g \cdot v^+ \in \mathbb{S}^+$? If so, this yields a subrepresentation. If so, $\gamma_5 v^+ = v^+$ since we're in the $+1$ eigenspace, and on the other hand, $g \cdot v^+ = g \cdot \gamma_5 v^+ = g\omega_{\mathbb{C}} \cdot v^+$ where the last identification comes from the map $\gamma_5 \mapsto \omega_{\mathbb{C}}$, and this is equal to $\omega_{\mathbb{C}} g \cdot v^+$ using commutativity. So $g \cdot v^+$ is in the $+1$ eigenspace of γ_5.

Remark 44.0.13: Now take γ_i. This actually switches spinors: by anticommutativity of the γ_i with γ_5, we have

$$\gamma_i \cdot v^+ = \gamma_i \gamma_5 v^+ = -\gamma_5 \gamma_i v^+,$$

which means $\gamma_i v^+$ is in the -1 eigenspace for γ_5, i.e. $\gamma_i v^+ \in \mathbb{S}^-$.

Remark 44.0.14: Suppose one has a spin structure and $\tilde{P} \to X$ is a principal $\mathrm{Spin}(n)$ bundle. There are bundles over this of the form $\rho : \mathrm{Spin}(n) \to \mathrm{GL}(\mathbb{S}^{\pm})$, yielding the **spinor bundle**

$$\tilde{P} \underset{\mathrm{Spin}(n)}{\times} \mathbb{S} = \mathbb{S}_x^+ \oplus \mathbb{S}_x^-.$$

Remark 44.0.15: Let $G \xrightarrow{\rho} \mathrm{GL}(V)$ be any representation. If $\varphi \in \mathrm{End}(V)$ commutes with $\rho(G)$, then the eigenspaces of φ are subrepresentations. In other words, $G \curvearrowright V = \bigoplus_{i=1}^n V_{\lambda_i}$, then $G \curvearrowright V_{\lambda_i}$ is a subrepresentation, using that

$$\varphi(v) = \lambda v \implies gv = g\varphi(\lambda^{-1} v) = \varphi\rho(g)\lambda^{-1} v,$$

which says $\varphi(\rho(g) \cdot v) = \lambda(\rho(g) \cdot v) \implies \rho(g) \cdot v \in V_\lambda$. We used it here by This rephrases Schur's lemma!

45 Spin Bundles and Dirac Operators (Monday, May 03)

Remark 45.0.1: Last time: we defined a Spin structure on an oriented manifold M as a lift of the principal $\mathrm{SO}(n)$ bundle $P \to M$ (*unassociated* to TM) to a $\mathrm{Spin}(n)$ bundle $\tilde{P}$. There was a **spin representation** $\mathrm{Spin}(n) \curvearrowright \mathbb{S}$, which is irreducible for $\mathrm{Cl}(\mathbb{R}^n)$ and splits as $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$, which are $\mathrm{Spin}(n)$ subrepresentations. We defined **spinor bundles**

$$\tilde{P} \underset{\mathrm{Spin}(n)}{\times} \mathbb{S} = \mathbb{S}_M = \mathbb{S}_M^+ \oplus \mathbb{S}_M^-.$$

Example 45.0.2 ***(Dimension 4):*** If $\dim_{\mathbb{R}} M = 4$, then $\mathbb{S}_M^{\pm} \in \mathsf{Vect}_{\mathbb{C}}^{\text{rank}=2}$, i.e. they are complex vector bundles of rank 2. Consider the eigenspaces $-e_1e_2e_3e_4 \curvearrowright \mathbb{S}$, then $e_i \cdot (-) : \mathbb{S}^{\pm} \to \mathbb{S}^{\mp}$.

Remark 45.0.3: Principal bundle: fibers are left G-torsors. In the fiber product, the group sits in the middle and acts on each factor. So $\tilde{P}$ eats the right G-action, and $\mathbb{S}$ eats the left action. Remarkably, for Spin bundles, there is an action leftover.

Proposition 45.0.4 ***(The spin bundle is a Clifford module).***
The spin bundle $\mathbb{S}_M$ naturally has the structure of a $\mathrm{Cl}(M)$-module.

Proof (?).
We have a Clifford action

$$\begin{aligned} \mathrm{Cl}(\mathbb{R}^n) \otimes \mathbb{S} &\to \mathbb{S} \\ x \otimes s &\mapsto x \cdot s. \end{aligned}$$

Recall that we have a natural conjugation action $\mathrm{Spin}(n) \curvearrowright \mathrm{Cl}(\mathbb{R}^n)$ where $g \mapsto g(-)g^{-1}$, and similarly $\mathrm{Spin}(n) \curvearrowright \mathbb{S}$ by $g \mapsto g \cdot (-)$. Given any $V \to W$ of G-modules, any $P \in \mathrm{Bun}^{\mathrm{prin}}(G)$ yields an induced module

$$P \underset{G}{\times} V \to P \underset{G}{\times} W,$$

and moreover $\tilde{P} \underset{\mathrm{Spin}(n)}{\times} \mathrm{Cl}(\mathbb{R}^n) = \mathrm{Cl}(M)$. We then conclude that there is an action $\mathrm{Cl}(M) \otimes \mathbb{S}_M \to \mathbb{S}_M$, the **Clifford multiplication**.

■

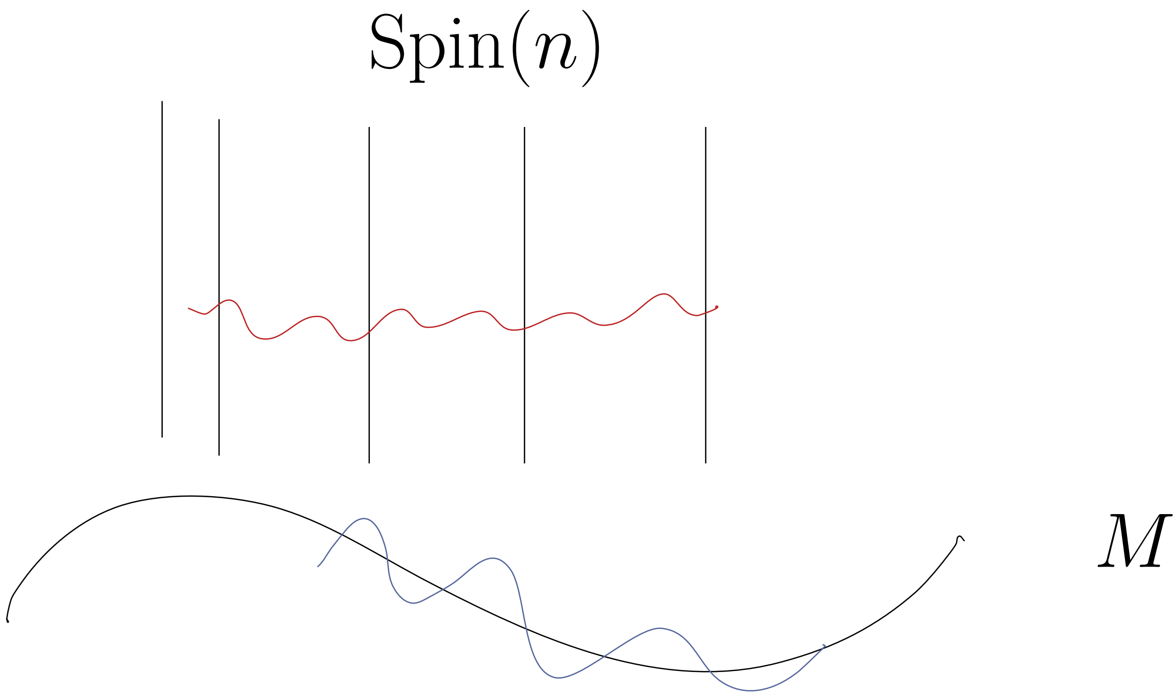
Remark 45.0.5: We have an isomorphism of bundles (not of algebras) $\mathrm{Cl}(M) \cong \bigwedge T^\vee M$, and any one form ω is an analogue of an element of $V^{\otimes 1}$, and $\omega \cdot (\mathbb{S}^+, \mathbb{S}^-) \in \mathbb{S}_M^- \oplus \mathbb{S}_M^+$.

Definition 45.0.6 (Clifford connection)
A connection ∇ on $\mathbb{S}$ is a **Clifford connection** if

$$\nabla(x \cdot s) = x \cdot \nabla(s) + d(x) \cdot s \qquad x \in H^0\mathrm{Cl}(M) = H^0\left(\bigwedge^* T^\vee M\right),\ s \in H^0(\mathbb{S}_M),$$

where d is the de Rham differential.

Remark 45.0.7: It is not obvious that a Clifford connection exists! We have $\mathbb{S}_M = \tilde{P} \underset{\mathrm{Spin}(n)}{\times} \mathbb{S}$, so it suffices to give a connection on $\tilde{P}$ which is $\mathrm{Spin}(n)$ invariant, since any associated bundle will inherit the connection. Idea: we need a notion of parallel transport. This is a principal $\mathrm{Spin}(n)$ bundle, so the fibers look like $\mathrm{Spin}(n)$, and we want to lift paths in M to paths in $\tilde{P}$:



It suffices to give a connection on P, and using that $\tilde{P} \to P$ is a 2 to 1 covering map, we can take a connecting on P coming from $\mathrm{OFrame}(T^\vee M, g^\vee)$. So it further suffices to produce a connection on $T^\vee M$ preserving orthogonality of frames under parallel transport, which is essentially the definition of the Levi-Cevita connection ∇^{LC}. Then the ∇ associated to ∇^{LC} on P is a Clifford connection, yielding existence.

Remark 45.0.8: The set of Clifford connections is a torsor over $\Omega^1(M)$. The association is $\nabla \mapsto \nabla - \nabla^{\mathrm{LC}}$, and one can compute

$$(\nabla - \nabla^{\mathrm{LC}})(x \cdot s) = x \cdot (\nabla - \nabla^{\mathrm{LC}})(s),$$

which exactly says that this is a $\mathrm{Cl}(M)$-linear map $\mathbb{S}_M \to \mathbb{S}_M \otimes \Omega^1$. We can write $\mathrm{Cl}(M) \cong \mathrm{End}(\mathbb{S}_M)$, and one can check that $[\mathrm{End}\,\mathbb{S}_M, \mathrm{End}\,\mathbb{S}_M]$ consists only of scalars.

Definition 45.0.9 (Dirac Operator)
Let ∇ be a Clifford connection on $\mathbb{S}_M$ and $s \in H^0(\mathbb{S}_M)$, so $\nabla(s) \in \mathbb{S}_M \otimes \Omega^1(M)$. Then the **Dirac operator** is defined as

$$\begin{aligned} \partial\!\!\!/ : H^0(\mathbb{S}) &\to H^0(\mathbb{S}) \\ s &\mapsto \sum_{e_i \in \mathrm{Fr}(T^\vee M)} e_i \cdot \nabla_{e_i^\vee}(s) \end{aligned}$$

where

- $\nabla(s) = H^0(\mathbb{S}_M \otimes \Omega^1)$
- $\nabla_{e_i^\vee}(s) = \nabla(s)(e_i{}^\vee) \in H^0(\mathbb{S}_M)$

Remark 45.0.10: This makes sense locally, and is well-defined independent of choice of frame. Henceforth, we'll take $\nabla = \nabla^{\text{LC}}$ – in this case, if $s^+ \in H^0(\mathbb{S}^\pm)$ then $\nabla_v^{\text{LC}}(s^\pm) \in H^0(\mathbb{S}^\pm)$. This is an order 1 differential operator:

$$\not{\partial}_{\nabla^{\text{LC}}} = \not{\partial} : H^0(\mathbb{S}^\pm) \to H^p(\mathbb{S}^\mp).$$

Proposition 45.0.11 ***(Relation between Dirac operator and Laplacian).***

$$\not{\partial}^2 = -\Delta.$$

Proof (?).

Given $\psi \in H^0(\mathbb{S})$, write $\psi = \sum_{i=1}^4 \psi_i s_i$ with the s_i forming a local frame of $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$. We can write

$$\not{\partial}\psi = \sum e_i \partial_{x_i}\psi = \sum_{i=1}^4 \gamma_i \psi_{x_i}.$$

where $\psi_{x_i} = [(\psi_1)_{x_i}, (\psi_2)_{x_i}, \cdots]$. We then have

$$\begin{aligned}
\not{\partial}^2\psi &= \sum_{i,j} \gamma_i\gamma_j\psi_{x_ix_j} \\
&= -\sum_{ij} 2(e_i \cdot_g e_j)\psi_{x_ix_j} \\
&= -2\sum_{ij} \delta_{ij}\psi_{x_ix_j} \\
&= -2\sum_{i} \psi_{x_ix_i} \\
&= -2\left(\sum_{i=1}^4 \partial_{x_i}^2\right)\psi \\
&= -2\Delta.
\end{aligned}$$

where we sum over *all* i, j and can pair terms, and we use that $\gamma_i\gamma_j + \gamma_j\gamma_i = -2e_1 \cdot e_j$

Upshot: $\not{\partial} \in \sqrt{\Delta}$, which is why the Dirac is an invariant in quantum mechanics. This reduces the 2nd order Schrödinger operator a 1st order operator. Note that $\not{\partial}\psi = 0$ is the equation for a massless particle.

■

See maybe Lawson's spin geometry? Or Salamon.

46 | Wednesday, May 05

46.1 Fun Physics Aside

Remark 46.1.1: Last time: we showed $\mathrm{Cl}(X) \coloneqq \mathrm{Cl}(T^{\vee}X, g^{\vee})$ acts on the spinor bundle $\mathbb{S}_X \coloneqq \tilde{P} \underset{\mathrm{Spin}(n)}{\times} \mathbb{S}$ by Clifford multiplication. For $\dim_{\mathbb{R}} X = 4$, we have a splitting $\mathbb{S}^+ \oplus \mathbb{S}^-$ as complex rank 2 vector bundles. If $\omega \in H^0\mathrm{Cl}(X)$ is a one form, then $\omega\mathbb{S}_X^{\pm} \subset \mathbb{S}^{\mp}$.

Definition 46.1.2 (Clifford Connection)
A **Clifford connection** is a map

$$\nabla : \mathbb{S}_X \to \mathbb{S}_X \otimes \Omega^1$$

.

where $\alpha \cdot s \mapsto \alpha \cdot \nabla s + dx \cdot s$.

Remark 46.1.3: There is a distinguished Clifford connection associated to ∇^{LC}. Also recall that we defined a Dirac operator $\partial\!\!\!/$ and showed $\partial\!\!\!/^2 = -2\Delta$.

Definition 46.1.4 (The Dirac Equation)
The **Dirac equation** is defined on $\psi \in H^0(X, \mathbb{S})$ as

$$(i\,\partial\!\!\!/ + m\omega)\psi = 0.$$

Here m denotes a mass, $\omega = \omega_{\mathbb{C}} = \displaystyle\prod_{i=1}^{4} \gamma_i$.

Remark 46.1.5: This describes fermions in a vacuum, e.g. an electron where ψ is its wave function. Applying this to $\mathbb{R}^4$ with $g = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$, then this equation in ψ is invariant under the Lorentz group $O(\mathbb{R}^4, g)$.

46.2 Rohklin's Theorem

Theorem 46.2.1 ***(Rohklin's Theorem).***
Let X be a smooth closed oriented spin 4-manifold. Then the signature $\sigma(X) \coloneqq b_2^+(X) - b_2^-(X)$ (the dimensions of positive/negative definite subspaces of $H^2(X; \mathbb{R})$ is divisible by 16.

Remark 46.2.2: This restricts what topological manifolds can admit smooth structures. Freedman constructed a topological manifold of dimension 4 with signature 8, which thus can not admit a smooth structure. Recall that having a spin structure was having a lift of a principal $\mathrm{SO}(n)$ bundle over $(T^\vee X, g)$ (namely $\mathrm{Frame}(X)$) to a $\mathrm{Spin}(n)$ bundle.

$$\begin{tikzcd}
\tilde{P}_{\mathrm{SO}(V)} \coloneqq P_{\mathrm{Spin}(V)} \arrow[d, "\tilde{p}"'] \arrow[dr, dashed, "\exists"] & \\
P_{\mathrm{SO}(V)} \arrow[r, "\pi"'] & X
\end{tikzcd}$$

Link to Diagram

Diagram doesn't match definition, check Phil's notes!

46.2.1 Proof

Consider $\mathbb{S}_X \coloneqq \tilde{P} \underset{\mathrm{Spin}(n)}{\times} \mathbb{S}$, then define

$$\not{\partial}^{\pm} : H^0(\mathbb{S}_X^{\pm}) \to H^0(\mathbb{S}^{\mp}).$$

Note that we can write $\not{\partial} = \not{\partial}^+ + \not{\partial}^-$;

- Step 1: Show $\operatorname{ind} \not{\partial}^+ = -\sigma(X)/8$,
- Step 2: Show $\operatorname{ind} \not{\partial}^+$ is even.

46.2.2 Step 1

What is the symbol $\mathrm{Symb}(\not{\partial})$? By definition

$$\mathrm{Symb}\, \not{\partial} : \pi^* \mathbb{S} \to \pi^* \mathbb{S}.$$

where $\pi : T^\vee X \to X$, and the symbol was defined by replacing $\dfrac{\partial}{\partial x_i}$ with a function $y_i : T^\vee X \to \mathbb{R}$. We can write

$$\not{\partial} \varphi = \sum_{e_i \in \mathrm{Fr}} e_i \cdot \nabla_{e_i^\vee} \psi,$$

and so

$$\mathrm{Symb}\, \not{\partial}(\psi) = \sum_i y_i e_i = \psi.$$

We have a tautological form $\alpha \in H^0(T^\vee X, \pi^*\Omega^1)$ where $(p, \alpha) \mapsto \alpha$, and so $\mathrm{Symb}(\not{\partial})(-) = \alpha \cdot (-)$.

Claim:

$$\not{\partial} : H^0(\mathbb{S})\circlearrowleft \qquad \text{is an elliptic operator.}$$

We need to check that the map $\alpha \cdot (-)$ is exact if $\alpha \neq 0$.

We have $\alpha \cdot (-) : \mathbb{S} \to \mathbb{S}$ and

$$(-\alpha)(-)\,\alpha(-) = (-\alpha \cdot \alpha) = \|\alpha\|^2 \neq 0,$$

which makes the operator invertible away from zero. Thus we can apply Atiyah-Singer.

Lemma 46.2.3 ***(Formula for Chern characters).***
There is a nice formula for Chern characters:

$$\mathrm{ch}\,\mathbb{S}^+ - \mathrm{ch}\,\mathbb{S}^- = \prod_{i=1}^{n} (e^{x_i/2} - e^{-x_i/2}).$$

where $\{\pm x_i\}$ are the Chern roots of $T^\vee X$.

Proof (?).
Use the splitting principle to write

$$T^\vee X \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{i=1}^{n} L_i \otimes L_i^{-1}.$$

Then $\mathbb{S}^+$ is a sum of all tensor products of $L_i \otimes L_i^{-1}$ where the number of -1s appearing is even.

■

Remark 46.2.4: Note there is ambiguity up to 2-torsion in the formula, but this gets moved into the choice of spin structure, which amounts to choice of a square root of each of these line bundles.

Setting $2n \coloneqq \dim X$, we have

$$\begin{aligned}
\operatorname{ind} \not{\partial}^+ &= (-1)^n \int_X \frac{\operatorname{ch} \mathbb{S}^+ - \operatorname{ch} \mathbb{S}^-}{\mathrm{eul} X} \operatorname{td}(TX \otimes \mathbb{C}) \\
&= \int_X \frac{\prod e^{x_i/2} - e^{-x_i/2}}{(-1)^n \prod x_i} \prod \frac{x_i}{1 - e^{x_i}} \prod \frac{x_i}{1 - e^{-x_i}} \\
&= \int_X \prod \frac{(e^{x_i/2} - e^{-x_i/2}) x_i}{(1 - e^{x_i})(1 - e^{-x_i})} \\
&= (-1)^n \int_X \prod_I \frac{x_i}{e^{x_i/2} - e^{-x_i/2}} \\
&= \int_X \left(1 - \frac{x_1^2}{24}\right)\left(1 - \frac{x_2^2}{24}\right) \\
&= -\frac{1}{24} \int_X x_1^2 + x_2^2 + (x_1 + x_2)^2 - 2x_1 x_2 \\
&= -\frac{1}{24}\left(c_1^2 - 2c_2\right).
\end{aligned}$$

Remark 46.2.5: See the $\widehat{A}$ genus.

Claim:

$$c_1^2 - 2c^2 = 3 \cdot \sigma(X).$$

This is another application of Atiyah-Singer, applied to a slightly different operator. Recall the Hodge star operator,

$$\star : \Omega^k(X) \to \Omega^{4-k}(X).$$

Defining $\tau \coloneqq i^{\frac{k(k-1)+4}{2}}$, we get $\tau^2 = 1$, so define an operator $\tau\star$. This yields a splitting into ± 1 eigenspaces:

$$\Omega(X) = \Omega^+(X) \oplus \Omega^-(X).$$

Recalling that $d^\dagger$ was the adjoint of d, one can check that $d + d^\dagger : \Omega^\pm(X) \to \Omega^\mp(X)$ interchanges these. It turns out that $\operatorname{ind}(d + d^\dagger) = \sigma(X)$, which by Atiyah-Singer and Hermite forms will equal $\frac{c_1^2 - 2c_2}{3}$. This yields the desired formula for step 1.

46.3 Step 2

We now want to show ind $\not{\partial}^+$ is divisible by 2. The key point is that ker $\not{\partial}^+$ and coker $\not{\partial}^+ = \ker \not{\partial}^-$ admit a quaternionic vector space structure. This comes from the fact that

$$\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \cong S^1(\mathbb{H}) \oplus S^1(\mathbb{H}) := \mathbb{S}^+ \oplus \mathbb{S}^-,$$

so we have a splitting into subspaces of unit quaternions. It turns out that $\not{\partial}$ is $\mathbb{H}$-linear. So we get an equality

$$-\sigma(X)/8 = \mathrm{ind}\ \not{\partial}^+ = 2\lambda$$

for some λ, yielding $8 \mid \sigma(X)$.

46.4 Remarks

Remark 46.4.1: If $H_1(X;\mathbb{Z})$ has no 2-torsion, e.g. if $\pi_1 X = 0$, then $w_2(X) = 0$ iff the intersection form on H^2 is even, where w_2 is the obstruction to existence of spin structures. Note that this makes sense for topological manifolds and not just smooth manifolds, and in this case $\sigma(X)$ is divisible by 8. This restriction comes from number theory: since we have a unimodular lattice, it breaks into sums of $E_8, -E_8$, and H if indefinite, and any even unimodular lattice has signature divisible by 8. So this can work as an obstruction to the existence of smooth structures.

Remark 46.4.2: Note that $\mathbb{CP}^2$ has no spin structure, and $\sigma(\mathbb{CP}^2) = 1$. There's a way to modify the invariant to set $\sigma(X)/8 =?$ (mod 2).

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