Aspects of motivic cohomology

Matthew Morrow, IAS/PCMI GSS 2021

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Last updated: 2021-07-27

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1.1 Intro

Abstract:

Motivic cohomology offers, at least in certain situations, a geometric refinement of algebraic K-theory or its variants (G-theory, KH-theory, étale K-theory, $\cdots$). We will overview some aspects of the subject, ranging from the original cycle complexes of Bloch, through Voevodsky's work over fields, to more recent p-adic developments in the arithmetic context where perfectoid and prismatic techniques appear.

References/Background:

- Algebraic geometry, sheaf theory, cohomology.
  - Comfort with derived techniques such as descent and the cotangent complex would be helpful.
  - Casual familiarity with K-theory, cyclic homology, and their variants would be motivational.
  - Infinity-categories and spectra will appear, though probably not in a very essential way.
- Lecture Notes

Remark 1.1.1: Some things we've already seen that will be useful:

- Motivic complexes
- Milnor K-theory
- Their relations to étale cohomology (e.g. Bloch-Kato)
- $\mathbb{A}^1$-homotopy theory
- Categorical aspects (e.g. presheaves with transfer)

These have typically been for $\mathsf{smVar}_{/k}$. Our goals will be to study

- Motivic cohomology as a tool to analyze algebraic K-theory.
- Recent progress in mixed characteristic, with fewer smoothness/regularity hypothesis

1.2 K_0 and K_1

Remark 1.2.1: Some phenomena of K-theory to keep in mind:

- It encodes other invariants.
- It breaks into "simpler" pieces that are motivic in nature.

Definition 1.2.2 (The Grothendieck group (Grothendieck, 50s))
Let $R \in \mathsf{CRing}$, then define the **Grothendieck group** $\mathsf{K}_0(R)$ as the free abelian group:

$$\mathsf{K}_0(R) = \mathsf{R\text{-}Mod}^{\text{proj,fg,}\cong} / \sim .$$

where $[P] \sim [P'] + [P'']$ when there is a SES

$$0 \to P' \to P \to P'' \to 0.$$

Remark 1.2.3: There is an equivalent description as a group completion:

$$\mathsf{K}_0(R) = \left(\mathsf{R\text{-}Mod}^{\text{proj,fg,}\cong}, \oplus\right)^{\text{gp}} .$$

The same definitions work for any $X \in \mathsf{Sch}$ by replacing $\mathsf{R\text{-}Mod}^{\text{proj,fg}}$ with $\mathsf{Bun}_{\mathrm{GL}_r/X}$, the category of (algebraic) vector bundles over X.

Example 1.2.4*(?)*: For $F \in \mathsf{Field}$, the dimension induces an isomorphism:

$$\begin{aligned} \dim_F : \mathsf{K}_0(F) &\to \mathbb{Z} \\ [P] &\mapsto \dim_F P. \end{aligned}$$

Example 1.2.5*(?)*: Let $\mathcal{O} \in \mathsf{DedekindDom}$, e.g. the ring of integers in a number field, then any ideal $I \trianglelefteq \mathcal{O}$ is a finite projective module and defines some $[I] \in \mathsf{K}_0(\mathcal{O})$. There is a SES

$$0 \to \mathrm{Cl}(\mathcal{O}) \xrightarrow{I \mapsto [I]-[\mathcal{O}]} \mathsf{K}_0(\mathcal{O}) \xrightarrow{\mathrm{rank}_{\mathcal{O}}(-)} \mathbb{Z} \to 0.$$

Thus $\mathsf{K}_0(\mathcal{O})$ breaks up as $\mathrm{Cl}(\mathcal{O})$ and $\mathbb{Z}$, where the class group is a classical invariant: isomorphism classes of nonzero ideals.

Example 1.2.6*(?)*: Let $X \in \mathsf{smAlgVar}^{\text{qproj}}_{/k}$ over a field, and let $Z \hookrightarrow X$ be an irreducible closed subvariety. We can resolve the structure sheaf $\mathcal{O}_Z$ by vector bundles:

$$0 \leftarrow \mathcal{O}_Z \leftarrow P_0 \leftarrow \cdots P_d \leftarrow 0.$$

We can then define

$$[Z] := \sum_{i=0}^{d} (-1)^i [P_i] \in \mathsf{K}_0(X),$$

which turns out to be independent of the resolution picked. This yields a filtration:

$$\mathrm{Fil}_j \mathsf{K}_0(X) := \left\langle [Z] \;\middle|\; Z \hookrightarrow X \text{ irreducible closed, } \operatorname{codim}(Z) \le j \right\rangle$$

$$\implies \mathsf{K}_0(X) \supseteq \mathrm{Fil}_d \mathsf{K}_0(X) \supseteq \cdots \supseteq \mathrm{Fil}_0 \mathsf{K}_0(X) \supseteq 0.$$

Theorem 1.2.7 ***(Part of Riemann-Roch).***
There is a well-defined surjective map

$$\begin{aligned} \mathrm{CH}_j(X) \coloneqq \{j\text{-dimensional cycles}\}/\text{rational equivalence} &\to \frac{\mathrm{Fil}_j \mathsf{K}_0(X)}{\mathrm{Fil}_{j-1}\mathsf{K}_0(X)} \\ Z &\mapsto [Z], \end{aligned}$$

and the kernel is annihilated by $(j-1)!$.

Slogan 1.2.8
Up to small torsion, $\mathsf{K}_0(X)$ breaks into Chow groups.

Definition 1.2.9 (Bass, 50s)
Set

$$\mathsf{K}_1(R) \coloneqq \mathrm{GL}(R)/E(R) \coloneqq \bigcup_{n\geq 1} \mathrm{GL}_n(R)/E_n(R)$$

where we use the block inclusion

$$\begin{aligned} \mathrm{GL}_n(R) &\hookrightarrow \mathrm{GL}_{n+1} \\ g &\mapsto \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

and $E_n(R) \subseteq \mathrm{GL}_n(R)$ is the subgroup of elementary row and column operations performed on I_n.

Example 1.2.10 ***(?):*** There exists a determinant map

$$\begin{aligned} \det : \mathsf{K}_1(R) &\to R^\times \\ g &\mapsto \det(g), \end{aligned}$$

which has a right inverse $r \mapsto \mathrm{diag}(r, 1, 1, \cdots, 1)$.

Example 1.2.11 ***(?):*** For $F \in \mathsf{Field}$, we have $E_n(F) = \mathrm{SL}_n(F)$ by Gaussian elimination. Since every $g \in \mathrm{SL}_n(F)$ satisfies $\det(g) = 1$, there is an isomorphism

$$\det : \mathsf{K}_1(F) \xrightarrow{\sim} F^\times.$$

Remark 1.2.12: We can see a relation to étale cohomology here by using Kummer theory to identify

$$\mathsf{K}_1(F)/m \xrightarrow{\sim} F^\times/m \xrightarrow{\text{Kummer},\sim} H^1_{\mathsf{Gal}}(F; \mu_m)$$

for m prime to $\mathrm{ch}\, F$, so this is an easy case of Bloch-Kato.

Example 1.2.13 ***(?):*** For $\mathcal{O}$ the ring of integers in a number field, there is an isomorphism

$$\det : \mathsf{K}_1(\mathcal{O}) \xrightarrow{\sim} \mathcal{O}^\times,$$

but this is now a deep theorem due to Bass-Milnor-Serre, Kazhdan.

Example 1.2.14 ***(?):*** Let $D := \mathbb{R}[x,y]/\left\langle x^2+y^2-1 \right\rangle \in \mathsf{DedekindDom}$, then there is a nonzero class

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix} \in \ker \det,$$

so the previous result for $\mathcal{O}$ is not a general fact about Dedekind domains. It turns out that

$$\mathsf{K}_1(D) \xrightarrow{\sim} D^\times \oplus \mathcal{L},$$

where $\mathcal{L}$ encodes some information about loops which vanishes for number fields.

1.3 Higher Algebraic K-theory

Remark 1.3.1: By the 60s, it became clear that $\mathsf{K}_0, \mathsf{K}_1$ should be the first graded pieces in some exceptional cohomology theory, and there should exist some $\mathsf{K}_n(R)$ for all $n \geq 0$ (to be defined). Quillen's Fields was a result of proposing multiple definitions, including the following:

Definition 1.3.2 (The K-theory spectrum (Quillen, 73))
Define a K-theory space or spectrum (infinite loop space) by deriving the functor $\mathsf{K}_0(-)$:

$$K(R) := \mathsf{BGL}(R)^+ \times \mathsf{K}_0(R)$$

where $\pi_* \mathsf{BGL}(R) = \mathrm{GL}(R)$ for $* = 1$. Quillen's plus construction forces π_* to be abelian without changing the homology, although this changes homotopy in higher degrees. We then define

$$\mathsf{K}_n(R) := \pi_n \mathsf{K}(R).$$

Remark 1.3.3: This construction is good for the (hard!) hands-on calculations Quillen originally did, but a more modern point of view would be

- Setting $\mathsf{K}(R)$ to be the ∞-group completion of the $\mathbb{E}_\infty$ space associated to the category $\mathsf{R\text{-}Mod}^{\mathsf{proj},\cong}$.
- Regarding $\mathsf{K}(-)$ as the universal invariant of $\underset{\infty}{\mathsf{StabCat}}$ taking exact sequences in $\underset{\infty}{\mathsf{StabCat}}$ to cofibers sequences in the category of spectra Sp, in which case one defines

$$\mathsf{K}(R) := \mathsf{K}(\mathsf{PerfCh}\,(\mathsf{R\text{-}Mod}))$$

as $\mathsf{K}(-)$ of perfect complexes of R-modules.

Both constructions output groups $\mathsf{K}_n(R)$ for $n \geq 0$.

Example 1.3.4 ***(Quillen, 73):*** The only complete calculation of K groups that we have is

$$\mathsf{K}_n(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \text{ even} \\ \mathbb{Z}/\left\langle q^{\frac{n+1}{2}-1} \right\rangle & n \text{ odd.} \end{cases}$$

Example 1.3.5 ***(?):*** We know K groups are hard because $\mathsf{K}_{n>0}(\mathbb{Z}) = 0 \iff$ the Vandiver conjecture holds, which is widely open.

Check content of conjecture, maybe 4n?

Conjecture 1.3.6.
If $R \in \mathsf{Alg}^{\mathrm{ft,reg}}_{/\mathbb{Z}}$ then $\mathsf{K}_n(R)$ should be a finitely generated abelian group for all n. This is widely open, but known when $\dim R \leq 1$.

Example 1.3.7 ***(?):*** For $F \in \mathsf{Field}$ with $\operatorname{ch} F$ prime to $m \geq 1$, ten

$$\text{TateSymb} : \mathsf{K}_2(F)/m \xrightarrow{\sim} H^2_{\mathsf{Gal}}(F; \mu_m^{\otimes 2}),$$

which is a specialization of Bloch-Kato due to Merkurjev-Suslin.

Example 1.3.8 ***(Lichtenbaum, Quillen 70s):*** Partially motivated by special values of zeta functions, for a number field F and $m \geq 1$, formulae for $\mathsf{K}_n(F; \mathbb{Z}/m)$ were conjectured in terms of $H_{\text{ét}}$.

Remark 1.3.9: Here we're using **K-theory with coefficients**, where one takes a spectrum and constructs a mod m version of it fitting into a SES

$$0 \to \mathsf{K}_n(F)/m \to \mathsf{K}_n(F; \mathbb{Z}/m) \to \mathsf{K}_{n-1}(F)[m] \to 0.$$

However, it can be hard to reconstruct $\mathsf{K}_n(-)$ from $\mathsf{K}_n(-, \mathbb{Z}/m)$.

1.4 Arrival of Motivic Cohomology

Question 1.4.1
K-theory admits a refinement in the form of motivic cohomology, which splits into simpler pieces such as étale cohomology. In what generality does this phenomenon occur?

Example 1.4.2 ***(?):*** This is always true in topology: given $X \in \mathsf{Top}$, $\mathsf{K}_0^{\mathsf{Top}}$ can be defined using complex vector bundles, and using suspension and Bott periodicity one can define $\mathsf{K}_n^{\mathsf{Top}}(X)$ for all n.

Theorem 1.4.3 ***(Atiyah-Hirzebruch).***
There is a spectral sequence which degenerates rationally:

$$E_2^{i,j} = H^{i-j}_{\mathrm{Sing}}(X;\mathbb{Z}) \Rightarrow \mathsf{K}^{\mathsf{Top}}_{-i-j}(X).$$

Remark 1.4.4: So up to small torsion, topological K-theory breaks up into singular cohomology. Motivated by this, we have the following

1.5 Big Conjecture

Conjecture 1.5.1 ***(Existence of motivic cohomology (Beilinson-Lichtenbaum, 80s)).***
For any $X \in \mathsf{smVar}_{/k}$, there should exist **motivic complexes**

$$\mathbb{Z}_{\mathrm{mot}}(j)(X), \qquad j \geq 0$$

whose homology, the **weight** j **motivic cohomology of** X, has the following expected properties:

- There is some analog of the Atiyah-Hirzebruch spectral sequence which degenerates rationally:

  $$E_2^{i,j} = H^{i-j}_{\mathrm{mot}}(X;\mathbb{Z}(-j)) \Rightarrow \mathsf{K}_{-i-j}(X),$$

  where $H^*_{\mathrm{mot}}(-)$ is taking kernels mod images for the complex $\mathbb{Z}_{\mathrm{mot}}(\bullet)(X)$ satisfying descent.

- In low weights, we have
  - $\mathbb{Z}_{\mathrm{mot}}(0)(X) = \mathbb{Z}^{\#\pi_0(X)}[0]$ in degree 0, supported in degree zero.
  - $\mathbb{Z}_{\mathrm{mot}}(1)(X) = \mathbb{R}\Gamma_{\mathrm{zar}}(X;\mathcal{O}_X^\times)[-1]$, supported in degrees 1 and 2 for a normal scheme after the right-shift.

- Range of support: $\mathbb{Z}_{\mathrm{mot}}(j)(X)$ is supported in degrees $0, \cdots, 2j$, and in degrees $\leq j$ if $X = \operatorname{Spec} R$ for R a local ring.

- Relation to Chow groups:

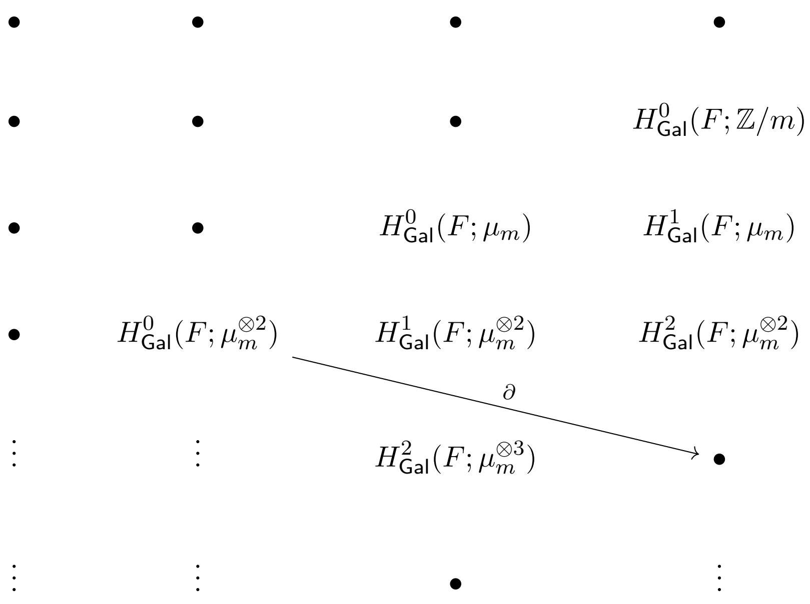
  $$H^{2j}_{\mathrm{mot}}(X;\mathbb{Z}(j)) \xrightarrow{\sim} \mathrm{CH}^j(X).$$

- Relation to étale cohomology (Beilinson-Lichtenbaum conjecture): taking the complex mod m and taking homology yields

  $$H^i_{\mathrm{mot}}(X;\mathbb{Z}/m(j)) \xrightarrow{\sim} H^i_{\text{ét}}(X;\mu_m^{\otimes j})$$

  if m is prime to $\operatorname{ch} k$ and $i \leq j$.

Example 1.5.2 ***(?):*** Considering computing $\mathsf{K}_n(F) \pmod{m}$ for m odd and for number fields F, as predicted by Lichtenbaum-Quillen. The mod m AHSS is simple in this case, since $\operatorname{cohdim} F \leq 2$:



Link to Diagram

The differentials are all zero, so we obtain

$$\mathsf{K}_{2j-1}(F;\mathbb{Z}/m) \xrightarrow{\sim} H^1_{\mathsf{Gal}}(F;\mu_m^{\otimes j})$$

and

$$0 \to H^2_{\mathsf{Gal}}(F,\mu_m^{\otimes j+1}) \to \mathsf{K}_{2j}(F;\mathbb{Z}/m) \to H^0_{\mathsf{Gal}}(F;\mu_m^{\otimes j}) \to 0.$$

Theorem 1.5.3 ***(Bloch, Levine, Friedlander, Rost, Suslin, Voevodsky, $\cdots$).*** The above conjectures are true **except** for Beilinson-Soulé vanishing, i.e. the conjecture that $\mathbb{Z}_{\mathrm{mot}}(j)(X)$ is supported in positive degrees $n \geq 0$.

Remark 1.5.4: Remarkably, one can write a definition somewhat easily which turns out to work in a fair amount of generality for schemes over a Dedekind domain.

Definition 1.5.5 (Higher Chow groups)
For $X \in \mathsf{Var}_{/k}$, let $z^j(X,n)$ be the free abelian group of codimension j irreducible closed subschemes of $X \underset{F}{\times} \Delta^n$ intersecting all faces properly, where

$$\Delta^n = \operatorname{Spec}\left(\frac{F[T_0,\cdots,T_n]}{\langle \sum T_i - 1\rangle}\right) \cong \mathbb{A}^n_{/F},$$

which contains "faces" Δ^m for $m \leq n$, and *properly* means the intersections are of the expected codimension. Then **Bloch's complex of higher cycles** is the complex $z^j(X,\bullet)$ where the boundary map is the alternating sum

$$z^j(X,n) \ni \partial(Z) = \sum_{i=0}^{n} (-1)^i [Z \cap \mathrm{Face}_i(X \times \Delta^{n-1})],$$

Bloch's higher Chow groups are the cohomology of this complex:

$$\mathsf{Ch}^j(X,n) \coloneqq H_n(z^j(X,\bullet)),$$

and then the following complex has the expected properties:

$$\mathbb{Z}_{\mathrm{mot}}(j)(X) \coloneqq z^j(X,\bullet)[-2j]$$

Remark 1.5.6: Déglise's talks present the machinery one needs to go through to verify this!

1.6 Milnor K-theory and Bloch-Kato

Remark 1.6.1: How is motivic cohomology related to the Bloch-Kato conjecture? Recall from Danny's talks that for $F \in \mathsf{Field}$ then one can form

$$\mathsf{K}_j^{\mathrm{M}}(F) = (F^\times)^{\otimes_F^j} / \langle \text{Steinberg relations} \rangle ,$$

and for $m \geq 1$ prime to $\operatorname{ch} F$ we can take Tate/Galois/cohomological symbols

$$\mathrm{TateSymb} : \mathsf{K}_j^{\mathrm{M}}(F)/m \to H^j_{\mathsf{Gal}}(F; \mu_m^{\otimes j}).$$

where $\mu_m^{\otimes j}$ is the jth Tate twist. Bloch-Kato conjectures that this is an isomorphism, and it is a theorem due to Rost-Voevodsky that the Tate symbol is an isomorphism. The following theorem says that a piece of H_{mot} can be identified as something coming from K^{M}:

Theorem 1.6.2 ***(Nesterenko-Suslin, Totaro).***
For any $F \in \mathsf{Field}$, for each $j \geq 1$ there is a natural isomorphism

$$\mathsf{K}_j^{\mathrm{M}}(F) \xrightarrow{\sim} H^j_{\mathrm{mot}}(F; \mathbb{Z}(j)).$$

Remark 1.6.3: Taking things mod m yields

$$\mathsf{K}_j^{\mathrm{M}}(F)/m \xrightarrow{\sim} H^j_{\mathrm{mot}}(F; \mathbb{Z}/m(j)) \xrightarrow{\sim,\mathrm{BL}} H^j_{\text{ét}}(F; \mu_m^{\otimes j}),$$

where the conjecture is that the obstruction term for the first isomorphism coming from H^{j+1} vanishes for local objects, and Beilinson-Lichtenbaum supplies the second isomorphism. The composite is the Bloch-Kato isomorphism, so Beilinson-Lichtenbaum $\implies$ Bloch-Kato, and it turns out that the converse is essentially true as well. This is also intertwined with the Hilbert 90 conjecture.

Tomorrow: we'll discard coprime hypotheses, look at p-adic phenomena, and look at what happens étale locally.

2 | Matthew Morrow, Talk 2 (Friday, July 16)

Remark 2.0.1: A review of yesterday:

- K-theory can be refined by motivic cohomology, i.e. it breaks into pieces. More precisely we have the Atiyah-Hirzebruch spectral sequence, and even better, the spectrum $\mathsf{K}(X)$ has a motivic filtration with graded pieces $\mathbb{Z}_{\text{mot}}(j)(X)[2j]$.
- The $\mathbb{Z}_{\text{mot}}(j)(X)$ correspond to algebraic cycles and étale cohomology mod m, where m is prime to $\operatorname{ch} k$, due to Beilinson-Lichtenbaum and Beilinson-Bloch.

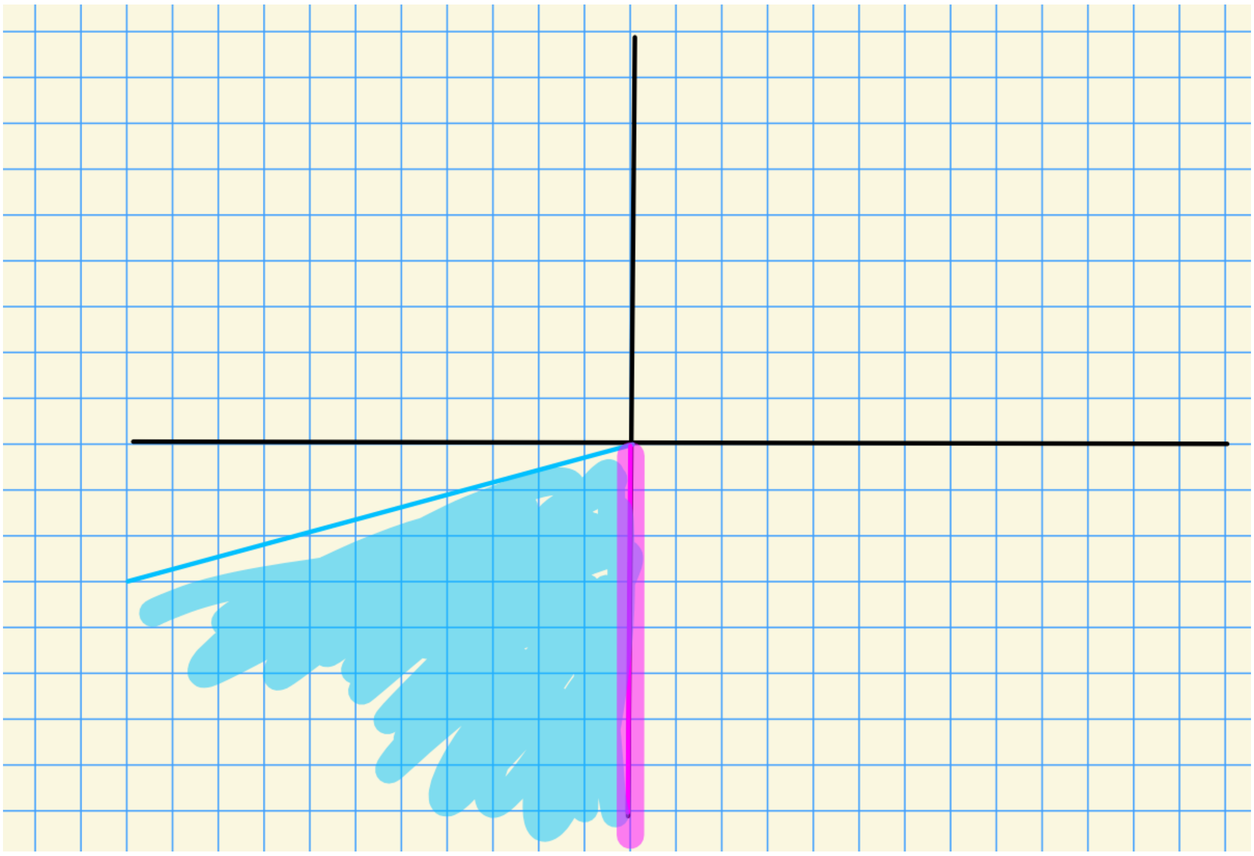
Today we'll look at the classical mod p theory, and variations on a theme: e.g. replacing K-theory with similar invariants, or weakening the hypotheses on X. We'll also discuss recent progress in the case of étale K-theory, particularly p-adically.

2.1 Mod p motivic cohomology in characteristic p

Remark 2.1.1: For $F \in \mathsf{Field}$ and $m \geq 1$ prime to $\operatorname{ch} F$, the Atiyah-Hirzebruch spectral sequence mod m takes the following form:

$$E_2^{i,j} = H^{i,j}_{\text{mot}}(F, \mathbb{Z}/m(-j)) \overset{BL}{=} \begin{cases} H^{i-j}_{\mathsf{Gal}}(F; \mu_m^{\otimes j}) & i \leq 0 \\ 0 & i > 0. \end{cases}.$$

Thus E_2 is supported in a quadrant four wedge:



We know the axis:

$$H^j(F;\mu_m^{\otimes j}) \xrightarrow{\sim} \mathsf{K}_j^{\mathrm{M}}(F)/m.$$

What happens if $m > p = \operatorname{ch} F$ for $\operatorname{ch} F > 0$?

Theorem 2.1.2 ***(Izhbolidin (90), Bloch-Kato-Gabber (86), Geisser-Levine (2000)).***
Let $F \in \mathsf{Field}^{\mathrm{ch}=p}$, then

- $\mathsf{K}_j^{\mathrm{M}}(F)$ and $\mathsf{K}_j(F)$ are p-torsionfree.
- $\mathsf{K}_j(F)/p \hookleftarrow \mathsf{K}_j^{\mathrm{M}}(F)/p \xhookrightarrow{\mathrm{dLog}} \Omega_F^j$

Definition 2.1.3 (dLog)
The dLog map is defined as

$$\begin{aligned} \mathrm{dLog} : \mathsf{K}_j^{\mathrm{M}}(F)/p &\to \Omega_f^j \\ \bigotimes_i \alpha_i &\mapsto \bigwedge\nolimits_i \frac{d\alpha_i}{\alpha_i}, \end{aligned}$$

and we write $\Omega^j_{F,\log} := \operatorname{im} \mathrm{dLog}$.

Remark 2.1.4: So the above theorem is about showing the injectivity of dLog. What Geisser-Levine really prove is that

$$\mathbb{Z}_{\text{mot}}(j)(F)/p \xrightarrow{\sim} \Omega^j_{F,\log}[-j].$$

Thus the mod p Atiyah-Hirzebruch spectral sequence, just motivic cohomology lives along the axis

$$E_2^{i,j} = \begin{cases} \Omega^{-j}_{F,\log} & i = 0 \\ 0 & \text{else} \end{cases} \Rightarrow \mathsf{K}_{i-j}(F; \mathbb{Z}/p)$$

and $\mathsf{K}_j(F)/p \xrightarrow{\sim} \Omega^j_{F,\log}$.

Remark 2.1.5: So life is much nicer in p matching the characteristic! Some remarks:

- The isomorphism remains true with F replaced any $F \in \mathsf{Alg}^{\text{reg},\mathsf{loc},\text{Noeth}}_{/\mathbb{F}_p}$:

$$\mathsf{K}_j(F)/p \xrightarrow{\sim} \Omega^j_{F,\log}.$$

- The hard part of the theorem is showing that mod p, there is a surjection $\mathsf{K}^{\text{M}}_j(F) \twoheadrightarrow \mathsf{K}_j(F)$. The proof goes through using $z^j(F, \bullet)$ and the Atiyah-Hirzebruch spectral sequence, and seems to necessarily go through motivic cohomology.

Question 2.1.6
Is there a direct proof? Or can one even just show that

$$\mathsf{K}_j(F)/p = 0 \text{ for } j > [F : \mathbb{F}_p]_{\text{tr}}?$$

Conjecture 2.1.7 ***(Beilinson).***
This becomes an isomorphism after tensoring to $\mathbb{Q}$, so

$$\mathsf{K}^{\text{M}}_j(F) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathsf{K}_j(F) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

This is known to be true for finite fields.

Conjecture 2.1.8.

$$H^i_{\text{mot}}(F; Z(j)) \text{ is torsion unless } i = j.$$

This is wide open, and would follow from the following:

Conjecture 2.1.9 ***(Parshin).***
If $X \in \mathsf{smVar}^{\mathsf{proj}}_{/k}$ over k a finite field, then

$$H^i_{\text{mot}}(X; Z(j)) \text{ is torsion unless } i = 2j.$$

2.2 Variants on a theme

Question 2.2.1
What things (other than K-theory) can be motivically refined?

2.2.1 G-theory

Remark 2.2.2: Bloch's complex $z^j(X, \bullet)$ makes sense for any $X \in \mathsf{Sch}$, and for X finite type over R a field or a Dedekind domain. Its homology yields an Atiyah-Hirzebruch spectral sequence

$$E_2^{i,j} = \mathrm{CH}^{-j}(X, -i-j) \Rightarrow \mathsf{G}_{-i-j}(X),$$

where G-theory is the K-theory of $\mathsf{Coh}(X)$. See Levine's work.

Then $z^j(X, \bullet)$ defines **motivic Borel-Moore homology**[1] which refines G-theory.

2.2.2 K^{H}-theory

Remark 2.2.3: This is Weibel's "homotopy invariant K-theory", obtained by forcing homotopy invariance in a universal way, which satisfies

$$\mathsf{K}^{\mathrm{H}}(R[T]) \xrightarrow{\sim} \mathsf{K}^{\mathrm{H}}(R) \qquad \forall R.$$

One defines this as a simplicial spectrum

$$\mathsf{K}^{\mathrm{H}}(R) := \left| q \mapsto \mathsf{K}\left(\frac{R[T_0, \cdots, T_q]}{1 - \sum_{i=0}^{q} T_i} \right) \right|.$$

Remark 2.2.4: One hopes that for (reasonable) schemes X, there should exist an $\mathbb{A}^1$-invariant motivic cohomology such that

- There is an Atiyah-Hirzebruch spectral sequence converging to $\mathsf{K}^{\mathrm{H}}_{i-j}(X)$.
- Some Beilinson-Lichtenbaum properties.
- Some relation to cycles.

For X Noetherian with $\operatorname{krulldim} X < \infty$, the state-of-the-art is that stable homotopy machinery can produce an Atiyah-Hirzebruch spectral sequence using representability of K^{H} in $\mathsf{SH}(X)$ along with the slice filtration.

[1] Note that this is homology and not cohomology!

2.2.3 Motivic cohomology with modulus

Remark 2.2.5: Let $X \in \mathsf{smVar}$ and $D \hookrightarrow X$ an effective (not necessarily reduced) Cartier divisor – thought of where $X \setminus D$ is an open which is compactified after adding D. Then one constructs $z^j(X|D, \bullet)$ which are complexes of cycles in "good position" with respect to the boundary D.

Conjecture 2.2.6.
There is an Atiyah-Hirzebruch spectral sequence

$$E_2^{i,j} = \mathrm{CH}^j(X|D, (-i-j)) \Rightarrow \mathsf{K}_{-i-j}(X, D),$$

where the limiting term involves *relative K-groups*. So there is a motivic (i.e. cycle-theoretic) description of relative K-theory.

2.3 Étale K-theory

Remark 2.3.1: K-theory is simple étale-locally, at least away from the residue characteristic.

Theorem 2.3.2 *(Gabber, Suslin)*.
If $A \in \mathsf{locRing}$ is strictly Henselian with residue field k and $m \geq 1$ is prime to $\operatorname{ch} k$, then

$$\mathsf{K}_n(A; \mathbb{Z}/m) \xrightarrow{\sim} \mathsf{K}_n(k; \mathbb{Z}/m) \xrightarrow{\sim} \begin{cases} \mu_m(k)^{\otimes \frac{n}{2}} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Remark 2.3.3: The problem is that K-theory does *not* satisfy étale descent!

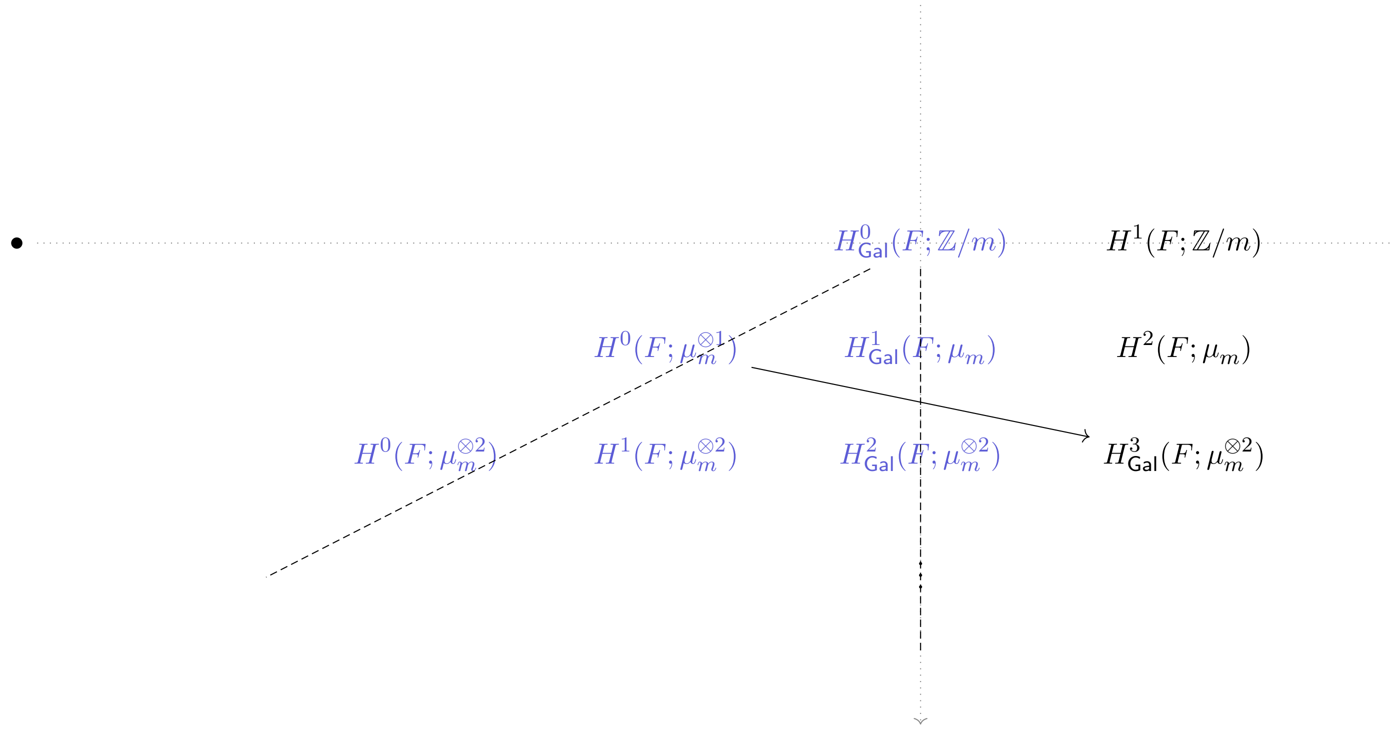
$$\text{For } B \in \mathsf{GalField}_{/A}^{\deg<\infty}, \qquad K(B)^{h\mathsf{Gal}(B_{/A})} \not\cong K(A).$$

View K-theory as a presheaf of spectra (in the sense of infinity sheaves), and define **étale K-theory** $K^{\text{ét}}$ to be the universal modification of K-theory to satisfy étale descent. This was considered by Thomason, Soulé, Friedlander.

Remark 2.3.4: Even better than $\mathsf{K}^{\text{ét}}$ is Clausen's **Selmer K-theory**, which does the right thing integrally. Up to subtle convergence issues, for any $X \in \mathsf{Sch}$ and m prime to $\operatorname{ch} X$ (the characteristic of the residue field) one gets an Atiyah-Hirzebruch spectral sequence

$$E_2^{i,j} = H_{\text{ét}}^{i-j}(X; \mu_m^{\otimes -j}) \Rightarrow \mathsf{K}_{i-j}^{\text{ét}}(X; \mathbb{Z}/m).$$

Letting F be a field and m prime to $\operatorname{ch} F$, the spectral sequence looks as follows:



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The whole thing converges to $\mathsf{K}^{\text{ét}}_{-i-j}(F;\mathbb{Z}/m)$, and the sector conjecturally converges to $\mathsf{K}_{-i-j}(F;\mathbb{Z}/m)$ by the Beilinson-Lichtenbaum conjecture.

2.4 Recent Progress

Remark 2.4.1: We now focus on

- Étale K-theory, $\mathsf{K}^{\text{ét}}$
- mod p coefficients, even period
- p-adically complete rings

The last is not a major restriction, since there is an arithmetic gluing square

$$\begin{array}{ccc} R & \longrightarrow & R\left[\frac{1}{p}\right] \\ \big\downarrow & & \big\downarrow \\ \widehat{R} & \longrightarrow & \widehat{R}\left[\frac{1}{p}\right] \end{array}$$

Link to Diagram

Here the bottom-left is the p-adic completion, and the right-hand side uses classical results when p is prime to all residue characteristic classes.

Theorem 2.4.2 ***(Bhatt-M-Scholze, Antieau-Matthew-M-Nikolaus, Lüders–M, Kelly-M).***
For any p-adically complete ring R (or in more generality, derived p-complete simplicial rings) one can associate a theory of **p-adic étale motivic cohomology** – p-complete complexes $\mathbb{Z}_p(j)(R)$ for $j \geq 0$ satisfying an analog of the Beilinson-Lichtenbaum conjectures:

1. An Atiyah-Hirzebruch spectral sequence:
$$E_2^{i,j} = H^{i-j}(\mathbb{Z}_p(j)(R)) \Rightarrow \mathsf{K}^{\text{ét}}_{-i-j}(R;\mathbb{Z})_{\widehat{p}}.$$

2. Known low weights:
$$\mathbb{Z}_p(0)(R) \xrightarrow{\sim} \mathbb{R}\Gamma_{\text{ét}}(R;\mathbb{Z}_p)$$
$$\mathbb{Z}_p(1)(R) \xrightarrow{\sim} \widehat{\mathbb{R}\Gamma_{\text{ét}}(R;\mathbb{G}_m)}[-1].$$

3. Range of support: $\mathbb{Z}_p(j)(R)$ is supported in degrees $d \leq j+1$, and even in degrees $d \leq n+1$ if the R-module $\Omega^1_{R/pR}$ is generated by $n' < n$ elements. It is supported in non-negative degrees if R is **quasisyntomic**, which is a mild smoothness condition that holds in particular if R is regular.

4. An analog of Nesterenko-Suslin: for $R \in \mathsf{locRing}$,
$$\widehat{\mathsf{K}}^{\mathrm{M}}_j(R) \xrightarrow{\sim} H^j(\mathbb{Z}_p(j)(R)),$$
where $\widehat{\mathsf{K}}^{\mathrm{M}}$ is the "improved Milnor K-theory" of Gabber-Kerz.

5. Comparison to Geisser-Levine: if R is smooth over a perfect characteristic p field, then
$$\mathbb{Z}_p(j)(R)/p \xrightarrow{\sim} \mathbb{R}\Gamma_{\text{ét}}(\operatorname{Spec} R;\Omega^j_{\log})[-j],$$
where $[-j]$ is a right-shift.

Remark 2.4.3: For simplicity, we'll write $H^i(j) := H^i(\mathbb{Z}_p(j)(R))$. The spectral sequence looks like the following:

It converges to $K^{\text{ét}}_{-i-j}(R;\mathbb{Z}/p)$. The 0-column is $\widehat{\overbrace{\mathsf{K}^{\mathrm{M}}_{-j}(R)}}$, and we understand the 1-column: we have

$$H^{j+1} \xrightarrow{\sim} \varprojlim_r \tilde{v}_r(j)(R).$$

where $\tilde{v}_r(j)(R)$ are the mod p^r weight j Artin-Schreier obstruction. For example,

$$\tilde{v}_1(j)(R) \coloneqq \operatorname{coker}\left(1 - C^{-1} : \Omega^j_{R/pR} \to \frac{\Omega^j_{R/pR}}{\partial\Omega^{j-1}_{R/pR}}\right) = \frac{R}{pR + \left\{a^p - a \,\middle|\, a \in R\right\}}.$$

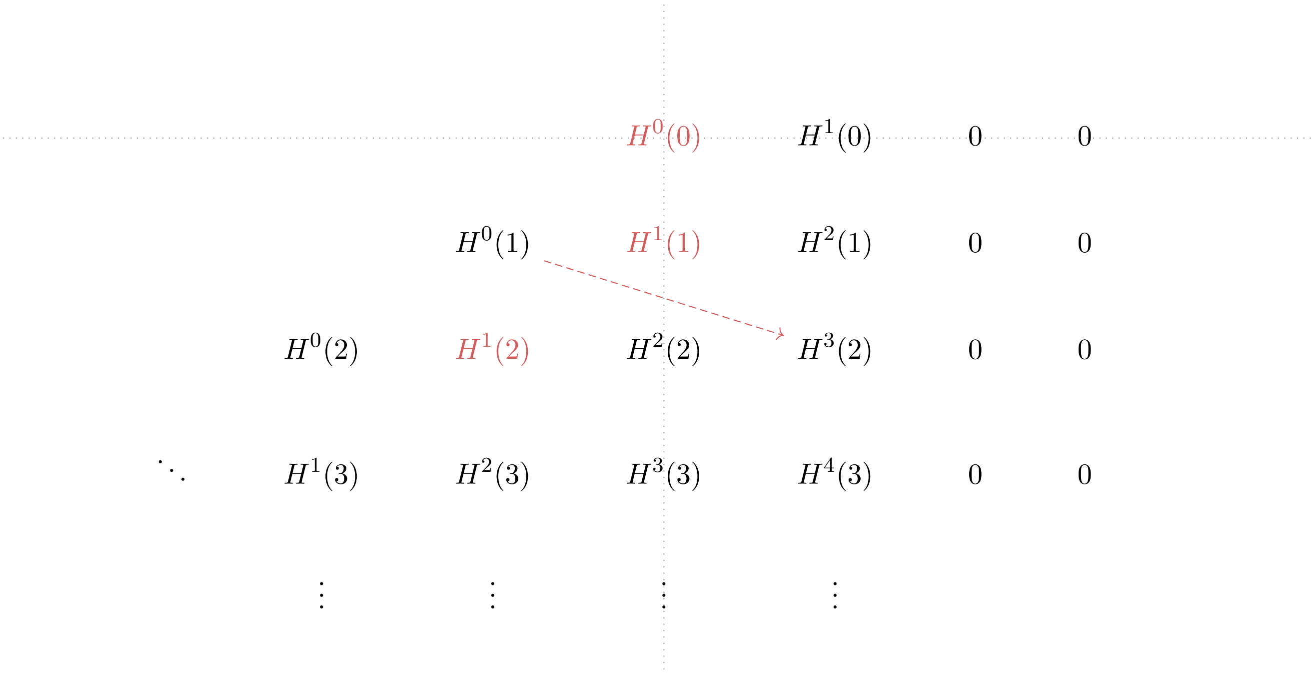
These are weird terms that capture some class field theory and are related to the Tate and Kato conjectures.

Theorem 2.4.4 ***((continued)).***
If R is local, then the 3rd quadrant of the above spectral sequence gives an Atiyah-Hirzebruch spectral sequence converging to $\mathsf{K}_{-i-j}(R;\mathbb{Z}_p)$.

Remark 2.4.5: So we get things describing étale K-theory, and after discarding a little bit we get something describing usual K-theory. Moreover, for any local p-adically complete ring R, we have broken $\mathsf{K}_*(R;\mathbb{Z}_p)$ into motivic pieces.

Example 2.4.6 ***(?):*** We same that for number fields, cohdim ≤ 2 yields a simple spectral sequence relating K groups to Galois cohomology. Consider now a truncated polynomial algebra $A = k[T]/T^r$ for $k \in \mathsf{PerfField}^{\mathrm{ch}=p}$ and let $r \geq 1$. Then by the general bounds given in the theorem, $H^i(j) = 0$ unless $0 \leq i \leq 2$, using that Ω can be generated by one element. Slightly more work will show H^0, H^2 vanish unless $i = j = 0$ (so higher weights vanish), since they're p-torsionfree and are killed by p.

So the spectral sequence collapses:



Link to Diagram

So the Atiyah-Hirzebruch spectral sequence collapses to

$$\mathsf{K}_n\left(\frac{K[T]}{\langle T^r\rangle}, \langle T\rangle\right) = \begin{cases} H^1\left(\mathbb{Z}_p\left(\frac{n+1}{2}\right)\right)(R) & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}.$$

When $r = 2$, one can even valuation these nontrivial terms.

Question 2.4.7
What is the motivic cohomology for regular schemes not over a field? We'd like to understand this in general.