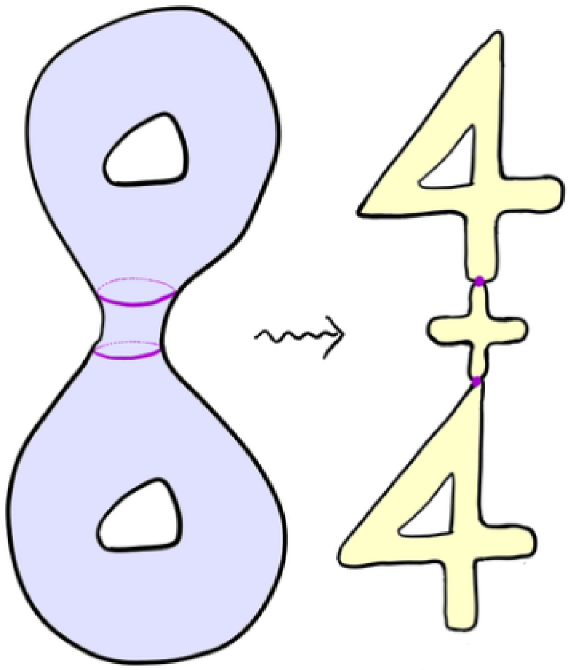

Notes: These are some sketchy personal notes from the ADDING 2022 conference at UGA.

ADDING 2022

Conference Notes

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1 Talk 1: Jordan Ellenberg, Sparsity of rational points on moduli spaces

ADDING Conference

Reference for this talk

Question 1.0.1
How many homogeneous forms in $\mathbb{Z}[x_0, \cdots, x_n]$ are there with discriminant equal to 1? More accurately, there is a PGL-action, so how many PGL orbits are there? More generally: how many are there with discriminant divisible only by primes in some finite set S?

Conjecture 1.0.2 ***(Shafarevich conjecture-esque).***
There are finitely many.

Remark 1.0.3: The general philosophy is that there should be only finitely many classes of X with good reduction outside of S – this is known e.g. for abelian varieties of dimension g, see Faltings' proof of Mordell.

Remark 1.0.4: In terms of rational points, there is a trichotomy:

- $g = 0$,
- $g = 1$,
- $g > 1$.

This in fact breaks into a dichotomy:

- $g \leq 1$,
- $g > 1$.

Definition 1.0.5 (Sparse points)
There are bounds on rational point counts on X with height at most B:

- $g = 0 \rightsquigarrow \ll_X B^2$
- $g = 1 \rightsquigarrow \ll_X \log B$ (Mordell)
- $g \geq 2 \rightsquigarrow \ll_X C_X$ where C is a constant depending on X (Faltings)

Here we'll dichotomize this in a different way: $g = 0$ and $g > 0$. We say X has **sparse points** if $\sharp X(\mathbb{Q}) \ll B^{\varepsilon}$.

Theorem 1.0.6 ***(E, Lawrence, Venkatesh 2021).***
Let $K \in \mathsf{Field}_{/\mathbb{Q}}, S \in \text{Places}(K)$ finite, $U_K \hookrightarrow \mathbb{P}^N$ a quasiprojective variety with a geometric variation of Hodge structure with finite-to-one period map, making U_X an interesting moduli space of something. Then the periods of $U(\mathcal{O}_K[\frac{1}{S}])$ are sparse.

Remark 1.0.7: The anabelian part: U has large π_1, so lots of etale covers. E.g. if U is a moduli of hypersurfaces, take the universal curve $\mathcal{H} \to U$, then $H^n(\mathcal{H}_t; C_m)$ varies and this can be interpreted as a moduli space with level structure. We try to show that large π_1 implies sparseness. This isn't quite true, since e.g. blowing up introduces lots of rational points, so a stronger condition is needed: $\pi_1(U)$ is infinite and for every finite-dimensional $V \subseteq U$, $\pi_1(V) \to \pi_1(U)$ has infinite image.

Remark 1.0.8: Why is it useful to have lots of etale covers? Consider $x - y = 1$ for $x, y \in \mathbb{Z}[\frac{1}{S}]^\times$ and $S = \{2, 3\}$. The S-units theorem of Siegel guarantees there are only finitely many solutions. One way to approach this: if $x = 2^a 3^b$, we know x up to squares: $x = s, 2s, 3s, 6s$ for s a square, as is y. This yields a system

$$\begin{aligned} m^2 - n^2 &= 1 \\ m^2 - 2n^2 &= 1 \\ 2m^2 - n^2 &= 1 \\ &\vdots \quad , \end{aligned}$$

which e.g. some can be solved using techniques for Pell equations. Having higher degree rigidifies the situation, so perhaps there are more techniques to solve them. Strategy: trade one hard equation for a finite list of higher degree easier equations.

Fact 1.0.9
A partitioning trick for integral points: $Y \to U$ is a finite etale cover, then there are a finite number of twists $\{Y_1, \cdots, Y_m\}$ (all isomorphic over $\overline{\mathbb{Q}}$) with $Y_i \to U$ such that every point in $U(\mathbb{Z}[\frac{1}{S}])$is in the image of $Y(\mathbb{Z}[\frac{1}{S}])$ for some i. So

$$\coprod_i Y_i(\mathbb{Z}[\tfrac{1}{S}]) \rightrightarrows U(\mathbb{Z}[\tfrac{1}{S}]).$$

Theorem 1.0.10 ***(Heath-Brown 2004 (determinantal methods)).***
Let $X \subset \mathbb{P}^2$ be a plane curve of degree d, then there is a uniform bound:

$$\sharp \left\{ p \in X(\mathbb{Q}) \;\middle|\; \text{ht}(p) \leq B \right\} \leq C_{d,\varepsilon} B^{2d+\varepsilon}.$$

Note: missed the exponent on B, need to fix.

Remark 1.0.11: Useful to control numbers of points for curves you know nothing about. Walsh removes the ε in all terms, Selberg, Brolog generalized to higher dimensions, CCDN make the constant effective. Pitch: these theorems are useful for other theorems which are not ostensibly about uniformity!

Remark 1.0.12: All we can control in this situation is the degree of $Y_i \to U$ and $U \subseteq \mathbb{P}^N$ has a degree, so we can control the degree of the Y_i. Broberg gives a bound $\ll B^{\frac{n+1}{d^{\frac{1}{n}}}}$ where $n = \dim U$. It doesn't actually matter what this is, just that it decreases in d, and we can take higher degree covers.

Remark 1.0.13: "Anabelian": π_1 somehow tells the entire story.

Remark 1.0.14: Heath-Brown's technique uses p-adic repulsion of points for $X(\mathbb{Q}) \to X(\mathbb{Q}_p)$ where low-height points do not end up nearby. Recall that there is a SES

$$1 \to \pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}) \to \pi_1^{\text{ét}}(X_{\mathbb{Q}}) \to G_{\mathbb{Q}} \to 1,$$

and any point $p \in X(\mathbb{Q})$ gives a section, thought of as $X(\mathbb{Q}) \to H^1(G_{\mathbb{Q}}; \pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}))$. Anabelian-ness: embed into some large interesting geometric space like this.

This cohomology group has a topology where $p, q \in X(\mathbb{Q})$ are nearby iff there exists a higher degree etale cover $Y \to X$ (small subsets correspond to large index subgroups in the profinite topology) such that p, q both lift to $Y(\mathbb{Q})$.

Remark 1.0.15: How this goes for curves: $C(\mathbb{Q}) \to \text{Jac}(\mathbb{Q})$, and one can tensor up to $\text{Jac}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z}_{\widehat{p}}$. Modern take: points are close if they differ by a power of p in the Mordell-Weil group. Interpretation of the main theorem: Heath-Brown in more general profinite topologies.

2 | Talk 2: Wanlin Li, Ceresa cycle and hyperellipticity

Remark 2.0.1: A hyperbolic curve is determined by its $\pi_1^{\text{ét}}$.

Remark 2.0.2: Recall $y^2 = f(x)$ defines a hyperelliptic curve C, which admits an involution $(x, y) \to (x, -y)$ and produces a degree 2 map

$$\begin{aligned} C &\to \mathbb{P}^1 \\ (x, y) &\mapsto x. \end{aligned}$$

Let $\bar{k}$ be a separable closure of k. There is a fibration induced by taking a geometric point of $\operatorname{Spec} k$ and pulling back:

$$\begin{array}{ccc} C_{\bar{k}} & \longrightarrow & C \\ \downarrow & \lrcorner & \downarrow \\ \operatorname{Spec} \bar{k} & \longrightarrow & \operatorname{Spec} k \end{array}$$

Link to Diagram

As in topology, this induces a LES in homotopy, which here splits into SESs. In particular,

$$1 \to \pi_1(C_{\bar{k}}) \to \pi_1(C_k) \to \mathsf{Gal}(\bar{k}/k).$$

A point induces a section and thus a map $\mathsf{Gal}(\bar{k}/k) \to \operatorname{Aut} \pi_1(C_{\bar{k}})$. Take the lower central series of $\pi := \pi_1(C_{\bar{k}})$, this induces

$$1 \to L^2\pi/L^3\pi \to \pi/L^3\pi \to \pi/L^1\pi = \pi^{\mathrm{ab}} \to 1$$

where the first term is abelian.

See Davis-Pries-Wickelgren for applications to Fermat curves.

Question 2.0.3
This extension corresponds to an element in $\mu(C) \in H^1(G_{\bar{k}}; \mathrm{Hom}(\pi^{\mathrm{ab}}, L^2\pi/L^3\pi))$, and when C is hyperelliptic $\mu(C) = 0$. Does the converse hold?

Theorem 2.0.4 ***(Bisogno-L-Litt-Srinivasan).***
There exist non-hyperelliptic curves C over k such that $\mu(C)$ is torsion – in particular, the *Fricke-Macbeath* curve, which is genus 7 Hurwitz. Moreover if $C_1 \to C_2$ then $\mu(C_1)$ torsion implies $\mu(C_2)$ torsion.

See https://arxiv.org/abs/2004.06146?

Theorem 2.0.5 ***(Harris-Pulte (Hain-Matsumoto)).***
$\mu(C)$ is the ℓ-adic cycles class associated to the **Ceresa cycle**.

See Hain-Matsumoto and Pulte

Remark 2.0.6: For $C_{/k}$ and $p \in C(K)$, the Abel-Jacobi map yields

$$\begin{aligned} \mathrm{AJ} : C &\hookrightarrow \mathrm{Jac}(C) \\ q &\mapsto [q-p]. \end{aligned}$$

So define the **Ceresa cycle** as

$$\tilde{c} := \mathrm{AJ}(C) - \overline{\mathrm{AJ}(C)} := [q-p] - [p-q].$$

Note that $\tilde{c}$ is homologically trivial in Chow, but algebraically *nontrivial* for a very general $C_{/\mathbb{C}}$ with $g \geq 3$.

Theorem 2.0.7 ***(Beauville).***
There is an explicit non-hyperelliptic curve C with $\tilde{c}$ torsion:

$$x^4 + xz^3 + y^3 z = 0 \subseteq \mathbb{P}^2, \qquad p = [0,0,1].$$

See https://arxiv.org/abs/2105.07160?

Remark 2.0.8: Consider curves over the local field $K = \mathbb{C}[[t]]$. Note $\mathsf{Gal}(\overline{\mathbb{C}[[t]]}/\mathbb{C}[[t]]) = \widehat{\mathbb{Z}}$, so this resembles a circle, and one can degenerate a family over the punctured disc. Apply nonabelian Picard-Lefschetz due to Asada-Matsumoto-Oda: if C has semistable reduction then the monodromy of $C/\mathbb{C}[[t]]$ is given by a multi-twist, i.e. a product of Dehn twists about simple closed curves. One can explicitly compute the Ceresa class in this situation. The degeneration data can be encoded as a tropical curve (essentially the dual graph of the special fiber).

See *Asada-Matsumoto-Oda*

Theorem 2.0.9 *(?).*
$\mu(C)$ is always torsion for C defined over $\mathbb{C}[[t]]$.

Remark 2.0.10: There is a notion of "hyperelliptic" for tropical curves: quotienting by the involution yields a tree.

3 Misc Notes

Remark 3.0.1: See Iwasawa group.

Conjecture 3.0.2.
Section conjecture: $\mathrm{Sec}(Y/K) \cong Y(K)$, i.e. every section comes from a rational point.

Remark 3.0.3: See the recent Lawrence-Venkatesh proof of Mordell. See *Selmer section set* and *adelic sections.*

Remark 3.0.4: Hyperbolic curves:

- $g = 0 \rightsquigarrow \mathbb{P}^1 \setminus Z$ where $\sharp Z \geq 3$
- $g = 1 \rightsquigarrow$ affine
- $g \geq 2$: anything.

For $X_{/K}$ for $K \in \mathsf{Field}_{/\mathbb{Q}}$ smooth hyperbolic with good reduction away from S, $\sharp X(\mathcal{O}_{K,S}) < \infty$ by Faltings. See Bloch-Kato and Fontaine-Mazur conjectures.

4 | Sunday, May 01

Remark 4.0.1: I missed the first two talks 😦

5 | Kiran Kedlaya: Crystalline companions as an anabelian phenomenon

Remark 5.0.1: Setup: $k = \mathbb{F}_q, q = p^n, X \in \mathsf{Sch}_{/k}$ smooth geometrically connected, $\ell \neq p$ arbitrary. Recall

$$1 \to \pi_1 X_{\bar{k}} \to \pi_1 X \to G_k = \widehat{\mathbb{Z}} \to 1.$$

Anabelian philosophy: everything you'd want to know about X is contained in $\pi_1 X$.

Theorem 5.0.2 ***(Tamagawa).***
If X is an affine curve, then $\pi_1^{\text{tame}} X$ determines X.

Remark 5.0.3: Problem: if X is an affine genus g curve with m punctures, $\pi_1^{\text{prime-to-}p} X_{\bar{k}}$ is the prime-to-p completion of $\mathsf{Free}(2g + m - 1)$, independently of X. Since sections induce $G_k \to \operatorname{Out}(\pi_1 X_{\bar{k}})$, we have lots of tame continuous $\overline{\mathbb{Q}_\ell}$ reps of $\pi_1 X_{\bar{k}}$, but very few are fixed by Frobenius.

Conjecture 5.0.4 ***(Deligne).***
Such representations only have "geometric origins", i.e. if $\mathcal{E}$ is a lisse $\mathbb{Q}_\ell$-sheaf, i.e. a lisse F-sheaf with $[F : \mathbb{Q}_\ell] < \infty$, which is irreducible with determinant of finite order, then it appears on relative etale cohomology of $\pi : Y \to X$ for some Y.

Remark 5.0.5: This is known in $\dim X = 1$, due to Deligne, around the same time Drinfeld proved Langlands for $\mathrm{GL}_2(k(X))$ for $k(X)$ a function field (or really the adeles). So all arithmetic reps of π_1 *come from geometry.*

Remark 5.0.6: Note that Y will eventually not even be a scheme. The determinant condition rules out transcendental twists. Galois side: lisse sheaves on X; automorphic side: reps values in $\mathrm{GL}_n(\mathbb{A}_K)$ for K a field. The proof above involves exhibiting the Galois objects as coming from relative etale cohomology in moduli of shtukas. A priori one only knows Frobenius traces, but this turns out to be enough to uniquely characterize things in this situation.

Conjecture 5.0.7.
Later Lafforgue did this for GL_n, but the corresponding statement about arithmetic reps is wide open.

Remark 5.0.8: If $\mathcal{E}$ as above, the Frobenius traces at all closed points $x \in |X|$ are algebraic over $\mathbb{Q}$.

Definition 5.0.9 (Companion sheaves)
Fix an algebraic closure $\overline{\mathbb{Q}}$ and two primes $\ell, \ell' \neq p$ and fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_{\ell'}$. Let $\mathcal{E}, \mathcal{E}'$ be lisse $\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_{\ell'}$ sheaves (resp.) on X. Say $\mathcal{E}, \mathcal{E}'$ are **companions** iff for every $x \in |X|$ the Frobenius traces at x are equal in $\overline{\mathbb{Q}}$.

Remark 5.0.10: Note that $\mathcal{E}$ determines $\mathcal{E}'$ up to semisimplification, using L-function techniques. Moreover properties like being irreducible or having finite determinant hold simultaneously for them.

Theorem 5.0.11 ***(Drinfeld, L. Lafforgue, Deligne).***
With this setup, a lisse $\overline{\mathbb{Q}}_\ell$ sheaf $\mathcal{E}_\ell$ admits a compantion $\mathcal{E}'$ which is a $\overline{\mathbb{Q}}_{\ell'}$ for chosen embeddings of $\mathbb{Q}$ for which all traces are in $\overline{\mathbb{Q}}$, i.e. irreducible and finite determinant. This is true in arbitrary dimension.

Remark 5.0.12: What about when $\ell = p$? There are somehow too many and too few lisse $\overline{\mathbb{Q}}_p$ sheaves! E.g. the Lefschetz trace formula doesn't hold. Instead use the Riemann-Hilbert correspondence – the p-adic analogues of lisse sheaves are certain p-adic integrable connections. How to construct: start with X affine and glue. Choose a smooth affine formal scheme over $W(k)$ with special fiber $P_k \cong X$. Let $K = \operatorname{ff} W(k)$ and P_K be the Raynaud generic fiber. See Tate model, Berkovich model, etc for rigid analytic geometry. Let $\mathbb{A}_K^n \rightsquigarrow \widehat{\mathbb{A}^n}_{W(k)}$, a closed unit disc over K.

Definition 5.0.13 (Convergent isocrystal)
Some definitions:

- A **convergent isocrystal** is a vector bundle with an integrable connection on P_K.
- A **convergent F-isocrystal** is this and a compatible action of (a lift of) Frobenius.
- An **overconvergent F-isocrystal** is this on some structural enlargment of P_K which takes the closed unit disc to the disc of radius $1 + \varepsilon$.

Remark 5.0.14: These are similar to $\pi_1(X_{\bar{k}})$-representations. Issue: p-adic antidifferentiation is hard; the integral of a formal power series converging on the closed disc only converges on the open disc.

Remark 5.0.15: If X is smooth this recovers Berthelot's rigid cohomology, which is a refinement of crystalline cohomology if X is proper. This yields a 6 functor formalism, and has the same moving parts as etale cohomology.

Theorem 5.0.16 ***(Abe).***
The Langlands correspondence extends to these when $\dim X = 1$.
The content here: Lafforgue's original proof can now be run with p-adic coefficients instead of just ℓ-adic coefficients.

Theorem 5.0.17 ***(Abe-?, Drinfeld-Abe-Esmult, K).***
The companion theorem extends to both $\ell = p$ and $\ell' = p$.

Remark 5.0.18: Deligne posited the existence of a *petit camarade crystalline*, a little crystalline friend. ❇️🙂

5.1 Applications

Remark 5.1.1: A partial result toward a conjecture of Simpson. Let $X_{/\mathbb{C}}$ be a smooth cohomologically rigid local system (so no nontrivial deformations) which is irreducible with finite determinant. Are these of geometric origin? In particular, is there a $\mathbb{Z}$VHS? Esnault-Groecherig show that the monodromy representation factors through $\mathrm{GL}_n(\mathcal{O}_K) \to \mathrm{GL}_n(\mathbb{C})$ for some $K \in \mathsf{Field}_{/\mathbb{Q}}$.

Idea: start from complex geometry, go to p-adic geometry, yields an overconvergent F-crystal. This yields integrality at p; use companions to go back to a lisse ℓ-adic sheaf, then back to $\mathbb{C}$ to get integrality at ℓ.

Remark 5.1.2: Going the other way, $\ell \to p$: one can prove "of geometric origin" results when $\operatorname{rank} \mathcal{E} = 2$ (Krishnmorthy-Pal). Idea: go from $\ell \to p$, make a candidate for the crystalline Dieudonne module for some family of AVs. One will have a bound on the motivic weight, which is at most $\operatorname{rank} \mathcal{E} - 1$.

A word on the proof: define a moduli stack M_n of mod p^n F-crystals, which is a horrendous algebraic stack. These are roughly coherent sheaves with extra data. Study some finite-type pieces using slops, and is universally closed since one can take flat limits along curves. Take the Zariski closure of companion points, then take stable images to get some M_n''.

Show that every point in each component of it is a companion point using horizontal companions (as opposed to vertical in the fiber direction). Then show each component maps isomorphically to S, which is a pointwise condition on S. This only uses the companion on the fiber, which is easier to study.

6 Alex Smith: Simple abelian varieties over finite fields with extreme point counts

Theorem 6.0.1 ***(Howe-Kedlaya).***
Given $n > 0$, there is an $A \in \mathsf{AbVar}_{/\mathbb{F}_2}$ with $\sharp A(\mathbb{F}_2) = n$.

Remark 6.0.2: Recall the Weil bounds: given $A_{/\mathbb{F}_q}$,

$$(q - 2q^{\frac{1}{2}} + 1)^g \leq \sharp A(\mathbb{F}_q) \leq (q + 2q^{\frac{1}{2}} + 1)^g.$$

Let $\{\alpha_i\}_{1\leq i\leq 2g}$ be the eigenvalues of Frob $\curvearrowright H^1_{\text{ét}}(A \underset{\mathbb{F}_q}{\times} \overline{\mathbb{F}}_q; \mathbb{Z}_\ell)$. Recall the Weil conjectures:

- All embeddings $\mathbb{Q}(\alpha_i) \hookrightarrow \mathbb{C}$ satisfy $|\alpha_i| = q^{\frac{1}{2}}$.
- Lefschetz trace: $\sharp A(\mathbb{F}_q) = \prod_{1\leq i\leq 2g} (\alpha_i - 1) = \prod (q + 1 - \alpha_i - \overline{\alpha_i})^{\frac{1}{2}}$ Note that these real numbers sit in $[-2q^{\frac{1}{2}}, 2q^{\frac{1}{2}}] \subseteq \mathbb{R}$.

Definition 6.0.3 (Totally Σ)
Given $\Sigma \subseteq \mathbb{C}$ we say α is **totally** Σ iff all conjugates of α are contained in Σ.

Remark 6.0.4: Remarkably for AVs, the α_i tell the entire story! Honda-Tate theory gives a correspondence

$$\{\text{Abelian varieties over } \mathbb{F}_q\} / \sim_{\text{isogeny}} \rightleftharpoons \{\text{Totally } I_q \text{ algebraic integers}\} / \sim_{\text{conjgacy}}$$

where $I = [-2\sqrt{q}, 2\sqrt{q}]$. Tate showed injectivity, Honda used CM theory to show surjectivity.

Theorem 6.0.5 ***(von Bomel-Costa-Li-Poonen-S.).***
Given any q and any $n \gg_q 0$, there is an $A \in \mathsf{AbVar}_{/\mathbb{F}_q}$ with $\sharp A(\mathbb{F}_q) = n$.

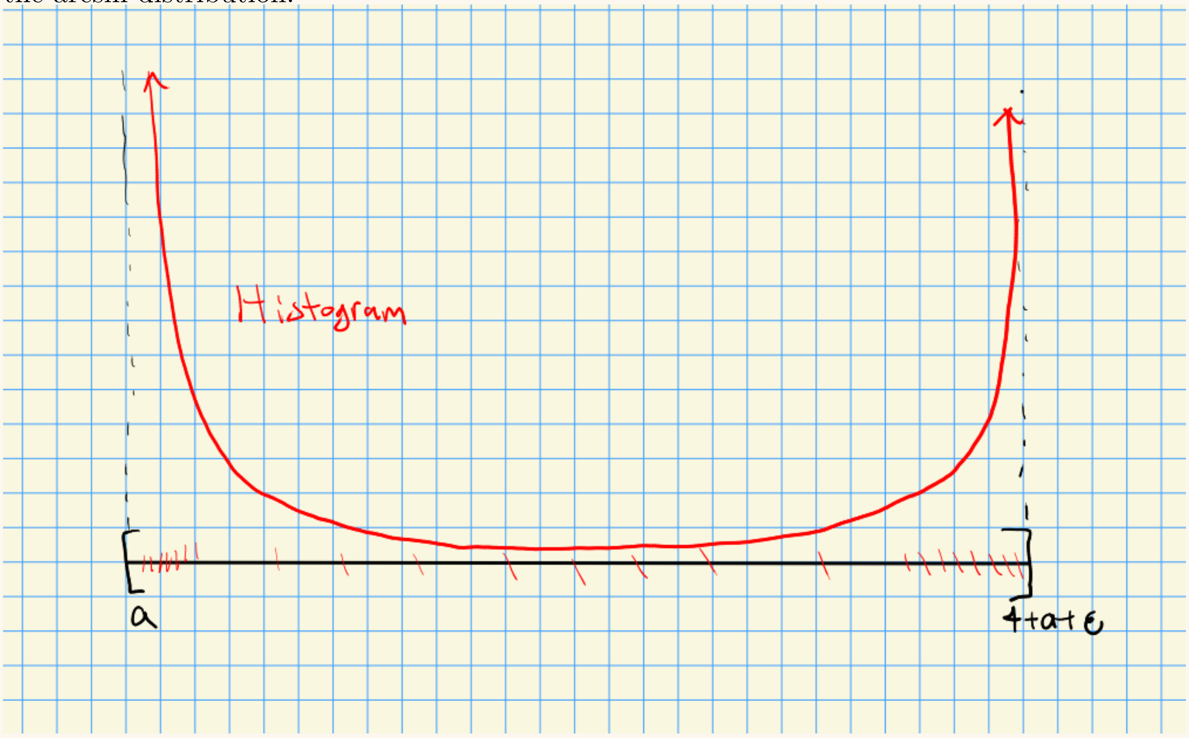
Remark 6.0.6: Idea: Howe-Kedlaya works infinitely often. One can't attain *every* integer in the Weil bound interval, but you can get pretty close:

Theorem 6.0.7 ***(?).***
Given $g \gg_q 0$ and

$$n \in [(q - 2\sqrt{q} + 3)^g, (q + 2\sqrt{q} - 1 - q^{-1})^g],$$

there exists A with $\sharp A(\mathbb{F}_q) = n$.

Proof (Sketch).
Given $\alpha \in \mathbb{R}$ and ε, one can produce totally I algebraic integers for $I = [\alpha, 4 + \alpha + \varepsilon]$ fitting the arcsin distribution:



For $\varepsilon = \alpha = 0$, choose a large cyclotomic polynomial, whose roots are roughly equidistributed in S^1, then map to $[-2, 2]$. Solving this for non-rational α and $\varepsilon = 0$ is a big open problem. ■

6.1 Schur-Segal-Smyth trace problem

Problem 6.1.1 (?)
Find the minimal t such that for any $\varepsilon > 0$, infinitely many totally positive algebraic integers satisfy

$$\mathrm{tr}(\alpha)/\deg(\alpha) < t + \varepsilon.$$

Remark 6.1.1: Idea: all conjugates greater than zero, how can you minimize the average trace? The cyclotomic method above infinitely many whose trace is at most 2. So using the arcsin distribution yields $t < 2$, open question: is $t = 2$? Progress has been slow and revolves around an old trick.

Proposition 6.1.2*(?)*.
$t \geq 1$.

Proof (?).
Take a totally positive algebraic integer with conjugates $\{\alpha_i\}_{i\leq n}$ with minimal polynomial p. Now apply AMGM:

$$1 \leq |p(0)| = \left(\prod a_i\right)^{\frac{n}{n}} \leq_{\text{AMGM}} \left(\frac{\sum a_i}{n}\right)^n.$$

■

Remark 6.1.3: We can do slightly better. If $|p(1)| \geq 1$, then $t \geq 1.05$. If $|p(a)p(\bar{a})| \geq 1$ for $a := \dfrac{3+\sqrt{5}}{2}$, then $t \geq 1.1$. More generally this is written as a resultant, i.e. $\text{res}(p, x^3 + 3x + 1)$.

Smyth shows that $t = \text{tr}(\alpha)/\deg(\alpha) \geq 1.771$ with 14 exceptions; Wang-Wu-Wu shows $t \geq 1.793$. Serre showed this argument can never show $t \geq 1.899$, Alex showed it can not show $t \geq 1.81$, so we're approaching the limit of Smyth's method.

Theorem 6.1.4 ***(Alex).***
Smyth's method limits to the right answer, and thus $t \leq 1.81$. In particular, $t \neq 2$.

Remark 6.1.5: Consequence: there are things that work better than the arcsin distribution.

Theorem 6.1.6 ***(?).***
Given sufficiently large square q, there are infinitely many $A_{/\mathbb{F}_q}$ with

$$\sharp A(\mathbb{F}_q) \geq (q + 2\sqrt{q} - 0.81)^{\dim A},$$

but only finitely many with

$$\sharp A(\mathbb{F}_q) \geq (q + 2\sqrt{q} - 0.8)^{\dim A}.$$

Definition 6.1.7 (?)
Given algebraic integers α with conjugates $\{\alpha_i\}_{i\leq n}$, let

$$\mu_\alpha = \frac{1}{n}\sum \delta_{\alpha_i}.$$

Remark 6.1.8: There is a weak-$*$ topology on the space of such measures.

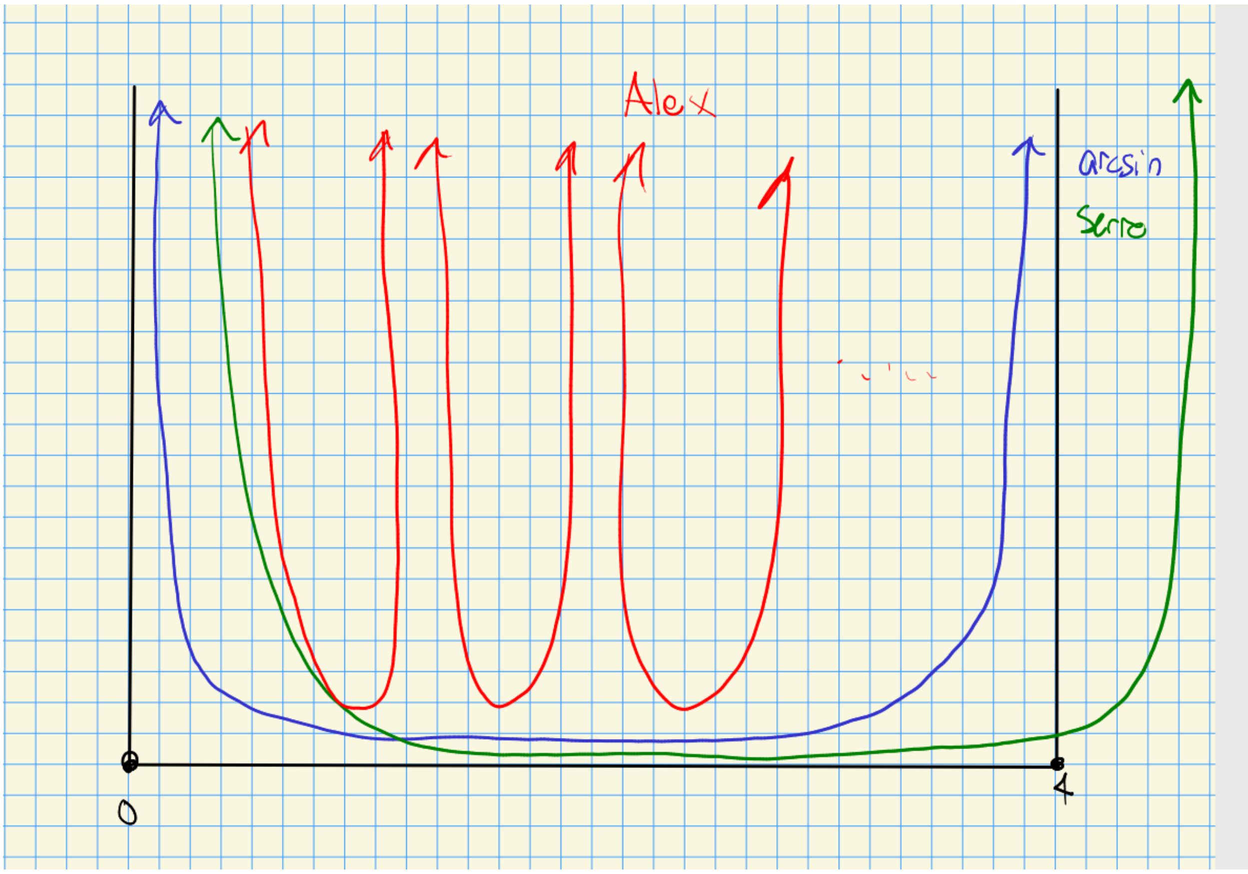
Theorem 6.1.9 ***(?).***
Choose $\Sigma \subseteq \mathbb{R}$ with countably many components (e.g. excluding Cantor sets) of *capacity* $c > 1$. TFAE are equivalent for a probability measure μ on Σ:

- There are totally Σ algebraic integers α_i whose distributions μ_{α_i} as above conver to μ.

- For any integer polynomial $Q \neq 0$,

$$\int_{\Sigma} \log |Q| \, d\mu \geq 0.$$

Remark 6.1.10: Idea of proof: apply Minkowski's 2nd theorem as a source of promising polynomials. Use an optimized distribution that avoids the 14 exceptions, whose average traces beat the previous averages:



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